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All the supersymmetric configurations of $N = 4, d = 4$ supergravity

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Abstract

All the supersymmetric configurations of pure, ungauged, $N = 4, d = 4$ supergravity are classified in a formalism that keeps manifest the S and T dualities of the theory. We also find simple equations that need to be satisfied by the configurations to be classical solutions of the theory. While the solutions associated to null Killing vectors were essentially classified by Tod (a classification that we refine), we find new configurations and solutions associated to timelike Killing vectors that do not satisfy Tod's rigidity hypothesis (hence, they have a non-trivial $U(1)$ connection) and whose supersymmetry projector is associated to 1-dimensional objects (strings), although they have a trivial axion field.

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1 Introduction and main results

Classical supersymmetric solutions of supergravity theories have played a very important role in many advances in string theory for the past 15 years and are still the subject of much interest since they include, for example, backgrounds (possibly with branes and fluxes) for string model-building and supersymmetric objects such as black holes, supertubes and, as it has been discovered recently in Ref. [1], black rings.

It is, thus, a very interesting problem to try to find or at least classify and characterize the supersymmetric solutions of (ideally all) supergravity theories. There have been many interesting results in the literature on this program, starting with the work of Gibbons and Hull in $N = 2, d = 4$ supergravity [2], completed in the seminal paper Ref. [3] by Tod, who, starting from the Killing spinor equations (KSEs) of that theory and using all the integrability conditions and properties derived from them, assuming the existence on one Killing spinor, was able to find, for the first time, all the field configurations (metric and vector field strength) for which the KSEs could be solved. His classification included field configurations which may or may not satisfy the classical equations of motion.

It was only 12 years later that a similar task was undertaken again by Tod, who in Ref. [4] studied the supersymmetric solutions of pure, ungauged, $N = 4, d = 4$ supergravity, achieving a complete classification of the degenerate case (in which the Killing spinor gives rise to a null Killing vector) and only a partial classification of the non-degenerate case (in which the Killing spinor gives rise to a timelike Killing vector), since he had to assume a hypothesis of internal rigidity that he could not prove. The internally rigid cases were very interesting, though, since, as shown in Ref. [5] they included all known the supersymmetric black-hole solutions of the theory, constructed by different methods and studied in Refs. [6]-[19]. By deformation of the supersymmetric black-hole solutions, the most general non-extremal black-hole solutions of the theory were constructed in Ref. [20].

The program enjoyed a revival when a new maximally supersymmetric solution of $N = 2B, d = 10$ supergravity was discovered in Ref. [21], analogous maximally supersymmetric solutions of 11-dimensional and $N = 2, d = 4$ supergravity [22, 23] were rediscovered and additional maximally supersymmetric solutions of the same kind were found in 5 and 6 dimensions in Ref. [24]. The classification of the maximally supersymmetric vacua of the 11- and 10-dimensional theories was completed in Refs. [25, 26]. It was then realized that we still had a very incomplete knowledge of the *landscape* of supersymmetric solutions of even the simplest supergravity theories and that new interesting supersymmetric solutions could be found by a systematic study of the solutions of the KSEs.

This was done in Ref. [27] for the minimal 5-dimensional supergravity, using a technique different from Tod's, who used the Newman-Penrose formalism. In this work, the KSEs were translated into a set of differential equations on all the tensors that could be constructed as bilinears of the Killing spinors, which can be managed by more standard techniques. Several of the new solutions found in this work have had a great impact: a new maximally supersymmetric solution of Gödel type and the supersymmetric black rings

[1, 28, 29][3] and generalizations that lead, for instance, to supersymmetric 4-dimensional rotating two- and one-black-hole solutions [31, 32, 33].

This work was generalized to minimal gauged 5-dimensional supergravity in Ref. [34] and then analogous results were obtained for minimal 6-dimensional supergravity in Refs. [35, 36] and for $N = 2, d = 4$ $U(1)$ gauged supergravity in Ref. [37]. There is also extensive work on the 11-dimensional and $N = 2A, B, d = 10$ supergravities (see e.g. Refs. [38]-[48]), although a complete classification is still lacking.

In this paper we return to the problem of finding all the supersymmetric configurations of $N = 4, d = 4$ supergravity, partially solved by Tod in Ref. [4]. We use tensor methods, based on the bilinears of complex chiral spinors with $SU(4)$ indices, which allows us to keep manifest the S and T dualities of the theory at all stages in our analysis and in the field configurations, as it happens in the solutions studied in Ref. [5]. The formalism used here can be used as starting point for the study of more complicated theories such as gauged and matter-coupled $N = 4, d = 4$ theories and there is work in progress in these directions.

We are going to describe our main results in a moment but, before, it is worth explaining why $N = 4, d = 4$ supergravity is an interesting theory from the string theory point of view. The toroidal compactification of the heterotic string effective action ($N = 1, d = 10$ supergravity coupled to 16 vector multiplets) gives ungauged $N = 4, d = 4$ supergravity coupled to 22 (matter) vector multiplets [50] and a consistent truncation of the matter vector multiplets gives the pure theory that we study here. Thus, all the solutions we will find are also solutions of the heterotic string effective action. The truncation preserves some of the $SO(6, 22; \mathbb{Z})$ T duality symmetry and the theory is invariant under the continuous group $SO(6) \sim SU(4)$ which naturally occurs as a *hidden symmetry* of the theory[4] [51]. The theory also has an S duality which manifest itself as a continuous $SL(2, \mathbb{R})$ *hidden symmetry*. It was this symmetry which lead to the S duality conjectures in the corresponding superstring theory [54]-[60]. We will also keep this symmetry manifest at all stages in our analysis.

Let us now describe our results for supersymmetric solutions, leaving the more general conditions for supersymmetric configurations which may or may not be solutions of the equations of motion.

There are two types of supersymmetric solutions in $N = 4, d = 4$ supergravity admitting at least one Killing spinor ϵ_I, that can be characterized by the causal nature of the vector bilinear $V^a = i\bar{\epsilon}^I\gamma^a\epsilon_I$, which is always a non-spacelike Killing vector.

Timelike V^a: Supersymmetric solutions are determined by a choice of 6 time-independent complex scalars M_{IJ} and a complex scalar τ that in general may depend on the spatial coordinates x, z, z^*. The M_{IJ}s have to satisfy two conditions:

1. Their matrix must have vanishing Pfaffian

[3] More general black ring solutions have also been found in Refs. [30]

[4] The first $N = 4, d = 4$ theory, constructed in Ref. [52] had only $SO(4)$ invariance. We will work with the $SU(4)$ theory of Ref. [53].

$$\varepsilon^{IJKL}M_{IJ}M_{KL} = 0\,. \tag{1.1}$$

2. They must be such that the 1-form ξ defined in Eq. (4.11) takes the form

$$\xi = \pm\tfrac{i}{2}(\partial_{\underline{z}}U dz - \partial_{\underline{z}^*}U dz^*) + \tfrac{1}{2}d\lambda\,, \tag{1.2}$$

for some real functions $U(z, z^*)$ and $\lambda(x, z, z^*)$[5]. Observe that it is the function U that makes ξ non-trivial.

τ and M_{IJ} must satisfy the 3-dimensional differential equations

$$\nabla_{\underline{i}}(e^{2i\lambda}A^{\underline{i}}) - e^{2i\lambda}[\partial_{\underline{z}}(e^{-2U})A_{\underline{z}^*} - \partial_{\underline{z}^*}(e^{-2U})A_{\underline{z}}] = 0\,, \tag{1.3}$$

both for

$$A = \frac{d\tau}{\Im\mathrm{m}\tau|M|^2}\,, \quad \text{and} \quad A = \frac{d[(\Im\mathrm{m}\tau)^{1/2}M^{IJ}]}{\Im\mathrm{m}\tau|M|^2}\,, \qquad |M|^2 = M^{IJ}M_{IJ}\,, \tag{1.4}$$

relative to the 3-dimensional metric

$$\gamma_{\underline{ij}}dx^{\underline{i}}dx^{\underline{j}} = dx^2 + 2e^{2U(z,z^*)}dzdz^*\,, \tag{1.5}$$

whose triviality is associated to that of the connection ξ. Then, the metric is given by

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}(dx^2 + 2e^{2U}dzdz^*)\,, \tag{1.6}$$

where $\omega = \omega_{\underline{i}}dx^i$ satisfies

$$f_{ij} = 4|M|^{-2}\epsilon_{ijk}\left(\xi_k - \frac{\partial_k\Re\mathrm{e}\tau}{4\Im\mathrm{m}\tau}\right)\,, \qquad f_{\underline{ij}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{1.7}$$

again relative to the above 3-dimensional metric and the vector field strengths are given by

$$F_{IJ} = \frac{1}{2|M|^2}\left\{\hat{V}\wedge dE_{IJ} - {}^\star\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dE_{IJ} - \frac{1}{\Im\mathrm{m}\,\tau}dB_{IJ}\right)\right]\right\}\,, \tag{1.8}$$

where

[5] A general *Ansatz* that satisfies these two conditions is given in Eq. (4.62).

$$\begin{aligned}
\hat{V} &= \sqrt{2}|M|^2(dt+\omega)\,,\\
E_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(M_{IJ}+\tilde{M}_{IJ})\,, \qquad (1.9)\\
B_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(\tau M_{IJ}+\tau^*\tilde{M}_{IJ})\,,
\end{aligned}$$

Examples of solutions corresponding to specific choices of M_{IJ} and τ are given in Section 4.4, but it is clear that there are two different kinds of solutions which differ by the triviality of the connection ξ and the 3-dimensional metric. The case in which ξ is trivial was completely solved by Tod in Ref. [4].

Null V^a This case (called *degenerate* by Tod) was essentially solved by Tod in Ref. [4], but we study it here again for the sake of completeness and to refine his results. There are two subcases which we call A and B and which are associated to $U(1)$ holonomy in a null direction and in a pair of spacelike directions, respectively, and describe *pp*-waves and the *stringy cosmic strings* of Ref. [63].

Case A: Each solution in this class is determined by 5 arbitrary functions of u: ϕ_I, τ. Given these functions, the metric and vector field strengths are given by

$$\begin{aligned}
ds^2 &= 2du[dv+K(u,z,z^*)du]-2dzdz^*\,,\\
F_{IJ} &= \tfrac{1}{2}(\mathcal{F}_{IJ}+\tfrac{1}{2}\varepsilon_{IJKL}\mathcal{F}^{KL})du\wedge dz^*\,,
\end{aligned} \qquad (1.10)$$

where

$$\begin{aligned}
\mathcal{F}_{IJ} &= \frac{8\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}}\dot{\phi}_{[I}\phi_{J]}\,,\\
2\partial_{\underline{z}}\partial_{\underline{z}^*}K &= \frac{|\dot{\tau}|^2}{(\Im\mathrm{m}\,\tau)^2}+\tfrac{1}{16}\Im\mathrm{m}\,\tau\,\mathcal{F}^2\,.
\end{aligned} \qquad (1.11)$$

Case B: These are well-known solutions determined by a choice of (in this case) antiholomorphic function $\tau=\tau(z^*)$. The vector field strengths vanish[6] and the metric takes the form

$$ds^2 = 2dudv-2e^{2U}dzdz^*\,, \qquad e^{2U}=\Im\mathrm{m}(\tau)\,. \qquad (1.12)$$

As for the unbroken supersymmetries of these solutions, they all preserve generically 1/4 of the supersymmetries. It is not easy to find generic conditions for the solutions to preserve 1/2 (although this has been studied in special cases, see Ref. [5]). As for maximally

[6]These solutions are given in Ref. [4] in different coordinates in which the metric functions have dependence on u, but this dependence can be eliminated.

supersymmetric solutions, we only expect Minkowski spacetime, since, otherwise, there would be another maximally supersymmetric solution of $N = 1, d = 10$ supergravity different from 10-dimensional Minkowski spacetime.

The rest of this paper is devoted to proof these results. In Section 2 we describe in detail pure, ungauged, $N = 4, d = 4$ supergravity. In Section 3 we define the problem and equations that we want to solve and find the first consistency conditions. To go on, one has to consider separately the timelike and null cases. This is done in Sections 4 and 5, respectively. Our conventions are described in Appendix A and Appendix B contains all the algebraic identities satisfied by the products of tensors constructed as bilinears of chiral spinors, derived by Fierzing.

2 Pure, ungauged, $N = 4, d = 4$ supergravity

The bosonic fields of $N = 4, d = 4$ supergravity multiplet are:

1. The Einstein metric $g_{\mu\nu}$.

2. The complex scalar τ that parametrizes an $SL(2,\mathbb{R})/U(1)$ coset space. In terms of its real and imaginary parts (the axion a and the dilaton ϕ) it is written $\tau = a + ie^{-\phi}$.

3. The 6 $U(1)$ vector fields whose complex combinations we label with an antisymmetric pair of $SU(4)$ indices $A_{IJ\,\mu}$, $I, J = 1, \cdots, 4$ and are subject to the reality constraint

$$A_{IJ\,\mu} = \tfrac{1}{2}\varepsilon_{IJKL}A^{KL}{}_{\mu}\,, \tag{2.1}$$

where we rise and lower all $SU(4)$ indices by complex conjugation: $A^{IJ}{}_{\mu} \equiv (A_{IJ\,\mu})^*$. Their field strengths are $F_{IJ} = dA_{IJ}$ and are subject to the same reality constraint.

The fermionic fields of this supermultiplet, which are always 4-component (complex) Weyl spinors, are

1. The 4 dilatini χ_I, which, with lower $SU(4)$ indices, have positive chirality.

2. The 4 gravitini $\psi_{I\,\mu}$ which, with lower $SU(4)$ indices, have negative chirality.

Complex conjugation raises the $SU(4)$ indices and reverses the chiralities.

There are two global (*hidden*) symmetries in the ungauged theory: $SU(4) \sim SO(6)$, associated to stringy T dualities [56] and $SL(2,\mathbb{R})$, which is associated to a stringy S duality [54]-[60] and leaves invariant the equations of motion but not the action. $SU(4)$ acts on all the fields in the obvious way:

$$\chi^{I\,\prime} = U^I{}_J\chi^J\,, \qquad \chi_I{}^{\prime} = \chi_J(U^\dagger)^J{}_I\,, \tag{2.2}$$

etc. The matrix $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ acts on τ via fractional-linear transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}\,. \tag{2.3}$$

An alternative, linear, description of the action of $\Lambda \in SL(2,\mathbb{R})$ on τ can be made using the symmetric $SL(2,\mathbb{R})$ matrix

$$\mathcal{M} \equiv \frac{1}{\Im\mathrm{m}\,\tau} \begin{pmatrix} |\tau|^2 & \Re\mathrm{e}\,\tau \\ \Re\mathrm{e}\,\tau & 1 \end{pmatrix}\,. \tag{2.4}$$

The fractional-linear transformations of τ are equivalent to the rule

$$\mathcal{M}' = \Lambda \mathcal{M} \Lambda^T\,. \tag{2.5}$$

Observe that the matrix $S \equiv i\sigma^2$ is invariant under $SL(2,\mathbb{R})$ transformations:

$$\Lambda S \Lambda^T = S\,. \tag{2.6}$$

The action of $\Lambda \in SL(2,\mathbb{R})$ on the vector fields is best described by defining the $SL(2,\mathbb{R})$-dual $\tilde{F}_{IJ}$ of the field strength by

$$\tilde{F}_{IJ} \equiv \tau F_{IJ}{}^{+} + \tau^* F_{IJ}{}^{-} = \Re\mathrm{e}\tau F_{IJ} - \Im\mathrm{m}\tau \,{}^{\star}F_{IJ}\,. \tag{2.7}$$

Then, the pair $\tilde{F}_{IJ}$, F_{IJ} transforms as an $SL(2,\mathbb{R})$ doublet, i.e.

$$\vec{F}_{IJ} \equiv \begin{pmatrix} \tilde{F}_{IJ} \\ F_{IJ} \end{pmatrix}\,, \qquad \vec{F}'_{IJ} = \Lambda \vec{F}_{IJ}\,. \tag{2.8}$$

This implies for $F_{IJ}{}^{\pm}$

$$F'_{IJ}{}^{+} = (c\tau + d) F_{IJ}{}^{+}\,, \qquad F'_{IJ}{}^{-} = (c\tau^* + d) F_{IJ}{}^{-}\,. \tag{2.9}$$

Defining the phase of $c\tau + d$ by

$$e^{2i\varphi} \equiv \frac{c\tau + d}{c\tau^* + d}\,, \tag{2.10}$$

we find that, under $SL(2,\mathbb{R})$ several fields and combinations of fields get a local $U(1)$ phase

$$\begin{array}{ll} \chi'_I = e^{-3i\varphi/2}\chi_I\,, & \psi'_{I\,\mu} = e^{i\varphi/2}\psi_{I\,\mu}\,, \\ \left(\dfrac{\partial_\mu \tau}{\Im\mathrm{m}\,\tau}\right)' = e^{-2i\varphi}\left(\dfrac{\partial_\mu \tau}{\Im\mathrm{m}\,\tau}\right)\,, & \left[(\Im\mathrm{m}\,\tau)^{1/2} F_{IJ}{}^{\pm}{}_{\mu\nu}\right]' = e^{\pm i\varphi}\left[(\Im\mathrm{m}\,\tau)^{1/2} F_{IJ}{}^{\pm}{}_{\mu\nu}\right]\,, \end{array} \tag{2.11}$$

corresponding to $U(1)$ charges $-3, 1, -4$ and ± 2 respectively. The combination

$$Q_\mu \equiv \tfrac{1}{4}\frac{\partial_\mu \Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}\,, \tag{2.12}$$

transforms as a $U(1)$ gauge field, $Q'_\mu = Q_\mu + \frac{1}{2}\partial_\mu\varphi$ and this allows us to define a $U(1)$-covariant derivative

$$\mathcal{D}_\mu = \nabla_\mu - iqQ_\mu\,, \tag{2.13}$$

acting on fields with $U(1)$ charge q. Complex conjugation reverses chirality and these $U(1)$ charges.

The action for the bosonic fields is

$$S = \int d^4x\sqrt{|g|}\left[R + \tfrac{1}{2}\frac{\partial_\mu\tau\,\partial^\mu\tau^*}{(\Im\mathrm{m}\,\tau)^2} - \tfrac{1}{16}\Im\mathrm{m}\,\tau F^{IJ\,\mu\nu}F_{IJ\,\mu\nu} - \tfrac{1}{16}\Re\mathrm{e}\,\tau F^{IJ\,\mu\nu\star}F_{IJ\,\mu\nu}\right]. \tag{2.14}$$

It is useful to introduce the following notation for the equations of motion of the bosonic fields:

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \qquad \mathcal{E} \equiv -\frac{2\Im\mathrm{m}\tau}{\sqrt{|g|}}\frac{\delta S}{\delta\tau}\,, \qquad \mathcal{E}^{IJ\,\mu} \equiv \frac{8}{\sqrt{|g|}}\frac{\delta S}{\delta A_{IJ\,\mu}}\,. \tag{2.15}$$

Then, the equations of motion take the form

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + \tfrac{1}{2}(\Im\mathrm{m}\,\tau)^{-2}[\partial_{(\mu}\tau\partial_{\nu)}\tau^* - \tfrac{1}{2}g_{\mu\nu}\partial_\rho\tau\partial^\rho\tau^*] - \tfrac{1}{4}\Im\mathrm{m}\,\tau F_{IJ}{}^+{}_\mu{}^\rho F^{IJ-}{}_{\nu\rho}\,, \tag{2.16}$$

$$\mathcal{E} = \mathcal{D}_\mu\left(\frac{\partial^\mu\tau^*}{\Im\mathrm{m}\,\tau}\right) - \tfrac{i}{8}\Im\mathrm{m}\,\tau F^{IJ\,+\,\rho\sigma}F_{IJ}{}^+{}_{\rho\sigma}\,, \tag{2.17}$$

$$\mathcal{E}^{IJ\,\mu} = \nabla_\nu{}^\star\tilde{F}^{IJ\,\nu\mu}\,. \tag{2.18}$$

The Maxwell equation $\mathcal{E}^{IJ\,\mu}$ transforms as an $SL(2,\mathbb{R})$ doublet together with the Bianchi identity which we denote for convenience $\mathcal{B}^{IJ\,\mu}$

$$\mathcal{B}^{IJ\,\mu} \equiv \nabla_\nu{}^\star F^{IJ\,\nu\mu}\,. \tag{2.19}$$

It is easy to see that the combinations

$$\frac{\mathcal{E}_{IJ}{}^\mu - \tau^*\mathcal{B}_{IJ}{}^\mu}{(\Im\mathrm{m}\,\tau)^{1/2}}\,, \qquad \frac{\mathcal{E}_{IJ}{}^\mu - \tau\mathcal{B}_{IJ}{}^\mu}{(\Im\mathrm{m}\,\tau)^{1/2}}\,, \tag{2.20}$$

have $U(1)$ charges $+2$ and -2, respectively. The equation of motion of the complex scalar $\mathcal{E}$ has $U(1)$ charge $+4$ and the Einstein equation is neutral.

For vanishing fermions, the supersymmetry transformation rules of the gravitini and dilatini, generated by 4 spinors ϵ_I of negative chirality and $U(1)$ charge $+1$, are

$$\delta_\epsilon \psi_{I\,\mu} \;=\; \mathcal{D}_\mu \epsilon_I - \tfrac{i}{2\sqrt{2}} (\Im\mathrm{m}\,\tau)^{1/2} F_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J\,, \tag{2.21}$$

$$\delta_\epsilon \chi_I \;=\; \tfrac{1}{2\sqrt{2}} \frac{\not{\partial}\tau}{\Im\mathrm{m}\,\tau}\epsilon_I - \tfrac{1}{8}(\Im\mathrm{m}\,\tau)^{1/2}\,\not{F}_{IJ}{}^{-}\epsilon^J\,. \tag{2.22}$$

We also need the supersymmetry transformation rules of the bosonic bosonic fields, which take the form

$$\delta_\epsilon e^a{}_\mu \;=\; -\tfrac{i}{4}(\bar{\epsilon}^I\gamma^a\psi_{I\,\mu} + \bar{\epsilon}_I\gamma^a\psi^I{}_\mu)\,, \tag{2.23}$$

$$\delta_\epsilon \tau \;=\; -\tfrac{i}{\sqrt{2}}\Im\mathrm{m}\tau\bar{\epsilon}^I\chi_I\,, \tag{2.24}$$

$$\delta_\epsilon A_{IJ\,\mu} \;=\; \frac{\sqrt{2}}{(\Im\mathrm{m}\tau)^{1/2}}\left[\bar{\epsilon}_{[I}\psi_{J]\,\mu} + \tfrac{i}{\sqrt{2}}\bar{\epsilon}_{[I}\gamma_\mu\chi_{J]} + \tfrac{1}{2}\epsilon_{IJKL}\left(\bar{\epsilon}^K\psi^L{}_\mu + \tfrac{i}{\sqrt{2}}\bar{\epsilon}^K\gamma_\mu\chi^L\right)\right]. \tag{2.25}$$

3 Supersymmetric configurations: general setup

Our goal is to find all the purely bosonic field configurations of $N = 4, d = 4$ supergravity $\{g_{\mu\nu}, A_{IJ\,\mu}, \tau, \psi_{I\,\mu} = 0, \chi_I = 0\}$ which are supersymmetric, i.e. invariant under, at least, one supersymmetry transformation generated by a supersymmetry parameter $\epsilon_I(x)$. Since the supersymmetry variations of the bosonic fields are odd in fermion fields, these transformations will always vanish, but the supersymmetry variations of the fermions, for vanishing fermions, Eqs. (2.21), may only vanish for special supersymmetry parameters $\epsilon_I(x)$ (*Killing spinors*) that solve the *Killing spinor equations* (KSEs)

$$\delta_\epsilon \psi_{I\,\mu} = \mathcal{D}_\mu \epsilon_I - \tfrac{i}{2\sqrt{2}} (\Im\mathrm{m}\,\tau)^{1/2} F_{IJ}{}^{+}{}_{\mu\nu}\gamma^\nu \epsilon^J \;=\; 0\,, \tag{3.1}$$

$$2\sqrt{2}\,\delta_\epsilon \chi_I = \frac{\not{\partial}\tau}{\Im\mathrm{m}\,\tau}\epsilon_I - \tfrac{1}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}\,\not{F}_{IJ}{}^{-}\epsilon^J \;=\; 0\,. \tag{3.2}$$

For a known bosonic field configuration these are, respectively differential and algebraic equations for the Killing spinor, which may or may not exist. We want to find precisely for which bosonic field configurations these equations do have at least one solution ϵ_I. Our procedure will consist in assuming the existence of such a solution and derive consistency conditions for the field configurations.

We shall be talking most of the time about supersymmetric *field configurations*. These may or may not be solutions of the classical equations of motion. There are several conceptual and practical advantages in doing so. First of all, we would like to emphasize the fact that supersymmetry does not imply by itself that the equations of motion are solved, although in general it considerably simplifies the task of solving them. Secondly, it is sometimes useful to consider that there are external sources for the fields, out of the regions in which we are solving the equations of motion. Including those regions with sources implies staying off-shell. Finally, the off-shell equations of motion of theories with gauge symmetries obey certain gauge identities. In theories with local supersymmetry and for field configurations admitting Killing spinors, the gauge identities are known as *Killing spinor identities* (KSIs) [61, 62] and can be used either to reduce the number of equations to be explicitly checked or, having at hands all the off-shell equations of motion of certain field configuration as we will, they can be used as a consistency check that it is a supersymmetric field configuration.

Since these identities are the first consistency conditions that can be derived from the KSEs, we are going to derive them in the next section. We are also going to see that they are related to the integrability conditions of the KSEs. then, in Section 3.2 we are going to explain the strategy that we will follow to find all the supersymmetric configurations.

3.1 Killing Spinor Identities (KSIs) and integrability conditions of the Killing spinor equations

Using the supersymmetry transformation rules of the bosonic fields Eqs. (2.23,2.24) and (2.25) we can derive relations between the (off-shell) equations of motion of the bosonic fields that are satisfied by any field configuration $\{e^a{}_\mu, A_{IJ\,\mu}, \tau\}$ admitting Killing spinors [61, 62]. These KSIs take, for this theory, the form

$$i\bar{\epsilon}^I\gamma^a\mathcal{E}_a{}^\mu + \frac{1}{\sqrt{2}(\Im\mathrm{m}\tau)^{1/2}}\bar{\epsilon}_J\mathcal{E}^{\mu JI} \;=\; 0\,, \tag{3.3}$$

$$\bar{\epsilon}^I\mathcal{E} + \frac{1}{\sqrt{2}(\Im\mathrm{m}\tau)^{1/2}}\bar{\epsilon}_J\, \not{\mathcal{E}}^{JI} \;=\; 0\,. \tag{3.4}$$

Observe that it is implicitly assumed that the Bianchi identities are identically satisfied, i.e.

$$\mathcal{B}_{IJ}{}^\mu = 0\,, \tag{3.5}$$

and, therefore, these identities are not $SL(2,\mathbb{R})$-covariant. We may have to take this point into account when comparing with the equations that we will actually find, but we can also find (with considerably more effort) the $SL(2,\mathbb{R})$-covariant relations between the equations of motion from the integrability conditions of the Killing spinor equations (3.1) and (3.2).

Thus, acting with $\mathcal{D}_\mu$ on the Eq. (3.1) using both Eq. (3.1) and Eq. (3.2) and antisymmetrizing on the vector indices we get

$$\begin{aligned}
\mathcal{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} &= \tfrac{1}{8}\frac{\partial_{[\mu}\tau\partial_{\nu]}\tau^*}{(\Im\mathrm{m}\,\tau)^2}\epsilon_I \\
&\quad -\tfrac{1}{8}\left\{R_{\mu\nu}{}^{ab}\delta_I{}^K - \Im\mathrm{m}\,\tau F_{IJ}{}^+{}_{[\mu}{}^a F^{KJ\,-}{}_{\nu]}{}^b\right\}\gamma_{ab}\epsilon_K \\
&\quad +\tfrac{1}{4\sqrt{2}}(\Im\mathrm{m}\,\tau)^{-1/2}\left\{F_{IJ}{}^+{}_{\rho[\nu}\partial_{\mu]}\tau - 2i\Im\mathrm{m}\,\tau\nabla_{[\mu|}F_{IJ}{}^+{}_{\rho|\nu]}\right\}\gamma^\rho\epsilon^J \\
&= 0\,.
\end{aligned} \tag{3.6}$$

To extract from this integrability condition a relation between the equations of motion we act with γ^ν from the left. We get

$$4\gamma^\nu\mathcal{D}_{[\mu}\delta_\epsilon\psi_{I\,\nu]} = (\mathcal{E}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{E}_\sigma{}^\sigma)\gamma^\nu\epsilon_I - \frac{i}{2\sqrt{2}(\Im\mathrm{m}\,\tau)^{1/2}}(\not{\mathcal{E}}_{IJ} - \tau^*\,\not{\mathcal{B}}_{IJ})\gamma_\mu\epsilon^J = 0\,. \tag{3.7}$$

Acting now with γ^μ and using the result to eliminate $\mathcal{E}_\sigma{}^\sigma$ we get, finally the $SL(2,\mathbb{R})$-covariantization of the KSIs Eq. (3.3)

$$\mathcal{E}^\mu{}_a\gamma^a\epsilon_I - \frac{i}{\sqrt{2}(\Im\mathrm{m}\,\tau)^{1/2}}(\mathcal{E}_{IJ}{}^\mu - \tau^*\mathcal{B}_{IJ}{}^\mu)\epsilon^J = 0\,. \tag{3.8}$$

Similarly, the $SL(2,\mathbb{R})$-covariantization of the KSIs Eq. (3.3) can be obtained by calculating $2\sqrt{2}\ \not{\mathcal{D}}\delta_\epsilon\chi_I = 0$ and takes the form

$$\mathcal{E}^*\epsilon_I - \frac{1}{\sqrt{2}(\Im\mathrm{m}\,\tau)^{1/2}}(\not{\mathcal{E}}_{IJ} - \tau\,\not{\mathcal{B}}_{IJ})\epsilon^J = 0\,. \tag{3.9}$$

These two identities are now manifestly $SL(2,\mathbb{R})$-covariant[7]. The comparison with our results will be easier if we multiply these equations by gamma matrices and conjugate spinors $\bar\epsilon_K$ and $\bar\epsilon^K$ from the left, to derive relations involving spinor bilinears. In the case in which the vector V^a is timelike, we get

$$\begin{aligned}
\mathcal{E}^{ab} - \tfrac{1}{2}\Im\mathrm{m}\,\mathcal{E}V^aV^b - \frac{1}{\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}\Im\mathrm{m}\,(M^{IJ}\mathcal{B}_{IJ}{}^a)V^b &= 0\,, &(3.10)\\
\mathcal{E}^*V^a - \frac{i}{\sqrt{2}(\Im\mathrm{m}\,\tau)^{1/2}}M^{IJ}(\mathcal{E}_{IJ}{}^a - \tau\mathcal{B}_{IJ}{}^a) &= 0\,, &(3.11)\\
\Im\mathrm{m}[M^{IJ}(\mathcal{E}_{IJ}{}^a - \tau^*\mathcal{B}_{IJ}{}^a)] &= 0\,. &(3.12)
\end{aligned}$$

Observe that the first equation implies the off-shell vanishing of all the Einstein equations with one or two spacelike components. Further, the Einstein equation is automatically satisfied when the Maxwell, Bianchi and complex scalar equations are satisfied.

[7] See the paragraph after Eq. (2.20).

When V^a is null (we denote it by l^a), all the spinors ϵ_I are proportional and we can use the parametrization of Eq. (B.26) in Eqs. (3.8) and (3.9). Contracting with ϕ^I using the normalization Eq. (B.27) and with the conjugate spinors $\bar{\epsilon}, \bar{\epsilon^*}, \bar{\eta}, \bar{\eta^*}$, where η is an auxiliary spinor with normalization Eq. (B.33), we arrive at the identities

$$(\mathcal{E}^\mu{}_a - \tfrac{1}{2}e_a{}^\mu \mathcal{E}^\rho{}_\rho)\, l^a = (\mathcal{E}^\mu{}_a - \tfrac{1}{2}e_a{}^\mu \mathcal{E}^\rho{}_\rho)\, m^a = 0\,, \tag{3.13}$$

$$\mathcal{E} = 0\,, \tag{3.14}$$

$$(\mathcal{E}_{IJ}{}^\mu - \tau^* \mathcal{B}_{IJ}{}^\mu)\phi^J = 0\,. \tag{3.15}$$

where the null complex vectors are defined in Eq. (B.34). Observe that in this case supersymmetry implies that the scalar equations of motion must be automatically satisfied.

3.2 Solving the Killing spinor equations

The procedure we will follow to find the field configurations for which the KSEs admit at least one solution will be the following:

1. In Section 3.3 we are going to reexpress the KSEs as differential and algebraic equations for the bilinears (scalars, vectors and 2-forms, see Appendix B) built with the Killing spinors. Solving the equations for all the bilinears is essentially equivalent to solving the KSEs.

2. In Section 3.4 we are going to find, among the bilinears, a Killing vector V^μ and decompose the vector field strengths w.r.t. to it computing $V^\rho F_{IJ}{}^+{}_{\mu\rho}$ or $V^\rho F_{IJ}{}^-{}_{\mu\rho}$ in terms of the scalar bilinears and τ and then using, Eqs. (A.16) if V is timelike and Eqs. (A.19) if V is null. These two cases have to be studied separately. One of the reasons is that, in the null case, the field strength is not completely determined by its contractions with V, but there are more differences that we are going to explain shortly and require a completely separate analysis.

3. In the timelike case, studied in Section 4 we will

   (a) Substitute the expressions of the field strengths in the algebraic KSEs ($\delta_\epsilon \chi_I = 0$) to check that it is completely solved.

   (b) Substitute into the equations of motion and we will check whether the KSIs Eqs. (3.10,3.11,3.12) are indeed satisfied or there are additional conditions to be imposed. This is done in two steps: first we substitute into the equations of motion of the vector fields and the complex scalar which we have already expressed in terms of the bilinears in Section 4.1 and then, after we specify the form of the metric in terms of the bilinears, we substitute into the Einstein equations in Section 4.2. Then we check the KSIs.

(c) Substitute, finally, into the differential KSEs ($\delta_\epsilon \psi_{I\,\mu} = 0$) to solve it finding additional conditions on the bilinears and the form of the Killing spinors in Section 4.3.

The timelike case will be completely solved by then and we will study some examples.

In the null case, which was completely solved by Tod,

(a) As explained in Appendix B all the spinors ϵ_I are proportional $\epsilon_I = \phi_I \epsilon$ and we use first this information in the KSEs to obtain separate equations for the coefficients ϕ_I and the spinor ϵ. This requires the introduction of a $U(1)$ connection ζ that covariantizes the equations with respect to (opposite) local changes of phase of ϕ_I and ϵ.

(b) All the vectors bilinears are also proportional to the Killing vector V^a which we rename here l^a. It is convenient to introduce an auxiliary spinor to build independent vector bilinears that constitute a null tetrad. The KSEs only give partial information about the derivatives of these vectors, except for l^a, which is built with ϵ and is always covariantly constant, the very definition of a *pp*-wave space [64, 65].

(c) Although the vector field strengths and the derivatives of the vector bilinears are not completely determined, it is possible to extract information constructing the equations of motion and imposing the KSI. In particular we find that the $U(1)$ connection ζ is trivial.

(d) There are two different cases to be considered (A and B) which are essentially solved by solving first the integrability constraints.

3.3 Killing equations for the bilinears

We start with the equations $\delta_\epsilon \chi_I = 0$. We just have to multiply the from the right with gamma matrices and Dirac conjugates of Killing spinors. We have, in particular, from $\bar{\epsilon}^K \delta_\epsilon \chi_I = 0$

$$V^K{}_I \cdot \partial\tau - \tfrac{i}{2\sqrt{2}} (\Im\mathrm{m}\,\tau)^{3/2} F_{IJ}{}^- \cdot \Phi^{KJ} = 0\,, \tag{3.16}$$

and, from $\bar{\epsilon}_K \gamma_\rho \delta_\epsilon \chi_I = 0$

$$F_{IJ}{}^-{}_{\rho\sigma} V^J{}_K{}^\sigma + \tfrac{i}{\sqrt{2}} (\Im\mathrm{m}\,\tau)^{-3/2} \left(M_{IK} \partial_\rho \tau - \Phi_{IK\,\rho}{}^\mu \partial_\mu \tau \right) = 0\,. \tag{3.17}$$

It is possible to derive more Killing equations for the bilinears from the dilatini supersymmetry rule, but it will not be necessary.

Let us turn to the gravitini supersymmetry rules. Now we apply $SL(2,\mathbb{R})$-covariant derivative on the bilinears and use $\delta_\epsilon \psi_{I\,\mu} = 0$ to reexpress $\mathcal{D}_\mu \epsilon_I$. We get

$$\mathcal{D}_\mu M_{IJ} = \tfrac{1}{\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2} F_{K[I|}{}^{+}{}_{\mu\nu} V^{K}{}_{|J]}{}^{\nu}\,, \tag{3.18}$$

$$\begin{aligned}\mathcal{D}_\mu V^{I}{}_{J\,\nu} = & -\tfrac{1}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}\left[M_{KJ}F^{KI\,-}{}_{\mu\nu} + M^{IK}F_{JK}{}^{+}{}_{\mu\nu}\right.\\ & \left.-\Phi_{KJ\,(\mu}{}^{\rho}F^{KI\,-}{}_{\nu)\rho} - \Phi^{IK}{}_{(\mu|}{}^{\rho}F_{KJ}{}^{+}{}_{|\nu)\rho}\right]\,,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\mathcal{D}_\mu \Phi_{IJ\,\nu\rho} = & -\tfrac{1}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}\left[2g_{\mu[\nu|}F_{KI}{}^{+}{}_{|\rho]\alpha}V^{K}{}_{J}{}^{\alpha} + 2F_{KI}{}^{+}{}_{\nu\rho}V^{K}{}_{J\,\mu}\right.\\ & \left.-3F_{KI}{}^{+}{}_{[\mu\nu|}V^{K}{}_{J\,|\rho]} + (I \leftrightarrow J)\right]\,.\end{aligned} \tag{3.20}$$

3.4 First consequences and general results

Contracting the free indices in Eqs. (3.19) and (3.16) it is immediate to see that $V^\mu \equiv V^{I}{}_{I}{}^{\mu}$ is a (non-spacelike, Eq. (B.15)) Killing vector and

$$V^\mu \partial_\mu \tau = 0\,. \tag{3.21}$$

It is also immediate to prove that

$$\nabla_\mu V^{I}{}_{J}{}^{\mu} = 0\,. \tag{3.22}$$

Let us now consider the implications of the reality constraint of the vector field strengths on the contraction $F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}$:

$$F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu} = \tfrac{1}{2}\varepsilon_{KIML}(F_{ML}{}^{-}{}_{\mu\nu})^{*}V^{K}{}_{J}{}^{\nu}\,. \tag{3.23}$$

Taking the $SU(4)$ dual in both sides of this equation and taking into account the reality properties of the vectors $V^{K}{}_{J}{}^{\nu}$, we get

$$\tfrac{1}{2}\varepsilon^{SRIJ}F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu} = -\tfrac{1}{2}\left[F_{SR}{}^{-}{}_{\mu\nu}V^{\nu} + 2F_{J[S|}{}^{-}{}_{\mu\nu}V^{J}{}_{|R]}{}^{\nu}\right]^{*}\,, \tag{3.24}$$

from which we get

$$F_{SR}{}^{-}{}_{\mu\nu}V^{\nu} = -2F_{J[S|}{}^{-}{}_{\mu\nu}V^{J}{}_{|R]}{}^{\nu} - \left[\varepsilon^{SRIJ}F_{KI}{}^{+}{}_{\mu\nu}V^{K}{}_{J}{}^{\nu}\right]^{*}\,. \tag{3.25}$$

The first and second terms in the r.h.s. of this equation can be rewritten in terms of scalars using the antisymmetric part of Eq. (3.17) and the complex conjugate of Eq. (3.18). We get, at last,

$$F_{SR}{}^{-}{}_{\mu\nu}V^{\nu} = -\frac{\sqrt{2}i}{(\Im\mathrm{m}\,\tau)^{3/2}}M_{SR}\partial_\mu\tau - \frac{\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}}\varepsilon_{SRIJ}\mathcal{D}_\mu M^{IJ}\,. \tag{3.26}$$

The complex conjugate of this equation gives us $F^{SR+}{}_{\mu\nu}V^{\nu}$ and, taking the $SU(4)$-dual we get $F_{IJ}{}^{+}{}_{\mu\nu}V^{\nu}$ etc.

From this equation, contracting the free index with V^{μ} and using Eq. (3.21) we get immediately

$$V^{\mu}\partial_{\mu}M_{IJ} = 0\,. \tag{3.27}$$

Now, the use that we make of this result and the subsequent analysis will depend on the causal nature if the non-spacelike vector V^{μ}. We must distinguish between two cases: the case in which it is timelike, which we consider in section 4 and the case in which it is null (and we rename it l^{μ}), which we consider in section 5.

4 The timelike case

If $V^2 = 2M^{IJ}M_{IJ} \equiv 2|M|^2 \neq 0$ we can use Eq. (3.26) to express $F_{IJ}{}^{-}$ entirely in terms of scalars, their derivatives, and V_{μ} using Eq. (A.16):

$$F_{SR}{}^{-} = -\frac{1}{\sqrt{2}|M|^2(\Im\mathrm{m}\,\tau)^{1/2}}\left\{\left[i\frac{M_{SR}}{(\Im\mathrm{m}\,\tau)}d\tau + \varepsilon_{SRIJ}\mathcal{D}M^{IJ}\right]\wedge\hat{V} - i\,{}^{\star}[\cdots]\right\}\,. \tag{4.1}$$

Here we have added a hat to V to denote the differential form $\hat{V} \equiv V_{\mu}dx^{\mu}$ and distinguish its norm.

It can be seen that this form of $F_{SR}{}^{-}$ satisfies identically all the Killing spinor equations $\delta_{\epsilon}\chi_I = 0$, that we can consider solved.

To solve the equations of motion it is convenient to have directly F_{IJ} and its $SL(2,\mathbb{R})$-dual $\tilde{F}_{IJ}$. Their expressions are, actually, somewhat simpler due to the following property: if $dF = 0$ (which is the equation satisfied by F_{IJ} and $\tilde{F}_{IJ}$) and $\pounds_V F = 0$ then $\nabla_{[\mu}(F_{\nu]\rho}V^{\rho}) = 0$ and, locally, $F_{\nu\rho}V^{\rho} = \nabla_{\nu}E$ for some scalar potential E. Thus, following Tod [4], we define

$$\nabla_{\mu}E_{IJ} \equiv V^{\nu}F_{IJ\,\nu\mu}\,, \qquad \nabla_{\mu}B_{IJ} \equiv V^{\nu}\tilde{F}_{IJ\,\nu\mu}\,, \tag{4.2}$$

and, using the above form of $F_{IJ}{}^{-}$ Eq. (4.1) we find

$$\begin{aligned} E_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(M_{IJ} + \tilde{M}_{IJ})\,, \\ B_{IJ} &= 2\sqrt{2}(\Im\mathrm{m}\,\tau)^{-1/2}(\tau M_{IJ} + \tau^{*}\tilde{M}_{IJ})\,, \end{aligned} \tag{4.3}$$

where

$$\tilde{F}_{IJ} = V^{-2}\left\{\hat{V}\wedge dB_{IJ} + {}^{\star}\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dB_{IJ} - \frac{|\tau|^2}{\Im\mathrm{m}\,\tau}dE_{IJ}\right)\right]\right\}\,, \tag{4.4}$$

$$F_{IJ} = V^{-2}\left\{\hat{V}\wedge dE_{IJ} - {}^{\star}\!\left[\hat{V}\wedge\left(\frac{\Re\mathrm{e}\,\tau}{\Im\mathrm{m}\,\tau}dE_{IJ} - \frac{1}{\Im\mathrm{m}\,\tau}dB_{IJ}\right)\right]\right\}\,. \tag{4.5}$$

It is worth spending a moment in checking the consistency of these results. By definition, B_{IJ} and E_{IJ} must transform under $SL(2,\mathbb{R})$ as $\tilde{F}_{IJ}$ and F_{IJ}, i.e. as a doublet:

$$\vec{E}_{IJ} \equiv \begin{pmatrix} B_{IJ} \\ E_{IJ} \end{pmatrix}, \qquad \vec{E}'_{IJ} = \Lambda \vec{E}_{IJ}\,. \tag{4.6}$$

We can check that this is consistent with Eqs. (4.4) and (4.5) by rewriting the last two equations in the manifestly $SL(2,\mathbb{R})$-covariant form

$$\vec{F}_{IJ} = V^{-2}\left\{\hat{V}\wedge d\vec{E}_{IJ} - {}^{\star}\!\left[\hat{V}\wedge(\mathcal{M}Sd\vec{E}_{IJ})\right]\right\}, \tag{4.7}$$

on account of Eqs. (2.4,2.5) and (2.6).

On the other hand, it is easy to check that the fact that $\vec{E}_{IJ}$ transforms as a doublet is consistent with the transformations rules of τ and M_{IJ} alone and Eqs. (4.3).

4.1 Vector and scalar equations of motion

Our next step consists in finding equations for M_{IJ} and τ from the equations of motion using the decompositions of $\tilde{F}_{IJ}$ and F_{IJ} Eqs. (4.4) and (4.5) in which these fields are written entirely in terms of those scalars and the Killing vector (1-form) V. In this process we are going to find derivatives of V, and we need to express these in terms of the scalars and V itself.

From Eq. (3.19) we find that V satisfies the equation

$$d\hat{V} = -\tfrac{1}{\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}[M^{IJ}F_{IJ}{}^{+} + M_{IJ}F^{IJ\,-}]\,. \tag{4.8}$$

Since

$$M^{IJ}F_{IJ}{}^{+} = -\frac{\sqrt{2}M^{IJ}}{(\Im\mathrm{m}\,\tau)^{1/2}|M|^{2}}[\mathcal{D}M_{IJ}\wedge\hat{V} + i{}^{\star}(\mathcal{D}M_{IJ}\wedge\hat{V})]\,, \tag{4.9}$$

we get

$$d\hat{V} = \frac{1}{|M|^{2}}\left\{d|M|^{2}\wedge\hat{V} + i\,{}^{\star}\!\left[(M^{IJ}\mathcal{D}M_{IJ} - M_{IJ}\mathcal{D}M^{IJ})\wedge\hat{V}\right]\right\}. \tag{4.10}$$

It is also convenient to define the 1-form ξ and the 2-form Ω

$$\xi \equiv \tfrac{i}{4}|M|^{-2}(M_{IJ}dM^{IJ} - M^{IJ}dM_{IJ})\,, \tag{4.11}$$

$$\Omega \equiv 2|M|^{-2\,\star}\!\left[(Q-\xi)\wedge\hat{V}\right]. \tag{4.12}$$

ξ transforms under $SL(2,\mathbb{R})$ as

$$\xi' = \xi + \tfrac{1}{2}d\varphi\,, \tag{4.13}$$

i.e. as the $U(1)$ connection Q, which makes Ω invariant. The connection ξ is also orthogonal to V and invariant under local rescalings of the scalar matrix M_{IJ}:

$$\xi(\Lambda(x)M_{IJ}) = \xi(M_{IJ})\,, \tag{4.14}$$

a property that we will exploit later on. Further, using Eq. (B.24) we can write the curvature of this connection in the form

$$d\xi = -\tfrac{i}{2}d\frac{M^{IJ}}{|M|}\wedge d\frac{M_{KL}}{|M|}[\delta_{IJ}{}^{KL} - \mathcal{J}^{K}{}_{[I}\mathcal{J}^{L}{}_{J]}]\,, \tag{4.15}$$

that relates the triviality of ξ with the constancy of the projection $\mathcal{J}^{I}{}_{J}$.

Finally, it is convenient to rewrite the equations of motion of the vector and scalar fields in differential-form language[8]:

$$\begin{aligned} \hat{\vec{\mathcal{E}}}^{IJ} &\equiv \vec{\mathcal{E}}^{IJ}{}_{\mu}dx^{\mu} = -{}^{\star}d\vec{F}^{IJ} = \begin{pmatrix} \hat{\mathcal{E}}^{IJ} \\ \hat{\mathcal{B}}^{IJ} \end{pmatrix}\,, && (4.16)\\ \hat{\mathcal{E}} &\equiv \mathcal{E}\hat{V}\,, && (4.17)\end{aligned}$$

where $\vec{\mathcal{E}}^{IJ}{}_{\mu}$ is the $SL(2,\mathbb{R})$ doublet formed by the Maxwell and Bianchi identities:

$$\vec{\mathcal{E}}^{IJ\,\mu} \equiv \begin{pmatrix} \mathcal{E}^{IJ\,\mu} \\ \mathcal{B}^{IJ\,\mu} \end{pmatrix} = \begin{pmatrix} \nabla_{\nu}{}^{\star}\tilde{F}^{IJ\,\nu\mu} \\ \nabla_{\nu}{}^{\star}F^{IJ\,\nu\mu} \end{pmatrix}\,. \tag{4.18}$$

Using the expressions that we have found for the Maxwell fields and their $SL(2,\mathbb{R})$ duals and using the above equation for dV rewritten in the form

$$d\hat{V} = \frac{d|M|^2}{|M|^2}\wedge\hat{V} + 2|M|^2\Omega\,, \tag{4.19}$$

we find the following two equations for M_{IJ} and τ:

$$\begin{aligned} {}^{\star}\hat{\vec{\mathcal{E}}}^{IJ} &= \tfrac{1}{2}d^{\star}\left[\frac{\mathcal{M}Sd\vec{E}_{IJ}}{|M|^2}\wedge\hat{V}\right] + d\vec{E}_{IJ}\wedge\Omega\,, && (4.20)\\ \frac{{}^{\star}\hat{\mathcal{E}}^{*}}{|M|^2} &= -\mathcal{D}^{\star}\left[\frac{d\tau}{|M|^2\Im\mathrm{m}\,\tau}\wedge\hat{V}\right] + 2i\frac{d\tau}{\Im\mathrm{m}\,\tau}\wedge\Omega + 2i\frac{\tilde{M}_{IJ}}{|M|^2}d^{\star}\left(\frac{dM^{IJ}}{|M|^2}\wedge\hat{V}\right)\,. && (4.21)\end{aligned}$$

These equations can be now be combined (this is the reason behind the introduction of V into the equation for τ and the use of differential forms) and simplified. Using the new variables N_{IJ} defined by

$$N_{IJ} = \sqrt{\Im\mathrm{m}\tau}M_{IJ}\,, \qquad |N|^2 = N^{IJ}N_{IJ} = \Im\mathrm{m}\tau|M|^2\,, \tag{4.22}$$

[8] We add hats to denote differential forms.

we construct a new combination of equations that we call $\hat{a}^{IJ}$

$$\hat{a}^{IJ} \equiv \frac{1}{2\sqrt{2}\Im\mathrm{m}\tau}(\tau\hat{\mathcal{B}}^{IJ} - \hat{\mathcal{E}}^{IJ}) - \tfrac{i}{2}\frac{(N^{IJ}+\tilde{N}^{IJ})}{|N|^2}\hat{\mathcal{E}}^*\,, \tag{4.23}$$

and, which, after some massaging, is going to have a much simpler form. To present in compact form the equations of motion we define these two equations

$$n^{IJ} \equiv (\nabla_\mu + 4i\xi_\mu)\left(\frac{\partial^\mu N^{IJ}}{|N|^2}\right)\,, \tag{4.24}$$

$$e^* \equiv (\nabla_\mu + 4i\xi_\mu)\left(\frac{\partial^\mu \tau}{|N|^2}\right)\,, \tag{4.25}$$

and, in terms of them, we have, switching again from differential form notation to tensor notation,

$$a^{IJ} = n^{IJ} - \frac{N^{IJ}+\tilde{N}^{IJ}}{|N|^2}\tilde{N}_{KL}n^{KL}\,, \tag{4.26}$$

$$\mathcal{B}^{IJ\,a} = \sqrt{2}V^a\left\{\frac{N^{IJ}+\tilde{N}^{IJ}}{|N|^2}\Re\mathrm{e}\,\mathcal{E} - i(a^{IJ}-\tilde{a}^{IJ})\right\}\,, \tag{4.27}$$

$$\mathcal{E}^{IJ\,a} = \sqrt{2}V^a\left\{\frac{N^{IJ}+\tilde{N}^{IJ}}{|N|^2}\Re\mathrm{e}\,(\tau\mathcal{E}) - i(\tau^* a^{IJ}-\tau\tilde{a}^{IJ})\right\}\,. \tag{4.28}$$

$$\mathcal{E} = |M|^2 e + 2i\tilde{N}^{KL}n_{KL}\,. \tag{4.29}$$

The combination $|N|^{-2}d\tau$ has $U(1)$ charge -4 and, thus, the second equation is just a $U(1)$-covariant divergence, the covariant derivative being constructed with the ξ connection. The first equation has a similar form and, although $\frac{dN_{IJ}}{|N|^2}$ does not transform covariantly under $SL(2,\mathbb{R})$, the equation is $SL(2,\mathbb{R})$-covariant up to terms proportional to the second equation.

4.2 Metric equations of motion

These are equations for the scalars M_{IJ} and τ and involve implicitly the spacetime metric, which is the only field not determined by them. We need to study now the Einstein equations and, to do it, it is convenient to choose coordinates adapted to the timelike Killing vector V. We define a time coordinate by

$$V^\mu\partial_\mu \equiv \sqrt{2}\partial_t\,, \tag{4.30}$$

and the metric takes the "conformastationary" form

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{i}\underline{j}}dx^i dx^j\,, \qquad i,j=1,2,3\,, \tag{4.31}$$

where $\omega = \omega_{\underline{i}}dx^i$ is a time-independent 1-form and $\gamma_{\underline{i}\underline{j}}$ is a time-independent (positive-definite!) metric on constant t hypersurfaces[9]. Since $|M|$ is in principle determined by the above equations, we only need to find equations for ω and γ. As usual, the equation for ω can be derived by comparing Eq. (4.19) for the 1-form $\hat{V}$, with the exterior derivative of the expression for $\hat{V}$ in the coordinates chosen

$$\hat{V} = \sqrt{2}|M|^2(dt+\omega)\,. \tag{4.32}$$

The result is the equation

$$d\omega = \frac{1}{\sqrt{2}}\Omega = \tfrac{i}{\sqrt{2}}|M|^{-4\,\star}\Big[(M^{IJ}\mathcal{D}M_{IJ} - M_{IJ}\mathcal{D}M^{IJ})\wedge\hat{V}\Big]\,. \tag{4.33}$$

Using the conformastationary metric we can reduce all the equations to equations in the 3 spatial dimensions with the metric γ. To start with, the equations n^{IJ} and e defined in Eqs. (4.24) and (4.25) can be expressed in terms of

$$n^{IJ}_{(3)} \;\equiv\; (\nabla_{\underline{i}} + 4i\xi_{\underline{i}})\left(\frac{\partial^{\underline{i}}N^{IJ}}{|N|^2}\right)\,, \tag{4.34}$$

$$e^{*}_{(3)} \;\equiv\; (\nabla_{\underline{i}} + 4i\xi_{\underline{i}})\left(\frac{\partial^{\underline{i}}\tau}{|N|^2}\right)\,, \tag{4.35}$$

where all the objects are now 3-dimensional with metric γ, by

$$n^{IJ} = -|M|^2 n^{IJ}_{(3)}\,, \qquad e = -|M|^2 e_{(3)}\,. \tag{4.36}$$

The equation (4.33) for the 1-form ω that enters the conformastationary metric reduces to

$$f_{ij} = 4|M|^{-2}\epsilon_{ijk}(\xi_k - Q_k)\,, \qquad f_{\underline{i}\underline{j}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{4.37}$$

Then, we can express all the equations of motion in terms of these two equations plus the equation[10]

$$e_{ij} \equiv R_{ij}(\gamma) - 2\partial_{(i}\left(\frac{N^{IJ}}{|N|}\right)\partial_{j)}\left(\frac{N_{KL}}{|N|}\right)(\delta^{KL}{}_{IJ} - \mathcal{J}^K{}_I\mathcal{J}^L{}_J)\,, \tag{4.38}$$

as follows:

[9] The components of the connection and curvature of this metric can be found in Appendix C.

[10] This equation should be compared with Eq. (4.15) in which the antisymmetric part of the same combination appears.

$$\mathcal{E}_{00} = |M|^2 \left[|M|^2 \Im\mathrm{m} e^*_{(3)} - 2\Re\mathrm{e}\,(N_{KL} n^{KL}_{(3)}) + \tfrac{1}{2} e_k{}^k\right] , \tag{4.39}$$

$$\mathcal{E}_{0i} = 0\,, \tag{4.40}$$

$$\mathcal{E}_{ij} = |M|^2 (e_{ij} - \tfrac{1}{2}\delta_{ij} e_k{}^k)\,, \tag{4.41}$$

$$\mathcal{B}^{IJ\,a} = -\sqrt{2}|M|^2 V^a \left\{ \frac{N^{IJ} + \tilde{N}^{IJ}}{\Im\mathrm{m}\tau} \Re\mathrm{e}\, e_{(3)} - i(n^{IJ}_{(3)} - \tilde{n}^{IJ}_{(3)}) \right\} , \tag{4.42}$$

$$\mathcal{E}^{IJ\,a} = -\sqrt{2}|M|^2 V^a \left\{ \frac{N^{IJ} + \tilde{N}^{IJ}}{\Im\mathrm{m}\tau} \Re\mathrm{e}\,(\tau e_{(3)}) - i(\tau^* n^{IJ}_{(3)} - \tau \tilde{n}^{IJ}_{(3)}) \right\} . \tag{4.43}$$

$$\mathcal{E} = -|M|^2 \left[|M|^2 e_{(3)} + 2i N_{KL} \tilde{n}^{KL}_{(3)}\right] . \tag{4.44}$$

We are now ready to check whether these equations satisfy the relations expressed in Eqs. (3.10-3.12). It is immediate to see that they do if the following conditions are satisfied off-shell:

$$e_{ij} = 0\,, \tag{4.45}$$

$$|M|^2 \Re\mathrm{e}\,(e_{(3)}) - 2\Im\mathrm{m}(N_{IJ} n^{IJ}_{(3)}) = 0\,. \tag{4.46}$$

The first equation determines the 3-dimensional matric γ as a function of the scalars N^{IJ} and says that γ is Ricci-flat is the projection $\mathcal{J}^I{}_J$ is constant. The second equation can be rewritten in the form

$$\nabla_{\underline{i}} \left(\frac{Q^{\underline{i}} - \xi^{\underline{i}}}{|M|^2} \right) = 0\,, \tag{4.47}$$

and is the integrability condition of Eq. (4.37) for the 1-form ω, whose existence we have assumed throughout all this analysis. Thus, it is not so much a necessary condition for supersymmetry as it is a necessary condition for the whole problem to be well defined.

Let us summarize the results of this section: we have seen that, in the timelike case at hands, field configurations with a metric of the form Eq. (4.31), vector field strengths of the form Eq. (4.1) and any complex scalar τ, and satisfying Eqs. (4.45) and (4.46) satisfy all the integrability conditions of the Killing spinor equations.

On the other hand, all the equations of motion, including the Bianchi identities, are satisfied if the equations

$$e^*_{(3)} = 0\,, \qquad n^{IJ}_{(3)} = 0\,, \qquad e_{ij} = 0\,, \tag{4.48}$$

(were $e^*_{(3)}$ and $n^{IJ}_{(3)}$ are defined in Eq. (4.35) and Eq. (4.34)) are satisfied, and automatically the integrability conditions are also satisfied.

We are now ready to check whether the Killing spinor equations always admit solutions for those field configurations. Thus will help us in solving the integrability conditions Eqs. (4.45) and (4.46).

4.3 Solving the Killing spinor equations

We have already checked that the equation $\delta_\epsilon \chi_I = 0$ is automatically solved by our field configurations, and we only have to check that the equations $\delta_\epsilon \psi_{a\,I} = 0$ can also be solved for them.

Let us consider the timelike component first. It can be put in this form:

$$|M|^{-1}\left\{\partial_t \epsilon_I - \tfrac{1}{2}M^{KL}\mathcal{D}_i M_{KL}\gamma_{0i}\left[\epsilon_I + i\sqrt{2}\gamma_0 \frac{M_{IJ}}{|M|}\epsilon^J\right] + \tfrac{i}{\sqrt{2}}|M|\tilde{\mathcal{J}}^K{}_I \mathcal{D}_i M_{KJ}\gamma_i \epsilon^J\right\} = 0\,. \tag{4.49}$$

Using the time-independent projector $\mathcal{J}^I{}_J$ we can split this equation into two equations:

$$\begin{aligned} \partial_t \epsilon_I - \tfrac{1}{2}M^{KL}\mathcal{D}_i M_{KL}\gamma_{0i}\left[\epsilon_I + i\sqrt{2}\gamma_0 \frac{M_{IJ}}{|M|}\epsilon^J\right] &= 0\,, && (4.50)\\ \tilde{\mathcal{J}}^K{}_I \mathcal{D}_i M_{KJ}\gamma_i \epsilon^J &= 0\,. && (4.51) \end{aligned}$$

The first equation is solved by a time independent spinor because

$$\epsilon_I + i\sqrt{2}\gamma_0 \frac{M_{IJ}}{|M|}\epsilon^J = 0\,, \tag{4.52}$$

due to the Fierz identity

$$M_{IJ}\epsilon^J = \tfrac{i}{2}V^a \gamma_a \epsilon_I\,, \tag{4.53}$$

and our choice of Vierbeins. For generic (i.e. not built from already-known Killing spinors) scalars M_{IJ} the above relation would be a constraint breaking 1/2 of the supersymmetries to be imposed on the Killing spinors whenever $M^{KL}\mathcal{D}_i M_{KL} \neq 0$. The counting of unbroken supersymmetries is, however, a bit more subtle and depends on the triviality of the $U(1)$ connection ξ: if ξ is a total derivative the projection $\mathcal{J}^I{}_J$ is constant and a global $SU(4)$ rotation suffices to set to zero two of the chiral Killing spinors. This is the procedure followed by Tod in Ref. [4], where he solved the constant $\mathcal{J}^I{}_J$ (*internally rigid*) case by setting to zero two of the spinors, breaking the explicit $SU(4)$ covariance of the solutions. The solutions found by Tod preserve, then, generically, 1/4 of the supersymmetries[11]. If $\mathcal{J}^I{}_J$ is not constant, ξ is non-trivial and the 4 Killing spinors cannot be related by global

[11] The conditions under which 1/2 of the supersymmetries are preserved were studied in Ref. [5].

$SU(4)$ rotations, but we are now going to see that this case can also be solved introducing a new projection on the Killing spinors which also reduces the amount of generically preserved supersymmetries to 1/4.

Now, using time-independence of the Killing spinors and Eq. (4.52), the spacelike components of $\delta_\epsilon \psi_{a\,I} = 0$ take the form

$$\left[\nabla_i - \tfrac{1}{2}\frac{M^{KL}\partial_i M_{KL}}{|M|^2}\right]\epsilon_I = 0\,, \tag{4.54}$$

which can be rewritten in the form

$$(\nabla_i - i\xi_i)(|M|^{-1/2}\epsilon_I) = 0\,. \tag{4.55}$$

The integrability condition for this equation is

$$[R_{ijkl}\gamma^{kl} + 4i(d\xi)_{ij}]\epsilon_I = 0\,. \tag{4.56}$$

This equation can be solved in essentially one way, up to local Lorentz transformations:

$$R_{12}{}^{12} = \pm 2(d\xi)_{12}\,, \qquad\qquad \tfrac{1}{\sqrt{2}}(1 \mp i\gamma_{12})\epsilon_I = 0\,, \tag{4.57}$$

the remaining components of the curvatures being zero. In terms of the connections we should have, in the appropriate Lorentz frame, the following relation between the 3-dimensional spin connection o^{ij} and the $U(1)$ connection ξ:

$$\xi = \pm\tfrac{1}{2}o^{12}(x^1, x^2) + \tfrac{1}{2}d\lambda(x^1, x^2, x^3)\,, \tag{4.58}$$

for some 3-dimensional 1-form ξ and some real scalar function λ. If complex scalars M^{IJ} and 3-dimensional metric $\gamma_{\underline{ij}}$ exist such that the above condition is met, then there are Killing spinors of the form

$$\epsilon_I = e^{\frac{i}{2}\lambda}|M|^{1/2}\epsilon_I\,, \qquad (\xi - \tfrac{1}{2}d\lambda)\tfrac{1}{\sqrt{2}}(1 \mp i\gamma_{12})\epsilon_I = 0\,. \tag{4.59}$$

The relation between the spin connection and the $U(1)$ connection is just the requirement that the 3-dimensional metric has $U(1)$ holonomy, which implies that it is reducible to the direct product of a 2- and a 1-dimensional metric and, thus, can always be written in the form

$$\gamma_{\underline{ij}}dx^{\underline{i}}dx^{\underline{j}} = dx^2 + 2e^{2U(z,z^*)}dzdz^*\,, \tag{4.60}$$

which, in turn, implies that ξ is given by

$$\xi = \pm\tfrac{i}{2}(\partial_{\underline{z}}U dz - \partial_{\underline{z}^*}U dz^*) + \tfrac{1}{2}d\lambda(x, z, z^*)\,. \tag{4.61}$$

Let us summarize the results of this section. We have found that, to construct a supersymmetric configuration (not necessarily a solution) of pure, ungauged, $N = 4, d = 4$ supergravity amounts, now, to

1. Find a set of time-independent complex scalars M^{IJ} satisfying $\varepsilon^{IJKL}M_{IJ}M_{KL} = 0$ such that the $U(1)$ connection ξ defined in Eq. (4.11) can be written in the form Eq. (4.61). The integrability condition Eq. (4.45) should automatically be solved by this choice.

2. Find τ by solving the integrability condition Eq. (4.47) of the defining equation of the 1-form ω (4.33).

If we want the supersymmetric configuration to be a solution of the equations of motion, we also need to impose Eqs. (4.34) and (4.35), but we do not need to check the integrability condition Eq. (4.47).

In the next section we study different solutions to these equations.

4.4 Supersymmetric configurations and solutions

According to the recipe of the previous section, our first step in finding supersymmetric configurations and solutions is to find the complex scalars M^{IJ} satisfying $\varepsilon^{IJKL}M_{IJ}M_{KL} = 0$ and such that ξ can be written in the form Eq. (4.61). The first condition can be easily met, for instance, by taking only M_{12}, M_{13} and M_{23} non-vanishing, but we prefer not to make any specific choice that would break $SU(4)$ covariance. The second condition can be solved by the following *Ansatz*

$$M_{IJ} = e^{i\lambda(x,z,z^*)}M(x,z,z^*)k_{IJ}(z)\,, \quad M = M^*\,, \quad \lambda = \lambda^*\,, \quad \varepsilon^{IJKL}k_{IJ}k_{KL} = 0\,, \quad (4.62)$$

which give a connection ξ of the form Eq. (4.61) with

$$U = +\ln|k|\,, \qquad |k|^2 \equiv k^{IJ}(z^*)k_{IJ}(z)\,, \quad (4.63)$$

and satisfies automatically the integrability condition Eq. (4.45).

Solving the integrability condition Eq. (4.47) is considerably more difficult and considering solutions (instead of general configurations) simplifies the problem. We have found three families of solutions.

1. If the k_{IJ} are constants, then, normalizing $|k|^2 = 1$ for simplicity, $\xi = \frac{1}{2}d\lambda$ and $U = 0$. This is the case considered by Tod in Ref. [4] and studied in detail in Ref. [5]. Tod took advantage of the fact that $d\xi = 0$ implies that $\mathcal{J}^I{}_J$ is constant and a global $SU(4)$ rotation can be used to set to zero two of the ϵ_Is. We will not do so, as this breaks the explicit $SU(4)$ covariance, but our results are, of course, equivalent.

   Eq. (4.34) takes the form

$$\partial_{\underline{i}}\partial_{\underline{i}}\mathcal{H}_1 = 0\,, \qquad \mathcal{H}_1 \equiv [(\Im\mathrm{m}\,\tau)^{1/2}e^{-i\lambda}M]^{-1}\,, \quad (4.64)$$

   and is solved by any arbitrary complex harmonic function $\mathcal{H}_1$.

Using the above equation, Eq. (4.35) takes the form

$$\partial_{\underline{i}}\partial_{\underline{i}}(\mathcal{H}_1\tau) = 0\,, \tag{4.65}$$

which is solved by

$$\tau = \mathcal{H}_1/\mathcal{H}_2\,, \qquad \partial_{\underline{i}}\partial_{\underline{i}}\mathcal{H}_2 = 0\,, \tag{4.66}$$

another arbitrary complex harmonic function. The pair of harmonic functions and the constants determine completely the solutions. In particular

$$|M|^{-2} = M^{-2} = \Im\mathrm{m}(\bar{\mathcal{H}}_2\mathcal{H}_1)\,. \tag{4.67}$$

2. If $e^{i\lambda} = M = 1$, the integrability condition Eq. (4.47) can be solved by taking τ constant. The only non-trivial equation of motion, Eq. (4.34) is solved using the holomorphicity of the k_{IJ}s. The metric takes the form

$$ds^2 = |k|^2(dt + \omega_{\underline{x}}dx)^2 - |k|^{-2}dx^2 - 2dzdz^*\,, \tag{4.68}$$

where $\omega_{\underline{x}}$ satisfies

$$\partial_{\underline{z}}\omega_{\underline{x}} - \partial_{\underline{x}}\omega_{\underline{z}} = \partial_{\underline{z}^*}|k|^{-2}\,, \quad \partial_{\underline{z}^*}\omega_{\underline{x}} - \partial_{\underline{x}}\omega_{\underline{z}^*} = \partial_{\underline{z}}|k|^{-2}\,, \quad \partial_{\underline{z}^*}\omega_{\underline{z}} - \partial_{\underline{z}}\omega_{\underline{z}^*} = 0\,. \tag{4.69}$$

The metric and the supersymmetry projectors indicate that these solutions describe stationary strings lying along the coordinate x, in spite of the trivial axion field, which is the dual of the Kalb-Ramond 2-form B that couples to strings. Observe, however, that the duality relation is not simply $dB = {}^\star da$: there are terms quadratic in the field strengths involved in the duality which must render B non-trivial.

The metric and the vector fields involved depends strongly on the choice of holomorphic k_{IJ}s. It is instructive to have an example completely worked out.

Let us consider the simplest case: only $k_{12} = \frac{1}{\sqrt{2}z}$ non-trivial. This allows us to set $\omega_{\underline{z}} = \omega_{\underline{z}^*} = 0$. Then, $|k|^2 = |z|^{-2}$ and $\omega_{\underline{x}} = 2\Re\mathrm{e}(z^2)$ and the full solution is given by

$$\begin{aligned} ds^2 &= \frac{1}{|z|^2}[dt + 2\Re\mathrm{e}(z^2)dx]^2 - |z|^2dx^2 - 2dzdz^*\,, \\ F_{12} &= -\frac{\sqrt{2}e^{\phi_0/2}}{z^2}\{[dt + 2\Re\mathrm{e}(z^2)dx]\wedge dz - i^\star[[dt + 2\Re\mathrm{e}(z^2)dx]\wedge dz]\} = (F_{34})^*\,, \\ \tau &= \tau_0\,. \end{aligned} \tag{4.70}$$

3. The only solutions that we have found with λ and the $k_{IJ}(z)$s simultaneously non-trivial have just $\lambda = \lambda(x)$ and $M = M(x)$ and are a superposition of the solutions with constant k_{IJ} and the solutions with constant λ in which these functions depend only on mutually transversal directions.

   Thus, these solutions depend on holomorphic functions $k_{IJ}(z)$ chosen with the same criteria as in the previous case, and a pair of complex functions $\mathcal{H}_1, \mathcal{H}_2$ linear in x such that $\Im\mathrm{m}\,\tau > 0$, and the metric is given by

$$ds^2 = (M|k|)^2(dt + \omega_{\underline{x}}dx) - (M|k|)^{-2}dx^2 - 2M^{-2}dzdz^*\,, \tag{4.71}$$

   where M is again given by Eq. (4.67).

5 The null case

As we have mentioned before, the null case was completely solved by Tod in Ref. [4], but we include it her for the sake of completeness.

As explained in Appendix B, in the null case all the spinors a proportional $\epsilon_I = \phi_I\epsilon$. In the $N = 4, d = 4$ case at hands, ϵ_I has a $U(1)$ charge under $SL(2,\mathbb{R})$ transformations that has to be distributed between ϕ_I and ϵ. We choose to have the ϕ_I uncharged. Had we chosen to have ϕ_I is charged with charge $q_\phi \neq 0$, then the real 1-form

$$\zeta \equiv i\phi_I d\phi^I\,, \tag{5.1}$$

would transform as a $U(1)$ connection under $SL(2,\mathbb{R})$ transformations as well and would play a role analogous to that of the connection ξ in the timelike case. With our choice, ζ is just a $U(1)$ connection under the transformations Eq. (B.28) and covariantizes with respect to them the expressions that involve ϵ.

We are now going to substitute $\epsilon_I = \phi_I\epsilon$ into the KSEs and we are going to use the normalization condition to split the KSEs into three algebraic and one differential equation for ϵ. One of the algebraic equations for ϵ will be a differential equation for ϕ_I.

The substitution yields immediately

$$\mathcal{D}_\mu\phi_I\epsilon + \phi_I\mathcal{D}_\mu\epsilon - \tfrac{i}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}F_{IJ}{}^{+}{}_{\mu\nu}\phi^J\gamma^\nu\epsilon^* \;=\; 0\,, \tag{5.2}$$

$$\phi_I\frac{\partial\!\!\!/\tau}{\Im\mathrm{m}\,\tau}\epsilon - \tfrac{1}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2}\;F\!\!\!/_{IJ}{}^{-}\phi^J\epsilon^* \;=\; 0\,. \tag{5.3}$$

Acting on Eq. (5.2) with ϕ^I leads to

$$\mathcal{D}_\mu\epsilon = -\phi^I\mathcal{D}_\mu\phi_I\epsilon\,, \tag{5.4}$$

which takes the form

$$\tilde{\mathcal{D}}_\mu \epsilon \equiv (\mathcal{D}_\mu + i\zeta_\mu)\epsilon = 0\,, \tag{5.5}$$

and becomes the only differential equation for ϵ. We have defined the derivative $\tilde{\mathcal{D}}$ covariant with respect to $SL(2,\mathbb{R})$ and $U(1)$ local rotations under which ϵ and ϕ_I have charges $+1$ and -1, respectively. Using Eq. (5.5) into Eq. (5.2) to eliminate $\mathcal{D}_\mu \epsilon$ we obtain

$$\tilde{\not{\mathcal{D}}}\phi_I \epsilon - \tfrac{i}{2\sqrt{2}}(\Im\mathrm{m}\,\tau)^{1/2} F_{IJ}{}^{+}{}_{\mu\nu}\phi^J \gamma^\nu \epsilon^* = 0\,, \tag{5.6}$$

which is one of the algebraic constraints for ϵ and is a differential equation for ϕ_I.

Acting with ϕ^I on Eq. (5.3) we see that it splits into two algebraic constraints for ϵ:

$$\not{\partial}\tau\epsilon \;=\; 0\,, \tag{5.7}$$

$$\not{F}_{IJ}{}^{-}\phi^J \epsilon^* \;=\; 0\,. \tag{5.8}$$

Finally, we add to the system an auxiliary spinor η, introduced in Appendix B, with charges opposite to those of ϵ. The normalization condition Eq. (B.33) will be preserved if and only if η satisfies a differential equation of the form

$$\tilde{D}_\mu \eta + a_\mu \epsilon = 0\,, \tag{5.9}$$

where a_μ is, in principle, an arbitrary vector with the right charges that transforms under the redefinitions Eqs. (B.36) and (B.37) as a connection

$$a'_\mu = a_\mu + \partial_\mu \delta\,. \tag{5.10}$$

In practice, however, a_μ cannot be completely arbitrary since the integrability conditions of the differential equation of η have to be compatible with those of the differential equation for ϵ and this requirement will determine a_μ.

Before we start a systematic analysis of these equations, it is worth comparing Eq. (5.5) to Eq. (4.55) and their integrability conditions which have the same structure except for the important detail of the dimensionality and signature. Therefore, we expect two main types of solutions: configurations with $U(1)$ holonomy on a 2-dimensional (spacelike) subspace and configurations with $U(1)$ holonomy in a null direction, which is the new possibility allowed by the Lorentzian signature. These expectations are also supported by the Fierz identities

$$\not{m}\epsilon \;=\; -i\epsilon\,, \tag{5.11}$$

$$\not{l}\epsilon^* \;=\; 0\,, \tag{5.12}$$

which are satisfied automatically here, but will be interpreted as projections.

We will call these two possibilities B and A respectively.

5.1 Killing equations for the vector bilinears and first consequences

We are now ready to derive equations involving the bilinears, in particular the vector bilinears which we construct with ϵ and the auxiliary spinor η introduced in Appendix B. First we deal with the equations that do not involve derivative of the spinors. Acting with $\bar{\epsilon}$ on Eq. (5.6) and with $\bar{\epsilon^*}\gamma^\mu$ on the complex conjugate of Eq. (5.8) we get

$$\phi^I F_{IJ}{}^{+}{}_{\mu\nu} l^\nu \;=\; 0\,, \tag{5.13}$$

$$\epsilon^{IJKL}\phi_J F_{KL}{}^{+}{}_{\mu\nu} l^\nu \;=\; 0\,. \tag{5.14}$$

Acting with $\bar{\epsilon^*}$ and $\bar{\eta^*}$ on Eq. (5.7) we get[12]

$$l\cdot\partial\tau \;=\; 0\,, \tag{5.15}$$

$$m^*\cdot\partial\tau \;=\; 0\,. \tag{5.16}$$

Now, from Eqs. (5.5) and (5.9) we find

$$\nabla_\mu l_\nu \;=\; 0\,, \tag{5.17}$$

$$\tilde{\mathcal{D}}_\mu n_\nu \;=\; -a^*_\mu m_\nu - a_\mu m^*_\nu\,, \tag{5.18}$$

$$\tilde{\mathcal{D}}_\mu m_\nu \;=\; -a_\mu l_\nu\,. \tag{5.19}$$

Let us now find the simplest implications of these equations.
To start with,Eqs. (5.13) and (5.14), together, imply for nonvanishing ϕ_I[13]

$$F_{IJ}{}^{+}{}_{\mu\nu} l^\nu = 0\,. \tag{5.20}$$

Using Eq. (A.19), we see that the vector field strengths must take the form

$$F_{IJ}{}^{+} \;=\; \tfrac{1}{2}\mathcal{F}_{IJ}\, l\wedge m^*\,, \tag{5.21}$$

$$F_{IJ}{}^{-} \;=\; \tfrac{1}{2}\tilde{\mathcal{F}}_{IJ}\, l\wedge m\,, \tag{5.22}$$

[12]The first of these equations had already been obtained in the general case Eq. (3.21).
[13]This equation also follows from the general result Eq. (4.1) for vanishing scalars M_{IJ}.

where $\mathcal{F}_{IJ}$ is a skew-symmetric $SU(4)$ matrix of scalars to be determined and $\tilde{\mathcal{F}}_{IJ}$ is its $SU(4)$ dual.

This solves completely Eq. (5.8), as can be seen using the Fierz identity

$$l_\mu \gamma^{\mu\nu} \epsilon^* = 3 l^\nu \epsilon^* \,, \tag{5.23}$$

and we can substitute Eq. (5.21) into Eq. (5.6) the only remaining equation in which vector field strengths occur. Using the Fierz identities

$$\not{l}\epsilon^* \;=\; 0\,, \tag{5.24}$$

$$\not{m}^*\epsilon^* \;=\; -i\epsilon\,, \tag{5.25}$$

it takes the form

$$\tilde{\mathcal{D}}_\mu \phi_I - \tfrac{1}{4\sqrt{2}} (\Im\mathrm{m}\,\tau)^{1/2} \mathcal{F}_{IJ} \phi^J l_\mu = 0\,, \tag{5.26}$$

from which we find

$$\mathcal{F}_{IJ}\phi^J = \frac{4\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}} n^\mu \tilde{\mathcal{D}}_\mu \phi_I\,. \tag{5.27}$$

On the other hand, from Eqs. (5.15) and (5.16) we find that

$$d\tau = A\hat{l} + B\hat{m}^*\,. \tag{5.28}$$

There are two cases to be considered here: case A ($B = 0$) and case B ($B \neq 0$). In case B, we can write

$$d\tau = B\left(\hat{m}^* + \frac{A}{B}\,\hat{l}\right) = B\hat{m}^{*\prime}\,, \tag{5.29}$$

after a redefinition of the type Eqs. (B.36) and (B.37). All the equations that we have written so far are covariant with respect to this kind of transformations and we just have to add primes (which we suppress immediately afterwards) everywhere. Thus, the case B is equivalent to $A = 0$ and we can always assume that either A or B is always zero. Since the connection Q depends on τ, the holonomy is different in these two cases. These are the two cases we mentioned at the end of the previous section and we will deal with them separately afterwards.

5.2 Equations of motion and integrability constraints

Although we have not yet discussed the form of the metric, we already have enough information to study the equations of motion and check whether they satisfy the integrability conditions Eqs. (3.13)-(3.15).

Using the results of the previous section, we can write the equations of motion in the form[14]

$$\begin{aligned}
\mathcal{E}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{E}^{\rho}{}_{\rho} &= R_{\mu\nu} + \left[\frac{|A|^2}{2(\Im\mathrm{m}\,\tau)^2} + \tfrac{1}{16} \Im\mathrm{m}\,\tau\, \mathcal{F}^2 \right] l_\mu l_\nu + \frac{|B|^2}{2(\Im\mathrm{m}\,\tau)^2} m_{(\mu} m^*_{\nu)} \,, && (5.30) \\
\mathcal{E} &= \frac{1}{\Im\mathrm{m}\,\tau} \left[l^\mu \partial_\mu A^* - B^* l^\mu a_\mu + m^\mu \partial_\mu B^* + \tfrac{i}{4} \frac{|B|^2}{\Im\mathrm{m}\,\tau} \right] , && (5.31) \\
\hat{\mathcal{E}}_{IJ} - \tau^* \hat{\mathcal{B}}_{IJ} &= -i (\Im\mathrm{m}\,\tau)\, d(\mathcal{F}_{IJ} \hat{l} \wedge \hat{m}^*) \,. && (5.32)
\end{aligned}$$

Substituting into Eqs. (3.13)-(3.15) and operating, we get

$$\begin{aligned}
R_{\mu\nu} l^\nu &= 0 \,, && (5.33) \\
R_{\mu\nu} m^\nu - \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2} m_\mu &= 0 \,, && (5.34) \\
l^\mu \partial_\mu A^* - B^* l^\mu a_\mu + m^\mu \tilde{\mathcal{D}}_\mu B^* + \tfrac{i}{4} \frac{|B|^2}{\Im\mathrm{m}\,\tau} &= 0 \,, && (5.35) \\
B^* \mathcal{F}_{IJ} \phi^J &= 0 \,. && (5.36)
\end{aligned}$$

We do not have a metric yet, but we can find $R_{\mu\nu} l^\nu$ and $R_{\mu\nu} m^\nu$ from the integrability conditions of Eqs. (5.5) and (5.9). Commuting the derivative and projecting with gamma matrices and spinors in the usual way, it is easy to find from Eq. (5.5)

$$\begin{aligned}
R_{\mu\nu} l^\nu &= -2i (d\zeta)_{\mu\nu} l^\nu \,, && (5.37) \\
R_{\mu\nu} m^\nu &= +2i (d\zeta)_{\mu\nu} m^\nu - 2i (dQ)_{\mu\nu} m^\nu \\
&= +2i (d\zeta)_{\mu\nu} m^\nu + \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2} m_\mu \,, && (5.38)
\end{aligned}$$

and from Eq. (5.9)

$$R_{\mu\nu} m^\nu = 2i (d\zeta)_{\mu\nu} m^\nu - 2i (dQ)_{\mu\nu} m^\nu - 2 (da)_{\mu\nu} l^\nu$$

[14] We have ignored all the terms that contain products AB etc.

$$= +2i(d\zeta)_{\mu\nu}m^{\nu} + \frac{|B|^2}{4(\Im\mathrm{m}\,\tau)^2}m_{\mu} - 2(da)_{\mu\nu}l^{\nu}\,, \tag{5.39}$$

$$R_{\mu\nu}n^{\nu} = 2i(d\zeta)_{\mu\nu}n^{\nu} - 2i(dQ)_{\mu\nu}n^{\nu} - 2(da)_{\mu\nu}m^{*\,\nu}$$

$$= 2i(d\zeta)_{\mu\nu}n^{\nu} - 2(da)_{\mu\nu}m^{*\,\nu}\,. \tag{5.40}$$

Comparing now these three sets of equations, we get

$$(d\zeta)_{\mu\nu}l^{\nu} = (d\zeta)_{\mu\nu}m^{\nu} = 0\,, \quad \Rightarrow d\zeta = 0\,, \quad \Rightarrow \zeta = d\alpha\,, \tag{5.41}$$

locally, and, eliminating ζ by a local phase redefinition,

$$(da)_{\mu\nu}l^{\nu} = 0\,, \tag{5.42}$$

$$(da)_{\mu\nu}m^{*\,\nu} = -\tfrac{1}{2}R_{\mu\nu}n^{\nu}\,, \tag{5.43}$$

which tell us that

$$da = -\tfrac{1}{2}R_{z^*u}\hat{m}\wedge\hat{m}^* + \tfrac{1}{2}R_{uu}\hat{l}\wedge\hat{m} + C\hat{l}\wedge\hat{m}^*\,, \tag{5.44}$$

where C is a function to be chosen so as to make this equation (and, hence, Eq. (5.9)) integrable.

Once ζ has been eliminated, we can solve Eq. (5.27) of $\mathcal{F}_{IJ}$ as follows:

$$\mathcal{F}_{IJ} = \frac{8\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}}n^{\mu}(\partial_{\mu}\phi_{[I})\phi_{J]}\,. \tag{5.45}$$

5.3 Metric

At this point we need information about the exact form of the metric. The most important piece of information comes from the covariant constancy of the null vector l^{μ}. Metrics admitting a covariantly constant null vector are known as *pp*-wave metrics and were first described by Brinkmann in Refs. [64, 65]. Since l^{μ} is a Killing vector and $d\hat{l} = 0$ we can introduce the coordinates u and v

$$l_{\mu}dx^{\mu} \equiv du\,, \tag{5.46}$$

$$l^{\mu}\partial_{\mu} \equiv \frac{\partial}{\partial v}\,. \tag{5.47}$$

The previous results imply that all the objects we are dealing with $(\tau, \phi_I, \mathcal{F}_{IJ})$ are independent of v.

Using these coordinates, a 4-dimensional *pp*-wave metric takes the form[15]

$$ds^2 = 2du(dv + Kdu + \omega) - 2e^{2U}dzdz^*\,, \qquad \omega = \omega_{\underline{z}}dz + \omega_{\underline{z}^*}dz^*\,, \tag{5.48}$$

where all the functions in the metric are independent of v and where either K or the 1-form ω could, in principle, be removed by a coordinate transformation. In this case, however, we have to be very careful because we have already used part of the freedom we had to redefine the spinors, and, therefore, the null tetrad, and we have to check that the tetrad integrability equations (5.17)-(5.19) are satisfied by our choices of e^U, K and ω.

We are now ready to study and solve each case separately.

5.4 Case A

This is the $B = 0$ case. $d\tau = A\hat{l}$ implies that $\tau = \tau(u)$ and $A = \dot{\tau}$. The connection Q can be integrated

$$Q = d\beta(u)\,, \tag{5.49}$$

and can be eliminated from all the equations by absorbing a phase into the spinors:

$$e^{-i\beta}\epsilon = \epsilon'\,, \qquad e^{i\beta}\eta = \eta'\,, \tag{5.50}$$

and similarly on the null tetrad.

To fix the form of the metric, we study the antisymmetric part of Eq. (5.19)

$$d\hat{m} + \hat{a}\wedge\hat{l} = dU\wedge\hat{m} + \hat{a}\wedge\hat{l} = 0\,, \tag{5.51}$$

which implies that U only depends on u and

$$\hat{a} = \dot{U}\hat{m} + D\hat{l}\,, \tag{5.52}$$

where D is a function to be found. Substituting into the antisymmetric part of Eq. (5.18) we find

$$d\hat{n} + \hat{a}^*\wedge\hat{m} + \hat{a}\wedge\hat{m}^* = d\hat{n} + D^*\hat{l}\wedge\hat{m} + D\hat{l}\wedge\hat{m}^* = 0\,, \tag{5.53}$$

which is solved by

$$n = dv + Kdu\,, \qquad D^* = e^{-U}\partial_{\underline{z}}K\,. \tag{5.54}$$

Now, comparing Eq. (5.52) with Eq. (5.44) we find that $R_{uz} = 0$ which implies (since $\omega = 0$) that $\dot{U} = 0$.

Finally, to ensure supersymmetry, the integrability conditions Eqs. (5.33)-(5.36) have to be satisfied, and, with constant U all of them are automatically satisfied.

[15]The components of the connection and the Ricci tensor of this metric can be found in Appendix D.

It also follows from the previous equations that the ϕ_Is can only depend on u and $\mathcal{F}_{IJ}$ is given by

$$\mathcal{F}_{IJ} = \frac{8\sqrt{2}}{(\Im\mathrm{m}\,\tau)^{1/2}}\dot{\phi}_{[I}\phi_{J]}\,. \tag{5.55}$$

Now, let us consider the equations of motion. The scalar, Maxwell and Bianchi equations are automatically satisfied and the Einstein equation can be solved by a K satisfying

$$2\partial_{\underline{z}}\partial_{\underline{z}^*}K = \frac{|\dot{\tau}|^2}{(\Im\mathrm{m}\,\tau)^2} + \tfrac{1}{16}\Im\mathrm{m}\,\tau\,\mathcal{F}^2\,. \tag{5.56}$$

These solutions preserve generically 1/4 of the supersymmetries.

5.5 Case B

This is the $A = 0$ case. If we choose $m^* = e^U dz^*$, then $d\tau = Bm^*$ implies $\tau = \tau(z^*)$ and $Be^U = \partial_{\underline{z}^*}\tau$. Substituting the corresponding connection 1-form Q into Eq. (5.19) one finds

$$B^* = \frac{g(z,u)}{(\Im\mathrm{m}\,\tau)^{1/2}}\,, \tag{5.57}$$

$$\hat{a} = -\partial_{\underline{u}}\ln g\,\hat{m} + D\hat{l}\,, \tag{5.58}$$

where g is a holomorphic function of z and D is a function to be determined. The first of these relations tells us that

$$\partial_{\underline{z}}\tau^* = \frac{e^U}{(\Im\mathrm{m}\,\tau)^{1/2}}g(z,u)\,, \tag{5.59}$$

is a holomorphic function of z, independent of u, and taking the derivative of both sides with respect to z^* we get

$$\frac{e^U}{(\Im\mathrm{m}\,\tau)^{1/2}} = f(u)\,, \qquad g(z,u) = \frac{h(z)}{f(u)}\,, \tag{5.60}$$

where $f(u)$ is a real function of u.

Substituting now $\hat{a}$ into the antisymmetric part of Eq. (5.18) we find that $\hat{n}$ is given by

$$\hat{n} = dv + \omega\,, \tag{5.61}$$

(so $K = 0$ in the metric Eq. (5.48)) where the 1-form ω satisfies

$$f_{\underline{zz}^*} = e^{2U}\partial_{\underline{u}}\ln(B/B^*) = 0\,, \tag{5.62}$$

and D is given by

$$D^* = -\dot{\omega}_{\underline{z}} e^{-U}\,. \tag{5.63}$$

Now that we have determined $\hat{a}$ we have to check that it satisfies the integrability condition Eq. (5.44). This requires the following equations to be satisfied:

$$R_{uz^*} + \tfrac{i}{2}\frac{\partial_{\underline{u}} \ln fB}{\Im\mathrm{m}\,\tau} = 0\,, \tag{5.64}$$

$$R_{uu} - 2[\partial^2_{\underline{u}} \ln f + \partial_{\underline{u}} \ln f \partial_{\underline{u}} \ln f] + 2e^{-U}\partial_{\underline{z}} D = 0\,, \tag{5.65}$$

$$C + e^{-U}\partial_{\underline{z}^*} D = 0\,. \tag{5.66}$$

Comparing with the integrability conditions Eqs. (5.33)-(5.36), we conclude that f must be a constant that we normalize $f = 1$ and that ω must be exact, and we can eliminate it. Further, the ϕ_Is must be constant and the vector field strengths must vanish.

All the equations of motion are automatically satisfied in these conditions, and the solutions are the *stringy cosmic strings* of Ref. [63].

Our result differs from Tod's, who used τ and τ^* as coordinates and found very similar solutions with nontrivial ω that depend in a very complicated way on a function $g(\tau, u)$ an its complex conjugate. This function could be eliminated by a coordinate change in which all the u dependence and the 1-form ω disappear, recovering the stringy cosmic string solutions.

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A Conventions

A.1 Tensors

We use Greek letters $\mu, \nu, \rho, \ldots$ as (*curved*) tensor indices in a coordinate basis and Latin letters $a, b, c \ldots$ as (*flat*) tensor indices in a tetrad basis. Underlined indices are always curved indices. We symmetrize () and antisymmetrize [] with weight one (i.e. dividing by $n!$). We use mostly minus signature $(+ - --)$. η is the Minkowski metric and a general metric is denoted by g. Flat and curved indices are related by tetrads $e_a{}^\mu$ and their inverses $e^a{}_\mu$, satisfying

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}\,, \qquad e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}\,. \tag{A.1}$$

∇ is the total (general- and Lorentz-) covariant derivative, whose action on tensors and spinors (ψ) is given by

$$\begin{aligned} \nabla_\mu \xi^\nu &= \partial_\mu \xi^\nu + \Gamma_{\mu\rho}{}^\nu \xi^\rho\,, \\ \nabla_\mu \xi^a &= \partial_\mu \xi^a + \omega_{\mu b}{}^a \xi^b\,, \\ \nabla_\mu \psi &= \partial_\mu \psi - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\psi\,, \end{aligned} \tag{A.2}$$

where γ_{ab} is the antisymmetric product of two gamma matrices (see next section), $\omega_{\mu b}{}^a$ is the spin connection and $\Gamma_{\mu\rho}{}^\nu$ is the affine connection. The respective curvatures are defined through the Ricci identities

$$\begin{aligned} [\nabla_\mu, \nabla_\nu]\ \xi^\rho &= R_{\mu\nu\sigma}{}^\rho(\Gamma)\,\xi^\sigma + T_{\mu\nu}{}^\sigma \nabla_\sigma \xi^\rho\,, \\ [\nabla_\mu, \nabla_\nu]\ \xi^a &= R_{\mu\nu b}{}^a(\omega)\xi^b\,, \\ [\nabla_\mu, \nabla_\nu]\ \psi &= -\tfrac{1}{4}R_{\mu\nu}{}^{ab}(\omega)\gamma_{ab}\psi\,. \end{aligned} \tag{A.3}$$

and given in terms of the connections by

$$\begin{aligned} R_{\mu\nu\rho}{}^\sigma(\Gamma) &= 2\partial_{[\mu}\Gamma_{\nu]\rho}{}^\sigma + 2\Gamma_{[\mu|\lambda}{}^\sigma \Gamma_{\nu]\rho}{}^\lambda\,, \\ R_{\mu\nu a}{}^b(\omega) &= 2\partial_{[\mu}\,\omega_{\nu]a}{}^b - 2\omega_{[\mu|a}{}^c\,\omega_{|\nu]c}{}^b\,. \end{aligned} \tag{A.4}$$

These two connections are related by the tetrad postulate

$$\nabla_\mu e_a{}^\mu = 0\,, \tag{A.5}$$

by

$$\omega_{\mu a}{}^b = \Gamma_{\mu a}{}^b + e_a{}^\nu \partial_\mu e_\nu{}^b\,, \tag{A.6}$$

which implies that the curvatures are, in turn, related by

$$R_{\mu\nu\rho}{}^\sigma(\Gamma) = e_\rho{}^a e^\sigma{}_b R_{\mu\nu a}{}^b(\omega)\,. \tag{A.7}$$

Finally, metric compatibility and torsionlessness fully determine the connections to be of the form

$$\begin{aligned} \Gamma_{\mu\nu}{}^\rho &= \tfrac{1}{2}g^{\rho\sigma}\left\{\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right\}\,, \\ \omega_{abc} &= -\Omega_{abc} + \Omega_{bca} - \Omega_{cab}\,, \qquad \Omega_{ab}{}^c = e_a{}^\mu e_b{}^\nu \partial_{[\mu} e^c{}_{\nu]}\,. \end{aligned} \tag{A.8}$$

The 4-dimensional fully antisymmetric tensor is defined in flat indices by tangent space by

$$\epsilon^{0123} = +1\,, \qquad \Rightarrow \epsilon_{013} = -1\,, \tag{A.9}$$

and in curved indices by

$$\epsilon^{\mu_1 \cdots \mu_3} = \sqrt{|g|}\, e^{\mu_1}{}_{a_1} \cdots e^{\mu_3}{}_{a_3} \epsilon^{a_3 \cdots a_3}\,, \tag{A.10}$$

so, with upper indices, is independent of the metric and has the same value as with flat indices.

We define the (Hodge) dual of a completely antisymmetric tensor of rank k, $F_{(k)}$ by

$${}^{\star}F_{(k)}{}^{\mu_1 \cdots \mu_{(d-k)}} = \frac{1}{k!\sqrt{|g|}} \epsilon^{\mu_1 \cdots \mu_{(d-k)} \mu_{(d-k+1)} \cdots \mu_d} F_{(k) \mu_{(d-k+1)} \cdots \mu_d}\,. \tag{A.11}$$

Differential forms of rank k are normalized as follows:

$$F_{(k)} \equiv \tfrac{1}{k!} F_{(k)}{}^{\mu_1 \cdots \mu_k} dx^1 \wedge \cdots dx^k\,. \tag{A.12}$$

For any 4-dimensional 2-form, we define

$$F^{\pm} \equiv \tfrac{1}{2}(F \pm i\, {}^{\star}F)\,, \qquad \pm i {}^{\star}F^{\pm} = F^{\pm}\,. \tag{A.13}$$

For any two 2-forms F, G, we have

$$F^{\pm} \cdot G^{\mp} = 0\,, \qquad F^{\pm}{}_{[\mu}{}^{\rho} \cdot G^{\mp}{}_{\nu]\rho} = 0\,. \tag{A.14}$$

Given any 2-form $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ and a non-null 1-form $\hat{V} = V_\mu dx^\mu$, we can express F in the form

$$F = V^{-2}[E \wedge \hat{V} - {}^{\star}(B \wedge \hat{V})]\,, \qquad E_\mu \equiv F_{\mu\nu} V^\nu\,, \qquad B_\mu \equiv {}^{\star}F_{\mu\nu} V^\nu\,. \tag{A.15}$$

For the complex combinations $F^{\pm}$ we have

$$F^{\pm} = V^{-2}[C^{\pm} \wedge \hat{V} \pm i\, {}^{\star}(C^{\pm} \wedge \hat{V})]\,, \qquad C^{\pm}_\mu \equiv F^{\pm}_{\mu\nu} V^\nu\,. \tag{A.16}$$

If we have a (real) null vector l^μ, we can always add three more null vectors $n^\mu, m^\mu, m^{*\,\mu}$ to construct a complex null tetrad such that the local metric in this basis takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.17}$$

with the ordering (l, n, m, m^*). For the local volume element we obtain $\epsilon^{lnmm^*} = i$. The general expansion in the dual basis of 1-forms $\left(\hat{l}, \hat{n}, \hat{m}, \hat{m}^*\right)$ of F^+ depends on three arbitrary complex functions a, b, c

$$F^+ = a\left(\hat{l}\wedge\hat{n} + \hat{m}\wedge\hat{m}^*\right) + b\hat{l}\wedge\hat{m}^* + c\hat{n}\wedge\hat{m}\,, \qquad F^- = (F^+)^*\,. \tag{A.18}$$

Then, in this case, F is not completely determined by its contraction with the null vector l, but

$$F^+ = L^\pm\wedge\hat{n} \pm {}^\star(L^\pm\wedge\hat{n}) + b\hat{l}\wedge\hat{m}\,, \qquad L^\pm_\mu \equiv F^\pm{}_{\mu\nu}l^\nu = al_\mu - cm_\mu\,. \tag{A.19}$$

A.2 Gamma matrices and spinors

We work with a purely imaginary representation

$$\gamma^{a\,*} = -\gamma^a\,, \tag{A.20}$$

and our convention for their anticommutator is

$$\{\gamma^a,\gamma^b\} = +2\eta^{ab}\,. \tag{A.21}$$

Thus,

$$\gamma^0\gamma^a\gamma^0 = \gamma^{a\,\dagger} = \gamma^{a\,-1} = \gamma_a\,. \tag{A.22}$$

The chirality matrix is defined by

$$\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \tfrac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \tag{A.23}$$

and satisfies

$$\gamma_5{}^\dagger = -\gamma_5{}^* = \gamma_5\,, \qquad (\gamma_5)^2 = 1\,. \tag{A.24}$$

With this chirality matrix, we have the identity

$$\gamma^{a_1\cdots a_n} = \frac{(-1)^{[n/2]}i}{(4-n)!}\epsilon^{a_1\cdots a_n b_1\cdots b_{4-n}}\gamma_{b_1\cdots b_{4-n}}\gamma_5\,. \tag{A.25}$$

Our convention for Dirac conjugation is

$$\bar\psi = i\psi^\dagger\gamma_0\,. \tag{A.26}$$

Using the identity Eq. (A.25) the general $d=4$ Fierz identity for *commuting* spinors takes the form

$$\begin{aligned}(\bar\lambda M\chi)(\bar\psi N\varphi) \;=\;& \tfrac14(\bar\lambda MN\varphi)(\bar\psi\chi) + \tfrac14(\bar\lambda M\gamma^a N\varphi)(\bar\psi\gamma_a\chi) - \tfrac18(\bar\lambda M\gamma^{ab}N\varphi)(\bar\psi\gamma_{ab}\chi)\\ &-\tfrac14(\bar\lambda M\gamma^a\gamma_5 N\varphi)(\bar\psi\gamma_a\gamma_5\chi) + \tfrac14(\bar\lambda M\gamma_5 N\varphi)(\bar\psi\gamma_5\chi)\,.\end{aligned} \tag{A.27}$$

We use 4-component chiral spinors whose chirality is related to the position of the $SU(4)$ index:

$$\gamma_5 \chi_I = +\chi_I \,, \qquad \gamma_5 \psi_{\mu\, I} = -\psi_{\mu\, I} \,, \qquad \gamma_5 \epsilon_I = -\epsilon_I \,. \tag{A.28}$$

Both (chirality and position of the $SU(4)$ index) are reversed under complex conjugation:

$$\gamma_5 \chi_I^* \equiv \gamma_5 \chi^I = -\chi^I \,, \qquad \gamma_5 \psi_{\mu\, I}^* \equiv \gamma_5 \psi_\mu{}^I = +\psi_\mu{}^I \,, \qquad \gamma_5 \epsilon_I^* \equiv \gamma_5 \epsilon^I = +\epsilon^I \,. \tag{A.29}$$

We take this fact into account when Dirac-conjugating chiral spinors:

$$\bar{\chi}^I \equiv i(\chi_I)^\dagger \gamma_0 \,, \quad \bar{\chi}^I \gamma_5 = -\bar{\chi}^I \,, \quad \text{etc.} \tag{A.30}$$

The sum of the two chiral spinors related by complex conjugation gives a standard (real) Majorana spinor with an $SU(4)$ index with the complicated transformation rule of Ref. [53].

B Fierz identities for bilinears

Here we are going to work with an arbitrary number N of chiral spinors, although we are ultimately interested in the $N = 4$ case only. Whenever there are special results for particular values of N, we will explicitly say so. We should bear in mind that the maximal number of independent chiral spinors is 2 and, for $N > 2$ (in particular for $N = 4$) N spinors cannot be linearly independent at a given point. This trivial fact has important consequences.

Given N chiral commuting spinors ϵ_I and their complex conjugates ϵ^I we can constructed the following bilinears that are not obviously related via Eq. (A.25):

1. A complex matrix of scalars

$$M_{IJ} \equiv \bar{\epsilon}_I \epsilon_J \,, \qquad M^{IJ} \equiv \bar{\epsilon}^I \epsilon^J = (M_{IJ})^* \,, \tag{B.1}$$

   which is antisymmetric $M_{IJ} = -M_{JI}$.

2. A complex matrix of vectors

$$V^I{}_{J\,a} \equiv i\bar{\epsilon}^I \gamma_a \epsilon_J \,, \qquad V_I{}^J{}_a \equiv i\bar{\epsilon}_I \gamma_a \epsilon^J = (V^I{}_{J\,a})^* \,, \tag{B.2}$$

   which is Hermitean:

$$(V^I{}_{J\,a})^* = V_I{}^J{}_a = V^J{}_{I\,a} = (V^I{}_{J\,a})^T \,. \tag{B.3}$$

3. A complex matrix of 2-forms

$$\Phi_{IJ\,ab} \equiv \bar{\epsilon}_I \gamma_{ab} \epsilon_J \,, \qquad \Phi^{IJ}{}_{ab} \equiv \bar{\epsilon}^I \gamma_{ab} \epsilon^J = (\Phi_{IJ\,ab})^* \,, \tag{B.4}$$

which is symmetric in the $SU(N)$ indices $\Phi_{IJ\,ab} = \Phi_{JI\,ab}$ and, further,

$$^\star\Phi_{IJ\,ab} = -i\Phi_{IJ\,ab} \;\;\Rightarrow\;\; \Phi_{IJ\,ab} = \Phi_{IJ}{}^+{}_{ab} \,. \tag{B.5}$$

As we are going to see, this matrix of 2-forms can be expressed entirely in terms of the scalar and vector bilinears.

It is straightforward to get identities for the products of these bilinears using the Fierz identity Eq. (A.27). First, the products of scalars:

$$M_{IJ} M_{KL} \;=\; \tfrac{1}{2} M_{IL} M_{KJ} - \tfrac{1}{8} \Phi_{IL} \cdot \Phi_{KJ} \,, \tag{B.6}$$

$$M_{IJ} M^{KL} \;=\; -\tfrac{1}{2} V^L{}_I \cdot V^K{}_J \,. \tag{B.7}$$

From Eq. (B.6) immediately follows

$$M_{I[J} M_{KL]} = 0 \,, \tag{B.8}$$

which is a particular case of the Fierz identity

$$\epsilon_{[J} M_{KL]} = 0 \,. \tag{B.9}$$

For $N = 4, 8, \ldots$, Eq. (B.8) implies, in turn

$$\mathrm{Pf}\, M = 0 \;\;\Rightarrow \det M = 0 \,. \tag{B.10}$$

For $N = 4$ we can define the $SU(4)$-dual of M_{IJ}

$$\tilde{M}_{IJ} \equiv \tfrac{1}{2} \varepsilon_{IJKL} M^{KL} \,, \qquad \varepsilon^{1234} = \varepsilon_{1234} = +1 \,, \tag{B.11}$$

and the vanishing of the Pfaffian implies

$$\tilde{M}_{IJ} M^{IJ} = 0 \,. \tag{B.12}$$

From Eq. (B.7) and the antisymmetry of M immediately follows

$$V^I{}_L \cdot V^K{}_J = -V^I{}_J \cdot V^K{}_L = -V^K{}_L \cdot V^I{}_J \,, \tag{B.13}$$

which implies that all the vector bilinears $V^I{}_{J\,a}$ are null:

$$V^I{}_J \cdot V^I{}_J = 0 \,. \tag{B.14}$$

On the other hand, from Eqs. (B.13) and (B.7) follows the real $SU(N)$-invariant combination of vectors $V_a \equiv V^I{}_{I\,a}$ is always non-spacelike:

$$V^2 = -V^I{}_J \cdot V^J{}_I = 2M^{IJ}M_{IJ} \geq 0\,. \tag{B.15}$$

The products of M with the other bilinears[16] give

$$\begin{aligned} M_{IJ}V^K{}_{L\,a} &= \tfrac{1}{2}M_{IL}V^K{}_{J\,a} + \tfrac{1}{2}\Phi_{IL\,ba}V^K{}_J{}^b\,, && (\text{B.16})\\ M_{IJ}\Phi^{KL}{}_{ab} &= V^L{}_{I\,[a|}V^K{}_{J\,|b]} - \tfrac{i}{2}\epsilon_{ab}{}^{cd}V^L{}_{I\,c}V^K{}_{J\,d}\,. && (\text{B.17})\end{aligned}$$

Now, let us consider the product of two arbitrary vectors[17]:

$$V^I{}_{J\,a}V^K{}_{L\,b} = \tfrac{i}{2}\epsilon_{ab}{}^{cd}V^I{}_{L\,c}V^K{}_{J\,d} + V^I{}_{L\,(a|}V^K{}_{J\,|b)} - \tfrac{1}{2}g_{ab}V^I{}_L \cdot V^K{}_J\,. \tag{B.18}$$

For V^2 this identity allows us to write the metric in the form

$$g_{ab} = 2V^{-2}[V_aV_b - V^I{}_{J\,a}V^J{}_{I\,b}]\,. \tag{B.19}$$

Following Tod [4], for $V^2 \neq 0$ we introduce

$$\mathcal{J}^I{}_J \equiv \frac{2M^{IK}M_{JK}}{|M|^2} = \frac{2V\cdot V^I{}_J}{V^2}\,, \qquad |M|^2 \equiv M^{LM}M_{LM} = \tfrac{1}{2}V^2\,. \tag{B.20}$$

Using Eq. (B.6) we can show that it is a Hermitean projector whose trace equals 2:

$$\mathcal{J}^I{}_J\mathcal{J}^J{}_K = \mathcal{J}^I{}_K\,, \qquad \mathcal{J}^I{}_I = +2\,. \tag{B.21}$$

Further, using the general Fierz identity we find

$$\mathcal{J}^I{}_J\epsilon^J = \epsilon^I\,, \qquad \epsilon_I\mathcal{J}^I{}_J = \epsilon_J\,, \tag{B.22}$$

which should be understood for $N > 2$ of the fact that the ϵ^I are not linearly independent[18]. As a consequence of the above identity, the contraction of $\mathcal{J}$ with any of the bilinears is the identity. Using this result and Eq. (B.17), we find

$$\Phi^{KL}{}_{ab} = \frac{2M^{IK}M_{IJ}}{|M|^2}\Phi^{JL}{}_{ab} = \frac{2M^{IK}}{|M|^2}V^L{}_{I\,[a}V_{b]} - i\frac{M^{IK}}{|M|^2}\epsilon_{ab}{}^{cd}V^L{}_{I\,c}V_d\,. \tag{B.23}$$

Other useful identities are

$$\frac{M_{IJ}M^{KL}}{|M|^2} = \mathcal{J}^K{}_{[I}\mathcal{J}^L{}_{J]}\,, \tag{B.24}$$

[16] We omit the product $M_{IJ}\Phi_{KL\,ab}$ which will not be used.
[17] The product $V^I{}_{J\,a}V_L{}^K{}_b$ gives a different identity that will not be used
[18] For $N = 2$ $\mathcal{J}^I{}_J = \delta^I{}_J$. See later on.

and

$$\frac{2\tilde{M}^{IK}\tilde{M}_{JK}}{|M|^2} = \delta^I{}_J - \mathcal{J}^I{}_J \equiv \tilde{\mathcal{J}}^I{}_J\,, \tag{B.25}$$

which is the complementary projector.

In the null case $V^2 = |M|^2 = 0$ it is customary to write $l_a \equiv V^I{}_{I\,a}$. Since $|M|^2$ is a sum of positive numbers, each of them must vanish independently, i.e. $M^{IJ} = 0$. This implies that all spinors ϵ^I are proportional and one can write

$$\epsilon_I = \phi_I \epsilon\,, \tag{B.26}$$

for some complex functions ϕ_I which transform as an $SU(4)$ vector, and some negative-chirality spinor ϵ. These are defined up to a rescaling by a complex function and opposite weights. Part of this freedom can be fixed by normalizing

$$\phi_I \phi^I = 1\,, \qquad \phi^I \equiv \phi_I^*\,. \tag{B.27}$$

Then, the only freedom that remains in the definition of ϕ^I is a change by a local phase $\theta(x)$

$$\phi_I \longrightarrow e^{i\theta}\phi_I\,, \qquad \epsilon \longrightarrow e^{-i\theta}\epsilon\,. \tag{B.28}$$

In this case on can construct another Hermitean projector $\mathcal{K}^I{}_J$ that plays a role analogous to that of $\mathcal{J}^I{}_J$ in the non-null case:

$$\mathcal{K}^I{}_J \equiv \phi^I \phi_J\,, \tag{B.29}$$

which satisfies

$$\mathcal{K}^I{}_J \mathcal{K}^J{}_K = \mathcal{K}^I{}_K\,, \qquad \mathcal{K}^I{}_I = +1\,, \tag{B.30}$$

and

$$\mathcal{K}^I{}_J \epsilon^J = \epsilon^I\,, \qquad \epsilon_I \mathcal{K}^I{}_J = \epsilon_J\,, \tag{B.31}$$

which expresses the known fact that only one spinor is linearly independent in this case.

In the null case, all the vector bilinears are also proportional to the null vector l:

$$V^I{}_{J\,a} = \mathcal{K}^I{}_J l_a\,. \tag{B.32}$$

Once ϵ is given, we may introduce an auxiliary spinor with the same chirality and opposite $U(1)$ charge as ϵ and normalized against ϵ by

$$\bar{\epsilon}\eta = \frac{1}{2}\,, \tag{B.33}$$

where $\bar{\epsilon} = i\epsilon^T\gamma_0$. With both spinors we can construct a complex null tetrad with metric Eq. (A.17) as follows:

$$l_\mu = i\bar{\epsilon^*}\gamma_\mu\epsilon\,, \quad n_\mu = i\bar{\eta^*}\gamma_\mu\eta\,, \quad m_\mu = i\bar{\epsilon^*}\gamma_\mu\eta = i\bar{\eta}\gamma_\mu\epsilon^*\,, \quad m^*_\mu = i\bar{\epsilon}\gamma_\mu\eta^* = i\bar{\eta^*}\gamma_\mu\epsilon\,. \tag{B.34}$$

The normalization condition (B.27) does not fix completely the auxiliary spinor η and the freedom in the choice of η becomes a freedom in the null tetrad. First of all, there is a $U(1)$ freedom Eq. (B.28) under which $\eta' = e^{i\theta}\eta$ and

$$l' = l\,, \quad n' = n\,, \quad m' = e^{2i\theta}m\,. \tag{B.35}$$

Further, we can also shift η by terms proportional to ϵ preserving the normalization

$$\eta' = \eta + \delta\epsilon\,. \tag{B.36}$$

Under this redefinition of η, the null tetrad transforms as follows:

$$l' = l\,, \quad n' = n + \delta^* m + \delta m^* + |\delta|^2 l\,, \quad m' = m + \delta l\,. \tag{B.37}$$

B.1 The $N = 2$ case

Here we describe some of the peculiarities of the $N = 2$ case in which the number of spinors is precisely the necessary to construct a basis at each point.

In the $N = 2$ case there is only one independent (complex) scalar X since

$$\bar{\epsilon}_I \epsilon_J = X \epsilon_{IJ}\,, \tag{B.38}$$

where ϵ_{IJ} is the (constant) 2-dimensional totally antisymmetric tensor. It follows that

$$|M|^2 = 2|X|^2\,, \tag{B.39}$$

and, using $\epsilon_{IJ}\epsilon^{KL} = \delta_{IJ}{}^{KL}$ we can show that the projector

$$\mathcal{J}^I{}_J = \delta^I{}_J\,. \tag{B.40}$$

In the $|M|^2 \neq 0$ case, the four vector bilinears $V^I{}_{J\,\mu}$ can be used as a null tetrad

$$l_\mu = V^1{}_{1\,\mu}\,, \quad n_\mu = V^2{}_{2\,\mu}\,, \quad m_\mu = V^1{}_{2\,\mu}\,, \quad m^*_\mu = V^2{}_{1\,\mu}\,,. \tag{B.41}$$

Alternatively, one can use the four combinations

$$V^a{}_\mu \equiv \tfrac{1}{\sqrt{2}} V^I{}_{J\,\mu} (\sigma^a)^J{}_I\,, \tag{B.42}$$

with $\sigma^0 = 1$ and σ^i the three (traceless, Hermitean) Pauli matrices as an orthonormal tetrad in which V^0 is timelike and the V^i are spacelike.

C Connection and curvature of the conformastationary metric

A conformastationary metric has the general form

$$ds^2 = |M|^2(dt+\omega)^2 - |M|^{-2}\gamma_{\underline{ij}}dx^i dx^j\,, \qquad i,j=1,2,3\,, \tag{C.1}$$

where all components of the metric are independent of the time coordinate t. Choosing the Vielbein basis

$$(e^a{}_\mu) = \begin{pmatrix} |M| & |M|\omega_{\underline{i}} \\ & \\ 0 & |M|^{-1}v_{\underline{i}}{}^j \end{pmatrix}\,, \qquad (e^\mu{}_a) = \begin{pmatrix} |M|^{-1} & -|M|\omega_i \\ & \\ 0 & |M|v_i{}^{\underline{j}} \end{pmatrix}\,, \tag{C.2}$$

where

$$\gamma_{\underline{ij}} = v_{\underline{i}}{}^k v_{\underline{j}}{}^l \delta_{kl}\,, \qquad v_i{}^{\underline{k}} v_{\underline{k}}{}^j v_j\,, \qquad \omega_i = v_i{}^{\underline{j}}\omega_{\underline{j}}\,, \tag{C.3}$$

we find that the spin connection components are

$$\begin{aligned} \omega_{00i} &= -\partial_i|M|\,, & \omega_{0ij} &= \tfrac{1}{2}|M|^3 f_{ij}\,, \\ \omega_{i0j} &= \omega_{0ij}\,, & \omega_{ijk} &= -|M|o_{ijk} - 2\delta_{i[j}\partial_{k]}|M|\,, \end{aligned} \tag{C.4}$$

where $o_i{}^{jk}$ is the 3-dimensional spin connection and

$$\partial_i \equiv v_i{}^{\underline{j}}\partial_{\underline{j}}\,, \qquad f_{ij} = v_i{}^{\underline{k}}v_j{}^{\underline{l}}f_{\underline{kl}}\,, \qquad f_{\underline{ij}} \equiv 2\partial_{[\underline{i}}\omega_{\underline{j}]}\,. \tag{C.5}$$

The components of the Riemann tensor are

$$\begin{aligned} R_{0i0j} &= \tfrac{1}{2}\nabla_i\partial_j|M|^2 + \partial_i|M|\partial_j|M| - \delta_{ij}(\partial|M|)^2 + \tfrac{1}{4}\nabla i|M|^6 f_{ik}f_{jk}\,, \\ R_{0ijk} &= -\tfrac{1}{2}\nabla_i(|M|^4 f_{jk}) + \tfrac{1}{2}f_{i[j}\partial_{k]}|M|^4 - \tfrac{1}{4}\delta_{i[j}f_{k]l}\partial_l|M|^4\,, \\ R_{ijkl} &= -|M|^2 R_{ijkl} + \tfrac{1}{2}|M|^6(f_{ij}f_{kl} - f_{k[i}f_{j]l}) - 2\delta_{ij,kl}(\partial|M|)^2 + 4|M|\delta_{[i}{}^{[k}\nabla_{j]}\partial^{l]}|M|\,, \end{aligned} \tag{C.6}$$

where all the objects in the right-hand sides of the equations are referred to the 3-dimensional spatial metric. The components of the Ricci tensor are

$$\begin{aligned} R_{00} &= -|M|^2\nabla^2\log|M| - \tfrac{1}{4}|M|^6 f^2\,, \\ R_{0i} &= \tfrac{1}{2}\nabla_j(|M|^4 f_{ji})\,, \\ R_{ij} &= |M|^2\{R_{ij} + 2\partial_i\log|M|\partial_j\log|M| - \delta_{ij}\nabla^2\log|M| - \tfrac{1}{2}|M|^4 f_{ik}f_{jk}\}\,, \end{aligned} \tag{C.7}$$

and the Ricci scalar is

$$R = -|M|^2\{R - \tfrac{1}{4}|M|^4 f^2 - 2\nabla^2 \log|M| + 2(\partial \log|M|)^2\}\,, \tag{C.8}$$

D Connection and curvature of a Brinkmann pp-wave metric

We rewrite here for convenience the 4-dimensional form of these metrics:

$$ds^2 = 2du(dv + Kdu + \omega) - 2e^{2U}dzdz^*\,, \qquad \omega = \omega_{\underline{z}}dz + \omega_{\underline{z}^*}dz^*\,, \tag{D.1}$$

where all the functions in the metric are independent of v.

Using also light-cone coordinates in tangent space, a natural Vielbein basis is

$$\begin{array}{rclcl rclcl}
e^u &=& du &=& \hat{l}\,, & e_u &=& \partial_{\underline{u}} - K\partial_{\underline{v}} &=& n^\mu\partial_\mu\,,\\
e^v &=& dv + Kdu + \omega &=& \hat{n}\,, & e_v &=& \partial_{\underline{v}} &=& l^\mu\partial_\mu\,,\\
e^z &=& e^U dz &=& \hat{m}\,, & e_z &=& e^{-U}(\partial_{\underline{z}} - \omega_{\underline{z}}\partial_{\underline{v}}) &=& -m^{*\mu}\partial_\mu\,,\\
e^{z^*} &=& e^U dz^* &=& \hat{m}^*\,, & e_{z^*} &=& e^{-U}(\partial_{\underline{z}^*} - \omega_{\underline{z}^*}\partial_{\underline{v}}) &=& -m^\mu\partial_\mu\,.
\end{array} \tag{D.2}$$

The components of the spin connection are

$$\begin{array}{rclrcl}
\omega_{uzu} &=& e^{-U}(\partial_{\underline{z}}K - \dot{\omega}_{\underline{z}})\,, & \omega_{uzz^*} &=& \tfrac{1}{2}e^{-2U}f_{\underline{z}\underline{z}^*}\,,\\
\omega_{zz^*u} &=& -\tfrac{1}{2}e^{-2U}f_{\underline{z}\underline{z}^*} + \dot{U}\,, & \omega_{zzz^*} &=& -e^{-U}\partial_{\underline{z}}U\,,
\end{array} \tag{D.3}$$

where $f_{\underline{z}\underline{z}^*} = 2\partial_{[\underline{z}}\omega_{\underline{z}^*]}$ and a dot stands for partial derivation with respect to u.

The components of the Ricci tensor are

$$\begin{array}{rcl}
R_{zz^*} &=& 2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}U\,,\\
R_{zu} &=& \tfrac{1}{2}e^{-U}\partial_{\underline{z}}(e^{-2U}f_{\underline{z}\underline{z}^*}) + e^{-U}\partial_{\underline{z}}\dot{U}\,,\\
R_{uu} &=& -2e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}K + \tfrac{1}{2}e^{-4U}(f_{\underline{z}\underline{z}^*})^2 + e^{-2U}(\partial_{\underline{z}}\dot{\omega}_{\underline{z}^*} + \partial_{\underline{z}^*}\dot{\omega}_{\underline{z}}) + 2(\ddot{U} + \dot{U}\dot{U})\,,
\end{array} \tag{D.4}$$

and the Ricci scalar is just

$$R = -4e^{-2U}\partial_{\underline{z}}\partial_{\underline{z}^*}U\,. \tag{D.5}$$

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