Finite-time future singularities in modified Gauss-Bonnet and $\mathcal{F}(R,G)$ gravity and singularity avoidance

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Abstract

We study all four types of finite-time future singularities emerging in late-time accelerating (effective quintessence/phantom) era from $\mathcal{F}(R,G)$-gravity, where R and G are the Ricci scalar and the Gauss-Bonnet invariant, respectively. As an explicit example of $\mathcal{F}(R,G)$-gravity, we also investigate modified Gauss-Bonnet gravity, so-called $F(G)$-gravity. In particular, we reconstruct the $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity models where accelerating cosmologies realizing the finite-time future singularities emerge. Furthermore, we discuss a possible way to cure the finite-time future singularities in $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity by taking into account higher-order curvature corrections. The example of non-singular realistic modified Gauss-Bonnet gravity is presented. It turns out that adding such non-singular modified gravity to singular Dark Energy makes the combined theory to be non-singular one as well.

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I. INTRODUCTION

Recent observations have implied that the current expansion of the universe is accelerating [1, 2]. There exist two broad categories to explain this phenomena [3–12]. One is the introduction of "dark energy" in the framework of general relativity. The other is the investigation of a modified gravitational theory, e.g., $f(R)$-gravity, in which the action is described by the Ricci scalar R plus an arbitrary function $f(R)$ of R (for reviews, see [6–10]).

It is known that accelerating Friedmann-Robertson-Walker (FRW) universe is described by cosmological constant/quintessence/phantom Dark Energy. In principle, Dark Energy (DE) could be described by scalar field theories, fluid, modified gravity, etc. It is quite well-known that any of such DE models may be represented as the effective fluid with corresponding characteristics. At the late-time accelerating stage of the FRW universe, if the ratio of the effective pressure to the effective energy density of the universe, i.e., the effective equation of state (EoS) $w_{\text{eff}} \equiv p_{\text{eff}}/\rho_{\text{eff}}$, is larger than -1, it is the quintessence [13–15] (non-phantom) phase. On the other hand, if w_{eff} is less than -1, it is the phantom phase [16] while effective cosmological constant appears as DE when $w_{\text{eff}} = -1$. Note that (non-transient) phantom phase evolution usually ends up in Type I (Big Rip) future singularity. It is remarkable that many of the effective quintessence/phantom DEs may bring the future universe evolution to finite-time singularity. The classification of such finite-time future singularities has been made in Ref. [17]. Some of these four types future singularities are softer than other, for instance, not all characteristic quantities (scale factor, effective pressure and energy-density) diverge in rip time. There is not any qualitative difference between convenient DEs and modified gravity in this respect. For instance, the convenient parameter-dependent DE models may show all four possible types of finite-time future singularity as demonstrated in Refs. [17–19]. On the same time it was demonstrated in Refs. [20, 21] that $f(R)$-gravity DE model may also bring the universe evolution to all four possible future singularities (for first example of Big Rip (Type I) singularity in modified gravity, see [22–24]). Furthermore, it is interesting that $f(R)$ modified gravity may also provide the universal scenario to cure the finite-time future singularity by adding, say, R^2-term [20, 21, 23, 25] or non-singular viable $f(R)$-gravity [26] (for related discussion of Type II future singularity in particular $f(R)$-gravity and its curing by R^2-term, see Refs. [27–32]).

It is clear that singular dark energy may lead to various instabilities in the current universe

cosmology, including black holes and stellar astrophysics. In this respect it is very important to list the singular dark energy models as well as try to indicate the physical consequences of possible future singularity. Moreover, it is desirable to construct the universal scenario to cure such singularities. The interesting class of modified gravity models which may easily produce the late-time acceleration epoch is string-inspired modified Gauss-Bonnet gravity, so-called $F(G)$-gravity [33–36], where $F(G)$ is an arbitrary function of the Gauss-Bonnet invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$ ($R_{\mu\nu}$ and $R_{\mu\nu\xi\sigma}$ are the Ricci tensor and the Riemann tensor, respectively). It is known that such class of models may also lead to finite-time future singularity [21].

In the present paper, as a generalized investigation of Ref. [21], we explore the $\mathcal{F}(R,G)$-gravity models with realizing the finite-time future singularities by using the reconstruction method of modified gravity [21, 37], where $\mathcal{F}(R,G)$ is an arbitrary function of R and G. The $\mathcal{F}(R,G)$-gravity is a gravitational theory in a more general class of modified gravity and includes $f(R)$-gravity and $F(G)$-gravity. As an explicit example of $\mathcal{F}(R,G)$-gravity, we also investigate $F(G)$-gravity and reconstruct the $F(G)$-gravity models in which finite-time future singularities could appear in great detail. It is shown that all four types of finite-time future singularity may occur in such modified gravity. In addition, we examine the possibility of the finite-time future singularities in $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity being cured under higher-order curvature corrections. The explicitly non-singular modified Gauss-Bonnet models is proposed and it is shown that the finite-time future singularities may be easily protected combining a singular theory with the non-singular one. This suggests the universal scenario to cure the finite-time future singularity in the same line as it was proposed in Ref. [26].

The paper is organized as follows. In Sec. II, we briefly review the model of $\mathcal{F}(R,G)$-gravity and write down the gravitational field equations. In addition, we classify the four types of the finite-time future singularities. In Sec. III, as a first step, we investigate $F(G)$-gravity and reconstruct the $F(G)$-gravity models where finite-time future singularities may occur. We also examine the finite-time future singularities in realistic models of $F(G)$-gravity. Next, in Sec. IV we study the general $\mathcal{F}(R,G)$-gravity models where the finite-time future singularities occur. Moreover, we explore the finite-time future singularities in a realistic model of $\mathcal{F}(R,G)$-gravity. In Sec. V, we discuss a possible way to resolve the finite-time future singularities in $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity by taking into account

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higher-order curvature corrections. The non-singular theories are proposed. It is shown that the addition of such non-singular effective dark energy to the singular one may cure the singularity of the combined theory. Hence, modified Gauss-Bonnet gravity may appear as the effective universal regulator of finite-time future singularity not only for singular alternative gravity but also for convenient singular DE. Finally, conclusions are given in Sec. VI. The finite-time future singularities in a simple model of $f(R)$-gravity are also examined in Appendix A. Furthermore, a further argument on the asymptotic behavior of singular models is presented in Appendix B.

We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G_N$ by $\kappa^2 \equiv 8\pi/M_{\mathrm{Pl}}{}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}\mathrm{GeV}$. A note on notation is that throughout the present paper, α, γ, z, n, m, δ and ζ are constants unless we mention some conditions in regard to these expressions.

II. $\mathcal{F}(R, G)$-GRAVITY

In this section, we briefly review $\mathcal{F}(R, G)$-gravity and derive the gravitational field equations. Moreover, we classify the finite-time future singularities into four types following ref. [17].

A. The Model

The action of $\mathcal{F}(R, G)$-gravity is given by

$$S = \int d^4x \sqrt{-g}\left[\frac{\mathcal{F}(R, G)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}}\right],\tag{2.1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{\mathrm{matter}}$ is the matter Lagrangian.

From the action in Eq. (2.1), the gravitational field equation is derived as

$$\mathcal{F}'_R\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = \kappa^2 T_{\mu\nu}^{(\mathrm{matter})} + \frac{1}{2}g_{\mu\nu}\left(\mathcal{F} - \mathcal{F}'_R R\right) + \nabla_\mu\nabla_\nu\mathcal{F}'_R - g_{\mu\nu}\Box\mathcal{F}'_R$$
$$+\left(-2RR_{\mu\nu} + 4R_{\mu\rho}R_\nu{}^\rho - 2R_\mu{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} + 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma}\right)\mathcal{F}'_G$$
$$+2\left(\nabla_\mu\nabla_\nu\mathcal{F}'_G\right)R - 2g_{\mu\nu}\left(\Box\mathcal{F}'_G\right)R + 4\left(\Box\mathcal{F}'_G\right)R_{\mu\nu} - 4\left(\nabla_\rho\nabla_\mu\mathcal{F}'_G\right)R_\nu{}^\rho$$
$$-4\left(\nabla_\rho\nabla_\nu\mathcal{F}'_G\right)R_\mu{}^\rho + 4g_{\mu\nu}\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma\mathcal{F}'_G\right)g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta},\quad(2.2)$$

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where we have used the following expressions:

$$\mathcal{F}'_R = \frac{\partial \mathcal{F}(R,G)}{\partial R}, \quad \mathcal{F}'_G = \frac{\partial \mathcal{F}(R,G)}{\partial G}. \tag{2.3}$$

Here, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $T^{(\text{matter})}_{\mu\nu} = \text{diag}(\rho, p, p, p)$ is the contribution to the energy-momentum tensor from all ordinary matters with ρ and p being the energy density and pressure of all ordinary matters, respectively.

The most general flat FRW space-time is described by the metric

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\mathbf{x}^2, \tag{2.4}$$

where $a(t)$ is the scale factor and $N(t)$ is an arbitrary function of t. In what follows, we take $N(t) = 1$.

In the FRW background, from $(\mu, \nu) = (0, 0)$ and the trace part of $(\mu, \nu) = (i, j)$ $(i, j = 1, \cdots, 3)$ components in Eq. (2.2), we obtain the gravitational field equations:

$$\rho_{\text{eff}} = \frac{3}{\kappa^2}H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right), \tag{2.5}$$

where ρ_{eff} and p_{eff} are the effective energy density and pressure of the universe, respectively, and these are defined as

$$\rho_{\text{eff}} \equiv \frac{1}{\mathcal{F}'_R}\left\{\rho + \frac{1}{2\kappa^2}\left[(\mathcal{F}'_R R - \mathcal{F}) - 6H\dot{\mathcal{F}}'_R + G\mathcal{F}'_G - 24H^3\dot{\mathcal{F}}'_G\right]\right\}, \tag{2.6}$$

$$p_{\text{eff}} \equiv \frac{1}{\mathcal{F}'_R}\left\{p + \frac{1}{2\kappa^2}\left[-(\mathcal{F}'_R R - \mathcal{F}) + 4H\dot{\mathcal{F}}'_R + 2\ddot{\mathcal{F}}'_R - G\mathcal{F}'_G + 16H\left(\dot{H} + H^2\right)\dot{\mathcal{F}}'_G\right.\right.$$
$$\left.\left. + 8H^2\ddot{\mathcal{F}}'_G\right]\right\}. \tag{2.7}$$

Here, $H = \dot{a}(t)/a(t)$ is the Hubble parameter and the dot denotes the time derivative of $\partial/\partial t$. For general relativity with $\mathcal{F}(R,G) = R$, $\rho_{\text{eff}} = \rho$ and $p_{\text{eff}} = p$ and therefore Eqs. (2.6) and (2.7) are the FRW equations. Consequently, Eqs. (2.6) and (2.7) imply that the contribution of modified gravity can formally be included in the effective energy density and pressure of the universe.

B. Four types of the finite-time future singularities

We consider the case in which the Hubble parameter is expressed as

$$H = \frac{h}{(t_0 - t)^\beta} + H_0, \tag{2.8}$$

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where h, t_0 and H_0 are positive constants, β is a constant, and $t < t_0$. We can see that if $\beta > 0$, H becomes singular in the limit $t \to t_0$. Hence, t_0 is the time when a singularity appears. On the other hand, if $\beta < 0$, even for non-integer values of β some derivative of H and therefore the curvature becomes singular [21]. We assume $\beta \neq 0$ because $\beta = 0$ corresponds to de Sitter space, which has no singularity.

The finite-time future singularities can be classified in the following way [17]:

- Type I (Big Rip): for $t \to t_0$, $a(t) \to \infty$, $\rho_{\mathrm{eff}} \to \infty$ and $|p_{\mathrm{eff}}| \to \infty$. The case in which ρ_{eff} and p_{eff} are finite at t_0 is also included. It corresponds to $\beta = 1$ and $\beta > 1$.

- Type II (sudden [38]): for $t \to t_0$, $a(t) \to a_0$, $\rho_{\mathrm{eff}} \to \rho_0$ and $|p_{\mathrm{eff}}| \to \infty$. It corresponds to $-1 < \beta < 1$.

- Type III: for $t \to t_0$, $a(t) \to a_0$, $\rho_{\mathrm{eff}} \to \infty$ and $|p_{\mathrm{eff}}| \to \infty$. It corresponds to $0 < \beta < 1$.

- Type IV: for $t \to t_0$, $a(t) \to a_0$, $\rho_{\mathrm{eff}} \to 0$, $|p_{\mathrm{eff}}| \to 0$ and higher derivatives of H diverge. The case in which ρ and/or p tend to finite values is also included. It corresponds to $\beta < -1$ but β is not any integer number.

Here, $a_0 (\neq 0)$ and ρ_0 are constants. We note that in the present paper, we call singularities for $\beta = 1$ and those for $\beta > 1$ as the "Big Rip" singularities and the "Type I" singularities, respectively.

III. $F(G)$-GRAVITY

In this section, as an explicit example of $\mathcal{F}(R, G)$-gravity, we first study $F(G)$-gravity [33–36]. We reconstruct the $F(G)$-gravity models where finite-time future singularities may occur. In addition, we explore the finite-time future singularities in realistic models of $F(G)$-gravity.

A. The Model

The action of $F(G)$-gravity is given by [33]

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + F(G) \right) + \mathcal{L}_{\mathrm{matter}} \right] , \tag{3.1}$$

which corresponds to the action in Eq. (2.1) with $\mathcal{F}(R, G) = R + F(G)$.

In the FRW background in Eq. (2.4) with $N(t) = 1$, it follows from the action in Eq. (3.1) that the equations of motion (EOM) for $F(G)$-gravity are given by [35]

$$24H^3\dot{F}'(G) + 6H^2 + F(G) - GF'(G) = 2\kappa^2\rho\,, \tag{3.2}$$

$$8H^2\ddot{F}'(G) + 16H\dot{F}'(G)\left(\dot{H} + H^2\right) + \left(4\dot{H} + 6H^2\right) + F(G) - GF'(G) = -2\kappa^2 p\,, \tag{3.3}$$

where the prime denotes differentiation with respect to G. Moreover, we have

$$R = 6\left(2H^2 + \dot{H}\right)\,, \tag{3.4}$$

$$G = 24H^2\left(H^2 + \dot{H}\right)\,. \tag{3.5}$$

In this case, ρ_{eff} and p_{eff} in the FRW equations (2.5) take the form

$$\rho_{\mathrm{eff}} = \frac{1}{2\kappa^2}\left[-F(G) + 24H^2\left(H^2 + \dot{H}\right)F'(G) - 24^2H^4\left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)F''(G)\right]$$
$$+\rho\,, \tag{3.6}$$

$$p_{\mathrm{eff}} = \frac{1}{2\kappa^2}\Big\{F(G) - 24H^2\left(H^2 + \dot{H}\right)F'(G) + (24)8H^2\big[6\dot{H}^3 + 8H\dot{H}\ddot{H} + 24\dot{H}^2H^2 + 6H^3\ddot{H}$$
$$+8H^4\dot{H} + H^2\dddot{H}\big]F''(G) + 8(24)^2H^4\left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H}\right)^2 F'''(G)\Big\} + p\,, \tag{3.7}$$

where we have used Eq. (3.5).

We assume that the matter has a constant equation of state (EoS) parameter $w \equiv p/\rho$. By combining the two equations in Eq. (2.5), we obtain

$$\mathcal{G}(H, \dot{H}...) = -\frac{1}{\kappa^2}\left[2\dot{H} + 3(1 + w)H^2\right]\,, \tag{3.8}$$

where

$$\mathcal{G}(H, \dot{H}...) = p_{\mathrm{eff}} - w\rho_{\mathrm{eff}}\,. \tag{3.9}$$

When a cosmology is given by $H = H(t)$, the right-hand side of Eq. (3.8) is described by a function of t. If the function $\mathcal{G}(H, \dot{H}...)$ in Eq. (3.9), which is the combination of H, $\dot{H}$, $\ddot{H}$ and the higher derivatives of H, reproduce the above function of t, this cosmology could be realized. Hence, the function $\mathcal{G}(H, \dot{H}...)$ can be used to judge whether the particular cosmology could be realized or not [21]. The form of $\mathcal{G}(H, \dot{H}...)$ is determined by the gravitational theory which one considers. In the case of $F(G)$-gravity, by substituting Eqs. (3.6)

and (3.7) into Eq. (3.9), we find

$$
\begin{aligned}
\mathcal{G}(H, \dot{H}...) = \frac{1}{2\kappa^2} \Bigg\{ & (1+w)F(G) - 24(1+w)H^2 \left(H^2 + \dot{H} \right) F'(G) + 8(24)H^2 \Big[6\dot{H}^3 \\
& + 8H\dot{H}\ddot{H} + 6(4+w)\dot{H}^2 H^2 + 3(2+w)H^3\ddot{H} + 4(2+3w)H^4\dot{H} + H^2\dddot{H} \Big] F''(G) \\
& + 8(24)^2 H^4 \left(2\dot{H}^2 + H\ddot{H} + 4H^2\dot{H} \right)^2 F'''(G) \Bigg\} .
\end{aligned}
\tag{3.10}
$$

B. Finite-time future singularities in $F(G)$-gravity

We investigate the $F(G)$-gravity models in which the finite-time future singularities could occur, when the form of H is taken as Eq. (2.8). To find such $F(G)$-gravity models, we use the reconstruction method of modified gravity [21, 37]. By using proper functions $P(t)$ and $Q(t)$ of a scalar field t which we identify with the cosmic time, the action in Eq. (3.1) can be rewritten to

$$
S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} \left(R + P(t)G + Q(t) \right) + \mathcal{L}_{\text{matter}} \right] .
\tag{3.11}
$$

The variation with respect to t yields

$$
\frac{dP(t)}{dt}G + \frac{dQ(t)}{dt} = 0 ,
\tag{3.12}
$$

from which we can find $t = t(G)$. By substituting $t = t(G)$ into Eq. (3.11), we find the action in terms of $F(G)$

$$
F(G) = P(t)G + Q(t) .
\tag{3.13}
$$

We describe the scale factor as

$$
a(t) = \bar{a} \exp \left(g(t) \right) ,
\tag{3.14}
$$

where $\bar{a}$ is a constant and $g(t)$ is a proper function. By using Eqs. (3.2), Eq. (3.3), (3.14), the matter conservation law $\dot{\rho} + 3H(\rho + p) = 0$ and then neglecting the contribution from matter, we get the differential equation

$$
2\frac{d}{dt}\left(\dot{g}^2(t)\frac{dP(t)}{dt} \right) - 2\dot{g}^3(t)\frac{dP(t)}{dt} + \ddot{g}(t) = 0 .
\tag{3.15}
$$

By using the first EOM for $F(G)$-gravity in Eq. (3.2), $Q(t)$ is given by

$$
Q(t) = -24\dot{g}^3(t)\frac{dP(t)}{dt} - 6\dot{g}^2(t) .
\tag{3.16}
$$

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1. Big Rip singularity

First, we examine the Big Rip singularity. If $\beta = 1$ in Eq. (2.8) with $H_0 = 0$, H and G are given by

$$H = \frac{h}{(t_0 - t)} , \tag{3.17}$$

$$G = \frac{24h^3}{(t_0 - t)^4}(1 + h) . \tag{3.18}$$

The most general solution of Eq. (3.15) is given by

$$P(t) = \frac{1}{4h(h-1)}(2t_0 - t)t + c_1\frac{(t_0 - t)^{3-h}}{3 - h} + c_2 , \tag{3.19}$$

where c_1 and c_2 are constants. From Eq. (3.16), we get

$$Q(t) = -\frac{6h^2}{(t_0 - t)^2} - \frac{24h^3\left[\frac{(t_0-t)}{2h(h-1)} - c_1(t_0 - t)^{2-h}\right]}{(t_0 - t)^3} . \tag{3.20}$$

Furthermore, from Eq. (3.12) we obtain

$$t = \left[\frac{24(h^3 + h^4)}{G}\right]^{1/4} + t_0 , \tag{3.21}$$

which is consistent with Eq. (3.18). By solving Eq. (3.13), we find the most general form of $F(G)$ which realizes the Big Rip singularity

$$F(G) = \frac{\sqrt{6h^3(1 + h)}}{h(1 - h)}\sqrt{G} + c_1 G^{\frac{h+1}{4}} + c_2 G . \tag{3.22}$$

This is an exact solution of Eq. (3.8) in the case of Eq. (3.17). In general, if for large values of G, $F(G) \sim \alpha G^{1/2}$, where $\alpha(\neq 0)$ is a constant, the Big Rip singularity could appear for any value of $h \neq 1$. In the case of $h = 1$, the solution of $\mathcal{G}(H, \dot{H}...)$ is zero for $F(G) = \alpha G^{1/2}$. Note that $\alpha G^{(1+h)/4}$ is an invariant with respect to the Big Rip solution.

In the case of $h = 1$, it is possible to find another exact solution for $P(t)$

$$P(t) = \alpha(t_0 - t)^q \ln\left[\gamma(t_0 - t)^z\right] , \tag{3.23}$$

where $\gamma(> 0)$ is a positive constant and q and z are constants. The equation (3.15) is satisfied for the case of Eq. (3.17) if $q = 3 - h = 2$ (and therefore $h = 1$) and $z\alpha = -1/4$. From Eq. (3.16), we have

$$Q(t) = -\frac{12}{(t_0 - t)^2} \ln\left[\gamma(t_0 - t)\right] . \tag{3.24}$$

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The form of $F(G)$ is given by

$$F(G) = \frac{\sqrt{3}}{2}\sqrt{G}\ln(\gamma G).$$ (3.25)

This is another exact solution of Eq. (3.8) for $H = 1/(t_0 - t)$. In general, if for large values of G, $F(G) \sim \alpha\sqrt{G}\ln(\gamma G)$ with $\alpha > 0$ and $\gamma > 0$, the Big Rip singularity could appear. The same result is found for $F(G) \sim \alpha\sqrt{G}\ln(\gamma G^z + G_0)$ with $\alpha > 0$, $\gamma > 0$ and $z > 0$, where G_0 is a constant.

2. Other types of singularities

Next, we investigate the other types of singularities. If $\beta \neq 1$, Eq. (2.8) with $H_0 = 0$ implies that the scale factor $a(t)$ behaves as

$$a(t) = \exp\left[\frac{h(t_0 - t)^{1-\beta}}{\beta - 1}\right].$$ (3.26)

We consider the case in which H and G are given by

$$H = \frac{h}{(t_0 - t)^\beta}, \quad \beta > 1,$$ (3.27)

$$G \sim \frac{24h^4}{(t_0 - t)^{4\beta}}.$$ (3.28)

A solution of Eq. (3.15) in the limit $t \to t_0$ is given by

$$P(t) \simeq \frac{\alpha}{(t_0 - t)^z}$$ (3.29)

with $z = -2\beta$ and $\alpha = -1/4h^2$. The form of $F(G)$ is expressed as

$$F(G) = -12\sqrt{\frac{G}{24}}.$$ (3.30)

Hence, if for large values of G, $F(G) \sim -\alpha\sqrt{G}$ with $\alpha > 0$, a Type I singularity could appear.

When $\beta < 1$, the forms of H and G are given by

$$H = \frac{h}{(t_0 - t)^\beta}, \quad 0 < \beta < 1,$$ (3.31)

$$G \sim \frac{24h^3\beta}{(t_0 - t)^{3\beta+1}}.$$ (3.32)

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An asymptotic solution of Eq. (3.15) in the limit $t \to t_0$ is given by

$$P(t) \simeq \frac{\alpha}{(t_0 - t)^z} \tag{3.33}$$

with $z = -(1 + \beta)$ and $\alpha = 1/2h(1 + \beta)$. The form of $F(G)$ becomes

$$F(G) = \frac{6h^2}{(\beta + 1)}(3\beta + 1)\left(\frac{|G|}{24h^3|\beta|}\right)^{2\beta/(3\beta+1)}. \tag{3.34}$$

Hence, if for large values of G, $F(G)$ has the form

$$F(G) \sim \alpha|G|^\gamma, \quad \gamma = \frac{2\beta}{3\beta + 1}, \tag{3.35}$$

with $\alpha > 0$ and $0 < \gamma < 1/2$, we find $0 < \beta < 1$ and a Type III singularity could emerge.

If for $G \to -\infty$, $F(G)$ has the form in Eq. (3.35) with $\alpha > 0$ and $-\infty < \gamma < 0$, we find $-1/3 < \beta < 0$ and a Type II (sudden) singularity could appear. Moreover, if for $G \to 0^-$, $F(G)$ has the form in Eq. (3.35) with $\alpha < 0$ and $1 < \gamma < \infty$, we obtain $-1 < \beta < -1/3$ and a Type II singularity could occur.

If for $G \to 0^-$, $F(G)$ has the form in Eq. (3.35) with $\alpha > 0$ and $2/3 < \gamma < 1$, we obtain $-\infty < \beta < -1$ and a Type IV singularity could appear. We also require that $\gamma \neq 2n/(3n - 1)$, where n is a natural number.

We can generate all the possible Type II singularities as shown above except in the case $\beta = -1/3$, i.e., $H = h/(t_0 - t)^{1/3}$. In this case, we have the following form of G:

$$G = 24h^3\beta + 24h^4(t_0 - t)^{4/3} < 0. \tag{3.36}$$

To find t in terms of G, we must consider the whole expression of G by taking into account also the low term of $(t_0 - t)$. We obtain

$$F(G) \simeq \frac{1}{4\sqrt{6}h^3}G(G + 8h^3)^{1/2} + \frac{2}{\sqrt{6}}(G + 8h^3)^{1/2}, \tag{3.37}$$

which satisfies Eq. (3.8) in the limit $t \to t_0$. As a consequence, the specific model $F(G) = \sigma_1 G(G + c_3)^{1/2} + \sigma_2(G + c_3)^{1/2}$, where σ_1, σ_2 and c_3 are positive constants, can generate a Type II singularity.

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C. Realistic models of $F(G)$-gravity

Here, we study the realistic models of $F(G)$-gravity, which reproduce the current acceleration, namely [36]

$$F_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2},$$
(3.38)

$$F_2(G) = \frac{a_1 G^{n+N} + b_1}{a_2 G^n + b_2},$$
(3.39)

$$F_3(G) = a_3 G^n (1 + b_3 G^m),$$
(3.40)

where a_1, a_2, b_1, b_2, a_3, b_3, n, N and m are constants. In the following, we always assume $n > 0$. For the model (3.39), Types I, II and III singularities may be present. In fact, for $N = 1/2$, one could have Big Rip singularities, since in this case, in the limit large G, Eq. (3.39) gives $\alpha G^{1/2}$. Thus, as discussed in Subsection III. B, one has a Big Rip singularity. Moreover, again with $N = 1/2$, if $a_1/a_2 < 0$, Eq. (3.39) for large value of G, leads to $-\alpha G^{1/2}$ with $\alpha > 0$ and thus Type I singularity could appear. If n and N are integers and $n + N > 0$, for large and negative value of G, $F_2(G) \sim a_1/a_2 G^N$. As a result, a Type II singularity could appear, when $-n < N < 0$, N even and $a_1/a_2 > 0$ or N odd and $a_1/a_2 < 0$ (see Eq. (3.35) and the related discussion). If $0 < N < 1/2$ and $a_1/a_2 > 0$, we have the Type III singularity (see Eq. (3.35)). When $G \to 0^-$, we do not recover any example of singularity of the preceding subsection.

If there exists any singularity solution, it must be consistent with Eq. (3.8). The behavior of Eq. (3.10) takes two asymptotic forms which depend on the parameter of β as follows:

- Case of $\beta \geq 1$: In the limit $t \to t_0$, we find

$$\mathcal{G}(H, \dot{H}...) \sim \alpha F(G) + \frac{\gamma}{(t_0 - t)^{4\beta}} F'(G) + \frac{\delta}{(t_0 - t)^{7\beta+1}} F''(G) + \frac{\zeta}{(t_0 - t)^{10\beta+2}} F'''(G),$$
(3.41)

where δ and ζ are constants. To realize a singularity, from Eq. (3.8) we must have

$$\mathcal{G}(H, \dot{H}...) \sim -\frac{3(1+w)h^2}{\kappa^2 (t_0 - t)^{2\beta}}.$$
(3.42)

Hence, if for $G \sim 24h^4/(t_0 - t)^{4\beta}$ with $\beta \geq 1$, the highest term of Eq. (3.41) is proportional to $1/(t_0 - t)^{2\beta}$, it is possible to have a Type I singularity. This condition is necessary and not sufficient. Another very important condition that must be satisfied is the concordance of the signs in Eq. (3.42), which depends on the parameters of the model.

- Case of $\beta < 1$: In the limit $t \to t_0$ we obtain

$$\mathcal{G}(H, \dot{H}...) \sim \alpha F(G) + \frac{\gamma}{(t_0 - t)^{3\beta+1}} F'(G) + \frac{\delta}{(t_0 - t)^{5\beta+3}} F''(G) + \frac{\zeta}{(t_0 - t)^{8\beta+4}} F'''(G) \,. \tag{3.43}$$

To realize a singularity, from Eq. (3.8) we must have

$$\mathcal{G}(H, \dot{H}...) \sim -\frac{2\beta h}{\kappa^2 (t_0 - t)^{\beta+1}} \,. \tag{3.44}$$

Thus, if for $G \sim 24h^3\beta/(t_0 - t)^{3\beta+1}$ with $\beta < 1$, the highest term of Eq. (3.43) is proportional to $1/(t_0 - t)^{\beta+1}$, it is possible to have a Type II, III or IV singularity. Also this condition is necessary and not sufficient.

We see that the model in Eq. (3.39) with $n > 0$ and $N > 0$ is not able to realize a Type IV singularity because for $\beta < -1$ the right-hand side of Eq. (3.44) tends to zero and the left-hand side of Eq. (3.44) tends to a constant ($F_2(G) \sim b_1/b_2$). Nevertheless, it is possible to have a Type II singularity for $0 < \beta < -1/3$. If $n > 0$ and $N > 0$, we get

$$F_2(G) \sim \frac{b_1}{b_2}, \quad F_2'(G) \sim -n\frac{b_1 a_2}{b_2^2} G^{n-1}, \quad F_2''(G) \sim -n\frac{b_1 a_2}{b_2^2}(n-1)G^{n-2},$$

$$F_2'''(G) \sim -n\frac{b_1 a_2}{b_2^2}(n-1)(n-2)G^{n-3} \,. \tag{3.45}$$

It can be shown that, under the requirement $n > 1$ (the relation between n and β is $n = 2\beta/(3\beta + 1)$), the asymptotic behavior of Eq. (3.43) when $G \simeq 24h^3\beta/(t_0 - t)^{3\beta+1}$ is proportional to $1/(t_0 - t)^{\beta+1}$ and therefore it is possible to realize the Type II singularity.

- For $N = 1$ and $n = 2$, $\mathcal{G}(H, \dot{H}...) \sim (24h^5)b_1 a_2/b_2^2$ when $\beta = -1/2$. Hence, if $b_1 a_2 > 0$, the model can become singular when $G \to 0^-$ (Type II singularity).

- For $N = 1$ and $n = 3$, $\mathcal{G}(H, \dot{H}...) \sim -b_1 a_2/b_2^2$ when $\beta = -3/7$. Thus, if $b_1 a_2 < 0$, the model can become singular when $G \to 0^-$ (Type II singularity).

In a certain sense, the model $F_1(G)$ in Eq. (3.38) is a particular case of Eq. (3.39). For large values of G, it tends to a constant with velocity being zero, so that it is impossible to find singularities (it is well known that $R + constant$ is free of singularities, according to the ΛCDM model). Nevertheless, similarly to the above, a Type II singularity can occur when $G \to 0^-$ for $n > 1$. For example, if $n = 2$, $\mathcal{G}(H, \dot{H}) \sim (24h^5/b_2^2)(b_1 a_2 - a_1 b_2)$ for $\beta = -1/2$. If $b_1 a_2 - a_1 b_2 > 0$, the model can become singular when $G \to 0^-$.

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With regard to $F_3(G)$ in Eq. (3.40), it is interesting to find the conditions on m, n, a_3 and b_3 for which we do not have any type of singularities. When $G \to \pm\infty$ or $G \to 0^-$, it is possible to write this model in the form $F(G) \sim \alpha G^\gamma$, which we have investigated on in the preceding subsection. We do not consider the trivial case $n = m$. The no-singularity conditions follow directly from the results of the preceding subsection as complementary conditions to the singularity ones:

- Case (A): $n > 0$, $m > 0$, $n \neq 1$ and $m \neq 1$. We avoid any singularity if $0 < n+m < 1/2$ and $a_3 b_3 < 0$; $n + m > 1/2$, $n > 1$ and $a_3 > 0$; $n + m > 1/2$, $2/3 < n < 1$ and $a_3 < 0$; $n + m > 1/2$, $0 < n \leq 2/3$ and if $n = 1/2$, $a_3 > 0$.

- Case (B): $n > 0$, $m < 0$ and $n \neq 1$. We avoid any singularity if $0 < n < 1/2$ and $a_3 < 0$; $n > 1/2$, $n + m > 1$ and $a_3 b_3 > 0$; $n > 1/2$, $2/3 < n + m < 1$ and $a_3 b_3 < 0$; $n > 1/2$, $n + m \leq 2/3$ and if $n + m = 1/2$, $a_3 b_3 > 0$.

- Case (C): $n < 0$, $m > 0$ and $m \neq 1$. We avoid any singularity if $m + n > 1/2$; $m + n < 1/2$ and $a_3 b_3 < 0$.

- Case (D): $n < 0$ and $m < 0$. We avoid any singularity if $a_3 < 0$.

We end this subsection considering the following realistic model, again for $n > 0$,

$$F_4(G) = G^\alpha \frac{a_1 G^n + b_1}{a_2 G^n + b_2}. \tag{3.46}$$

Since for large G, one has $F_4(G) \simeq a_1/a_2 G^\alpha$ and for small G, one has $F_4(G) \simeq b_1/b_2 G^\alpha$, the analysis of Subsection III. B leads to the absence of any type of singularities for

$$\frac{1}{2} < \alpha < \frac{2}{3}. \tag{3.47}$$

In fact, for this range of values, the asymptotic behavior of the right-hand side of Eq. (3.8) is different from the asymptotic behavior of its left-hand side on the singularity solutions. Thus, Eq. (3.46) provides an example of realistic model free of all possible singularities when Eq. (3.47) is satisfied, independently of the coefficients. Moreover, this model suggests the universal scenario to cure finite-time future singularity. Adding above model to any singular Dark Energy (in the same way as adding R^2-term [20, 21, 23]) results in combined non-singular model. Hence, unlike to convenient DE which may be singular or not, (non-singular) modified gravity may suggest the universal recipe to cure the finite-time future

singularity. In this respect, modified gravity seems to be more fundamental theory than convenient DEs.

IV. FINITE-TIME FUTURE SINGULARITIES IN $\mathcal{F}(R, G)$-GRAVITY

In this section, we consider the finite-time future singularities in $\mathcal{F}(R, G)$-gravity. We reconstruct the $\mathcal{F}(R, G)$-gravity models with producing the finite-time future singularities. Furthermore, we examine the finite-time future singularities in a realistic model of $\mathcal{F}(R, G)$-gravity.

A. Formalism

We study the pure gravitational action of $\mathcal{F}(R, G)$-gravity, i.e., the action in Eq. (2.1) without $\mathcal{L}_{\mathrm{matter}}$. In this case, it follows from Eqs. (2.6) and (2.7) that the EOM of $F(R, G)$-gravity are given by [35]

$$24H^3 \dot{\mathcal{F}}'_G + 6H^2 \mathcal{F}'_R + 6H\dot{\mathcal{F}}'_R + (\mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G) = 0 \,, \tag{4.1}$$

$$8H^2 \ddot{\mathcal{F}}'_G + 2\ddot{\mathcal{F}}'_R + 4H\dot{\mathcal{F}}'_R + 16H\dot{\mathcal{F}}'_G(\dot{H} + H^2)$$
$$+ \mathcal{F}'_R(4\dot{H} + 6H^2) + \mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G = 0 \,. \tag{4.2}$$

In the case of pure gravity, these two equations are linearly dependent.

Now, similarly to the previous section, by using proper functions $P(t)$, $Z(t)$ and $Q(t)$ of a scalar field which is identified with the time t, we can rewrite the action in Eq. (2.1) without $\mathcal{L}_{\mathrm{matter}}$ to

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(P(t)R + Z(t)G + Q(t) \right) \,. \tag{4.3}$$

By the variation with respect to t, we obtain

$$P'(t)R + Z'(t)G + Q'(t) = 0 \,, \tag{4.4}$$

from which in principle it is possible to find $t = t(R, G)$. Here, the prime denotes differentiation with respect to t. By substituting $t = t(R, G)$ into Eq. (4.3), we find the action in terms of $\mathcal{F}(R, G)$

$$\mathcal{F}(R, G) = P(t)R + Z(t)G + Q(t) \,. \tag{4.5}$$

By using the conservation law and Eq. (4.1), we get the differential equation

$$P''(t) + 4\dot{g}^2(t)Z''(t) - \dot{g}(t)P'(t) + (8\dot{g}\ddot{g} - 4\dot{g}^3(t))Z'(t) + 2\ddot{g}(t)P(t) = 0\,, \qquad (4.6)$$

where we have used the expression of the scale factor in Eq. (3.14) and the Hubble parameter $H(t) = \dot{g}(t)$. By using Eq. (4.1), $Q(t)$ becomes

$$Q(t) = -24\dot{g}^3(t)Z'(t) - 6\ddot{g}^2(t)P(t) - 6\dot{g}(t)P'(t)\,. \qquad (4.7)$$

In general, if $P(t) \neq 0$, $\mathcal{F}(R, G)$ can be written in the following form:

$$\mathcal{F}(R, G) = Rg(R, G) + f(R, G)\,, \qquad (4.8)$$

where $g(R, G) \neq 0$ and $f(R, G)$ are generic functions of R and G. From Eqs. (4.1) and (4.2), we obtain

$$\rho_{\text{eff}} = -\frac{1}{2\kappa^2 g(R, G)}\left[24H^3\dot{\mathcal{F}}_G' + 6H^2\left(R\frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR}\right) + 6H\dot{\mathcal{F}}_R'\right.$$
$$\left. + (\mathcal{F} - R\mathcal{F}_R' - G\mathcal{F}_G')\right] \qquad (4.9)$$

and

$$p_{\text{eff}} = \frac{1}{2\kappa^2 g(R, G)}\left[8H^2\ddot{\mathcal{F}}_G' + 2\ddot{\mathcal{F}}_R' + 4H\dot{\mathcal{F}}_R' + 16H\dot{\mathcal{F}}_G'(\dot{H} + H^2)\right.$$
$$\left. + \left(R\frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR}\right)(4\dot{H} + 6H^2) + \mathcal{F} - R\mathcal{F}_R' - G\mathcal{F}_G'\right]\,, \qquad (4.10)$$

respectively, where ρ_{eff} and p_{eff} are given by the expressions in (2.5). As a consequence, we recover the same formalism of Sec. III as

$$\mathcal{G}(H, \dot{H}...) = p_{\text{eff}} - w\rho_{\text{eff}}$$
$$= \frac{1}{2\kappa^2 g(R, G)}\left\{(1 + w)(\mathcal{F} - R\mathcal{F}_R' - G\mathcal{F}_G')\right.$$
$$+ \left(R\frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR}\right)\left[6H^2(1 + w) + 4\dot{H}\right]$$
$$\left. + H\dot{\mathcal{F}}_R'(4 + 6w) + 8H\dot{\mathcal{F}}_G'\left[2\dot{H} + H^2(2 + 3w)\right] + 2\ddot{\mathcal{F}}_R' + 8H^2\ddot{\mathcal{F}}_G'\right\}\,, \qquad (4.11)$$

where w is the constant EoS parameter of matter. The use of this equation requires that $g(R, G) \neq 0$ on the solution. The equation for $\mathcal{G}(H, \dot{H}...)$ is given by Eq. (3.8).

B. Finite-time future singularities

We examine the $\mathcal{F}(R, G)$-gravity models in which the finite-time future singularities could appear.

1. Big Rip singularity

First, we investigate the Big Rip singularity. If $\beta = 1$ in Eq. (2.8) with $H_0 = 0$, we have

$$H = \frac{h}{t_0 - t}, \tag{4.12}$$

$$R = \frac{6h}{(t_0 - t)^2}(2h + 1), \tag{4.13}$$

$$G = \frac{24h^3}{(t_0 - t)^4}(1 + h), \tag{4.14}$$

with $h > 0$. A simple (trivial) solution of Eq. (4.6) is given by

$$P(t) = \alpha(t_0 - t)^z, \tag{4.15}$$

$$Z(t) = \delta(t_0 - t)^x, \tag{4.16}$$

with α and δ being constants, where $x = 3 - h$ and z is given by

$$z_\pm = \frac{1 - h \pm \sqrt{h^2 - 10h + 1}}{2}. \tag{4.17}$$

Thus, the most general solution of $P(t)$ is expressed as

$$P(t) = \alpha_1(t_0 - t)^{z_+} + \alpha_2(t_0 - t)^{z_-}, \tag{4.18}$$

where α_1 and α_2 are constants. From Eq. (4.7), we have

$$Q(t) = \frac{24h^3\delta(3 - h)}{(t_0 - t)^{h+1}} + \frac{6h\alpha_1(z_+ - h)}{(t_0 - t)^{2-z_+}} + \frac{6h\alpha_2(z_- - h)}{(t_0 - t)^{2-z_-}}. \tag{4.19}$$

Under the condition $0 < h < 5 - 2\sqrt{6}$ or $h > 2 + \sqrt{6}$, the solution of $\mathcal{F}(R, G)$ (by absorbing some factor into the constants) is given by

$$\mathcal{F}(R, G) = \alpha_1 R^{1-z_+/2} + \alpha_2 R^{1-z_-/2} + \delta G^{\frac{h+1}{4}}. \tag{4.20}$$

If $\delta = 0$, we find a well-known result of $f(R)$-gravity. $G^{\frac{h+1}{4}}$ is an invariant of the Big Rip solution in a $F(G)$-gravity theory and it is a solution in a general $\mathcal{F}(R, G)$-gravity theory. Note that $1 - z_\pm \neq 1$.

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Another exact solution of Eq. (4.6) is given by

$$P(t) = \frac{\alpha}{(t_0 - t)^z}, \qquad (4.21)$$

$$Z(t) = \frac{\delta}{(t_0 - t)^x}, \qquad (4.22)$$

where δ and x are constants, $z = x + 2$ and α is given by

$$\alpha = \frac{4h^2\delta x(h - x - 3)}{x^2 + (5 - h)x + 6}. \qquad (4.23)$$

From Eq. (4.7), we find

$$Q(t) = -\frac{6h}{(t_0 - t)^{x+4}} \left[4h^2\delta x + \alpha(x + 2 + h) \right]. \qquad (4.24)$$

The solution of Eq. (4.4) is given by

$$
\begin{aligned}
t_0 - t &= f(R, G) \\
&= \left\{ \frac{-\alpha(x + 2)R \pm \sqrt{\alpha^2(x + 2)^2 R^2 + 24h\left[4h^2\delta x + \alpha(x + 2 + h)\right](x + 4)\delta xG}}{2\delta xG} \right\}^{1/2}, \quad (4.25)
\end{aligned}
$$

with $x \neq 0$ and $\delta \neq 0$.

To have real solutions, we must require that the arguments of the roots in Eq. (4.25) are positive. For $h > 0$, the principal cases are as follows:

- Case (1): $x > 0$, $\delta > 0$, $1 + x \leq h < x + 5 + \frac{6}{x}$. We must use the sign $+$ in (4.25).

- Case (2): $-\frac{3}{2} \leq x < 0$, $\delta < 0$, $h \geq x + 1$. We must use the sign $+$.

- Case (3): $-4 < x < -\frac{3}{2}$, $\delta < 0$, $h > x + 5 + \frac{6}{x}$. We must use the sign $+$.

- Case (4): $x > 0$, $\delta < 0$, $x + 5 + \frac{6}{x} > h \geq 1 + x$. We must use the sign $-$.

- Case (5): $-\frac{3}{2} \leq x < 0$, $\delta > 0$, $h \geq x + 1$. We must use the sign $-$.

- Case (6): $-4 < x < -\frac{3}{2}$, $\delta > 0$, $h > x + 5 + \frac{6}{x}$. We must use the sign $-$.

- Case (7): $x = -4$, $\delta > 0$. We must use the sign $-$.

- Case (8): $x = -4$, $\delta < 0$. We must use the sign $+$.

The solution of $\mathcal{F}(R, G)$ is given by

$$\mathcal{F}(R, G) = \frac{\alpha}{(f(R,G))^{x+2}}R + \frac{\delta}{(f(R,G))^x}G - \frac{6h}{(f(R,G))^{x+4}}\left[4h^2\delta x + \alpha(x+2+h)\right], \quad (4.26)$$

where $f(R, G)$ is given by Eq. (4.25). This is an exact solution of EOM in Eqs. (4.1) and (4.2) for the Big Rip case.

We show several examples. In the case $\alpha = 1$ and $x = -2$, we find

$$\mathcal{F}(R, G) = R + \frac{\sqrt{6}\sqrt{h(1+h)}}{(1-h)}\sqrt{G}, \quad h \neq 1, \quad (4.27)$$

which is in agreement with the result of the previous section.

If $\alpha = 0$ and $x = h - 3$ (this case corresponds to the cases (1)–(6) presented above), we find

$$\mathcal{F}(R, G) = \delta G^{\frac{h+1}{4}}, \quad \delta \neq 0, \quad (4.28)$$

which is equivalent to Eq. (4.20) with $\alpha_1 = \alpha_2 = 0$.

If $x = -4$, the result is given by

$$\mathcal{F}(R, G) = \frac{16h^4\delta}{(1+2h^2)^2}\left[(9 + 21h + 6h^2) - (1+h)^2\frac{R^2}{G}\right], \quad \delta \neq 0. \quad (4.29)$$

Hence, if for large values of R and G, $\mathcal{F}(R, G) \sim \pm\alpha \mp \delta(R^2/G)$ with $\alpha > 0$ and $\delta > 0$, the Big Rip singularity could appear.

If $x = h - 1$, the solution becomes (by absorbing some constant)

$$\mathcal{F}(R, G) = \delta G\left(\frac{R}{G}\right)^{\frac{1-h}{2}}, \quad \delta \neq 0, \quad h \neq 1. \quad (4.30)$$

Thus, if for large values of R and G, $\mathcal{F}(R, G) \sim \delta G^\gamma/R^{\gamma-1}$ with $\delta \neq 0$ and $1/2 < \gamma < 1$ or $1 < \gamma < +\infty$, the Big Rip singularity could appear.

Furthermore, it is possible to verify that the model:

$$\mathcal{F}(R, G) = \gamma\frac{G^m}{R^n}, \quad (4.31)$$

with γ being a generic constant, is a solution of Eqs. (4.1) and (4.2) in the case of the Big Rip singularity ($\beta = 1$) for some value of h. In general, it is possible to obtain solutions for $h > 0$ if $m > 0$, $n > 0$ and $m > n$. For example, the case $n = 2$ and $m = 3$ realizes the singularity in $h = 5$; the case $n = 1$ and $m = 3$ realizes the singularity in $h = 4 + \sqrt{19}$ and so forth. This is a generalization of Eq. (4.30). Note that we do not recover a physical solution for $m = -1$ and $n = -2$ because in this case $h = -3$. For a similar kind of model $F(R^2/G)$ which produces the Big Rip singularity, see Eq. (4.29). For $m = 0$ or $n = 0$, we recover Eq. (4.20).

2. *Other types of singularities*

Next, we study the other types of singularities. We consider the case in which H is given by

$$H = \frac{h}{(t_0 - t)^\beta} \,. \tag{4.32}$$

An exact solution of Eq. (4.6) is given by

$$P(t) = -\lambda(4h^2)(t_0 - t) \,, \tag{4.33}$$

$$Z(t) = \lambda(t_0 - t)^{2\beta+1} \,, \tag{4.34}$$

where λ is a generic constant. The form of $Q(t)$ is given by

$$Q(t) = \frac{24h^4\lambda}{(t_0 - t)^{2\beta-1}} + \frac{48h^3\beta}{(t_0 - t)^\beta} \,. \tag{4.35}$$

For $\beta = 1$, we find a special case of Eq. (4.26). For $\beta > 1$, we obtain the asymptotic real solution of Eq. (4.4):

$$t_0 - t = f(R, G) = 2^{1/2\beta} \left[\frac{h^2 R + \sqrt{h^4 R^2 + 6h^4(4\beta^2 - 1)G}}{(1 + 2\beta)G} \right]^{1/2\beta} . \tag{4.36}$$

The form of $\mathcal{F}(R, G)$ is expressed as

$$\mathcal{F}(R, G) = -4h^2\lambda(f(R, G))R + \lambda(f(R, G)^{1+2\beta})G + 24h^4\lambda(f(R, G)^{1-2\beta}) \,, \quad \beta > 1 \,. \tag{4.37}$$

This is an asymptotic solution of Eq. (3.8) when (for $\beta > 1$)

$$-\frac{1}{\kappa^2} \left[2\dot{H} + 3(1 + w)H^2 \right] \sim -\frac{3(1 + w)h^2}{\kappa^2}(t_0 - t)^{-2\beta} \,. \tag{4.38}$$

In the case $\beta \gg 1$, the form of $\mathcal{F}(R, G)$ is written as

$$\mathcal{F}(R, G) \simeq \lambda \left(\frac{\alpha G}{R + \sqrt{R^2 + \gamma G}} - R \right) \,, \quad \alpha > 0 \,, \quad \gamma > 0 \,, \quad \lambda \neq 0 \,. \tag{4.39}$$

This is the asymptotic behavior of a $\mathcal{F}(R, G)$ model in which a "strong" Type I singularity ($\beta \gg 1$) could appear. By taking $g(R, G) = \gamma G^m/R^{n+1}$ and using Eqs. (4.9) and (4.10), it is possible to verify that for the model

$$\mathcal{F}(R, G) = \gamma \frac{G^m}{R^n} \,, \tag{4.40}$$

the function $\mathcal{G}(H, \dot{H}...)$ in Eq. (4.11) is given by

$$\mathcal{G}(H, \dot{H}...) \simeq -\frac{3h^2(2m - n - 1)(1 + w)}{\kappa^2(t_0 - t)^{2\beta}}, \tag{4.41}$$

which is, under the condition $2m - n - 1 > 0$, an asymptotic solution of Eq. (3.8) in the case of $\beta > 1$. Thus, in the model $\mathcal{F}(R, G) \simeq \gamma G^m/R^n$ with $m > (n+1)/2$ the Type I singularity could appear. This point has important consequences because it is possible to see that the theories $F(R) = R^n$ with $n > 1$ or $F(G) = G^m$ with $m > 1/2$ can become singular.

To find other models, we can consider the results of Sec. III. The Type I singularities correspond to the asymptotic limits for R and G

$$R \sim 12H^2, \quad G \sim 24H^4. \tag{4.42}$$

These are two functions of the Hubble parameter only, so that

$$\lim_{t \to t_0} 24 \left(\frac{R}{12}\right)^2 = \lim_{t \to t_0} G. \tag{4.43}$$

If we substitute G for R in Eq. (3.30) by taking into account Eq. (4.43), we obtain a zero function (this is because Eq. (3.30) is zero on the singularity solution). If we substitute G for G/R, however, we obtain the following model:

$$\mathcal{F}(R, G) = R - \frac{6G}{R}. \tag{4.44}$$

This is an asymptotic solution of Eq. (3.8) such as Eq. (4.38). Thus, there appears Type I singularity for $\mathcal{F}(R, G) \sim R - \alpha(G/R)$ with $\alpha > 0$.

In the case of $H = h/(t_0 - t)^\beta$ with $\beta < 1$, it is not possible to write G and R like functions of the same variable (H or the same combination of H and $\dot{H}$). Nevertheless, if we examine the asymptotic behavior of G and R, we have

$$R \simeq \frac{6h\beta}{(t_0 - t)^{\beta+1}}, \tag{4.45}$$

$$G \simeq \frac{24h^3\beta}{(t_0 - t)^{3\beta+1}}, \tag{4.46}$$

and

$$\frac{G}{R} \sim G^{\frac{2\beta}{3\beta+1}} \tag{4.47}$$

If we use G/R for G in Eq. (3.34) as in Eq. (4.47), we see that the asymptotic time dependence in Eq. (3.8) for $\beta < 1$ is the same:

$$-\frac{1}{\kappa^2}\left[2\dot{H} + 3(1 + w)H^2\right] \sim \frac{\alpha}{(t_0 - t)^{\beta+1}} + \frac{\gamma}{(t_0 - t)^{2\beta}}. \tag{4.48}$$

Under this consideration, it is possible to derive a $\mathcal{F}(R,G)$-gravity theory (by setting some parameters) from Eq. (3.34) as

$$\mathcal{F}(R,G) = R + \frac{3}{2}\frac{G}{R}, \tag{4.49}$$

in which the other types of singularities appear. Thus, in this model ($\mathcal{F}(R,G) \sim R+\alpha(G/R)$ with $\alpha > 0$) the Type II, III and IV singularities could appear. Then, by substituting G for R we get

$$\mathcal{F}(R,G) \simeq R - \delta\frac{(1+\beta)}{(\beta-1)}|R|^{\frac{2\beta}{1+\beta}}, \quad \delta > 0. \tag{4.50}$$

This is a well-know result. In the model $\mathcal{F}(R \to \infty) \sim R+\alpha R^\gamma$, for $0 < \gamma < 1$ and $\alpha > 0$, a Type III singularity could appear. In the model $\mathcal{F}(R \to -\infty) \sim R+\alpha|R|^\gamma$, for $-\infty < \gamma < 0$ and $\alpha > 0$, a Type II singularity could appear. In the model $\mathcal{F}(R \to 0^-) \sim R + \alpha|R|^\gamma$, for $2 < \gamma < +\infty$ ($\gamma \neq 2n/(n-1)$, where n is a natural number) and $\alpha < 0$, a Type IV singularity could appear. (In the Big Rip case, we have found exact solutions. This kind of reasoning is therefore inapplicable.)

C.   Realistic model of $\mathcal{F}(R,G)$-gravity

We study the following realistic model of $\mathcal{F}(R,G)$-gravity:

$$\mathcal{F}(R,G) = a_1 G^n + a_2 R^m + \frac{a_5}{a_3 G^n + a_4 R^m}, \tag{4.51}$$

where $a_i(i = 1, \cdots, 5)$ are constants and $n(> 0)$ and $m(> 0)$ are positive constants. For large values of R and G, we have

$$\mathcal{F}(R,G) \simeq a_1 G^n + a_2 R^m. \tag{4.52}$$

In the specific case $n \geq 3$ and $m = (1/2)(1 + 2n + \sqrt{3 - 12n + 4n^2})$ (in which, $m \geq (7 + \sqrt{3})/2$), from Eq. (4.20) we see that the Big Rip singularity could occur. To find other singularity solutions, we investigate the asymptotic form of $\mathcal{G}(H, \dot{H}...)$ in Eq. (4.11) and require the consistency with Eq. (3.8) of Eq. (4.8). The behavior of Eq. (4.11) takes two different asymptotic forms which depend on the parameter of β as follows:

- Case of $\beta \geq 1$: In the limit $t \to t_0$, we find

$$
\begin{aligned}
\mathcal{G}(H, \dot{H}...) \sim \frac{1}{g(R,G)} &\left\{ \alpha \left[\mathcal{F} + \frac{\mathcal{F}'_R}{(t_0 - t)^{2\beta}} + \frac{\mathcal{F}'_G}{(t_0 - t)^{4\beta}} \right] \right. \\
&+ \gamma \left[\frac{1}{(t_0 - t)^{2\beta}} \frac{dg(R,G)}{dR} + \frac{df(R,G)}{dR} \right] \frac{1}{(t_0 - t)^{2\beta}} \\
&\left. + \delta \frac{\dot{\mathcal{F}}'_R}{(t_0 - t)^{\beta}} + \epsilon \frac{\dot{\mathcal{F}}'_G}{(t_0 - t)^{3\beta}} + \zeta \ddot{\mathcal{F}}'_R + \eta \frac{\ddot{\mathcal{F}}'_G}{(t_0 - t)^{2\beta}} \right\},
\end{aligned}
\tag{4.53}
$$

where ϵ and η are constants. To realize a I Type singularity, from Eq. (3.8) we must have

$$
\mathcal{G}(H, \dot{H}...) \sim -\frac{3(1+w)h^2}{\kappa^2(t_0 - t)^{2\beta}} .
\tag{4.54}
$$

Hence, if for $G \sim 1/(t_0 - t)^{4\beta}$ and $R \sim 1/(t_0 - t)^{2\beta}$ with $\beta \geq 1$, the highest term of Eq. (4.53) is proportional to $1/(t_0 - t)^{2\beta}$, it is possible to have a Type I singularity. As in $F(G)$-gravity, this condition is necessary and not sufficient.

- Case of $\beta < 1$: In the limit $t \to t_0$, we obtain

$$
\begin{aligned}
\mathcal{G}(H, \dot{H}...) \sim \frac{1}{g(R,G)} &\left\{ \alpha \left[\mathcal{F} + \frac{\mathcal{F}'_R}{(t_0 - t)^{\beta+1}} + \frac{\mathcal{F}'_G}{(t_0 - t)^{3\beta+1}} \right] + \gamma \left[\frac{1}{(t_0 - t)^{\beta+1}} \frac{dg(R,G)}{dR} \right. \right. \\
&\left. \left. + \frac{df(R,G)}{dR} \right] \frac{1}{(t_0 - t)^{\beta+1}} + \delta \frac{\dot{\mathcal{F}}'_R}{(t_0 - t)^{\beta}} + \epsilon \frac{\dot{\mathcal{F}}'_G}{(t_0 - t)^{2\beta+1}} + \zeta \ddot{\mathcal{F}}'_R + \eta \frac{\ddot{\mathcal{F}}'_G}{(t_0 - t)^{2\beta}} \right\}.
\end{aligned}
\tag{4.55}
$$

To realize this kind of singularities, from Eq. (3.8) we must have

$$
\mathcal{G}(H, \dot{H}...) \sim -\frac{2\beta h}{\kappa^2(t_0 - t)^{\beta+1}} .
\tag{4.56}
$$

Thus, if for $G \sim 1/(t_0 - t)^{3\beta+1}$ and $R \sim 1/(t_0 - t)^{\beta+1}$ with $\beta < 1$, the highest term of Eq. (4.55) is proportional to $1/(t_0 - t)^{\beta+1}$, it is possible to have a Type II, III or IV singularity. Also this condition is necessary and not sufficient.

For the model in Eq. (4.51), if m and n are positive numbers ($\mathcal{F}(R \to \infty, G \to \infty) \simeq a_1 G^n + a_2 R^m$) and $m = 2n$, the asymptotic behavior of Eq. (4.53) (in this case $g(R,G) = a_2 R^{m-1}$) when $R \simeq 12h^2/(t_0 - t)^{2\beta}$ and $G \simeq 24h^4/(t_0 - t)^{4\beta}$ is proportional to $1/(t_0 - t)^{2\beta}$ and therefore it is possible to realize the Type I singularity. As a consequence, we get

$$
\mathcal{G}(H, \dot{H}...) \simeq -\frac{3(1+w)h^2}{\kappa(t_0 - t)^{2\beta}} \left[m - 1 + \frac{(m-2)a_1}{a_2} \right] .
\tag{4.57}
$$

To have the consistence with Eq. (4.54), we find that if $1 \leq m < 2$ and $a_1/a_2 < 0$, the Type I singularity could appear (for example, if $m = 1$, $n = 1/2$, $a_2 = 1$ and $a_1 < 0$, we recover the case of Eq. (3.30)). The same result is obtained if $m > 2$ and $a_1/a_2 > 0$.

We see that for $m > 0$, $n > 0$ and $-1 < \beta < 1$, $\ddot{\mathcal{F}}'_R/g(R,G) \sim (t_0 - t)^{-2}$ (like above, $g(R,G) = a_2 R^{m-1}$) and $\mathcal{G}(H, \dot{H}...)$ in Eq. (4.55) diverges faster than $(t_0 - t)^{-\beta-1}$, so that in order to find Type II or III singularities, we must have $\ddot{\mathcal{F}}'_R = 0$. In general, this is true if $m = 1$ and we can recover the results in Sec. III for $F(G)$-gravity.

When $R \to 0^-$ and $G \to 0^-$, for $m > 0$ and $n > 0$, $\mathcal{F}(R,G)$ behaves as

$$\mathcal{F}(R \to 0^-, G \to 0^-) \simeq \frac{a_5}{a_3 G^n + a_4 R^m} . \tag{4.58}$$

In this case, if $\beta < -1$, Eq. (4.55) ($g(R,G) = a_5/(a_3 G^n R + a_4 R^{m+1})$) diverges and Eq. (4.56) becomes inconsistent, so that the model is free of Type IV singularities.

V.   CURING THE FINITE-TIME FUTURE SINGULARITIES

In this section, we discuss a possible way to cure the finite-time future singularities in $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity. In the limit of large curvature, the quantum effects become important and lead to higher-order curvature corrections. It is therefore interesting to resolve the finite-time future singularities with some power function of G or R.

A. $F(G)$-gravity

First, we consider $F(G)$-gravity. If any singularity occurs, Eq. (3.8) behaves as

$$\mathcal{G}(H, \dot{H}...) \simeq \begin{cases} -\frac{3(1+w)h^2 + 2\beta h}{\kappa^2}(t_0 - t)^{-2} & \text{Big Rip} \\ -\frac{3(1+w)h^2}{\kappa^2}(t_0 - t)^{-2\beta} & \beta > 1 \text{ (Type I)} \\ -\frac{2\beta h}{\kappa^2}(t_0 - t)^{-\beta-1} & \beta < 1 \text{ (Types II, III, IV)} \end{cases} \tag{5.1}$$

The singularities appear in two cases: $G \to \pm\infty$ or $G \to 0^-$.

(i) *Case of* $G \to \pm\infty$

Suppose that for large values of G,

$$R + F(G \to \pm\infty) \longrightarrow R + \gamma G^m , \quad m \neq 1 , \tag{5.2}$$

with $\gamma \neq 0$. One way to prevent a singularity appearing could be that the function $\mathcal{G}(H, \dot{H}...)$ becomes inconsistent with the behavior of Eq. (5.1). In general, $\mathcal{G}(H, \dot{H}...)$ must tend to

infinity faster than Eq. (5.1). For $H = h/(t_0 - t)$ (Big Rip), we have

$$\mathcal{G}(H, \dot{H}...) \simeq \frac{\alpha}{(t_0 - t)^{4m}} \, . \tag{5.3}$$

Hence, if $m > 1/2$, we avoid the singularity. Nevertheless, there is one specific case in which the Big Rip singularity could occur. If $m = (1 + h)/4$, $\mathcal{G}(H, \dot{H}...)$ is exactly equal to zero, so that (for example) the following specific model admits the Big Rip singularity:

$$R + F(G) = R + \frac{\sqrt{24m(4m - 1)^3}}{2h(1 - 2m)} G^{1/2} + \gamma G^m \, . \tag{5.4}$$

This is because the power function G^m is an invariant with respect to the Big Rip singularity of $G^{1/2}$. If for large values of G, $F(G) \sim \alpha G^{1/2}$, we can eliminate the Big Rip singularity with a power function γG^m ($m \geq 2$) only if $\alpha > 0$.

For $H = h/(t_0 - t)^\beta$ with $\beta > 1$ (Type I) and the behavior in Eq. (5.2), we find

$$\mathcal{G}(H, \dot{H}...) \simeq \frac{\alpha}{(t_0 - t)^{4\beta m}} \, . \tag{5.5}$$

Also in this case, if $m > 1/2$, we avoid the singularity. For example, $R + F(G) = R + \alpha\sqrt{G} + \gamma G^2$ with $\alpha > 0$ is free of Type I singularities, while if $\alpha < 0$, the Big Rip singularity could appear.

For $H = h/(t_0 - t)^\beta$ with $0 < \beta < 1$ (Type III) and the behavior in Eq. (5.2), we obtain

$$\mathcal{G}(H, \dot{H}...) \simeq \frac{\alpha}{(t_0 - t)^{m(3\beta+1)+(1-\beta)}} \, . \tag{5.6}$$

If $m > 2\beta/(3\beta + 1)$ (i.e. $m > 1/2$), we avoid the singularity.

Also for $H = h/(t_0 - t)^\beta$ with $-1/3 < \beta < 0$ (Type II, $G \to -\infty$), we have to require the same condition. For example, $R + \alpha|G|^\zeta + \gamma G^2$ with $\zeta < 1/2$ is free of Type I, II and III singularities.

(ii) Case of $G \to 0^-$

Suppose that for small values of G,

$$R + F(G \to 0^-) \longrightarrow R + \gamma G^m \, , \tag{5.7}$$

with $\gamma \neq 0$ and m being an integer. For $H = h/(t_0 - t)^\beta$ with $\beta < -1/3$ (Type II and IV singularities), we get

$$\mathcal{G}(H, \dot{H}...) \simeq \frac{\alpha}{(t_0 - t)^{m(3\beta+1)+(1-\beta)}} \, , \tag{5.8}$$

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which diverges and hence becomes inconsistent with Eq. (5.1) if $m < 2/3$. For example, $\alpha|G|^\zeta + \gamma G^{-1}$ with $\zeta > 2/3$ is free of Types IV singularities.

As a result, the term γG^m with $m > 1/2$ and $m \neq 1$ cure the singularities occurring when $G \to \pm\infty$. Moreover, the term γG^m with $m \leq 0$ and m being an integer cure the singularities occurring when $G \to 0^-$.

In $f(R)$-gravity, by using the term γR^m, the same consequences are found. The term γR^m with $m > 1$ cures the Type II and III singularities. On the other hand, the term γR^m with $m < 2$ cures the Type IV singularity.

Note that γG^m or γR^m are invariants with respect to the Big Rip solution (see Eq. (4.20)), so it is necessary to pay attention to the whole form of the $F(G)$ or $f(R)$-gravity (see Eq. (5.4)).

It is also possible to cure the singularities in a $F(G)$-gravity theory with the power functions of R and a $f(R)$-gravity theory with the power functions of G. To do it, it is useful to take into account that G diverges as R^2 in the Type I singularity solutions, at least as R in the Type II, and as R^2 in the Type III, and G tends to zero at least as R^3 in the Type IV (proved by the fact that when $\beta < 1$, $G \sim R^{3\beta+1/\beta+1}$). We show several examples.

- $R + G^n + R^m \sim R + R^m$ on the asymptotic limit of the Types I and III singularity solutions (if they exist) when $m > 2n$.

- $R + G^n + R^m \sim R + R^m$ on the Types II singularity solutions if $m > n$ (on this kind of solutions R tends always to infinity, while in some cases, for $-1 < \beta < -1/3$, G tends to zero).

- $1/(G^n + R^m) \sim 1/R^m$ on the Types IV singularity solutions (for which $R \to 0^-$ and $G \to 0^-$) if $m < 3n$.

Thus, the singularity solutions found in Sec. III for $F(G)$-gravity can be cured by the term γR^m with $m > 1$ for Type I, II and III singularities and that with $m < 1$ for Type IV singularity.

We mention the Type I singularities with $\beta > 1$. We have shown that the model $R + \gamma G^m$ with $m > 1/2$ ($m \neq 1$) is free of Type I singularity because Eq. (3.8) becomes inconsistent. Nevertheless, it follows from Eq. (4.40) that the models G^m with $m > 1/2$ and R^n with $n > 1$ can show the Type I singularity in the asymptotic limit. This means that when t is

very close to t_0, the term G^m or R^n is dominant over R, $R + G^m \sim G^m$ (or $R + R^n \sim R^n$) and therefore the Type I singularity could appear. Hence, the important point is whether the model can approach to very large values of R or G with a non-singular metric (which is not admitted) and then become singular because R is negligible. It depends on the form and the dynamics of the model and the value of m or n. If $m \gg 1/2$ or $n \gg 1$, the singularity could appear more easily. Thus, in order to avoid the singularity solutions, it is better to choose m and n as $m > 1/2$ and as $n \gtrsim 1$, respectively, but m and n are not very large. This does not hold in the other types of singularities. The theory $R + G^m$ with $m \le 0$ and m being an integer (or $R + R^n$ with $n < 2$) is free of Type II, III and IV singularities as the theories G^m or R^n.

B. $\mathcal{F}(R, G)$-gravity

Next, we study $\mathcal{F}(R, G)$-gravity. In the general $\mathcal{F}(R, G)$-gravity, in order to avoid the singularities with power functions, we must require that the EOM (4.1) and (4.2) are inconsistent on the singularities solutions. Within the framework of $\mathcal{F}(R, G)$-gravity, we can use the terms such as G^m/R^n to cure the singularities. The singularities appear in the following three cases: (a) $R \to \pm\infty$, $G \to \pm\infty$ (Types I, II, III), (b) $R \to -\infty$, $G \to 0^-$ (Type II for $-1 < \beta < -1/3$), and (c) $R \to 0^-$, $G \to 0^-$ (Type IV).

We investigate general cases. Suppose that for large values of G and R,

$$\mathcal{F}(R \to \infty, G \to \infty) \longrightarrow R + \gamma \frac{G^m}{R^n}, \tag{5.9}$$

with $\gamma \neq 0$. In the case of the Big Rip singularity, in which H is given by Eq. (3.17), $\mathcal{G}(H, \dot{H}...)$ in Eq. (4.11) diverges as

$$\mathcal{G}(H, \dot{H}...) \sim \frac{\alpha}{(t_0 - t)^{4m-2n}}. \tag{5.10}$$

Thus, if $m > (n+1)/2$, we avoid the singularity. Nevertheless, there is the possibility that $\mathcal{G}(H, \dot{H}...)$ is exactly equal to zero and the Big Rip singularity could occur (see Eqs. (4.30) and (4.31) in the case of $m = n + 1$). Hence, the whole form of $\mathcal{F}(R, G)$ as well as its form in the asymptotic limit must be examined.

In the case of Eq. (3.27) (Type I), $\mathcal{G}(H, \dot{H}...)$ diverges as

$$\mathcal{G}(H, \dot{H}...) \sim \frac{\alpha}{(t_0 - t)^{4\beta m - 2\beta n}}. \tag{5.11}$$

Also in this case, if $m > (n+1)/2$, we avoid the singularity. Similarly to the above, however, if $m \gg 1$ and $n \ll 1$, the asymptotic limit of $\mathcal{F}(R,G)$ in Eq. (5.9) behaves as $\gamma G^m/R^n$ and therefore the Type I singularity could occur (see Eq. (4.40)).

As a consequence, we can avoid the Type I singularities if the asymptotic behavior of the model is given by Eq. (5.9) and its asymptotic form has the power functions

$$G^m R^n \,, \tag{5.12}$$

or

$$\frac{G^m}{R^n} \,, \quad m > \frac{n+1}{2} \,, \tag{5.13}$$

with m and n being positive integers.

Now, suppose that when $H = h/(t_0 - t)^\beta$ with $\beta < 1$, the asymptotic limit of $\mathcal{F}(R,G)$ becomes

$$\mathcal{F}(R,G) \longrightarrow \gamma \frac{G^m}{R^n} \,. \tag{5.14}$$

For $\beta < 1$, $\mathcal{G}(H, \dot{H}...)$ behaves as

$$\mathcal{G}(H, \dot{H}...) \sim \frac{\alpha}{(t_0 - t)^2} \,, \tag{5.15}$$

which diverges faster than $(t_0 - t)^{-\beta-1}$ and therefore the Type II, III and IV singularities are always avoided for any value of m and n. The same scenario to cure the future singularity by adding the non-singular modified gravity maybe applied here again.

VI.   CONCLUSION

In the present paper, we have investigated the finite-time future singularities in $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity. We have reconstructed the $F(G)$-gravity and $\mathcal{F}(R,G)$-gravity models in which the finite-time future singularities may occur. It has been demonstrated that all four types of finite-time future singularity may emerge for a variety of the above models with the effective quintessence/phantom EoS behavior in the same qualitative way as for convenient DEs where also all four types of future singularity may occur [17–19]. This provides the explicit demonstration that whatever the effective DE model (convenient one or modified gravity) is, it may lead to singular future universe. Moreover, the future singularity may manifest itself as radius singularity for spherically-symmetric spaces. This may cause

instabilities for black holes [18] and relativistic stars [27, 29, 31, 39]. Other imprints of the singular future universe to current cosmology may be searched as well.

However, there exists fundamental qualitative difference between convenient DE and modified gravity. It turns out that sometimes it is possible to solve the singularity issue taking account of quantum gravity effects (see Big Rip singularity resolution in Ref. [40]) or by the coupling of DE with Dark Matter (DM) (some fine-tuning of initial conditions may help to resolve Type II or Type IV future singularity [41]). Nevertheless, quantum gravity account is effectively the modification of gravity. Moreover, it is only modified gravity (actually, its additional modification as we have demonstrated on the example of non-singular $F(G)$-model in Subsection III. C) may suggest the universal scenario to cure any finite-time future singularity. This is achieved by adding such non-singular theory to any DE containing future singularity in its evolution. Furthermore, such additional modification may always be made by terms which are relevant only in the early universe and are typical as quantum gravity corrections. Hence, modified gravity suggests the universal scenario to protect the future universe from singularity while not destroying the attractive cosmological properties of specific DE alternative gravity like its viability with local/cosmological tests if exists. This may be considered as powerful theoretical argument in favor of the consideration of such theories as DEs.

Acknowledgments

Appendix A: Simple model of $f(R)$-gravity

There is a great diffusion of modified gravity models which for large values of curvature tend to a constant and imitate the ΛCDM model. Nevertheless, the existence of derivatives

of the modified function, which is very small but different from zero, involves the possibility of singularities for $\beta < 1$. In this Appendix, we explore the finite-time future singularities in a simple model of $f(R)$-gravity. The action of $f(R)$-gravity is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + f(R)\right) + \mathcal{L}_{\text{matter}}\right], \tag{A1}$$

which corresponds to the action in Eq. (2.1) with $\mathcal{F}(R,G) = R + f(R)$. We examine the following simple $f(R)$-gravity model [42] which reproduces the current accelerated expansion of the universe and imitate a cosmological constant for large values of curvature:

$$f(R) = \alpha(e^{-bR} - 1), \tag{A2}$$

where $b\,(> 0)$ is a positive constant. In what follows, we consider the pure gravitational action of $f(R)$-gravity in Eq. (A1) without $\mathcal{L}_{\text{matter}}$.

The finite-time future singularities in $f(R)$-gravity have been discussed in Ref. [21], from which we propose the asymptotic expression of $\mathcal{G}(H, \dot{H}, ...)$ when $\beta < 1$, similarly to Eq. (3.43):

$$\mathcal{G}(H, \dot{H}, ...) \sim \alpha f(R) + \frac{\gamma}{(t_0 - t)^{\beta+1}}f'(R) + \frac{\delta}{(t_0 - t)^{\beta+3}}f''(R) + \frac{\zeta}{(t_0 - t)^{2\beta+4}}f'''(R). \tag{A3}$$

Here, the prime denotes differentiation with respect to R. When $\beta < 0$, the asymptotic behavior of R is given by

$$R \sim 6h\beta(t_0 - t)^{-\beta-1}. \tag{A4}$$

We assume $R > 0$, so in this case h must be negative. If H behaves as

$$H \sim h(t_0 - t)^{-\beta} + H_0, \tag{A5}$$

as Eq. (2.8), H can be still positive in the limit $t \to t_0$. For large values of R $(-1 < \beta < 1)$, $f(R)$ in Eq. (A2) tend to $-\alpha$ and

$$f'(R) \sim -\alpha b e^{-bR}, \quad f''(R) \sim \alpha b^2 e^{-bR}, \quad f'''(R) \sim -\alpha b^3 e^{-bR}. \tag{A6}$$

To obtain a singularity, the highest term in Eq. (A3) must be divergent as $1/(t_0 - t)^{\beta+1}$. In order to check it, it is convenient to develop the exponential function in power-series. The third term of Eq. (A3) behaves as

$$\frac{\delta}{(t_0 - t)^{\beta+3}}f''(R) \sim \frac{\delta}{\Sigma_{(n=0\to\infty)}(t_0 - t)^{-n(\beta+1)+\beta+3}}. \tag{A7}$$

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For $n = 2/(\beta + 1)$, the highest term of the denominator behaves as $(t_0 - t)^{\beta+1}$. If $\beta \to -1^+$, $n \to \infty$ and this is just the asymptotic value of n of the highest term in Eq. (A7). A similar argument is valid for the last term of Eq. (A3), whereas the first and second terms of Eq. (A3) tend to a constant and zero, respectively. Thus, a Type II singularity could occur in the model in Eq. (A2) for large values of R. Note that $f(R)$-gravity unifying the early-time inflation with late-time acceleration as proposed in Ref. [43] turns out to be non-singular due to the presence of R^2 term.

Appendix B: Asymptotic behavior of singular models

In this Appendix, we discuss the asymptotic behavior of singular models. In Sec. IV A, we have shown that in principle it is possible to write a general $\mathcal{F}(R, G)$-gravity theory in the form in Eq. (4.8). In this case, from Eqs. (3.9) and (4.11) we obtain

$$\mathcal{G}(H, \dot{H}...) = -\frac{1}{\kappa^2}\left[2\dot{H} + 3(1 + w)H^2\right], \tag{B1}$$

where

$$\begin{aligned}
\mathcal{G}(H, \dot{H}...) = \frac{1}{2\kappa^2 g(R, G)}&\left\{(1 + w)(\mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G)\right.\\
&+ \left(R\frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR}\right)\left[6H^2(1 + w) + 4\dot{H}\right]\\
&\left.+ H\dot{\mathcal{F}}'_R(4 + 6w) + 8H\dot{\mathcal{F}}'_G\left[2\dot{H} + H^2(2 + 3w)\right] + 2\ddot{\mathcal{F}}'_R + 8H^2\ddot{\mathcal{F}}'_G\right\}. \tag{B2}
\end{aligned}$$

It is clear that in the case of $\mathcal{F}(R, G) = R + F(G)$, by taking $g(R, G) = 1$ and $f(R, G) = F(G)$ in Eq. (4.8), Eqs. (3.8) and (3.10) in Sec. III A is recovered.

To verify the existence of singularities, it is very useful to control the consistence of Eq. (B1). On the exact solutions (this is the case of the Big Rip solutions), this check is independent of the choice of $g(R, G)$ in Eq. (4.8). We must carefully verify only that $g(R, G) \neq 0$ on the singularity solution (or -equivalently- if we use this equation to find singularity solutions, it is possible to lost some solutions in which $g(R, G) = 0$). When we check asymptotic solutions, a problem could appear. It is based on the confrontation between the asymptotic behaviors of the right-hand and left-hand sides of Eq. (B1). Furthermore, in this case there is a problem if all the terms of $\mathcal{F}(R, G)$ have the same asymptotic behavior on the singularity solution. For example, in order to verify the existence of the Type I

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singularity solution in the model $R - \alpha G/R$ with $\alpha > 0$ (see Eq. (4.44)), it is indifferent to take $g(R,G) = 1$ and $f(R,G) = -\alpha G/R$ or $g(R,G) = -\alpha G/R^2$ and $f(R,G) = R$ because on the singularity solution the asymptotic behavior of R ($R \sim (t_0 - t)^{-2}$) is the same as G/R. Nevertheless, if the terms do not have the same asymptotic behavior (for example, $\mathcal{F}(R,G) = R + R^2 + R^3...$), it is necessary to be careful in the choice of $g(R,G)$ to substitute into Eq. (B2). The mechanism is the following: The right-hand side of Eq. (B1) behaves as R, while the left-hand side of Eq. (B1) is proportional to $1/g(R,G)$. Automatically, in the asymptotic limit all the terms smaller than $Rg(R,G)$ (and also their derivatives) are neglected. As a consequence, when we use the EOM in the asymptotic limit, we must choose $g(R,G)$ as the coefficient of the smallest term which we want to consider. This is easy to do when the terms are completely different in the limit. For example, $R^2 + 1/R \sim R^2$ when $R \to \infty$ (this is the principle that we have used in the study of the realistic models shown in the present paper). The question is trickier when the terms of $\mathcal{F}(R,G)$ tend together to infinity or to zero with different velocities. In this case, the choice of $g(R,G)$ depends on our target, if we want to verify the EOM in more or less strong limit. For example, let us consider the model $\mathcal{F}(R,G) = R + R^2$. If we choose $g(R,G) = 1$ and $f(R,G) = R^2$, we find that Eq. (B1) is inconsistent on the Type I singularity solution, so we can say that the model is free of this kind of singularity. Nevertheless, the choice of $g(R,G) = R$ and $f(R,G) = R$ is equivalent to neglecting the first term of $\mathcal{F}(R,G)$, so we are considering $\mathcal{F}(R,G) \sim R^2$ (strong limit when $R \to \infty$). In this case, we find that the model could be affected by the Type I singularity. The physical meaning has been discussed in Sec. V.

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