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A Note on Simple Applications of the Killing Spinor Identities

Jorge Bellorín[1] and Tomás Ortín[2]

Instituto de Física Teórica UAM/CSIC
Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049-Madrid, Spain

Abstract

We show how the Killing Spinor Identities (KSI) can be used to reduce the number of independent equations of motion that need to be checked explicitly to make sure that a supersymmetric configuration is a classical supergravity solution. We also show how the KSI can be used to compute BPS relations between masses and charges.

[1]E-mail: `Jorge.Bellorin@uam.es`
[2]E-mail: `Tomas.Ortin@cern.ch`

Supersymmetric solutions of supergravity theories play a very relevant role today. As classical superstring backgrounds, they are used in the search for phenomenologically viable superstring compactifications or, via the AdS/CFT correspondence, they are used to study new states of SCFTs.

To find these solutions it is customary to introduce first an *Ansatz* that incorporates the relevant fields and symmetries into the Killing spinor equations in order to constrain the form of the solution and make sure that the required amount of supersymmetry will be preserved. Then one still has to solve all the equations of motion, but this task is usually not too difficult once the supersymmetry test is passed. It it, however, possible, to use the Killing spinor equations in more efficient ways, as we are going to see.

For instance, recently, in Ref. [1] it has been proven that, for supersymmetric configurations of massive type IIA supergravity, if the equations of motion and Bianchi identities are satisfied for all the p-form potentials and the dilaton, then the Einstein equations (and also the dilaton equation) are also satisfied, under certain mild conditions. Similar results had been obtained earlier in the context of minimal $d = 5$ and $d = 11$ supergravity in Refs. [2, 3, 4]. In this short note we are going to show that this result is a simple consequence of the general *Killing Spinor Identities* derived in Ref. [5]. These identities are relations between equations of motion of the bosonic fields of supergravity theories and using them we can show that the results of Refs. [1, 2, 3, 4] hold in any theory of supergravity. These relations are the reason why supersymmetric solutions depend on a very reduced number of independent functions that solve simple equations. The advantage of this method is that it is conceptually more clear and it does not require the computation of the commutator of two supercovariant derivatives (the integrability conditions for the Killing spinor equation), which is often algebraically quite involved.

The Killing Spinor Identities (KSI) of any supergravity theory with bosonic and fermionic fields ϕ^b, ϕ^f, and invariant under local supersymmetry transformations $\delta_\epsilon\phi^b, \delta_\epsilon\phi^f$, can be derived as follows: from the supersymmetry variation of the action of the theory, which vanishes by hypothesis, we obtain the identity

$$\delta_\epsilon S = \int d^dx(S_{,b}\,\delta_\epsilon\phi^b + S_{,f}\,\delta_\epsilon\phi^f) = 0\,. \tag{1}$$

Here $S_{,b(f)}$ are the first variations of the action with respect to the bosonic (fermionic) fields, i.e. their equations of motion. Summation over the indices b, f is understood. Strictly speaking, the r.h.s. of this formula is a boundary term odd in fermion fields which we have assumed vanish on the boundary. This is an acceptable assumption since we are going to set all the fermionic fields to zero in the end.

Now we vary this equation w.r.t. the fermionic fields and evaluate the expression for vanishing fermionic fields, getting

$$\left\{S_{,bf_2}\,\delta_\epsilon\phi^b + S_{,b}\,(\delta_\epsilon\phi^b)_{,f_2} + S_{,f_1f_2}\,\delta_\epsilon\phi^{f_1} + S_{,f_1}\,(\delta_\epsilon\phi^{f_1})_{,f_2}\right\}_{\phi^f=0} = 0\,. \tag{2}$$

Since the bosonic equations of motion $S_{,b}$ and the supersymmetry variations of the fermions $\delta_\epsilon\phi^f$ are necessarily even in fermions

$$S_{,bf_2}\big|_{\phi^f=0} = (\delta_\epsilon \phi^{f_1})_{,f_2}\big|_{\phi^f=0} = 0\,, \tag{3}$$

and we are left with only two terms

$$\left\{S_{,b}\,(\delta_\epsilon \phi^b)_{,f_2} + S_{,f_1 f_2}\,\delta_\epsilon \phi^{f_1}\right\}_{\phi^f=0} = 0\,. \tag{4}$$

This expression is valid for any values of the bosonic fields ϕ^b and supersymmetry parameters ϵ, but it takes a most useful form when we specialize it for supersymmetry parameters which are Killing spinors which we denote by κ and which satisfy, by definition, the Killing spinor equation

$$\delta_\kappa \phi^f\big|_{\phi^f=0} = 0\,. \tag{5}$$

Thus, supersymmetric (i.e. admitting Killing spinors) bosonic configurations satisfy the following *Killing Spinor Identities* (KSI) found in Ref. [5] that relate their equations of motion

$$S_{,b}\,(\delta_\kappa \phi^b)_{,f}\big|_{\phi^f=0} = 0\,. \tag{6}$$

Of course, these equations are a particularly useful subset of the supersymmetric gauge identities which relate all the equations of motion of a locally supersymmetric theory, and their content is highly non-trivial even if each term vanishes separately on-shell. This is the reason behind the well-known fact that supersymmetric solutions are given in terms of a very small number of functions that satisfy certain equations: each equation of motion is a simple combination of the equations satisfied by those few functions and that is how the equations of motion are related by the KSI, on- or off-shell. For example, in simple p-brane solutions, all the equations of motion are proportional to the Laplacian of a single function.

The KSI can be used, for instance, to reduce the number of independent equations of motion that need to be solved explicitly[3] to make sure that a configuration satisfies them all. Let us consider a few examples.

The action of the bosonic sector of $d = 11$ supergravity is[4]

$$S = \int d^{11}x\,\sqrt{|g|}\,\left[R - \tfrac{1}{2\cdot 4!}G^2 - \tfrac{1}{(144)^2\sqrt{|g|}}\epsilon GGC\right]\,, \tag{7}$$

and the supersymmetry variations of the bosonic fields are

[3]The contracted Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ is used in General Relativity in a similar fashion: it implies $\sum_{\phi^b} \nabla_\mu T^{\mu\nu}(\phi^b) = 0$ and, given that $\nabla_\mu T^{\mu\nu}(\phi^b)$ is always proportional to the equation of motion of the field ϕ^b (it only vanishes on-shell), we get a relation between the equations of motion of all the matter fields ϕ^b. For a single minimally-coupled scalar field, for instance, if the Einstein equation is satisfied, we get $(\nabla^2\phi)(\nabla^\nu\phi) = 0$ and, if $\nabla^\nu\phi \neq 0$ we get $\nabla^2\phi = 0$, and if $\nabla^\nu\phi = 0$ we get the same result.

[4]Our notation and conventions are those of Refs. [6] and [7].

$$
\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\tfrac{i}{2}\bar{\epsilon}\,\Gamma^a\,\psi_\mu\,, \\
\delta_\epsilon C_{\mu\nu\rho} &= \tfrac{3}{2}\bar{\epsilon}\,\Gamma_{[\mu\nu}\,\psi_{\rho]}\,.
\end{aligned}
\tag{8}
$$

Defining

$$
\begin{aligned}
E_a{}^\mu(e) \equiv \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\bigg|_{\psi=0} &= -2\left\{G_a{}^\mu - \tfrac{1}{12}\left[G_{abcd}G^{\mu bcd} - \tfrac{1}{8}e_a{}^\mu G^2\right]\right\}\,, \\
E^{\mu\nu\rho}(C) \equiv \frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta C_{\mu\nu\rho}}\bigg|_{\psi=0} &= \tfrac{1}{3!}\left[\nabla_\sigma G^{\sigma\mu\nu\rho} - \frac{1}{9\cdot 2^7\sqrt{|g|}}\epsilon^{\mu\nu\rho\lambda_1\cdots\lambda_4\gamma_1\cdots\gamma_4}G_{\lambda_1\cdots\lambda_4}G_{\gamma_1\cdots\gamma_4}\right]\,,
\end{aligned}
\tag{9}
$$

we immediately get the KSI of $d=11$ supergravity

$$
\bar{\kappa}\left[E_a{}^\mu(e)\gamma^a + 3iE^{\mu ab}(C)\gamma_{ab}\right] = 0\,. \tag{10}
$$

If the equation of motion of the 3-form is satisfied (the Bianchi identity is always assumed to be satisfied in this formalism), then, a bosonic configuration always satisfies

$$
\bar{\kappa}E_a{}^\mu(e)\gamma^a = 0\,. \tag{11}
$$

This is the equation obtained in Ref. [2, 3, 4, 1] by computing the commutator of two supercovariant derivatives. Now we can follow the reasoning in Refs. [2, 3] to see under which conditions this equation implies Einstein's $E_a{}^\mu(e)=0$. Multiplying by $i\kappa$ on the right, we get

$$
E_a{}^\mu V^a = 0\,, \tag{12}
$$

where

$$
V^a \equiv i\bar{\kappa}\gamma^a\kappa\,, \tag{13}
$$

is always a non-spacelike vector. If we multiply by $E_b{}^\nu(e)\gamma^b$ and symmetrize in the free indices we get

$$
E_a{}^\mu(e)E_b{}^\nu(e)\eta^{ab} = 0\,. \tag{14}
$$

If V is spacelike, introducing a frame in which $e^0 = V$, Eq. (12) implies that all the components $E_0{}^\mu(e)$ vanish[5] and Eqs. (14) can be seen as positive- or negative-definite scalar products of vectors and one concludes that $E_a{}^\mu(e)=0$.
If V is null, we construct a frame

$$
ds^2 = 2e^+e^- - e^ie^i\,, \qquad i=1,\cdots,9\,. \tag{15}
$$

[5]In Ref. [1] this condition was imposed by hand. In this case, we see that it follows from Eq. (12). In the null case that we consider next, only part of this condition has to be imposed by hand.

with $e^+ = V$. Now Eq. (12) implies that all the components $E_-{}^{\mu}(e)$ vanish and Eqs. (14) imply that $E_+{}^i = E_j{}^i = 0$. The only component of the Einstein equation that one needs to impose independently is $E_+{}^+ = 0$.

Let us now consider the example directly studied in Ref. [1]: massive type IIA supergravity. The action of this theory is

$$
\begin{aligned}
S \;\; = \;\; & \int d^{10}x\,\sqrt{|g|}\Big\{e^{-2\phi}\left[R - 4\,(\partial\phi)^2 + \tfrac{1}{2\cdot 3!}H^2\right] - \tfrac{1}{2}m^2 - \tfrac{1}{4}G^{(2)\,2} - \tfrac{1}{2\cdot 4!}G^{(4)\,2} \\
& -\tfrac{1}{144}\tfrac{1}{\sqrt{|g|}}\;\epsilon\left[\partial C^{(3)}\partial C^{(3)}B + \tfrac{1}{2}m\partial C^{(3)}BBB + \tfrac{9}{80}m^2 BBBBB\right]\Big\}\,,
\end{aligned}
\tag{16}
$$

where the field strengths are given by

$$
H = 3\partial B\,, \qquad G^{(2)} = 2\partial C^{(1)} + mB\,, \qquad G^{(4)} = 4\partial C^{(3)} - 4HC^{(1)} + 3mBB\,, \tag{17}
$$

and the supersymmetry transformation rules of the bosonic fields are

$$
\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -i\bar{\epsilon}\Gamma^a\psi_\mu\,, \\
\delta_\epsilon B_{\mu\nu} &= -2i\bar{\epsilon}\Gamma_{[\mu}\Gamma_{11}\psi_{\nu]}\,, \\
\delta_\epsilon\phi &= -\tfrac{i}{2}\bar{\epsilon}\lambda\,, \\
\delta_\epsilon C^{(1)}{}_\mu &= -e^{\phi}\bar{\epsilon}\Gamma_{11}\left(\psi_\mu - \tfrac{1}{2}\Gamma_\mu\lambda\right)\,, \\
\delta_\epsilon C^{(3)}{}_{\mu\nu\rho} &= 3e^{\phi}\bar{\epsilon}\Gamma_{[\mu\nu}\left(\psi_{\rho]} - \tfrac{1}{3!}\Gamma_{\rho]}\lambda\right) + 3C^{(1)}{}_{[\mu}\delta_\epsilon B_{\nu\rho]}\,.
\end{aligned}
\tag{18}
$$

The equations of motion of the different fields, using the same notation as in the 11-dimensional case, are

$$
\begin{aligned}
E_{\mu\nu}(e) &= -2e^{-2\phi}\left\{R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\phi + \tfrac{1}{4}H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma} - \tfrac{1}{2}e^{2\phi}\textstyle\sum_{n=0,2,4}\frac{1}{(n-1)!}T^{(n)}{}_{\mu\nu}\right\} \\
&\quad -\tfrac{1}{2}g_{\mu\nu}E(\phi)\,, \\
E(\phi) &= -2e^{-2\phi}\left\{R + 4\left(\partial\phi\right)^2 - 4\nabla^2\phi + \tfrac{1}{2\cdot 3!}H^2\right\} \\
E^{\mu\nu}(B) &= -\tfrac{1}{2}\{\nabla_\rho(e^{-2\phi}H^{\rho\mu\nu}) + mG^{(2)\,\mu\nu} + \tfrac{1}{2}G^{(4)\,\mu\nu\alpha\beta}G^{(2)}{}_{\alpha\beta} \\
&\quad + \tfrac{1}{2\cdot(4!)^2\sqrt{|g|}}\epsilon^{\mu\nu\alpha_1\cdots\alpha_4\beta_1\cdots\beta_4}G^{(4)}{}_{\alpha_1\cdots\alpha_4}G^{(4)}{}_{\beta_1\cdots\beta_4}\} \\
&\quad -3E^{\mu\nu\alpha}(C^{(3)})C^{(1)}{}_\alpha\,, \\
E^\mu(C^{(1)}) &= \nabla_\nu G^{(2)\,\nu\mu} + \tfrac{1}{3!}H_{\alpha_1\cdots\alpha_3}G^{(4)\,\alpha_1\cdots\alpha_3\mu}\,, \\
E^{\mu\nu\rho}(C^{(3)}) &= \tfrac{1}{3!}\{\nabla_\sigma G^{(4)\,\sigma\mu\nu\rho} - \tfrac{1}{3!\cdot 4!\sqrt{|g|}}\epsilon^{\mu\nu\rho\alpha_1\cdots\alpha_3\beta_1\cdots\beta_4}H_{\alpha_1\cdots\alpha_3}G^{(4)}{}_{\beta_1\cdots\beta_4}\}\,,
\end{aligned} \tag{19}
$$

where $T^{(n)}{}_{\mu\nu}$ are the energy-momentum tensors of the RR fields:

$$
T^{(n)}{}_{\mu\nu} = G^{(n)}{}_\mu{}^{\rho_1\cdots\rho_{n-1}}G^{(n)}{}_{\nu\rho_1\cdots\rho_{n-1}} - \tfrac{1}{2n}g_{\mu\nu}G^{(n)\,2}\,, \tag{20}
$$

and, for $n = 0$

$$
T^{(0)}{}_{\mu\nu} = -\tfrac{1}{2}m^2 g_{\mu\nu}\,. \tag{21}
$$

The KSI of (massive) type IIA supergravity associated to the variations with respect to the gravitino and the dilatino take, then, the form

$$
\begin{aligned}
\bar\kappa\left\{E_a{}^\mu(e)\Gamma^a + 2E^{a\mu}(B)\Gamma_a\Gamma_{11} - ie^\phi E^\mu(C^{(1)})\Gamma_{11}\right. \\
\left. +3iE^{ab\mu}(C^{(3)})[e^\phi\Gamma_{ab} - 2iC^{(1)}{}_a\Gamma_b\Gamma_{11}]\right\} &= 0\,, \\
\bar\kappa\{E(\phi) + ie^\phi E^a(C^{(1)})\Gamma_{11}\Gamma_a - ie^\phi E^{abc}(C^{(3)})\Gamma_{abc}\} &= 0\,.
\end{aligned} \tag{22}
$$

The second equation tells us that, in presence of some unbroken supersymmetries, if the equations of motion of the RR potentials are satisfied, then the equation of motion of the dilaton is automatically solved. If also the equation of motion of the NSNS 2-form is solved, then we get $\bar\kappa E_a{}^\mu\Gamma^a = 0$ as in the 11-dimensional case and, following again the reasoning of Ref. [3] we arrive at the same results.

By now, given that the Vielbein supersymmetry transformation rule always has the same form, it should be clear that similar results are going to hold in all supergravity theories.

For the sake of completeness we can also compute the KSI of type IIB supergravity. The equations of motion can be derived from the non-self-dual (NSD) action of Ref. [8]

$$
\begin{aligned}
S_{\rm NSD} \;=\; & \int d^{10}x\, \sqrt{|\jmath|}\, \Big\{ e^{-2\varphi}\left[R(\jmath) - 4\,(\partial\varphi)^2 + \tfrac{1}{2\cdot 3!}\mathcal{H}^2\right] \\
& +\tfrac{1}{2}G^{(1)\,2} + \tfrac{1}{2\cdot 3!}G^{(3)\,2} + \tfrac{1}{4\cdot 5!}G^{(5)\,2} - \tfrac{1}{192}\tfrac{1}{\sqrt{|\jmath|}}\,\epsilon\,\partial C^{(4)}\partial C^{(2)}\mathcal{B}\Big\}\,,
\end{aligned} \tag{23}
$$

where the field strengths are given by

$$
\mathcal{H} = 3\partial\mathcal{B}\,, \quad G^{(1)} = \partial C^{(0)}\,, \quad G^{(3)} = 3\partial C^{(2)} - \mathcal{H}C^{(0)}\,, \quad G^{(5)} = 5\partial C^{(4)} - 10\mathcal{H}C^{(2)}\,. \tag{24}
$$

The NSD action has to be supplemented, after variation, with the self-duality of the 5-form field strength

$$
G^{(5)} = {}^{\star}G^{(5)}\,. \tag{25}
$$

The equations of motion that one derives from the NSD action are

$$
\begin{aligned}
E_{\mu\nu}(e) \;&=\; -2e^{-2\varphi}\Big\{R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\varphi + \tfrac{1}{4}\mathcal{H}_\mu{}^{\rho\sigma}\mathcal{H}_{\nu\rho\sigma} + \tfrac{1}{2}e^{2\varphi}\textstyle\sum_{n=1,3}\tfrac{1}{(n-1)!}T^{(n)}{}_{\mu\nu} \\
&\quad +\tfrac{1}{4\cdot 4!}e^{2\varphi}T^{(5)}{}_{\mu\nu}\Big\} - \tfrac{1}{2}\jmath_{\mu\nu}E(\varphi)\,, \\
E(\varphi) \;&=\; -2e^{-2\varphi}\left\{R + 4(\partial\varphi)^2 - 4\nabla^2\varphi + \tfrac{1}{2\cdot 3!}\mathcal{H}^2\right\}\,, \\
E^{\mu\nu}(\mathcal{B}) \;&=\; -\tfrac{1}{2}\left\{\nabla_\rho(e^{-2\varphi}\mathcal{H}^{\rho\mu\nu}) - G^{(3)\,\mu\nu\alpha}G^{(1)}{}_\alpha - \tfrac{1}{3!}G^{(5)+\,\mu\nu\alpha_1\alpha_2\alpha_3}G^{(3)}{}_{\alpha_1\alpha_2\alpha_3}\right\} \\
&\quad -C^{(0)}E^{\mu\nu}(C^{(2)}) - 3!E^{\mu\nu\alpha\beta}(C^{(4)})C^{(2)}{}_{\alpha\beta}\,, \\
E(C^{(0)}) \;&=\; -\left\{\nabla_\rho G^{(1)\,\rho} + \tfrac{1}{3!}G^{(3)\,\alpha\beta\gamma}\mathcal{H}_{\alpha\beta\gamma}\right\}\,, \\
E^{\mu\nu}(C^{(2)}) \;&=\; -\tfrac{1}{2}\{\nabla_\rho G^{(3)\,\rho\mu\nu} + \tfrac{1}{3!}G^{(5)+\,\mu\nu\alpha_1\cdots\alpha_3}\mathcal{H}_{\alpha_1\cdots\alpha_3}\}\,, \\
E^{\mu_1\cdots\mu_4}(C^{(4)}) \;&=\; -\tfrac{1}{2\cdot 4!}\{\nabla_\rho G^{(5)\,\rho\mu_1\cdots\mu_4} - \tfrac{1}{(3!)^2\sqrt{|\jmath|}}\epsilon^{\mu_1\cdots\mu_4\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}\mathcal{H}_{\alpha_1\alpha_2\alpha_3}G^{(3)}{}_{\beta_1\beta_2\beta_3}\}\,.
\end{aligned} \tag{26}
$$

The last equation is automatically satisfied once the self-duality of $G^{(5)}$ is taken into account, and we will eliminate it from now on. Taking the self-duality of $G^{(5)}$ into account the equation of $\mathcal{B}$ also takes a simpler form:

$$
\begin{aligned}
E^{\mu\nu}(\mathcal{B}) \;&=\; -\tfrac{1}{2}\left\{\nabla_\rho(e^{-2\varphi}\mathcal{H}^{\rho\mu\nu}) - G^{(3)\,\mu\nu\alpha}G^{(1)}{}_\alpha - \tfrac{1}{3!}G^{(5)\,\mu\nu\alpha_1\alpha_2\alpha_3}G^{(3)}{}_{\alpha_1\alpha_2\alpha_3}\right\} \\
&\quad -C^{(0)}E^{\mu\nu}(C^{(2)})\,.
\end{aligned} \tag{27}
$$

The supersymmetry variations of the bosonic fields are

$$\begin{aligned}
\delta_\varepsilon e_\mu{}^a &= -i\bar{\varepsilon}\Gamma^a\zeta_\mu\,,\\
\delta_\varepsilon\varphi &= -\tfrac{i}{2}\bar{\varepsilon}\chi\,,\\
\delta_\varepsilon\mathcal{B}_{\mu\nu} &= -2i\bar{\varepsilon}\sigma^3\Gamma_{[\mu}\zeta_{\nu]}\,,\\
\delta_\varepsilon C^{(0)} &= \tfrac{1}{2}e^{-\varphi}\bar{\varepsilon}\sigma^2\chi\,,\\
\delta_\varepsilon C^{(2)}{}_{\mu\nu} &= 2ie^{-\varphi}\bar{\varepsilon}\sigma^1\Gamma_{[\mu}\left(\zeta_{\nu]}-\tfrac{1}{4}\Gamma_{\nu]}\chi\right)+C^{(0)}\delta_\varepsilon\mathcal{B}_{\mu\nu}\,,\\
\delta_\varepsilon C^{(4)}{}_{\mu\nu\rho\sigma} &= -4e^{-\varphi}\bar{\varepsilon}\sigma^2\Gamma_{[\mu\nu\rho}\left(\zeta_{\sigma]}-\tfrac{1}{8}\Gamma_{\sigma]}\chi\right)+6C^{(2)}{}_{[\mu\nu}\delta_\varepsilon\mathcal{B}_{\rho\sigma]}\,,
\end{aligned}\tag{28}$$

and the KSI of type IIB supergravity are given by

$$\begin{aligned}
\bar{\kappa}\{E_a{}^\mu(e)\Gamma^a+E^{a\mu}(\mathcal{B})\sigma^3\Gamma_a-2E^{a\mu}(C^{(2)})[e^{-\varphi}\sigma^1-C^{(0)}\sigma^3]\Gamma_a\} &= 0\,,\\
\bar{\kappa}\{E(\varphi)+iE(C^{(0)})e^{-\varphi}\sigma^2+E^{ab}(C^{(2)})e^{-\varphi}\sigma^1\Gamma_{ab}\} &= 0\,.
\end{aligned}\tag{29}$$

If the equation of $C^{(2)}$ is satisfied, those of the two scalars $\varphi, C^{(0)}$ are automatically satisfied. Further, if the equation of $\mathcal{B}$ is satisfied, we arrive again at $\bar{\kappa}E_a{}^\mu(e)\Gamma^a=0$.

Another use (the one originally proposed in Ref. [5]) is to constrain the form of corrections (due to quantum effects or to the presence of external sources) to supersymmetric solutions. The main assumption here is that the supersymmetry transformation rules themselves do not get any corrections. Under these conditions, if the bosonic fields satisfy now the equations

$$S_{,b}=J_b\,,\tag{30}$$

then the sources J_b must satisfy

$$J_b\,(\delta_\kappa\phi^b)_{,f}\big|_{\phi^f=0}=0\,.\tag{31}$$

Since the integration of the sources gives the charges of the object that generates the fields of the solution, the KSI identities give BPS relations between those charges. Observe that this method does not allow for magnetic sources or charges, since the Bianchi identities are assumed to hold from the beginning, although perhaps it might be generalized to overcome this problem.

Let us consider a simple example: $N=2, d=4$ ungauged supergravity. The action for the bosonic fields $g_{\mu\nu}, A_\mu$ is

$$S=\int d^4x\sqrt{|g|}\,[R-\tfrac{1}{4}F^2]\,,\qquad F=2\partial A\,,\tag{32}$$

and the supersymmetry variations of the bosonic fields are

$$\begin{aligned} \delta_\epsilon e^a{}_\mu &= -i\bar{\epsilon}\gamma^a\psi_\mu + c.c.\,, \\ \delta_\epsilon A_\mu &= -2i\bar{\epsilon}\psi_\mu + c.c.\,. \end{aligned} \tag{33}$$

The equations of motion are

$$\begin{aligned} E_a{}^\mu(e) &= -2\{G_a{}^\mu - \tfrac{1}{2}\left[F_{ab}F^{\mu b} - \tfrac{1}{4}e_a{}^\mu F^2\right]\}\,, \\ E^\mu(A) &= \nabla_\alpha F^{\alpha\mu}\,, \end{aligned} \tag{34}$$

and the KSI are given by

$$\bar{\kappa}\{E_a{}^\mu(e)\gamma^a + 2E^\mu(A)\} = 0\,. \tag{35}$$

These equations lead to relations between sources as those found in Refs. [9] and [10] in which off-shell configurations of $N = 2, d = 4$ ungauged and gauged supergravity were considered.

Observe that, according to the standard argument, in the timelike case, these equations tell us that one only has to solve the Maxwell equations and Bianchi identities for the vector field strength in order to have a solution of the full set of equations of motion, and these equations reduce to just two equations for two real functions (combined into a complex function thanks to electric-magnetic duality). The same argument goes through in the gauged case, studied in Refs. [11, 12], where it can be seen that there are only two equations for two real functions becasue the extra real function and the equation that it satisfies can be deduced from the other two.

Defining sources for the fields $E_a{}^\mu(e) \equiv 2T_a{}^\mu$ and $E^\mu(A) = J^\mu$ and multiplying the KSI by $i\kappa$ from the right gives

$$T_a{}^\mu(e)V^a + aJ^\mu(A) = 0\,, \tag{36}$$

where we have defined the real bilinears

$$V^a = i\bar{\kappa}\gamma^a\kappa\,, \qquad a = i\bar{\kappa}\kappa\,. \tag{37}$$

Let us now make assume that

1. Our supersymmetric configuration satisfies the condition that all the components $E_a{}^{\underline{0}}(e)$, $a \neq 0$ vanish (which is valid for the kind of static configuration that we have in mind in this simple example). Then, taking $\mu = 0$ in the above equation, we get

$$T_0{}^{\underline{0}}(e)V^0 + aJ^{\underline{0}}(A) = 0\,, \tag{38}$$

2. The Killing spinor satisfies a projection condition of the form

$$(1 \pm \gamma^0)\kappa = 0\,. \tag{39}$$

Then, $V^0 = \mp a$ and we get a relation between gravitational and electric sources

$$T_0{}^{\underline{0}}(e) \mp J^{\underline{0}}(A) = 0\,, \tag{40}$$

that will give $M = |Q|$ upon integration.

Clearly, similar arguments a and use of projectors lead to the relation between mass and charge of the M2-brane in 11-dimensional supergravity.

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