On isotropic turbulence in the dark fluid universe

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(Dated: April 5, 2011)

As first part of this work, experimental information about the decay of isotropic turbulence in ordinary hydrodynamics, $\overline{\mathbf{u}^2(t)} \propto t^{-6/5}$, is used as input in FRW equations in order to investigate how an initial fraction f of turbulent kinetic energy in the cosmic fluid influences the cosmological development in the late, quintessence/phantom, universe. First order perturbative theory to the first order in f is employed. It turns out that both in the Hubble factor, and in the energy density, the influence from the turbulence fades away at late times. The divergences in these quantities near the Big Rip behave essentially as in a non-turbulent fluid. However, for the scale factor, the turbulence modification turns out to diverge logarithmically. As second part of our work, we consider the full FRW equation in which the turbulent part of the dark energy is accounted for by a separate term. It is demonstrated that turbulence occurrence may change the future universe evolution due to dissipation of dark energy. For instance, phantom-dominated universe becomes asymptotically a de Sitter one in the future, thus avoiding the Big Rip singularity.

PACS numbers: 98.80.-k,04.50.+h,11.10.Wx

I. INTRODUCTION

Consider a spatially flat FRW universe with $H = \dot{a}/a$ the Hubble parameter. In standard notation the FRW equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\kappa^2(\rho + 3p) \tag{1}$$

with $\kappa^2 = 8\pi G$, implies that for the scale factor $a(t)$ to depict a curve concave upwards when drawn as a function of t, i.e. $\ddot{a} > 0$, it is sufficient that the equation of state (EoS) satisfies the condition

$$p < -\frac{1}{3}\rho\,. \tag{2}$$

In order to obtain $\dot{H} > 0$ it follows, however, from the equation

$$\dot{H} = -\frac{1}{2}\kappa^2(\rho + p) \tag{3}$$

that p has to satisfy the stronger condition

$$p < -\rho\,. \tag{4}$$

This is the phantom region, corresponding to a positive tensile stress in the dark energy fluid. It is known that phantom-dominated universe usually enters to finite-time future singularity (Big Rip) (see Refs. [1–3]). The region $-\rho < p < -\rho/3$ is called the quintessence region, where the expansion of the universe is accelerated $\ddot{a} > 0$ but not super-accelerated $\dot{H} > 0$. Note that effective quintessence universe may also end up in one of three possible types of future singularity [3].

This paper analyzes the possibility whether there can be a *turbulent microstructure* superimposed on the dark fluid. Quite obviously it is the isotropic version of turbulence theory which then becomes most relevant, as one wishes not to disturb the macroscopic isotropy. Turbulence generally implies that there is a loss of kinetic energy into heat. Whereas this loss is often described macroscopically, in terms of a bulk viscosity ζ (cf., for instance, Refs. [4–6]), our program here is to replace ζ with a microscopic *shear* viscosity. In other words, we wish to replace the length scale associated with the macroscopic ζ by the Kolmogorov microlength scale, conventionally denoted by η.

The next section gives a brief overview of isotropic classical turbulence theory, in order to put our present approach in a proper context. Thereafter, we consider how the turbulence effect can be taken into account in the cosmological formalism. We focus on the following two possibilities:
1) The turbulence effect can be included by the addition of a constant *fraction*, called f, to the laminar ordinary energy density ρ in the first FRW equation. Assuming f to be a small quantity, a perturbative solution to the first order in f can conveniently be found.
2) Our second approach is to write the total energy density as a sum of four different parts: (i) a laminar dark energy part, (ii) a turbulent dark energy part, (iii) a radiation part, and (iv) an ordinary matter part. The fate of the universe is in principle predictable, on the basis of the weight given to each of the constituents of the total energy density.

Options 1) and 2) are considered in Sections III and IV, respectively.

We mention finally that our method of decomposing the fluid into turbulent and non-turbulent parts is in principle similar to the method recently used by Balakin and Bochkarev [7]. These authors divided the energy as well as the pressure of the cosmic fluid into two components, one component referring to dark matter, the other referring to dark energy.

II. EXTRACTS FROM KOLMOGOROV'S ISOTROPIC TURBULENCE THEORY [8, 9]

Turbulence generally implies a loss of kinetic energy into heat. As mentioned, we replace the macroscopic bulk viscosity with a microscopic shear viscosity, corresponding to the Kolmogorov length

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \tag{5}$$

Here ν is the kinematic microscopic shear viscosity and ϵ is the dissipation per unit time and unit mass.

Let l denote the external scale of the turbulence, with $1/l$ the corresponding wave number. The large eddies move around with only a little dissipation of energy. According to the so-called second hypothesis of Kolmogorov [8], in an isotropic region the motion is entirely determined by friction and inertia. There occurs a continuous flux of energy transferred by means of a hierarchy of eddies corresponding to the dissipation ϵ. Let λ characterize the size of an eddy, $k = 1/\lambda$ being the corresponding wave number. The *equilibrium range* is that for which all memory of the flow is lost,

$$k \gg 1/l\,. \tag{6}$$

If u_λ is the typical velocity of an eddy of size λ, the internal Reynolds number is $\mathrm{Re}_\lambda \sim \lambda u_\lambda/\nu$. For increasing values of k, Re_λ decreases. Dissipation becomes important when $\mathrm{Re}_\lambda \sim 1$. This is just the condition leading to the Kolmogorov length (5).

If the Reynolds number of the flow as a whole is high, the wave numbers $1/l$ and $1/\eta$ are widely separated, and there exists an inertial subrange characterized by

$$\frac{1}{l} \ll k \ll \frac{1}{\eta}, \tag{7}$$

in which the fluid behaves like a non-viscous fluid. A famous formula for the spectral energy density, conventionally called $E(k)$, in the inertial subrange is

$$E(k) = \alpha\epsilon^{2/3}\,k^{-5/3}\,, \tag{8}$$

where $\alpha \approx 1.5$ is the Kolmogorov constant.

For practical purposes the von Kármán interpolation formula for $E(k)$, linking the region of small k to the region of high k, is useful in order to calculate the total energy density E by integrating over all wave numbers (cf., for instance, Ref. [10]). We abstain from going into further detail here. Of interest for us here is, however, is the empirical decay law for isotropic turbulence (of course, under the assumption that it is left to itself; there are no external sources). Based on grid experiments in wind and water tunnels, it turns out that the mean kinetic energy $\frac{1}{2}\overline{\mathbf{u}^2(t)}$ decays as

$$\overline{\mathbf{u}^2(t)} \propto t^{-6/5}; \tag{9}$$

cf. Refs. [11, 12], as well as the theoretical treatment in [10]. We shall make use of this relationship in the following.

Consider now the classical equation of motion for a viscous fluid:

$$\partial_t(\rho_m u_i) + \partial_k \Pi_{ik} = 0, \tag{10}$$

where ρ_m is the mass density and Π_{ik} the momentum flux density tensor [9],

$$\Pi_{ik} = p\delta_{ik} + \rho_m u_i u_k - \mu(\partial_k u_i + \partial_i u_k), \tag{11}$$

μ being the shear viscosity. Taking the mean of this equation, observing that $\bar{u}_i = 0$ in homogeneous and isotropic turbulence, we get

$$\bar{\Pi}_{xx} = \bar{\Pi}_{yy} = \bar{\Pi}_{zz} \equiv p_{\rm eff} = p + \frac{2}{3}\rho_{\rm turb}, \tag{12}$$

where $p_{\rm eff}$ is the effective which takes into account that the thermodynamical pressure is augmented by a term $(2/3)\rho_{\rm turb}$ associated with the turbulent energy density,

$$\rho_{\rm turb} = \frac{1}{2}\rho_m \overline{u_i u_i} \equiv \frac{1}{2}\rho_m \overline{\mathbf{u}^2}. \tag{13}$$

Thus $\rho_{\rm turb}$ designates a mean quantity.

III. FRW EQUATIONS WITH A FRACTION f OF THE ENERGY AS TURBULENT ENERGY

We will now consider this fluid in a cosmological setting. As usual in turbulence theory we may start by decomposing the fluid velocity u_i into a mean component U_i and a fluctuating component u_i', $u_i = U_i + u_i'$. However, in a comoving reference frame $U_i = 0$, so that we can simply replace u_i' with u_i. In order to keep the turbulent part of the cosmic fluid separate from the non-viscous (non-turbulent) part, we shall from now on endow non-viscous quantities with a subscript zero. Thus Eq. (12) is rewritten as

$$p_{\rm eff} = p_0 + \frac{2}{3}\rho_{\rm turb}, \tag{14}$$

with $\rho_{\rm turb} = \frac{1}{2}\rho_m \overline{\mathbf{u}^2}$ as before. Note that the turbulent energy in the comoving frame is regarded as a nonrelativistic quantity. The total energy density can be written as

$$\rho = \rho_0 + \rho_{\rm turb}. \tag{15}$$

As mentioned above, we shall first look for a solution of the cosmological equations when a constant fraction f of the energy exists in the form of turbulent energy. As we are primarily interested in the dark energy epoch of the universe, we assume henceforth that the thermodynamical parameter w satisfies the inequality $w < -1$ (see Eq. (21) below). The initial instant for our considerations will be denoted by $t_{\rm in}$.

By assumption we can thus write the energy density at time $t = t_{\rm in}$ as

$$\rho(t_{\rm in}) = \rho_0(t_{\rm in})(1 + f). \tag{16}$$

We see that f can be interpreted as the ratio between the turbulent energy and the total energy at $t_{\rm in}$,

$$f = \frac{\rho_{\rm turb}}{\rho_0}\Big|_{t_{\rm in}}. \tag{17}$$

As ρ_0 includes the rest mass, this shows showing that the assumption $f \ll 1$ is a plausible one.

For $t > t_{\rm in}$ we now require $\rho_{\rm turb}$ to decay with time as

$$\rho_{\rm turb} \propto t^{-6/5}, \tag{18}$$

in accordance with Eq. (9) for ordinary turbulence. We can write $\rho(t)$ for $t \geq t_{\rm in}$ in the form

$$\rho(t) = \rho_0(t)[1 + f\rho_1(t)], \tag{19}$$

with

$$\rho_1(t) = \left(\frac{t_{\rm in}}{t}\right)^{6/5}. \tag{20}$$

When $t = t_{\rm in}$, this agrees with Eq. (16).

The equation of state (EoS) for the cosmic fluid is now to be introduced. This can be done in various ways. We choose write it in such a way that only non-turbulent quantities are involved,

$$p_0(t) = w\rho_0(t), \quad w: \text{ constant}. \tag{21}$$

The effective pressure will analogously to Eq. (19) be expanded as

$$p_{\rm eff}(t) = w\rho_0(t)[1 + fp_1(t)]. \tag{22}$$

At $t = t_{\rm in}$ it follows that $p_1(t_{\rm in})$ is actually a known quantity,

$$wp_1(t_{\rm in}) = \frac{2}{3}. \tag{23}$$

In the same way we can expand the scale factor as

$$a(t) = a_0(t)[1 + fa_1(t)]\,, \tag{24}$$

and analogously for the Hubble factor

$$H(t) = H_0(t)[1 + fH_1(t)]. \tag{25}$$

The correction terms $\{p_1, a_1, H_1\}$ are all of zeroth order in f.

From $H = \dot{a}/a$ we get at once

$$\dot{a}_1 = H_0 H_1, \tag{26}$$

whereas the first FRW equation $H^2 = \frac{1}{3}\kappa^2\rho$ yields the first order relationship

$$H_1 = \frac{1}{2}\rho_1. \tag{27}$$

The Hubble parameter $H(t)$ thus satisfies the equation

$$H(t) = H_0(t)\left[1 + \frac{1}{2}f\left(\frac{t_{\rm in}}{t}\right)^{6/5}\right]. \tag{28}$$

We still need to determine $H_0(t)$. It can be found from the non-turbulent FRW equations

$$H_0^2 = \frac{1}{3}\kappa^2\rho_0\,, \tag{29}$$

$$\frac{\ddot{a}_0}{a_0} + \frac{1}{2}H_0^2 = -\frac{1}{2}\kappa^2 p_0\,, \tag{30}$$

which, together with the conservation equation for energy, $T^{0\nu}{}_{;\nu} = 0$, yield

$$\dot{\rho}_0 + 3H_0(\rho_0 + p_0) = 0\,. \tag{31}$$

From Eq. (29), $\dot{H}_0 = (\sqrt{3}/6)\kappa\dot{\rho}_0/\sqrt{\rho_0}$, and as $\dot{H} = -H^2 + \ddot{a}/a$ we get

$$\dot{H}_0 + \frac{3}{2}\gamma H_0^2 = 0\,. \tag{32}$$

Here we have for convenience introduced the symbol γ, defined as

$$\gamma = 1 + w. \tag{33}$$

The solution of this equation is

$$H_0(t) = \frac{H_0(t_{\rm in})}{1 + \frac{3}{2}\gamma H_0(t_{\rm in})(t - t_{\rm in})} \, . \tag{34}$$

Thus

$$H(t) = \frac{H_0(t_{\rm in})}{1 + \frac{3}{2}\gamma H_0(t_{\rm in})(t - t_{\rm in})} \left[1 + \frac{1}{2} f \left(\frac{t_{\rm in}}{t} \right)^{6/5} \right] . \tag{35}$$

Correspondingly, we obtain

$$\rho(t) = \frac{3}{\kappa^2} \frac{H_0^2(t_{\rm in})}{[1 + \frac{3}{2}\gamma H_0(t_{\rm in})(t - t_{\rm in})]^2} \left[1 + f \left(\frac{t_{\rm in}}{t} \right)^{6/5} \right] . \tag{36}$$

We can now draw the following important conclusion: both for the Hubble factor, and the energy density, the influence from the turbulence fades away when $t \gg t_{\rm in}$. To a good approximation the future singularity occurs at the same instant $t = t_s$ as if turbulence were absent, i.e.

$$t_s = t_{\rm in} + \frac{2}{3|\gamma| H_0(t_{\rm in})} \, . \tag{37}$$

Both H and ρ diverge at $t = t_s$. Near t_s, as $t_{\rm in}/t_s \ll 1$,

$$H(t) \approx \frac{H_0(t_{\rm in})}{1 - t/t_s}, \quad t \to t_s \, , \tag{38}$$

$$\rho(t) \approx \frac{3}{\kappa^2} \frac{H_0(t_{\rm in})}{(t - t/t_s)^2} \, , \quad t \to t_s \, . \tag{39}$$

Consider next the correction a_1 to the scale factor. From Eqs. (26) and (27),

$$\dot{a}_1 = \frac{1}{2} H_0 \rho_1 \, , \tag{40}$$

from which we obtain by integration, setting $x = t/t_{\rm in} - 1$,

$$a_1(t) = \frac{1}{2} H_0(t_{\rm in}) t_{\rm in} \int_0^{t/t_{\rm in} - 1} \frac{dx}{(1+x)^{6/5}} \frac{1}{1 - (t_{\rm in}/t_s)x}. \tag{41}$$

We need not calculate this integral in full, but note that it diverges logarithmically at $x = t_s/t_{\rm in}$. Omitting multiplicative factors, we write for the dominant part

$$a_1(t) \sim \ln(1 - t/t_s), \quad t \to t_s \, . \tag{42}$$

Thus the modification coming from turbulence is in this case itself turbulent. As the solution in the case $f = 0$ is now

$$a_0(t) = \frac{a_0(t_{\rm in})}{[1 + \frac{3}{2}\gamma H_0(t_{\rm in})(t - t_{\rm in})]^{2/3|\gamma|}} \, , \tag{43}$$

it follows however from the expansion (24) that the divergence in $a_0(t)$ is much stronger than the turbulence modification. The dominant term near $t = t_s$ is thus

$$a(t) \approx \frac{a_0(t_{\rm in})}{(1 - t/t_s)^{2/3|\gamma|}} \, , \quad t \to t_s \, , \tag{44}$$

just as in the case $f = 0$.

So far, we have made use of the first FRW equation only; we have not considered the pressure in the cosmic fluid. To deal with the pressure, we have to take into account the second FRW equation also, for instance in the form

$$\frac{d}{dt}(\rho a^3) = -3H p_{\rm eff} a^3. \tag{45}$$

By expanding in the parameter f in the same way as above, we obtain to first order

$$p_1 = \rho_1 - H_1 - \frac{\dot{\rho}_1 + 3\dot{a}_1}{3H_0 w}\,. \tag{46}$$

Inserting for ρ_1, H_1, a_1, and H_0 we get

$$p_1(t) = \frac{1}{2}\left[1 - \frac{1}{w} + \frac{4}{5}\frac{1}{H_0 w t}\right]\rho_1(t)\,. \tag{47}$$

It is here to be observed that if one extrapolates this expression back in time, until the initial instant $t_{\rm in}$, the expression does not in general agree with the previous equation (23). The reason for this is that our condition (23) on the initial pressure makes the system mathematically over-determined. Of most physical interest is, however, the cosmic pressure at late times, H_0 and t large, in which case the last term in Eq. (47) fades away and we get

$$p_1(t) \approx \frac{1}{2}\left(1 - \frac{1}{w}\right)\left(\frac{t_{in}}{t_s}\right)^{6/5}\,, \quad t \to t_s\,. \tag{48}$$

When w lies between -1/3 and -1, i.e. in the quintessence region, the value of p_1 is actually higher than when $w < -1$.

IV. TURBULENT DARK ENERGY DENSITY COMPONENT AS A SEPARATE TERM IN THE FRW EQUATION

We now leave the perturbative approach, and consider instead the first FRW equation together with the energy conservation equation when the total energy density is written as a sum of four different parts: first, a dark energy contribution consisting of a laminar part $\rho_{\rm dark}$ and a turbulent energy part $\rho_{\rm turb}$ so that

$$\rho_{\rm dark\ energy} = \rho_{\rm dark} + \rho_{\rm turb}; \tag{49}$$

secondly, a radiation part $\rho_{\rm rad}$ and an ordinary matter part $\rho_{\rm matter}$. The FRW equation thus reads

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm dark} + \rho_{\rm turb} + \rho_{\rm rad} + \rho_{\rm matter}\,. \tag{50}$$

As in the previous section, we follow the development of the universe from the instant $t_{\rm in}$ onwards, and we adopt the same empirical law for the time development of the turbulent energy density,

$$\rho_{\rm turb} = \rho_{\rm turb}(t_{\rm in})\left(\frac{t}{t_{\rm in}}\right)^{-6/5}\,. \tag{51}$$

The time derivative will be written in the form

$$\dot{\rho}_{\rm turb} = -C\rho_{\rm turb}^{11/6}\,, \tag{52}$$

where

$$C = \frac{6}{5t_{\rm in}}\left[\rho_{\rm turb}(t_{\rm in})\right]^{-5/6}\,. \tag{53}$$

Now consider the energy balance equation in which $\dot{\rho}_{\rm turb}$ is considered as a source term,

$$\dot{\rho}_{\rm turb} + 3H(\rho_{\rm turb} + p_{\rm turb}) = -C\rho_{\rm turb}^{11/6}\,. \tag{54}$$

Here, we assume Eq. (52) should hold only in the flat universe. In the FRW universe, Eq. (52) should be changed as in (54). Because of the turbulence, the kinetic energy changes into heat and the heat then becomes radiation. Then we may consider the the conservation law for radiation in the form

$$\dot{\rho}_{\rm rad} + 3H(\rho_{\rm rad} + p_{\rm rad}) = C\rho_{\rm turb}^{11/6}\,. \tag{55}$$

Here, $p_{\rm rad} = \rho_{\rm rad}/3$. For definiteness let us consider the case where the turbulent part $\rho_{\rm turb}$ dominates. Then Eq. (50) gives

$$H \sim \frac{\kappa}{\sqrt{3}}\,\rho_{\rm turb}^{1/2}\,. \tag{56}$$

We shall now assume that the EoS parameter $w_{\rm turb}$ of the turbulent part is constant. The EoS for the turbulent quantities is written in conventional form,

$$p_{\rm turb} = w_{\rm turb}\,\rho_{\rm turb}\,. \tag{57}$$

Then, with the definition $\gamma_{\rm turb} = 1 + w_{\rm turb}$ we can write Eq. (54) as

$$0 = \dot{\rho}_{\rm turb} + \kappa\sqrt{3}\,\gamma_{\rm turb}\,\rho_{\rm turb}^{3/2} + C\rho_{\rm turb}^{11/6}\,. \tag{58}$$

We shall discuss three different options for this equation:
(i) If

$$\rho_{\rm turb} \gg \left(\frac{\kappa}{C}\right)^3\,, \tag{59}$$

the third term dominates compared with the second, and we recover the expression (52). It means that $\rho_{\rm turb}$ behaves as in flat spacetime, $\rho_{\rm turb} \propto t^{-6/5}$.
(ii) By contrast, if

$$\rho_{\rm turb} \ll \left(\frac{\kappa}{C}\right)^3\,, \tag{60}$$

the second term dominates compared with the third, and the turbulent term becomes negligible. Then $\rho_{\rm turb}$ behaves as a usual perfect fluid, giving $\rho_{\rm turb} \propto a^{-3\gamma_{\rm turb}}$ and $H \sim \frac{2}{3\gamma_{\rm turb}}\frac{1}{t}$ (it is here assumed that $w_{\rm turb} > -1$).
(iii) If

$$w_{\rm turb} < -1\,, \tag{61}$$

there exists remarkably enough a solution where $\rho_{\rm turb}$ is a constant,

$$\rho_{\rm turb} = \left(-\frac{C}{3\gamma_{\rm turb}}\right)^{-6/5}\,. \tag{62}$$

We now consider the case where the scale factor a is given as a function of the cosmological time, $a = a(t)$. Equation (54) may be rewritten as

$$\frac{d}{dt}\left(a^{3\gamma_{\rm turb}}\rho_{\rm turb}\right) = -Ca^{-5\gamma_{\rm turb}/2}\left(a^{3\gamma_{\rm turb}}\rho_{\rm turb}\right)^{11/6}\,, \tag{63}$$

which can be integrated to yield

$$\rho_{\rm turb}(t) = a(t)^{-3\gamma_{\rm turb}}\left(\frac{5C}{6}\int_{t_{\rm in}}^t dt' a(t')^{-5\gamma_{\rm turb}/2} + C_0\right)^{-6/5}\,. \tag{64}$$

Here C_0 is a constant of integration, which can be determined from the initial condition at $t = t_{\rm in}$. When $a(t)$ is a constant, Eq. (64) reproduces the standard result: $\rho_{\rm turb} \propto t^{-\frac{6}{5}}$ when t is large enough. In case of $w_{\rm turb} = -1$, even if $a(t)$ is not a constant, we obtain $\rho_{\rm turb} \propto t^{-\frac{6}{5}}$ for large t, again.

We can also integrate (55) to obtain

$$\rho_{\rm rad}(t) = Ca(t)^{-4}\left(\int_{t_{\rm in}}^t dt' a(t')^4\rho_{\rm turb}(t) + C_1\right)\,. \tag{65}$$

Here C_1 is a constant of integration.

When $w_{\rm turb} \neq -1$, if we consider the case of the de Sitter space: $a(t) = a_0 \mathrm{e}^{H_0 t}$ with constants a_0 and H_0, we obtain

$$\rho_{\rm turb} = \mathrm{e}^{-3\gamma_{\rm turb} H_0 t} \left(\frac{C}{3\gamma_{\rm turb}} \left(\mathrm{e}^{-\frac{5\gamma_{\rm turb} H_0}{2} t_{\rm in}} - \mathrm{e}^{-\frac{5\gamma_{\rm turb} H_0}{2} t} \right) + \frac{2 C_0 a_0^{\frac{5\gamma_{\rm turb} H_0}{2}}}{5\gamma_{\rm turb}} \right)^{-\frac{6}{5}} . \tag{66}$$

When $w > -1$ and t is large enough, we find

$$\rho_{\rm turb} \propto \mathrm{e}^{-3\gamma_{\rm turb} H_0 t} . \tag{67}$$

On the other hand, when $w < -1$ and $t \gg t_{\rm in}$, $\rho_{\rm turb}$ goes to a constant

$$\rho_{\rm turb} \to \left(- \frac{C}{3\gamma_{\rm turb}} \right)^{-\frac{6}{5}} , \tag{68}$$

which corresponds to (62).

When $w_{\rm turb} \neq -1$, if we consider the case of an effective quintessence-like power law expansion, $a(t) = a_0 t^{h_0}$ with constants a_0 and h_0, we find

$$\rho_{\rm turb} = t^{-3\gamma_{\rm turb} h_0} \left(\frac{5C}{6\left(1 - \frac{5}{2}\gamma_{\rm turb} h_0\right)} \left(t^{1-\frac{5}{2}\gamma_{\rm turb} h_0} - t_{\rm in}^{1-\frac{5}{2}\gamma_{\rm turb} h_0} \right) + C_0 a_0^{\frac{5\gamma_{\rm turb} H_0}{2}} \right)^{-\frac{6}{5}} . \tag{69}$$

If we consider the case of the phantom-like power law expansion, $a(t) = a_0 \left(t_s - t\right)^{-h_0}$, we find

$$\begin{aligned} \rho_{\rm turb} = &\left(t_s - t\right)^{3\gamma_{\rm turb} h_0} \\ &\times \left(\frac{5C}{6\left(1 + \frac{5}{2}\gamma_{\rm turb} h_0\right)} \left(\left(t_s - t\right)^{1+\frac{5}{2}\gamma_{\rm turb} h_0} - \left(t_s - t_{\rm in}\right)^{1+\frac{5}{2}\gamma_{\rm turb} h_0} \right) + C_0 a_0^{\frac{5\gamma_{\rm turb} H_0}{2}} \right)^{-\frac{6}{5}} . \end{aligned} \tag{70}$$

Especially when $h_0 = -\frac{2}{3\gamma_{\rm turb}}$ with $w_{\rm turb} < -1$, we find $\rho_{\rm turb} \propto \left(t_s - t\right)^{-\frac{6}{5}}$ when $t \to t_s$. In this case, from (65), we also obtain $\rho_{\rm rad} \propto \left(t_s - t\right)^{-\frac{6}{5}}$. Then both $\rho_{\rm turb}$ and $\rho_{\rm rad}$ increase rather rapidly, although the rate of increase is smaller than for the energy density of the phantom dark energy where $\rho_{\rm phantom} \propto \left(t_s - t\right)^{-2}$.

Just for simplicity, we now consider the turbulence of the dark energy with $w_{\rm turb} = -1$. Then Eq. (64) gives

$$\rho_{\rm turb} = \left(\frac{6}{5C} \right)^{\frac{6}{5}} \left(t - t_0\right) . \tag{71}$$

Here $t_0 \equiv t_{\rm in} - \frac{6C_0}{5C}$ and we assume $t_{\rm in} > t_0$. We now consider the simple case where the contribution from the matter (except radiation generated by the turbulence) can be neglected and the non-turbulent part of the dark energy has the constant EoS parameter $w_{\rm dark} = -1$, as for the cosmological constant. We write $\rho_{\rm dark} = \Lambda$.

By using (71), we may rewrite (55) as

$$\frac{d}{dt}\left(a^4 \rho_{\rm rad}\right) = \left(\frac{6}{5} \right)^{\frac{11}{5}} C^{-\frac{6}{5}} \left(t - t_0\right)^{-\frac{11}{5}} a^4 , \tag{72}$$

since $p_{\rm rad} = \rho_{\rm rad}/3$. Then by multiplying the FRW equation (50) ($\rho_{\rm dark} = \Lambda$ and $\rho_{\rm matter} = 0$) with a^4 and differentiating with respect to the cosmological time t, we obtain

$$\frac{6}{\kappa^2}\left(\dot H + 2H^2 \right) = 4\Lambda + 4 \left(\frac{6}{5C} \right)^{\frac{6}{5}} \left(t - t_0\right)^{-\frac{6}{5}} . \tag{73}$$

The second term decreases with time and therefore for large t, we obtain the asymptotic de Sitter universe, where H is a constant

$$H = H_{L0} \equiv \sqrt{\frac{\Lambda \kappa^2}{3}} . \tag{74}$$

Let assume the turbulence begins at $t = t_0$. Then since we assume $w_{\rm turb} = w_{\rm dark} = -1$, the total dark energy behaves as a cosmological constant and the de Sitter universe is realized, where

$$H = H_{I0} \equiv \sqrt{\frac{\kappa^2}{3}\left(\Lambda + \left(\frac{6}{5C}\right)^{\frac{6}{5}} (t_{\rm in} - t_0)^{-\frac{6}{5}}\right)} . \tag{75}$$

After the turbulence begins, the cosmological constant decays and H becomes smaller and at the late time, the universe reaches the asymptotic de Sitter universe with $H = H_{L0}$. Then we may assume $H_{I0} \gg H_{L0}$.

Thus, the situation considered is one where there is matter with vanishing EoS parameter and phantom dark energy. Without turbulence there could occur a phantom crossing. After such a crossing, the dark energy dominates. If there is no non-turbulent part, the density of dark energy is large and finally satisfies the condition (59), the dark energy dissipates and converts into radiation. Hence, the accelerated expansion will terminate. Or before satisfying the condition (59), the energy density goes to a constant (62), corresponding to the asymptotic de Sitter space.

Some remark is in order. Let us imagine the inflation ended by the turbulence and the turbulent part $\rho_{\rm turb}$ of the energy-density generated the reheating. $\rho_{\rm turb}$ could been converted to the radiation. Then the energy-density $\rho_{\rm rad}$ of the radiation after the reheating could be of almost the same order with $\rho_{\rm turb}$. Therefore, we may obtain

$$\rho_{\rm rad} \sim \rho_{\rm turb} = \left(\frac{6}{5C}\right)^{\frac{6}{5}} (t_{\rm in} - t_0) . \tag{76}$$

Here, Eq. (71) is used. Applying the Stefan-Boltzmann law

$$\rho_{\rm rad} = \sigma T^4 , \tag{77}$$

with the constant σ, one may evaluate the reheating temperature T as

$$T \sim \sigma^{-\frac{1}{4}} \left(\frac{6}{5C}\right)^{\frac{3}{10}} (t_{\rm in} - t_0)^{\frac{1}{4}} , \tag{78}$$

which may give a constraint on the parameters. This may indicate on observational manifestations of turbulence.

V. DISCUSSION

In summary, we have investigated the role which may be played by isotropic turbulence in a dark fluid universe at late times. Using experimental information about decay of isotropic turbulence in classical hydrodynamics, two different approaches to modification of FRW equations are proposed. The consequences of turbulence presence at future phantom/quintessence universe are studied. It is demonstrated that it may change the characteristics details of finite-time future singularity, for instance, the behavior of scale factor at Big Rip. However, it seems that turbulence cannot remove the future finite-time singularity in a perturbative approach. In a non-perturbative approach, the turbulence presence may change the evolution of dark fluid via its dissipation. In particular, it may terminate the accelerated evolution or convert the phantom-dominated universe into a future de Sitter space. Hence, inclusion of turbulence may suggest a way to resolve the future singularity problem.

In principle, one should not limit oneself only to the known decay law for isotropic turbulence. It is quite possible that at large scales more general laws should be implemented. In this case, the description is somewhat similar to the inhomogeneous equation of state of the universe introduced in Refs. [13–15]. Moreover, the explicit scenario of the (effective) turbulence emergence should be developed. Perhaps, the easiest way to realize it is to use of cosmological reconstruction in modified gravity (for a recent review, see [16]). In this case, the use of turbulence turns out to be an effective description due to a corresponding modification of gravity. Alternatively, the origin of turbulence in a dark fluid universe may be related to an inhomogeneous equation of state for the universe.

The important question in connection with our proposal of including turbulence in a dark energy universe is of course related to the search of observational evidence for turbulence signatures. This is expected to be possible in the near future when observational data will give us more accurate information about the cosmological equation of state.

Acknowledgments

This research has been supported in part by MEC (Spain) project FIS2006-02842 and AGAUR(Catalonia) 2009SGR-994 (SDO), by Global COE Program of Nagoya University (G07) provided by the Ministry of Education, Culture, Sports, Science & Technology and by the JSPS Grant-in-Aid for Scientific Research (S) # 22224003 (SN). The support of the ESF Casimir Network ia also acknowledged.

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