7-Branes and Higher Kaluza-Klein Branes

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Abstract

We present and study a new chain of 10-dimensional T duality related solutions and their 11-dimensional parents whose existence had been predicted in the literature based in U duality requirements in 4 dimensions. The first link in this chain is the S dual of the D7-brane. The next link has 6 spatial worldvolume dimensions, it is charged w.r.t. the RR 7-form but depends only on 2 transverse dimensions since the third has to be compactified in a circle and is isometric and hence is similar in this respect to the KK monopole. The next link has 5 spatial worldvolume dimensions, it is charged w.r.t. the RR 6-form but, again, depends only on 2 transverse dimensions since the third and fourth have to be compactified in circles and are isometric and so on for the following links.

All these solutions are identical when reduced over the p spatial worldvolume dimensions and preserve a half on the available supersymmetries. Their masses depend on the square of the radii of the isometric directions, just as it happens for the KK monopole. We give a general map of these branes and their duality relations and show how they must appear in the supersymmetry algebra.

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Introduction

In the last few years there has been a lot of interest in discovering classical solutions of effective superstring theories (supergravity theories) with such properties that one could argue that they represent the fields produced by solitonic objects present in the superstring spectrum. The interplay between the knowledge of the superstring spectrum and the knowledge of classical solutions has been very fruitful since each of them has contributed to the increase of the other. The two most important tools used in this field have been supersymmetry and duality. Unbroken supersymmetry ensures in many cases the absence of corrections of the classical solutions and the lack of quantum corrections to the mass of the corresponding objects in the string theory spectrum. Hence, more effort has been put in finding supersymmetric (i.e. admitting Killing spinors) solutions, associated to BPS string states. Duality transformations preserve in general supersymmetry, relating different states in dual theories. In general [1], but not always [2] duality relations between different higher-dimensional theories manifest themselves as non-compact global symmetries of the compactified supergravity theory that leave invariant its equations of motion so one can use them to transform known solutions into new solutions, preserving their supersymmetry properties.

Thus, it so happens that most classical solutions of superstring effective field theories belong to chains or families of solutions related by duality transformations. The best known chain of solutions is that of the Dp-branes, with $p = 0, \ldots, 8$ in 10 dimensions. They belong to two different theories: 10-dimensional type IIA for p even and 10-dimensional type IIB for p odd. All of them preserve $1/2$ of the supersymmetries available, represent objects with p spatial worldvolume dimensions and $9 - p$ transverse dimensions (Dirichlet branes), carrying charge associated to the RR $(p + 1)$-form $\hat{C}^{(p+1)}$ whose existence was discovered by Polchinski [3], and are related by generalized Buscher type II T duality transformations [2, 4].

Sometimes it is possible to find families of solutions that are, by themselves, representations of the duality group in the sense that they are invariant, as families, under the full duality group. This is the case, for instance, of the SWIP solutions of $N = 4, d = 4$ supergravity constructed in Ref. [5, 6]. In that case one can argue that all the solitonic objects of a given type (charged, stationary, black holes) and preserving a certain amount of supersymmetry are described by particular solutions, with particular values of the parameters of that general family. More interesting cases are $N = 8$, $N = 4$ with 22 vector multiplets and general $N = 2$ theories, all in $d = 4$, but fully general solutions in their duality-invariant form are not available. A great deal is, however, known of the solitonic spectrum of the 4-dimensional theories due to our knowledge of their duality groups (the so-called U duality group in the $N = 8$ case). All these theories can be obtained from 10-dimensional theories by compactification (toroidal or more general) and the compactification of the solitonic 10-dimensional objects gives rise to 4-dimensional solitonic objects of different kinds, depending on how the 10-dimensional objects are wrapped in the internal dimensions and one can study if these objects fill 4-dimensional duality multiplets. It has been realized that this is not the case if one considers only the standard 10-dimensional solitons: the

Dp-branes, KK monopole, gravitational wave (W), fundamental string (F1) and solitonic 5-brane (S5) [7, 8, 9, 10]. More 10-dimensional solitons are needed to give rise to all the 4-dimensional solitons predicted by duality and some of the properties they should exhibit, in particular the dependence of the mass in the radii of the internal dimensions and the coupling constant, have been deduced.

In this paper we present candidates for some of the missing 10-dimensional solitons and study them. The key to their construction is the realization that there are 4-dimensional duality symmetries which are neither present in 10 dimensions nor are a simple consequence of reparametrization invariance in the internal coordinates. These are, in general, S duality (i.e. electric-magnetic) transformations which only exist in certain dimensions and that enable us to use the mechanism *reduction-S dualization-oxidation* to generate new solutions in higher dimensions.

Let us consider a familiar example: 5 dimensional gravity compactified in a circle. The 4-dimensional theory has electric-magnetic duality and one expects an S duality symmetric spectrum. However, if we only considered the 5-dimensional plane wave solution we would only find electrically charged 4-dimensional solitons. To find the magnetically charged ones we S dualize these and, oxidizing the solutions to 5 dimensions we find the Kaluza-Klein (KK) monopole [11, 12]. In principle, this is a solution one would not expect in 5 dimensions since it has one dimension necessarily compactified in a circle.

At this point it is useful to remember why the KK monopole has necessarily one compact dimension. First of all, it is known that only if the special isometric direction is compact with the right periodicity the solution is free from string singularities. Secondly, the mass (tension per unit (world-) volume) of the KK monopole depends quadratically on the radius of that direction. The masses of standard branes are always proportional to the volume of the internal manifold in which they are compactified and tend to infinity when those volumes tend to infinity. Thus, when considering uncompactified standard branes (all D-branes, the fundamental string and the solitonic 5-brane) the quantity of interest is not the mass, but the brane tension which is finite. In the KK monopole case, though, due to the qudratic dependence of the mass on the radius of the special isometric direction, the tension is proportional to the radius of that direction and diverges in the decompactification limit.

The solutions we present here can also be generated by a mechanismo similar to the one we have explained for the KK monopole, exploiting S dualities present in dimensions lower than 10 and 11 and also have properties similar to those of the KK monopole: there are dimensions that cannot be decompactified because the masses of these objects depend quadratically on the radii of those dimensions. Somehow this is consistent with the fact that they are generated using dualities that only exist if some of the dimensions are compact.

One of the problems raised by the need to consider new 10- and 11-dimensional solutions was that fact that the 10- and 11-dimensional supersymmetry algebras did not contain central charges associated to those possible new objects. In our opinion the predictive power of the supersymmetry algebras has been overestimated and we will propose a way to include in them these new objects.

The rest of this paper is organized as follows: in Section 1 we present our family of T duality-related solutions whose construction via the *reduction-S dualization-oxidation*

mechanism is explained in Section 2. In Section 3 we find other duality-related solutions in 10 and 11 dimensions. In Section 4 we calculate the dependence of the masses of these objects on compactification radii and coupling constants and in Section 5 we calculate the Killing spinors of all the solutions we have presented. Our conclusions are in Section 6. In Appendix B we derive the $SL(2,\mathbb{R})/SO(2)$ sigma model from toroidal compactification and explain how $SL(2,\mathbb{R})$ is broken to $SL(2,\mathbb{Z})$ and in Appendix A we briefly review holomorphic $(d-3)$-brane solutions of the $SL(2,\mathbb{R})/SO(2)$ sigma model to clarify certain points.

1 The Basic Family of Solutions

The basic family of solutions are solutions of the type II supergravity theories in $d = 10$ and are a sort of deformation of the family of Dp-brane solutions for $0 \leq p \leq 7$. As such, they have $p+1$ worldvolume coordinates $t, \vec{y}_p = (y^1, \ldots, y^p)$ and $9-p$ transverse coordinates. We combine two of them into the complex coordinate ω and the remaining $7-p$ we denote by $\vec{x}_{7-p} = (x^1, \ldots, x^{7-p})$. The solutions can collectively be written in the string-frame metric in the form[3]

$$
\begin{cases}
d\hat{s}_s^2 = \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1/2} \left(dt^2 - d\vec{y}_p^{\,2}\right) - \left(H\mathcal{H}\bar{\mathcal{H}}\right)^{1/2} d\omega d\bar{\omega} - \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{1/2} d\vec{x}_{7-p}^{\,2}\,, \\[4mm]
\hat{C}^{(p+1)}{}_{ty^1\ldots y^p} = (-1)^{\left[\frac{(p+1)}{2}\right]} \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\[4mm]
\hat{C}^{(7-p)}{}_{x^1\ldots x^{7-p}} = -\dfrac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \\[4mm]
e^{\hat{\phi}} = \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{\frac{3-p}{4}}\,,
\end{cases}
\tag{1.1}
$$

where we function $\mathcal{H} = \mathcal{H}(\omega)$ is a complex, holomorphic, (multivalued) function of ω, i.e. $\partial_{\bar{\omega}}\mathcal{H} = 0$ with the behavior $\mathcal{H} \sim \frac{1}{2\pi i} \log \omega$ around $\omega = 0$, where we assume the object is placed[4]. Its real and imaginary parts are

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[3]For convenience, we give the form of the potential to which the p-brane naturally couples $\hat{C}^{(p+1)}$ and the dual one $\hat{C}^{(7-p)}$. In the $p = 3$ case, these are the two non-vanishing sets of components of the 4-form potential with self-dual field strength. (Our conventions are those of Ref. [4] whose type II T duality rules, generalizing those of Ref. [2], we use.) Since the solutions we will be dealing with are not asymptotically flat, we do not write explicitly the asymptotic values of the scalars (for example, $\hat{\phi}_0$ for the dilaton).

[4]The analytic extension of $\mathcal{H}$ to the whole ω space (for which the above expression is clearly not valid) is a non-trivial problem that depends, among other things, on the topology assumed for the ω space. In general, it requires the introduction of other singularities around which $\mathcal{H}$ is also multivalued so that one gets a consistent monodromy. This problem was first considered in Ref. [21].

$$\mathcal{H} = A + iH \, . \tag{1.2}$$

These solutions have the same form as the standard Dp-brane solutions if we delete everywhere the combination $\mathcal{H}\bar{\mathcal{H}}$, but they are clearly different. In particular we can understand them as having $7 - p$ extra isometric directions that should be considered compact[5]. Our goal will be to understand how they arise, their M theoretic origin and their supersymmetry properties and explore the implications of it all. We will also find other solutions related by dualities with them or belonging to the same class. Since we will find that all these solutions preserve a half of the symmetries, we are going to argue that they describe the long range fields of elementary, non-perturbative objects of string theory and we will calculate their masses.

2 Construction of the Solutions

The solutions (1.1) can be obtained by successive T duality transformations in worldvolume directions of the $p = 7$ solution. The $p = 7$ solution is nothing but the type IIB solitonic 7-brane (S7) that was obtained by S duality from the D7-brane and called Q7-brane in Ref. [4]. The worldvolume directions are transformed into transverse isometric directions that should be considered compact[6]. Thus, we obtain a chain of T dual solutions of both type II theories.

There is an alternative way of constructing these solutions that also helps to understand them. Let us consider a piece of the 10-dimensional type II supergravity theories in which we only keep the metric, the dilaton and the field strength $\hat{G}^{(8-p)}$ of the RR $(7 - p)$-form $\hat{C}^{(7-p)}$. The action is

$$\hat{S} = \int d^{10}\hat{x} \sqrt{|\hat{g}|} \left\{ e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 \right] + \frac{(-1)^{7-p}}{2 \cdot (8-p)!} \left(\hat{G}^{(8-p)} \right)^2 \right\} \, . \tag{2.1}$$

Now, let us compactify it over a $(7 - p)$-torus using a simplified Kaluza-Klein Ansatz that only takes into account the volume modulus of the internal torus, the dilaton (both rewritten in terms of two convenient scalars φ and η), the internal volume mode of the RR $(7 - p)$-form, a and the $(3 + p)$-dimensional Einstein metric $g_{\mu\nu}$:

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[5]It seems difficult (it is perhaps impossible) to extend the dependence of the function $\mathcal{H}$ to those coordinates. Furthermore, the construction procedure *reduction-S dualization-oxidation* and the dependence of the masses on the radii of those dimensions that we are going to calculate later on suggest that those coordinates should be compactified on a torus.

[6]This is somewhat analogous to what happens in the well-known duality between the solitonic fivebrane S5 and the KK monopole in which a transverse direction of the S5 is T dualized into an isometric, compact, direction of the KK monopole.

$$\begin{cases} d\hat{s}^2 = e^{\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\frac{7-p}{p+1}}\eta} g_{\mu\nu} dx^\mu dx^\nu - e^{-\frac{1}{2}\varphi + \frac{1}{2}\sqrt{\frac{p+1}{7-p}}\eta} d\vec{x}_{7-p}^{\,2} \,, \\\\ \hat{C}^{(7-p)}{}_{x^1 \dots x^{7-p}} = a \,, \\\\ e^{\hat{\phi}} = e^{\frac{p-3}{4}\varphi + \frac{7-p}{4}\sqrt{\frac{p+1}{7-p}}\eta} \,. \end{cases} \tag{2.2}$$

After some straightforward calculations one obtains, in all cases, the reduced action

$$S = \int d^{p+3}x \sqrt{|g|} \left\{ R + \tfrac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{(\Im m\tau)^2} + \tfrac{1}{2}(\partial\eta)^2 \right\} \,, \tag{2.3}$$

where

$$\tau = a + ie^{-\varphi} \,, \tag{2.4}$$

i.e. gravity coupled to an $SL(2,\mathbb{R})/SO(2)$ sigma model parametrized in the standard form by the complex scalar (sometimes known as *axidilaton* although here this name could be misleading since in some cases ($p = 3$) the string dilaton simply does not contribute to it) τ and another scalar, η, decoupled from τ. In the $p = 7$ case ($d = 10$) this is the well-known piece of the type IIB supergravity action. In lower dimensions, it is integrated in much bigger sigma models associated to much bigger U-duality groups[7] but it is a most interesting part of it.

There is a very general solution of this model

$$\begin{cases} ds^2 = dt^2 - d\vec{y}_p^{\,2} - Hd\omega d\bar{\omega} \,, \\\\ \tau = \mathcal{H} \,, \\\\ \eta = 0 \,, \end{cases} \tag{2.5}$$

with $\partial_{\bar{\omega}}\mathcal{H} = 0$. In $d = 10$ ($p = 7$) this is just the general D7-brane solution. Choosing $\mathcal{H} \sim \log\omega$ we get the single D7-brane solution. In lower dimensions, these solutions are just compactifications of the standard general Dp-brane solution in which we have assumed that the harmonic function only depends on two transverse directions (ω) and we have dualized the RR ($p + 1$)-potential, giving rise to the real part of $\mathcal{H}$. Thus, this is a well-known solution.

We can now perform an $SL(2,\mathbb{R})$ duality rotation of this solution[8] $\tau \to -1/\tau$, since

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[7]In $d = 6$ dimensions, this model was studied in Ref. [13] and in $d = 8$ it was studied in Ref. [14].

[8]Continuous duality symmetries are usually broken to their discrete subgroups, for instance $SL(2,\mathbb{R})$ is usually broken to $SL(2,\mathbb{Z})$. This can be clearly seen in the case in which the $SL(2,\mathbb{R})/SO(2)$ sigma model originates in a toroidal compactification and is explained in Appendix B. In other cases one has to study the quantization of charges to arrive to the same conclusion. We will loosely use the continuous of the discrete form of the duality group in the understanding that in some contexts only the discrete one is really a symmetry of the theory.

this is a symmetry of the dimensionally reduced action[9] that leaves the Einstein metric invariant. This is not a symmetry of the 10-dimensional action and one really needs extra compact dimensions to establish it. The resulting solutions[10]

$$\begin{cases} ds^2 & = & dt^2 - d\vec{y}_p{}^2 - H d\omega d\bar{\omega} \,, \\ \\ \tau & = & -1/\mathcal{H} \,, \\ \\ \eta & = & 0 \,, \end{cases} \tag{2.6}$$

are nothing but the solutions Eqs. (1.1) reduced according to the above KK Ansatz

What we are doing here is similar to what one does in standard KK theory: reducing to 4 dimensions the 5-dimensional pp wave one obtains the electric, extreme KK black hole. Since the $d = 4$ theory has electric-magnetic duality as a symmetry, one can find the magnetic, extreme KK black hole and then uplift it to $d = 5$ to find the KK monopole [11, 12] that has a special isometric direction that cannot be decompactified. The symmetry between the pp wave and the KK monopole cannot be established without assuming one compact direction. It is only natural, by analogy, to consider here that the dimensions that we have compactified cannot be decompactified after the duality transformations. We will support this assumption not by geometrical arguments but calculating the masses of these objects and finding its dependence on the radii of those dimensions.

3 Duality-related Solutions and M-theoretic Origin

Since we are dealing with many new solutions, we first propose to denote them by "Dp_i" where "$p + 1$" is the worldvolume and "i" is the number of isometric directions. According to this notation, the solutions described by Eq. (1.1) are in the $p = 7$ case $D7_0$ (the type IIB S dual of the D7-brane, called Q7 in Ref. [4]), $D6_1$ for $p = 6$, and $D5_2$, $D4_3$, $D3_4$, $D2_5$, $D1_6$, $D0_7$ for the remaining cases.

For all the type IIB solutions in the class (1.1) we can find an S dual using the 10-dimensional type IIB S duality symmetry. While in the $p = 7$ case the S dual solution is just the well-known D7-brane, and in the $p = 3$ case the solution is self-dual, in the $p = 5, 1$ cases we find genuinely new solutions. For $D5_2$ we get a solution which is a deformation of the solitonic fivebrane, and we call $S5_2$

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[9]In general, it is only a symmetry of the equations of motion of the complete, untruncated, type II theory.

[10]In Appendix A we discuss these general solutions and in which sense they are new. We stress that we are considering only the choice holomorphic function $\mathcal{H} \sim \frac{1}{2\pi i} \log \omega$.

$$
\mathbf{S5_2} \quad
\begin{cases}
d\hat{s}_s^2 &= dt^2 - d\vec{y}_5{}^2 - H d\omega d\bar{\omega} - \dfrac{H}{\mathcal{H}\bar{\mathcal{H}}} d\vec{x}_2{}^2 \,, \\[3mm]
\hat{\mathcal{B}}_{x^1 x^2} &= -\dfrac{A}{\mathcal{H}\bar{\mathcal{H}}} \,, \\[3mm]
\hat{\tilde{\mathcal{B}}}_{ty^1\dots y^5} &= \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1} \,, \\[3mm]
e^{\hat{\phi}} &= \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{\frac{1}{2}} \,,
\end{cases}
\tag{3.1}
$$

and for $D1_6$, we get a sort of deformation of the fundamental string solution that we call $F1_6$

$$
\mathbf{F1_6} \quad
\begin{cases}
d\hat{s}_s^2 &= \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1} (dt^2 - dy^2 - H d\omega d\bar{\omega}) - d\vec{x}_6^2 \,, \\[3mm]
\hat{\mathcal{B}}_{ty} &= -\left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1} \,, \\[3mm]
\hat{\tilde{\mathcal{B}}}_{x^1\dots x^6} &= \dfrac{A}{\mathcal{H}\bar{\mathcal{H}}} \,, \\[3mm]
e^{\hat{\phi}} &= \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-\frac{1}{2}} \,.
\end{cases}
\tag{3.2}
$$

These two solutions only have non-trivial common sector NSNS fields and therefore they are also solutions of the heterotic string effective field theory. We can also understand these solutions by appealing to the existence in both cases of a reduced action of the form Eq. (2.3) that arises from the 10-dimensional actions

$$
\hat{S} = \int d^{10}\hat{x} \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2 \right] \,,
\tag{3.3}
$$

and

$$
\hat{S} = \int d^{10}\hat{x} \sqrt{|\hat{g}|} \left\{ e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 \right] + \tfrac{1}{2\cdot 7!} e^{2\hat{\phi}} \hat{\tilde{H}}^2 \right\} \,,
\tag{3.4}
$$

where $\hat{H}$ is the NSNS 3-form field strength and $\hat{\tilde{H}} = e^{2\hat{\phi}} \star \hat{H}$ is the dual 7-form field strength. Reducing the first action to 8 dimensions with the Ansatz

$$\begin{cases} d\hat{s}^2 &=& e^{\frac{1}{\sqrt{3}}\eta}g_{\mu\nu}dx^\mu dx^\nu - e^{-\varphi}d\vec{x}_2^{\,2}\,, \\[2mm] \hat{B}_{x^1x^2} &=& a\,, \\[2mm] e^{\hat{\phi}} &=& e^{\frac{\sqrt{3}}{2}\eta-\frac{1}{2}\varphi}\,, \end{cases} \tag{3.5}$$

and the second action down to 4 dimensions with the Ansatz

$$\begin{cases} d\hat{s}^2 &=& e^{\varphi}g_{\mu\nu}dx^\mu dx^\nu - e^{\frac{1}{\sqrt{3}}\eta}d\vec{x}_6^{\,2}\,, \\[2mm] \hat{\tilde{B}}_{x^1\cdots x^6} &=& -a\,, \\[2mm] e^{\hat{\phi}} &=& e^{\frac{1}{2}\varphi+\frac{\sqrt{3}}{2}\eta}\,, \end{cases} \tag{3.6}$$

we get in both cases Eq. (2.3) in 8 and 4 dimensions.

As for the M-theoretic origin of the type IIA solutions, they can be derived from the following 11-dimensional solutions through compactification of the 11th dimension (z): a pp wave with 7 extra isometries

$$\mathbf{WM_7} \quad d\hat{s}^2 = -2dtdz - \frac{H}{\mathcal{H}\bar{\mathcal{H}}}dz^2 - \mathcal{H}\bar{\mathcal{H}}d\omega d\bar{\omega} - d\vec{x}_7^{\,2}\,, \tag{3.7}$$

a deformation of the M2-brane

$$\mathbf{M2_6} \quad \begin{cases} d\hat{s}^2 &=& \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-2/3}(dt^2 - d\vec{y}_2^{\,2}) - H^{1/3}\left(\mathcal{H}\bar{\mathcal{H}}\right)^{2/3}d\omega d\bar{\omega} - \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{1/3}d\vec{x}_6^{\,2}\,, \\[4mm] \hat{\tilde{C}}_{ty^1y^2} = -\left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\[4mm] \hat{\tilde{\tilde{C}}}_{x^1\cdots x^6} = \dfrac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \end{cases} \tag{3.8}$$

a deformation of the M5-brane

$$
\mathbf{M5_3} \begin{cases} d\hat{s}^2 = \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1/3}(dt^2 - d\vec{y}_5{}^2) - H^{2/3}\left(\mathcal{H}\bar{\mathcal{H}}\right)^{1/3}d\omega d\bar{\omega} - \left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{2/3}d\vec{x}_3{}^2\,, \\[3mm] \hat{\hat{C}}_{ty^1\dots y^5} = -\left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)^{-1}\,, \\[3mm] \hat{\hat{C}}_{x^1x^2x^3} = -\dfrac{A}{\mathcal{H}\bar{\mathcal{H}}}\,, \end{cases}
$$

$$(3.9)$$

and the KK monopole (with no dependence on the 11th dimension)

$$\mathbf{KK7M} \quad d\hat{s}^2 = dt^2 - d\vec{y}_6{}^2 - H(d\omega d\bar{\omega} + dz^2) - H^{-1}\left(dy^7 - Adz\right)^2\,. \qquad (3.10)$$

In these four cases we can also trace the origin of the solution to the existence of a sector like that in Eq. (2.3) in the reduced action of 11-dimensional supergravity. In the purely gravitational cases, the action Eq. (2.3) can be derived from the dimensional reduction of the Einstein term alone as shown in detail in Appendix B. In the second and third cases, one needs the 6-form or the 3-form dual potential respectively.

In some cases the dimensional reduction of these 11-dimensional solutions in isometric directions different from z produce new 10-dimensional solutions. In particular, we get two purely gravitational solutions

$$\mathbf{W_6} \quad d\hat{s}^2 = -2dtdz - \frac{H}{\mathcal{H}\bar{\mathcal{H}}}dz^2 - \mathcal{H}\bar{\mathcal{H}}d\omega d\bar{\omega} - d\vec{x}_6{}^2\,, \qquad (3.11)$$

and the Kaluza-Klein monopole with no dependence in z

$$\mathbf{KK6} \quad d\hat{s}^2 = dt^2 - d\vec{y}_5{}^2 - H(d\omega d\bar{\omega} + dz^2) - H^{-1}\left(dy^7 - Adz\right)^2\,. \qquad (3.12)$$

In all cases (see Figure 1) we see that whenever we reduce the same 11-dimensional solution over 2 directions to 9 dimensions and we do it in different order, we get a pair of 9-dimensional solutions that form an $SL(2,\mathbb{R})$ $(SL(2,\mathbb{Z}))$ doublet and also originate from a type IIB $SL(2,\mathbb{R})$ $(SL(2,\mathbb{Z}))$ doublet as it must [2].

4 Masses

The mass of the Dp_i solutions can be calculated using S and T duality rules from the standard $D7$-brane and can be written in a general formula:

$$M_{Dp_i} = \frac{R_3\dots R_{p+2}(R_{p+3}\dots R_9)^2}{g^3\ell_s^{p+2i+1}}\,. \qquad (4.1)$$

The masses of the NSNS solutions found by S duality from the $D5_2$ and the $D1_6$ are

$$M_{S5_2} = \frac{R_3 \ldots R_7 (R_8 R_9)^2}{g^2 \ell_s^{10}},$$

$$M_{F1_6} = \frac{R_3 (R_4 \ldots R_9)^2}{g^4 \ell_s^{14}}.$$

(4.2)

The masses of the 11-dimensional objects from which the type IIA objects can be derived can be calculated using the relations between the 11-dimensional Planck length $\ell_{\text{Planck}}^{(11)}$ and the radius of the 11th dimension[11] R_{10} and the type IIA string coupling constant g_A and the string length ℓ_s $\ell_{\text{Planck}}{}^{(11)} = 2\pi \ell_s g_A^{1/3}$ and $R_{10} = \ell_s g_A$:

$$M_{M2_6} = \frac{R_3 R_4 (R_5 \ldots R_{10})^2}{(\ell_{\text{Planck}}^{(11)})^{15}},$$

$$M_{M5_3} = \frac{R_3 \ldots R_6 (R_7 R_8 R_9)^2 R_{10}}{(\ell_{\text{Planck}}^{(11)})^{12}}.$$

(4.3)

where $\ell_{\text{Planck}}^{(11)}$ is the reduced 11-dimensional Planck length $\ell_{\text{Planck}}^{(11)} = \ell_{\text{Planck}}^{(11)}/2\pi$.

These expressions should be compared with the well-known expression of the mass of the 11-dimensional KK monopole $KK7M$ when the special isometric direction is x^{10}

$$M_{KK7M} = \frac{R_4 \ldots R_9 R_{10}^2}{(\ell_{\text{Planck}}^{(11)})^9},$$

(4.4)

or the 10-dimensional KK monopole $KK6$ (A or B) when the special isometric direction is x^9

$$M_{KK6} = \frac{R_4 \ldots R_8 R_9^2}{g^2 \ell_s^8}.$$

(4.5)

In both cases the mass is not simply proportional to the volume of the brane which is assumed wrapped on a torus but depends quadratically on the radius of the special isometric direction. The same happens to the masses of all the Dp_i branes: they depend quadratically on the radii of the directions that we have argued are isometric, which supports our assumption.

Apart from the dependence on the radii we see that in general these objects are highly non-perturbative since their masses are proportional to g^{-3} and g^{-4} except for $S5_2$, whose mass goes like g^{-2}, as for any standard solitonic object.

The momentum of the WM_7 solution is

$$M_{WM_7} = \frac{(R_3 \ldots R_9)^2 R_{10}^3}{(\ell_{\text{Planck}}^{(11)})^{18}}.$$

(4.6)

---

[11] R_{11} is the conventional name in the literature. Here we use R_m for the radius of the coordinate x^m.

5   Killing Spinors and Unbroken Supersymmetries

It is important to find the amount of supersymmetry preserved by our solutions since, if they preserve less than one half of the total supersymmetry available, one could argue that they correspond to composite objects. Since all these solutions are related by S and T duality transformations to the $D7$-brane, which preserves exactly $1/2$ of the supersymmetries, it is to be expected that they will do so as well. Nevertheless, a direct calculation of the Killing spinors should always be performed since it will confirm our expectations and it will also provide us with projectors that will help us to associate the solutions to central charges in the supersymmetry algebra and therefore to identify them with supersymmetric states in the string spectrum.

We first calculate the Killing spinors of the Dp_i family of solutions with the obvious choice for the Vielbein basis[12]

$$e_{\underline{i}}{}^{i} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4} , \qquad e_{\underline{m}}{}^{m} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{-1/4} , \qquad e_{\underline{8}}{}^{8} = e_{\underline{9}}{}^{9} = \left(\frac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4} H^{1/2} . \qquad (5.1)$$

For the Type IIA solutions we use the supersymmetry transformation rules for the gravitino and dilatino which, in the purely bosonic background we are considering, take the form[13]

$$\begin{cases} \delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} & = \; \left[\partial_{\hat{\mu}} - \frac{1}{4} \; \hat{\omega}\!\!\!/_{\hat{\mu}} + \frac{i}{8} \frac{1}{(8-p)!} e^{\hat{\phi}} \; \hat{\mathcal{G}}^{(8-p)}\hat{\Gamma}_{\hat{\mu}}(-\hat{\Gamma}_{11})^{\frac{8-p}{2}}\right] \hat{\epsilon} \,, \\[2mm] \delta_{\hat{\epsilon}}\hat{\tau} & = \; \left[\partial\!\!\!/\hat{\phi} + \frac{i}{4} \frac{p-3}{(8-p)!} e^{\hat{\phi}} \; \hat{\mathcal{G}}^{(8-p)}(-\hat{\Gamma}_{11})^{\frac{8-p}{2}}\right] \hat{\epsilon} \,, \end{cases} \qquad (5.2)$$

Imposing the vanishing of dilatino transformation rule we obtain the following constraint in the Killing spinor:

$$\left[1 - i\hat{\Gamma}^{p+1}\cdots\hat{\Gamma}^{8}\hat{\Gamma}^{9}(-\hat{\Gamma}_{11})^{\frac{8-p}{2}}\right] \hat{\epsilon} = 0 \,, \qquad (5.3)$$

or, equivalently

$$\left[1 - (-1)^{[p/2]}i\hat{\Gamma}^{0}\cdots\hat{\Gamma}^{p}(-\hat{\Gamma}_{11})^{\frac{10-p}{2}}\right] \hat{\epsilon} = 0 \,. \qquad (5.4)$$

This constraint automatically sets to zero the worldvolume (t, y^i) and transverse, isometric (x^m) components of the supersymmetry variation of the gravitino. The remaining transverse components (x^8, x^9) give in *all* cases, the following coupled partial differential equations

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[12]Underlined indices are world indices and non-underlined indices are tangent space indices. They take values in the ranges $i = 0, 1, \ldots, p$, $m = p + 1, \ldots, 7$.

[13]Our type IIA spinors are full 32-component Majorana spinors.

$$\begin{cases} \delta_{\hat{\epsilon}}\hat{\psi}_{\underline{8}} = \left[\partial_{\underline{8}} - \tfrac{1}{4}\hat{\Gamma}^8\hat{\Gamma}^9\,\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}}) + \tfrac{1}{8}\,\partial_{\underline{8}}\log\left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)\right]\hat{\epsilon} = 0\,, \\[4mm] \delta_{\hat{\epsilon}}\hat{\psi}_{\underline{9}} = \left[\partial_{\underline{9}} - \tfrac{1}{4}\hat{\Gamma}^9\hat{\Gamma}^8\,\partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}}) + \tfrac{1}{8}\,\partial_{\underline{9}}\log\left(\dfrac{H}{\mathcal{H}\bar{\mathcal{H}}}\right)\right]\hat{\epsilon} = 0\,. \end{cases} \tag{5.5}$$

Now, using the Cauchy-Riemann equations for the holomorphic function $\mathcal{H}$, i.e.:

$$\partial_{\underline{8}}A = +\partial_{\underline{9}}H\,, \qquad \partial_{\underline{9}}A = -\partial_{\underline{8}}H\,, \tag{5.6}$$

we can express $\partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}})$ and $\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}})$ in the following way:

$$\partial_{\underline{9}}\log(\mathcal{H}\bar{\mathcal{H}}) = -2\partial_{\underline{8}}(\arg\mathcal{H})\,, \qquad \partial_{\underline{8}}\log(\mathcal{H}\bar{\mathcal{H}}) = +2\partial_{\underline{9}}(\arg\mathcal{H})\,, \tag{5.7}$$

and the Killing spinor equations are easily seen to be solved by

$$\begin{cases} \left[1 - (-1)^{[p/2]}i\hat{\Gamma}^0\cdots\hat{\Gamma}^p(-\hat{\Gamma}_{11})^{\frac{10-p}{2}}\right]\hat{\epsilon}_0 = 0\,, \\[4mm] \hat{\epsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\Gamma}^8\hat{\Gamma}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/8}\hat{\epsilon}_0\,. \end{cases} \tag{5.8}$$

$\hat{\epsilon}_0$ being any constant spinor satisfying the above constraint.

In the type IIB cases we use the relevant supersymmetry transformation laws[14]

$$\begin{cases} \delta_{\hat{\varepsilon}}\hat{\zeta}_{\hat{\mu}} = \left[\partial_{\hat{\mu}} - \tfrac{1}{4}\,\slashed{\omega}_{\hat{\mu}} + \tfrac{1}{8}\dfrac{1}{(8-p)!}e^{\hat{\varphi}}\,\hat{\mathcal{G}}^{(8-p)}\hat{\Gamma}_{\hat{\mu}}\mathcal{P}_{\frac{9-p}{2}}\right]\hat{\varepsilon}\,, \\[4mm] \delta_{\hat{\varepsilon}}\hat{\chi} = \left[\slashed{\partial}\hat{\varphi} + \tfrac{1}{4}\dfrac{3-p}{(8-p)!}e^{\hat{\varphi}}\,\hat{\mathcal{G}}^{(8-p)}\mathcal{P}_{\frac{9-p}{2}}\right]\hat{\varepsilon}\,, \end{cases} \tag{5.9}$$

where $\mathcal{P}_n$ is

$$\mathcal{P}_n\begin{cases} \sigma^1\,, & n \text{ even}\,, \\[2mm] i\sigma^2\,, & n \text{ odd}\,. \end{cases}$$

Proceeding as in the type IIA case, we find the Killing spinors

$$\begin{cases} \left[1 + (-1)^{[p/2]}\hat{\Gamma}^0\cdots\hat{\Gamma}^p\mathcal{P}_{\frac{p+1}{2}}\right]\hat{\varepsilon}_0 = 0\,, \\[4mm] \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\Gamma}^8\hat{\Gamma}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/8}\hat{\varepsilon}_0\,, \end{cases} \tag{5.10}$$

where, now, $\hat{\varepsilon}_0$ is any pair of constant positive-chirality Majorana-Weyl spinors satisfying the above constraint.

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[14]Our type IIB spinors are pairs (whose indices 1,2 are not explicitly shown of 32-component, positive chirality, Majorana-Weyl spinors. Pauli matrices act on the indices not shown.

The Killing spinors of the $S5_2$ and the $F1_6$ can be found in a similar fashion and are, respectively

$$\begin{cases} \left[1 - \hat{\Gamma}^6\hat{\Gamma}^7\hat{\Gamma}^8\hat{\Gamma}^9\sigma^3\right]\hat{\varepsilon}_0 = 0\,, \\[2ex] \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat{\Gamma}^8\hat{\Gamma}^9}\hat{\varepsilon}_0\,, \end{cases}$$

and

$$\begin{cases} \left[1 + \hat{\Gamma}^0\hat{\Gamma}^1\sigma^3\right]\hat{\varepsilon}_0 = 0\,, \\[2ex] \hat{\varepsilon} = e^{-\frac{1}{2}\arg(\mathcal{H})\,\hat{\Gamma}^8\hat{\Gamma}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4}\hat{\varepsilon}_0\,, \end{cases}$$

Before discussing these results it is worth finding the Killing spinors of the 11-dimensional solutions. The only relevant supersymmetry transformation rule is that of the gravitino, which with our conventions is:

$$\delta_{\hat{\epsilon}}\hat{\psi}_{\hat{\mu}} = \left[2\partial_{\hat{\mu}} - \tfrac{1}{2}\,\hat{\slashed{\omega}}_{\hat{\mu}} + \tfrac{i}{144}\left(\hat{\hat{\Gamma}}^{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}{}_{\hat{\mu}} - 8\hat{\hat{\Gamma}}^{\hat{\beta}\hat{\gamma}\hat{\delta}}\hat{\eta}_{\hat{\mu}}{}^{\hat{\alpha}}\right)\hat{\hat{G}}_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}}\right]\hat{\hat{\epsilon}}\,. \tag{5.11}$$

In the obvious Vielbein basis we find, for the WM_7 solution

$$\begin{cases} \left[1 - \hat{\hat{\Gamma}}^0\hat{\hat{\Gamma}}^{10}\right]\hat{\hat{\epsilon}}_0 = 0\,, \\[2ex] \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^8\hat{\hat{\Gamma}}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/4}\hat{\hat{\epsilon}}_0\,. \end{cases} \tag{5.12}$$

for the $M2_6$ solution

$$\begin{cases} \left[1 + i\hat{\hat{\Gamma}}^0\hat{\hat{\Gamma}}^1\hat{\hat{\Gamma}}^2\right]\hat{\hat{\epsilon}}_0 = 0\,, \\[2ex] \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^8\hat{\hat{\Gamma}}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/6}\hat{\hat{\epsilon}}_0\,, \end{cases} \tag{5.13}$$

for the $M5_3$ solution

$$\begin{cases} \left[1 - \hat{\hat{\Gamma}}^0\cdots\hat{\hat{\Gamma}}^4\hat{\hat{\Gamma}}^{10}\right]\hat{\hat{\epsilon}}_0 = 0\,, \\[2ex] \hat{\hat{\epsilon}} = e^{-\frac{1}{2}\arg(\mathcal{H})\hat{\hat{\Gamma}}^8\hat{\hat{\Gamma}}^9}\left(\dfrac{\mathcal{H}\bar{\mathcal{H}}}{H}\right)^{1/12}\hat{\hat{\epsilon}}_0\,, \end{cases} \tag{5.14}$$

and for the $KK7M$ solution, as it is well known, the Killing spinor is any constant spinor $\hat{\hat{\epsilon}}_0$ satisfying the constraint

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$$\left[1 + i\hat{\hat{\Gamma}}^0 \cdots \hat{\hat{\Gamma}}^6\right] \hat{\hat{\epsilon}}_0 = 0 \,. \tag{5.15}$$

In all cases one can see that these solutions preserve one half of the supersymmetries.

6 Conclusions

In this paper we have presented new 10-dimensional solutions of the type IIB theories that can be thought of as having a certain number of isometric, compact, dimensions, that cannot be decompactified (one could say that these are really solutions of lower-dimensional theories) and which we have referred generically to as "KK-branes". We have described how they can be obtained via the *reduction-S dualization-oxidation* which could explain why some of the directions have to be compactified in circles since S duality only exists in the compactified theory. Furthermore, we have computed the masses of these solutions and we have found that they depend on the square of the radii of the directions that we have identified as compact, just as it happens in the KK monopole case, which is consistent with our identification. The mass formula are also coincident with what is needed to complete the U duality invariant spectrum of $N = 8, d = 4$ supergravity [8, 9, 10]. It has also been recently argued that the presence of certain KK-branes is necessary to explain from the M theory point of view the existence of some massive/gauged type II supergravities in lower dimensions [15].

Perhaps the only element that does not seem to fit in the picture we are putting forward is the supersymmetry algebra since there seems to be no place in it for the new objects. For the sake of concreteness we will focus in the 11-dimensional supersymmetry algebra ("M algebra") but the problems and the solutions we propose can be applied in the obvious way to other cases.

The M algebra is usually written, up to convention-dependent numerical factors c, c_n, in the form[15]

$$\left\{Q^\alpha, Q^\beta\right\} = c\left(\Gamma^a \mathcal{C}^{-1}\right)^{\alpha\beta} P_a + \frac{c_2}{2}\left(\Gamma^{a_1 a_2} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(2)}_{a_1 a_2} + \frac{c_5}{5!}\left(\Gamma^{a_1 \cdots a_5} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(5)}_{a_1 \cdots a_5} \,. \tag{6.1}$$

A lightlike component of the momentum is then associated to the gravitational waves moving in that direction, the spatial components of $\mathcal{Z}^{(2)}$ and $\mathcal{Z}^{(5)}$ are associated respectively to $M2$- and $M5$-branes wraped in those directions. The timelike components have more complicated interpretations: in the $\mathcal{Z}^{(5)}$ case, they are associated to the KK monopole in a complicated way and in the $\mathcal{Z}^{(2)}$ case they are associated to an object that we would call the $KK9$-brane of which we only know that it should give the $D8$-brane upon dimensional reduction. All these objects break (preserve) a half of the available supersymmetries and strict relations between their masses and charges can be derived from the algebra.

Clearly the M algebra contains a good deal of information about the solitons of the theory that realizes it (11-dimensional supergravity or M theory). However, it is clear that

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[15]See e.g. [16].

it does not contain all the information about them. To start with, it does not tell us why some branes are fundamental and some are solitonic, it does not tell us why some objects exist in the uncompactified theory (the wave, $M2$ and $M5$) while other objects only exist when one dimension is compactified in a circle (the KK monopole and the $KK9$-brane). Furthermore, all solitonic objects should be associated to spacelike components of central charges: that is the result we would always get if we performed the calculation. All this is not so surprising: the M algebra is not derived from the theory and their solutions but just by imposing consistency of the possible central charges. If we were able to derive the algebra from M theory and its solitonic solutions, the central charges would be associated to specific objects and we would know whether they have compact dimensions or not. Since we do know many things about the solitonic solutions, we can try to reflect what we know in a form of the M algebra mathematically consistent and then we can check if the results are consistent with dualities.

To start with, we consider the M algebra with the most general central extensions allowed:

$$\{Q^\alpha, Q^\beta\} = c\left(\Gamma^a \mathcal{C}^{-1}\right)^{\alpha\beta} P_a + \sum_{n=2,5,6,9,10} \frac{c_n}{n!} \left(\Gamma^{a_1\cdots a_n} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(n)}_{a_1\cdots a_n}. \tag{6.2}$$

We know the wave is associated to P, the $M2$-brane to $\mathcal{Z}^{(2)}$ and the M5-brane to $\mathcal{Z}^{(5)}$. We also know [17] that the KK monopole is a sort of 6-brane with one of the 4 possible transverse dimensions wrapped in a circle. We are going to reflect this fact by writing, instead of just the $\mathcal{Z}^{(6)}$ term as above, the term

$$\frac{c_6}{6!} \left(\Gamma^{a_1\cdots a_6} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(7)}_{a_1\cdots a_6 a_7} k^{a_7}, \tag{6.3}$$

where k^a is a vector pointing in the compact direction.

We also know that the $KK9$-brane (or $M9$-brane) [18] has 9 spacelike worldvolume dimensions one of which is always wrapped on a circle. We reflect this fact by writing, instead of just the $\mathcal{Z}^{(9)}$ term as above, the term

$$\frac{c_9}{9!} \left(\Gamma^{a_1\cdots a_9} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(8)}_{a_1\cdots a_8} l_{a_9}, \tag{6.4}$$

where l_a is a vector pointing in the direction around which the $KK9$-brane is wrapped.

We do not know of any brane associated to $\mathcal{Z}^{(10)}$ and so we will not consider it in the M algebra, which takes the form

$$\begin{aligned} \{Q^\alpha, Q^\beta\} &= c\left(\Gamma^a \mathcal{C}^{-1}\right)^{\alpha\beta} P_a + \frac{c_2}{2}\left(\Gamma^{a_1 a_2} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(2)}_{a_1 a_2} + \frac{c_5}{5!}\left(\Gamma^{a_1\cdots a_5} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(5)}_{a_1\cdots a_5} \\ &\quad + \frac{c_6}{6!}\left(\Gamma^{a_1\cdots a_6} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(7)}_{a_1\cdots a_6 a_7} k^{a_7} + \frac{c_9}{9!}\left(\Gamma^{a_1\cdots a_9} \mathcal{C}^{-1}\right)^{\alpha\beta} \mathcal{Z}^{(8)}_{a_1\cdots a_8} l_{a_9}. \end{aligned} \tag{6.5}$$

We could certainly write more general central charges by allowing more vectors to be present in the algebra, meaning allowing objects with more isometric directions such as

the $M2_6$ or the $M5_3$ branes presented in this paper. However, considering objects with just one special isometry will be enough to present our ideas.

Let us now reduce this algebra in one dimension. From each of the standard central charges we get two central charges in one dimension less, namely $P, \mathcal{Z}^{(0)}$ from P, $\mathcal{Z}^{(1)}, \mathcal{Z}^{(2)}$ from $\mathcal{Z}^{(2)}$ and $\mathcal{Z}^{(4)}, \mathcal{Z}^{(5)}$ from $\mathcal{Z}^{(5)}$, corresponding to the known reductions of M theory solitons: wave and $D0$-brane from the wave, $F1$ and $D2$-brane from the $M2$-brane and $D4$- and $S5$-brane from the $M5$-brane. From each of the new charges we have introduced we get instead three lower dimensional central charges: from the contraction $\mathcal{Z}^{(7)}k$ associated to the KK monopole we get a $\mathcal{Z}^{(6)}$ associated to the $D6$-brane when k points in the direction we are reducing, we get a contraction $\mathcal{Z}^{(6)}k$ associated to the type IIA KK monopole ($KK6A$) if we reduce on the KK monopole worldvolume and we get a $\mathcal{Z}^{(7)}k$ associated to the $D6_1$ (called $KK7A$ in Ref. [4], also studied in Ref. [19]) if we reduce in a transverse direction. From the product $\mathcal{Z}^{(8)}l$ we get a $\mathcal{Z}^{(8)}$, associated to the $D8$-brane when we reduce the KK9-brane in the isometric direction l points to, we get a product $\mathcal{Z}^{(7)}l$ associated to an object with the same features of the M theory $KK9$-brane but in one dimension less and a product $\mathcal{Z}^{(8)}l$ associated to a type IIA spacetime filling $KK9$-brane referred to as $NS-9A$-brane in Ref. [20]. The result is the following form of the type IIA supersymmetry algebra:

$$
\begin{aligned}
\left\{Q^\alpha, Q^\beta\right\} \;=\;\; & c\,(\Gamma^a \mathcal{C}^{-1})^{\alpha\beta}\, P_a + \textstyle\sum_{n=0,1,4,8} \frac{c_n}{n!}\,(\Gamma^{a_1\cdots a_n}\Gamma_{11}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(n)}_{a_1\cdots a_n} \\[2mm]
& + \textstyle\sum_{n=2,5,6} \frac{c_n}{n!}\,(\Gamma^{a_1\cdots a_n}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(n)}_{a_1\cdots a_n} \\[2mm]
& + \frac{c_5}{5!}\,(\Gamma^{a_1\cdots a_5}\Gamma_{11}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(6)}_{a_1\cdots a_5 a_6}k^{a_6} + \frac{c_6}{6!}\,(\Gamma^{a_1\cdots a_6}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(7)}_{a_1\cdots a_6 a_7}l^{a_7} \\[2mm]
& + \frac{c_8}{8!}\,(\Gamma^{a_1\cdots a_8}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(7)}_{a_1\cdots a_7}m_{a_8} + \frac{c_9}{9!}\,(\Gamma^{a_1\cdots a_9}\mathcal{C}^{-1})^{\alpha\beta}\, \mathcal{Z}^{(8)}_{a_1\cdots a_8}n_{a_9}\,.
\end{aligned}
\tag{6.6}
$$

Every known solitonic solution of the type IIA supergravity theory has an associated charge in this algebra. If we now reduce again to nine dimensions we will get the algebra of the massive 9-dimensional theories presented in Ref. [4] with $SL(2,\mathbb{Z})$ covariance. This is possible only because we have allowed for charges corresponding to KK-branes in 11 dimensions. To get the same algebra from the type IIB side a charge has to be introduced for the $S7$ brane which, even though it does not carry any $SO(2)$ R-symmetry indices, is not invariant but is interchanged with the $D7$-brane charge under S duality. We will present these results elsewhere.

Acknowledgments

A Holomorphic $(d-3)$-Branes

In this Appendix we briefly discuss holomorphic $(d-3)$-brane solutions of the d-dimensional $SL(2, \mathbb{R})/SO(2)$ sigma model

$$S = \int d^d x \sqrt{|g|} \left\{ R + \tfrac{1}{2} \frac{\partial \tau \partial \bar\tau}{(\Im m \tau)^2} \right\} , \tag{A.1}$$

where τ lives in the complex upper half plane and is defined up to modular $PSL(2, \mathbb{Z})$ transformations, so multivalued solutions are allowed if the value of τ changes by a modular transformation.

$(d-3)$-brane-type solutions of this model were first considered in Ref. [21] in $d = 4$. In these dimensions $(d-3)$-branes are strings. In that reference, the following general solution of the above model was found[16]

$$\begin{cases} ds^2 &= dt^2 - d\vec{y}_{(d-3)}^2 - H d\omega d\bar\omega \, , \\[2mm] \tau &= \mathcal{H} \, , \end{cases} \tag{A.2}$$

where $\mathcal{H}$ is, in principle, any complex holomorphic or antiholomorphic function of the complex variable ω (i.e. either $\partial_{\bar\omega} \mathcal{H} = 0$ or $\partial_\omega \mathcal{H} = 0$) and $H = \Im m(\mathcal{H})$. H is, therefore, a real harmonic function of the 2-dimensional Euclidean spacetime transverse to the $(d-2)$-dimensional worldvolume directions. Only functions with $H \geq 0$ are admissible.

A few remarks are in order here: although $g_{\omega\bar\omega} = H$ is in this solution equal to the imaginary part of τ, it does not transform under $PSL(2, \mathbb{Z})$. Modular invariance of the metric is, therefore, not an issue. We could have wrongly concluded that in this solution, the metric is not modular invariant because $g_{\omega\bar\omega} = \Im m(\tau)$ but, by definition, it is, since the metric does not transform under $PSL(2, \mathbb{Z})$. Then, the l.h.s. if that equation does not transform, and the r.h.s. does, and we get a new solution (denoted by primes) with

$$\begin{aligned} \tau'(\omega) &= \frac{a\tau(\omega) + b}{c\tau(\omega) + d} = \frac{a\mathcal{H} + b}{c\mathcal{H} + d} \equiv \mathcal{H}' \, , \\[3mm] g'_{\omega\bar\omega} &= g_{\omega\bar\omega} = \Im m(\tau) = \frac{\Im m(\tau')}{|-c\tau' + a|^2} = \frac{\Im m(\mathcal{H}')}{|-c\mathcal{H}' + a|^2} \, . \end{aligned} \tag{A.3}$$

We could remove if we wished the extra factor by a conformal reparametrization:

$$d\omega' = \frac{d\omega}{-c\mathcal{H}'(\omega) + a} \, , \tag{A.4}$$

and we then could write again the new solution in a form similar to that of the original one Eq. (A.2) but with a new holomorphic function $\mathcal{H}'[\omega(\omega')]$. Thus, as in Ref. [21] we could have written from the beginning the general solution in the form

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[16]Here we write the obvious generalization to any dimension d (see also Refs. [22, 4]).

$$\begin{cases} ds^2 &=& dt^2 - d\vec{y}_{(d-3)}^{\,2} - H|f(\omega)|^2 d\omega d\bar{\omega}\,, \\ \\ \tau &=& \mathcal{H}\,, \end{cases} \tag{A.5}$$

where $f(\omega)$ is any holomorphic function of ω, but this function can always be reabsorbed into a holomorphic coordinate change $\omega' = F(\omega)\,,\quad dF/d\omega = f$ and $\tau(\omega') = \tau[F^{-1}(\omega')]$.

All this said, it must be acknowledged that, even though modular invariance of the metric is not an issue, its single-valuedness is. Since $\mathcal{H}$ will in general be a multivalued function with monodromies in G, its imaginary part will also be multivalued and it might be necessary to multiply it by $|f(\omega)|^2$, with $f(\omega)$ multivalued to make $g_{\omega\bar{\omega}}$ single valued.

A second remark we can make here is that there exists another form of the general solution which is manifestly $SL(2,\mathbb{R})$ invariant without having to invoke coordinate changes to show it:

$$\begin{cases} ds^2 &=& dt^2 - d\vec{y}_p^{\,2} - e^{-2U} d\omega d\bar{\omega}\,, \\ \\ \tau &=& \mathcal{H}_1/\mathcal{H}_2\,, \\ \\ e^{-2U} &=& \Im\mathrm{m}\left(\mathcal{H}_1 \bar{\mathcal{H}}_2\right)\,, \end{cases} \tag{A.6}$$

where $\mathcal{H}_{1,2}$ are two arbitrary complex functions of the complex variable ω transforming as a doublet under $SL(2,\mathbb{R})$, i.e.

$$\begin{pmatrix} \mathcal{H}_1' \\ \mathcal{H}_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{pmatrix}\,, \tag{A.7}$$

both in τ and in the metric (but e^{-2U} is invariant, as it must). The structure of this family is similar to that of the duality-invariant families of black-hole solutions of pure $N = 4, d = 4$ supergravity presented in Refs. [23, 5, 6], closely related to special geometry objects as discovered in [24]. We can relate this general solution either to the solution Eq. (A.2) as the particular case $\mathcal{H}_1 = \mathcal{H}\,,\ \mathcal{H}_2 = 1$ or to the solution Eq. (A.5) as the particular case $\mathcal{H}_1/\mathcal{H}_2 = \mathcal{H}\,,\ f = \mathcal{H}_2$ since $\Im\mathrm{m}(\mathcal{H}_1\bar{\mathcal{H}}_2) = |\mathcal{H}_2|^2\Im\mathrm{m}(\mathcal{H}_1/\mathcal{H}_2)$.

All this means that we cannot generate new solutions not in this classes via $SL(2,\mathbb{R})$ transformations.

Since all these solutions are equivalent, up to coordinate transformations, we take now Eq. (A.2) and now consider the choice of function $\mathcal{H}$. First, we have to choose between holomorphic and anti-holomorphic $\mathcal{H}$. This choice is related to the choice between $(d-3)$-branes and anti-$(d-3)$-branes with opposite charge with respect to the $(d-2)$-form potential dual to a. The impossibility of having $\mathcal{H}$ depending on both ω and $\bar{\omega}$ is due to the impossibility of having objects with opposite charges in equilibrium. We opt for holomorphy.

Which holomorphic function should one choose? As usual, the choice has to be based on local and global conditions. Local conditions are essentially related to the existence of

extended sources (with $(d-3)$ spatial dimensions) at given points in transverse (ω) space manifold. Global conditions are essentially related to the choice of global transverse space. Not all local conditions are possible for a given choice of transverse space. For instance, there is no holomorphic function for a single $(d-3)$-brane in the Riemann sphere[17].

To clarify these issues, let us consider the simplest solution in this class: let us couple the action Eq. (A.1) to a charged $(d-3)$-brane source. We first have to dualize the pseudoscalar a into a $(d-2)$-form potential $A_{(d-2)}$ with field strength $F_{(d-1)} = (d-1)\partial A_{(d-2)}$: $\partial a = e^{-2\varphi} \star F_{(d-1)}$. The bulk plus brane action is

$$
\begin{aligned}
S \;=\; & \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{|g|} \left\{ R + \tfrac{1}{2}(\partial\varphi)^2 + \frac{1}{2\cdot(d-1)!} F_{(d-1)}^2 \right\} \\[2mm]
& - \frac{T}{2} \int d^{d-2}\xi \sqrt{|\gamma|} \left\{ e^{\frac{2}{(d-2)}\varphi} \gamma^{ij} g_{ij} - (d-4) \right\} \\[2mm]
& - \alpha \frac{T}{(d-2)!} \int d^{d-2}\xi \, A_{(d-2)\; i_1\cdots i_{(d-2)}} \epsilon^{i_1\cdots i_{(d-2)}} \,,
\end{aligned}
\tag{A.8}
$$

where g_{ij} and $A_{(d-2)\; i_1\cdots i_{(d-2)}}$ are the pullbacks through the embedding coordinates $X^\mu(\xi)$ of the metric and $(d-2)$-form potential. T is the tension (in principle, a positive number) and $\alpha = \pm 1$ gives the sign of the charge (which is evidently proportional to the tension). The coupling to φ is the only one that allows for solutions of the form we want.

A solution is provided by

$$
\begin{cases}
ds^2 \;=\; dt^2 - d\vec{y}_{(d-3)}^{\,2} - H d\vec{x}_2^{\,2}\,, \\[2mm]
e^{-\varphi} \;=\; H\,, \\[2mm]
A_{(d-2)\; ty^1\cdots y^{(d-3)}} \;=\; \alpha H^{-1}\,, \\[2mm]
Y^i \;=\; \xi^i\,, \qquad \vec{X}_2 = 0\,,
\end{cases}
\tag{A.9}
$$

where H satisfies the equation

$$
\partial^2 H = -16\pi G_N^{(d)} T \delta^{(2)}(\vec{x}_2)\,,
\tag{A.10}
$$

i.e. it is a harmonic function with a pole at $\vec{x}_2 = 0$, where the brane is placed. The above equation is solved by a function H that behaves near $\vec{x}_2 = 0$

$$
H \sim -8 G_N^{(d)} T \log |\vec{x}_2|\,.
\tag{A.11}
$$

---

[17]of course, one meets the same situation for other branes. However, for smaller branes one can always find harmonic functions with a single pole (describing a single brane) that lead to spaces asymptotically flat in transverse directions. This is not true for higher ($(d-3)$- and $(d-2)$-) branes.

It is clear that this solution cannot be globally correct as H becomes negative for $|\vec{x}_2| > 1$, but the local behavior of the global solution has to be the same. Any solution behaving in this way at any given point will describe a $(d-3)$-brane placed there.

Let us now compute the charge. This is defined by

$$p = \oint_\gamma e^{-2\varphi} \star F_{(d-1)} = \oint_\gamma da \,, \tag{A.12}$$

where γ is a closed loop around the origin. a is given by

$$\partial_{\underline{n}} a = \alpha \epsilon_{nm} \partial_{\underline{m}} H \,, \tag{A.13}$$

i.e. combining $x^1 + ix^2 \equiv \omega$

$$\begin{cases} \partial_{\bar{\omega}}\tau = 0 \,, & \alpha = +1 \,, \\[2mm] \partial_{\bar{\omega}}\bar{\tau} = 0 \,, & \alpha = -1 \,, \end{cases} \tag{A.14}$$

that is: a is the real part of a holomorphic or antiholomorphic function of ω, whose imaginary part is the above function H. We find $a = \alpha 8 G_N^{(d)} T \, \mathrm{Arg}(\omega)$ and $p = \alpha \frac{1}{16\pi G_N^{(d)}} T$. The choice $\alpha = +1$ then, corresponds to a single $(d-3)$-brane with charge $p = +\frac{1}{16\pi G_N^{(d)}} T$ placed at the origin and corresponds to a holomorphic function $\tau = \mathcal{H}(\omega)$ that close to the origin is given by

$$\mathcal{H} \sim -8 G_N^{(d)} T i \log \omega \,. \tag{A.15}$$

Observe that the charge is given by the multivaluedness of τ around the source, which goes from τ to $\tau + 16\pi G_N^{(d)} T$ which should be identified with τ. The charge is usually quantized due to quantum-mechanical reasons in multiples of the unit of charge (e, say) which implies the identification $\tau \equiv \tau + ne$ and the breaking of $SL(2,\mathbb{R})$. If $e = 1$ (i.e. $16\pi G_N^{(d)} T = 1$ which we can always get by rescaling τ) then $SL(2,\mathbb{Z})$ is the unbroken symmetry of the theory and the above $(d-3)$-branes are associated to the modular group element T[18].

We see that in this context solutions (and charges) can be characterized by the non-trivial monodromies around singular points which, by hypothesis, are elements of the modular group.

We can clearly generate via modular (duality) transformations of this solution with T monodromy other solutions with different monodromies. it is easy to see that if we perform a transformation $\tau \to M(\tau)$ $M \in PSL(2\mathbb{Z})$ on the above solution, the monodromy of the new solution around the origin will be MTM^{-1}. The most interesting modular transformation is $S(\tau) = -1/\tau$ which in other contexts relates electric and magnetic ("S dual") objects. Then, the S dual of the above solution will have monodromy STS around the

---

[18]For 10-dimensional type IIB D7-branes $16\pi G_N^{(10)} = (2\pi)^7 \ell_s^8 g^2$ and $T = (2\pi)^{-7} \ell_s^{-8} g$, and, thus, $\mathcal{H} \sim -\frac{gi}{2\pi} \log \omega$. On the other hand, $C^{(8)}_{ty^1\ldots y^7} = g^{-1} H^{-1}$ ($\alpha = +1$) and we get $p = 1$ in a most natural way.

origin and will be given either by $\mathcal{H} = -\frac{2\pi i}{\log \omega}$ using the general solution in the form of Eq. (A.2) or with $\mathcal{H} = \frac{1}{2\pi i} \log \omega$ and the form (2.6) of the solution. This is the form we have used in the main text to stress that we are dealing with a solution different from the one with monodromy T, the difference being in the choice of holomorphic function since, as we have stressed at the beginning of this Appendix all homomorphic solutions can always be written in the form (A.2), no matter if the monodromy is T or STS.

B   The KK Origin of the $SL(2,\mathbb{R})/SO(2)$ Model

We are going to see how the modular group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\mathbb{I}_{2\times 2}\}$ and the $SL(2,\mathbb{Z})/SO(2)$ sigma model arise in standard Kaluza-Klein compactification on a 2-torus T^2.

B.1   The Modular Group

As usual in KK compactifications, we use two periodic coordinates x^m $m = 1, 2$ whose periodicity is fixed to $2\pi\ell$ where ℓ is some fundamental length. This means that we make the identifications

$$\vec{x} \sim \vec{x} + 2\pi\ell\vec{n}, \qquad \vec{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \qquad \vec{n} \in \mathbb{Z}^2. \tag{B.1}$$

The information on relative sizes and angles of the periods and the size of the torus is codified in the internal metric G_{mn},

$$ds_{\text{Int}}^2 = d\vec{x}^T G d\vec{x}, \tag{B.2}$$

which is, by hypothesis, independent on the torus coordinates $\vec{x}$, (but may depend on the remaining coordinates).

The KK Ansatz is invariant under global diffeomorphisms in the internal manifold. These are, generically, of the form

$$\vec{x}\,' = R^{-1}\vec{x} + \vec{a}, \qquad R \in GL(2,\mathbb{R})\vec{a} \in \mathbb{R}^2. \tag{B.3}$$

$\vec{a}$ simply shifts the coordinate origin and does not affect the metric. R acts on the internal metric according to

$$G' = R^T G R, \qquad (G_{mn} = R^p{}_m G_{pq} R^q{}_n). \tag{B.4}$$

We want to separate the volume part of the metric from the rest[19]. Thus, we define[20]

$$K \equiv |\det G_{mn}|, \qquad G_{mn} \equiv -K^{1/2}\mathcal{M}_{mn}. \tag{B.5}$$

---

[19]This is necessary, for instance, when we are interested in conformal classes of equivalence of metrics, as in string path integrals, but convenient in general.

[20]Remember that G has signature $(--)$.

$\mathcal{M}$ has determinant $+1$ and, therefore, it is a symmetric $SL(2,\mathbb{R})$ matrix and, in fact, it can be understood as an element of the coset $SL(2,\mathbb{R})/SO(2)$ with only two independent entries. If we factor out the determinant of the $GL(2,\mathbb{R})$ transformations too,

$$R \equiv |\det R^m{}_n|\,, \qquad s = \text{sign}(\det R^m{}_n)\,, \qquad R^m{}_n \equiv sR^{1/2}\mathcal{S}^m{}_n\,, \tag{B.6}$$

then the volume element K and the matrix $\mathcal{M}$ transform according to

$$\begin{aligned} \mathcal{M}' &= S^T\mathcal{M}S\,, \\[4pt] K' &= RK\,. \end{aligned} \tag{B.7}$$

$|K|$ is an element of the multiplicative group $\mathbb{R}^+$ and S is an element of $SL(2,\mathbb{R})$. This decomposition reflects the decomposition $GL(2,\mathbb{R}) = SL(2,\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2$. s does not act neither on K nor on $\mathcal{M}$.

We have not yet taken into account the periodic boundary conditions of the coordinates, that have to be preserved by the diffeomorphisms in the KK setting. Clearly the rescalings R do not respect the torus boundary conditions, but they rescale ℓ. The rotations S respect the boundary conditions only if $S^{-1}\vec{n} \in \mathbb{Z}^2$ the matrix entries are integer, i.e. $S \in SL(2,\mathbb{Z})$. Up to a reflection $S = -\mathbb{I}_{2\times 2}$, these diffeomorphisms are known as *Dehn twists* and are not connected with the identity (in fact, they constitute the mapping class group of torus diffeomorphisms) and they constitute the *modular group* $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm\mathbb{I}_{2\times 2}\}$. This is the group that acts on $\mathcal{M}$.

We are going to write the modular group matrices in the slightly unconventional form

$$S = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}\,, \tag{B.8}$$

to get the conventional form of the transformation of the modular parameter Eq. (B.15).

B.2 The Modular Parameter τ

We can define a complex modular-invariant coordinate ω on T^2 by

$$\omega = \tfrac{1}{2\pi\ell}\,\vec{\omega}^T \cdot \vec{x}\,, \qquad \vec{\omega} = \mathbb{C}^2\,, \tag{B.9}$$

where, under modular transformations, we assume that the complex vector $\vec{\omega}$ transforms according to

$$\vec{\omega}' = S^T\vec{\omega}\,. \tag{B.10}$$

The periodicity of ω is

$$\omega \sim \omega + \vec{\omega}^T \cdot \vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^2\,. \tag{B.11}$$

What we have done is to transfer the information contained in the metric (more precisely, in $\mathcal{M}$) into the complex periods $\vec{\omega}$. The relation between these two is

$$\mathcal{M} = \frac{1}{\Im m(\omega_1\bar{\omega}_2)} \begin{pmatrix} |\omega_1|^2 & \Re e(\omega_1\bar{\omega}_2) \\ \Re e(\omega_1\bar{\omega}_2) & |\omega_2|^2 \end{pmatrix}. \tag{B.12}$$

We can check that the transformation rules for the complex periods Eq. (B.10) and for the matrix $\mathcal{M}$ Eq. (B.7) are perfectly compatible.

In terms of the modular-invariant complex coordinate, the torus metric element takes the form

$$ds^2_{\text{Int}} = K^{1/2}\frac{1}{\Im m\omega_1\bar{\omega}_2}d\omega d\bar{\omega}. \tag{B.13}$$

Observe that $\Im m(\omega_1\bar{\omega}_2)$ is modular-invariant (and a quite important one).

It should be clear that not all pairs of complex periods characterize different tori. Recall that $\mathcal{M}$ only has 2 independent entries while $\vec{\omega}$ contains 4 real independent quantities. In particular, we can see that multiplying $\vec{\omega}$ by any complex number leaves the matrix $\mathcal{M}$ invariant. It is customary to multiply by ω_2^{-1} both the coordinate ω and define

$$\xi = \omega/\omega_2, \qquad \tau = \omega_1/\omega_2, \tag{B.14}$$

that can always be chosen to belong to the upper half complex plane $\mathbb{H}$ $\Im m(\tau) \geq 0$ ($-\omega_1$ defines the same torus as ω_1).

Under a modular transformation with S given by Eq. (B.8), the modular parameter undergoes a fractional-linear transformation

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}. \tag{B.15}$$

and the torus coordinate ξ transforms

$$\xi' = \frac{\xi}{(c\tau + d)}. \tag{B.16}$$

Finally, in terms of τ, the matrix $\mathcal{M}$ reads

$$\mathcal{M} = \frac{1}{\Im m(\tau)} \begin{pmatrix} |\tau|^2 & \Re e(\tau) \\ \Re e(\tau) & 1 \end{pmatrix}. \tag{B.17}$$

B.3 The $SL(2,\mathbb{R})/SO(2)$ Sigma-Model

In pure KK theory (with no higher-dimensional fields apart from the metric), the toroidal compactification of the Einstein-Hilbert action from $\hat{d}$ to d dimensions with the KK Ansatz

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}} \right) = \begin{pmatrix} e_{\mu}{}^{a} & e_{m}{}^{i} A^{m}{}_{\mu} \\ 0 & e_{m}{}^{i} \end{pmatrix} , \tag{B.18}$$

where the internal metric

$$G_{mn} = e_{m}{}^{i} e_{n\,j} = -e_{m}{}^{i} e_{n}{}^{j} \delta_{ij} . \tag{B.19}$$

gives, upon the rescaling

$$g_{E\,\mu\nu} = K^{\frac{2}{(d-2)}} g_{\mu\nu} , \tag{B.20}$$

$$\begin{aligned} S &= \int d^{\hat{d}}\hat{x} \sqrt{|\hat{g}|} \, \hat{R} \\[2mm] &= \int d^{d}x \sqrt{|g_E|} \left[R_E + \frac{(\hat{d}-2)(\hat{d}-d)}{4(d-2)} (\partial \log K)^2 + \tfrac{1}{4} \mathrm{Tr} \left(\partial \mathcal{M} \mathcal{M}^{-1} \right)^2 \right. \\[2mm] &\qquad \left. -\tfrac{1}{4} K^{\frac{(\hat{d}-2)}{(d-2)}} \mathcal{M}_{mn} F^{m\,\mu\nu} F^{n}{}_{\mu\nu} \right] . \end{aligned} \tag{B.21}$$

The kinetic term for the scalar matrix $\mathcal{M}$ is manifestly invariant under $SL(2,\mathbb{R})$ transformations (the action we started from is diffeomorphism-invariant). Using the parametrization Eq. (B.17), it takes the standard form

$$\tfrac{1}{2} \frac{\partial \tau \partial \bar{\tau}}{(\Im \mathrm{m}(\tau))^2} . \tag{B.22}$$

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Figure 1: Duality relations between KK branes. The numbers in parenthesis represent the worldvolume dimension, isometric and transverse directions. The arrows indicate dimensional reduction in the corresponding kind of direction. In the upper row we represent M-theory KK branes, below 10-dimensional type IIA branes and below them 9-dimensional branes. Type IIB KK branes are in the bottom row. Pairs of branes in boxes are S duality doublets. They are always related to reductions from 11 to 9 dimensions of the same object in two different orders. Sometimes there is an third object with the same numbers as those in a doublet, but transforming as a singlet and we denote it with (s).