Time-dependent matter instability and star singularity in $F(R)$ gravity

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Abstract

We investigate a curvature singularity appearing in the star collapse process in $F(R)$ gravity. In particular, we propose an understanding of the mechanism to produce the curvature singularity. Moreover, we explicitly demonstrate that R^α $(1 < \alpha \leq 2)$ term addition could cure the curvature singularity and viable $F(R)$ gravity models could become free of such a singularity. Furthermore, we discuss the realization process of the curvature singularity and estimate the time scale of its appearance. For exponential gravity, it is shown that in case of the star collapse, the time scale is much shorter than the age of the universe, whereas in cosmological circumstances, it is as long as the cosmological time.

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I.   INTRODUCTION

Modified gravity is expected to be the fundamental scenario to give the unified gravitational alternative to dark energy and inflation [1]. A number of viable modified gravities are known. Of course, there exist important criteria for viability, such as the fulfillment of Solar System tests. Among these criteria, one of the most important ones is related with so-called matter instability [2] in $F(R)$ gravity, which means that the curvature inside matter sphere becomes very large, i.e., strong gravity (for an introduction to $F(R)$ gravity and very recent reviews on it, see, e.g., [1, 3]). It was indicated that such matter instability may be dangerous in the relativistic star formation process [4] due to the appearance of corresponding singularity. On the other hand, the hydrostatic equilibrium of a stellar structure in the framework of $F(R)$ gravity has been explored by studying the modified Lané-Emden equation in Ref. [5].

Recently, the instability in $F(R)$ gravity has been discussed for a gravitating system with a time dependent mass density like astronomical massive objects in Ref. [6]. It has been shown that when a star shrinks and the mass density becomes larger, the time-dependent matter instability, in which the scalar curvature becomes very large, could occur for a class of $F(R)$ gravity models. It is interesting to understand how common such time-dependent matter instability is in $F(R)$ gravity and how viable $F(R)$ gravity models may be protected against it. In this paper, we study the generation mechanism of the time-dependent matter instability in the star collapse. We show that the time-dependent matter instability develops and consequently the curvature singularity could appear in a viable $F(R)$ gravity model [7] and some version of exponential gravity [8–10]. We note that the equivalent or very similar modification of gravity to Ref. [7] has been considered in Refs. [11, 12]. In addition, we demonstrate that the curvature singularity could be cured by adding the higher derivative term R^α ($1 < \alpha \leq 2$) and viable $F(R)$ gravity models could become free of such a singularity. Furthermore, we discuss the realization process of the curvature singularity in the viable $F(R)$ gravity model and estimate the time scale of its appearance in exponential gravity. We show that in case of the star collapse, the time scale is much shorter than the age of the universe, whereas in cosmological circumstances, it is as long as the cosmological time. We mention that the problem of singularity in the star collapse was studied in Ref. [13], in which it was stated that the curvature singularity in the future does not appear if the model

of $F(R)$ gravity is built very carefully (for an earlier proposal, see [14]). It is considered that the curvature singularity could emerge in a generic $F(R)$ gravity model unless fine-tuning is taken. We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G$ by $\kappa^2 \equiv 8\pi/M_{\mathrm{Pl}}{}^2$ with the Planck mass of $M_{\mathrm{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\mathrm{GeV}$.

The paper is organized as follows. In Sec. II, we first review the matter instability and study the generation mechanism of the time-dependent matter instability. In Sec. III, we examine how the curvature singularity occurs and analyze the time scale of its appearance. Finally, conclusions are given in Sec. IV.

II.   CURVATURE SINGULARITY IN THE COLLAPSE OF STARS

A.   Matter instability

We start with reviewing the matter instability issue [2] in $F(R)$ gravity. It is related with the fact that spherical body solution in general relativity may not be the solution in modified theory. It may appear when the energy density or the curvature is large compared with the average one in the universe, as is the case inside of a star.

We consider the following action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{F(R)}{2\kappa^2} + \mathcal{L}_{\mathrm{matter}} \right\} , \qquad (2.1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $F(R)$ is an arbitrary function of R, and $\mathcal{L}_{\mathrm{matter}}$ is the matter Lagrangian. The trace of the gravitational field equation derived from the action in Eq. (2.1) is given by

$$\Box R + \frac{F^{(3)}(R)}{3F^{(2)}(R)} \nabla_\rho R \nabla^\rho R + \frac{F'(R)R}{3F^{(2)}(R)} - \frac{2F(R)}{3F^{(2)}(R)} = \frac{\kappa^2}{3F^{(2)}(R)} T_{\mathrm{matter}} , \qquad (2.2)$$

where a prime denotes a derivative with respect to R, ∇_μ is the covariant derivative operator associated with $g_{\mu\nu}$, $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant d'Alembertian for a scalar field, and $T_{\mathrm{matter}} \equiv T_{\mathrm{matter}\,\rho}{}^\rho$ is the trace of the matter energy-momentum tensor $T_{\mathrm{matter}\,\mu\nu}$ defined as $T_{\mathrm{matter}\,\mu\nu} \equiv -\left(2/\sqrt{-g}\right)\left(\delta\mathcal{L}_{\mathrm{matter}}/\delta g^{\mu\nu}\right)$. We also denote $d^n F(R)/dR^n$ by $F^{(n)}(R)$. Let us now examine the perturbation from the solution of general relativity. We express the scalar curvature solution given by the matter density in the Einstein gravity by $R_{\mathrm{b}} \sim \kappa^2 \rho_{\mathrm{matter}} > 0$ with ρ_{matter} being the energy density of matter and separate the scalar curvature R into

the sum of R_b and the perturbed part R_p as $R = R_b + R_p$ ($|R_p| \ll |R_b|$). Substituting this expression into Eq. (2.2), we obtain the perturbed equation [15, 16]. It is convenient to consider the case that R_b and R_p are homogeneous and isotropic, that is, they do not depend on the spatial coordinates. Hence, the d'Alembertian can be replaced with the second derivative with respect to the time coordinate: $\Box R_p \to -\partial_t^2 R_p$. As a result, the perturbed equation has the following structure:

$$0 = -\partial_t^2 R_p + \mathcal{U}(R_b)R_p + \text{const.} . \tag{2.3}$$

Thus, if $\mathcal{U}(R_b) > 0$, R_p becomes exponentially large with time t: $R_p \sim e^{\sqrt{\mathcal{U}(R_b)}t}$ and the system is unstable.

For example, in the model of $F(R) = R - a/R + bR^2$ with a and b being positive constants [15], for $b \gg a/|R_b^3|$ one gets

$$\mathcal{U}(R_b) \sim \frac{R_b}{3} > 0 . \tag{2.4}$$

Therefore, the system could be unstable. Since R_b is estimated as [2], however,

$$R_b \sim \left(10^3 \text{sec}\right)^{-2} \left(\frac{\rho_{\text{matter}}}{\text{g cm}^{-3}}\right) , \tag{2.5}$$

the decay time is $\sim 1,000$ sec, i.e., macroscopic. On the other hand, in a viable model [7]

$$F(R) = R - m^2 \frac{c_1 \left(R/m^2\right)^n}{c_2 \left(R/m^2\right)^n + 1} , \tag{2.6}$$

where c_1 and c_2 are dimensionless parameters, $n(> 0)$ is a positive constant, and m is a mass scale, if one takes $c_1 > 0$, $\mathcal{U}(R_b)$ is negative as

$$\mathcal{U}(R_b) \sim -\frac{(n+2)c_2^2 m^2}{n(n+1)c_1} < 0 . \tag{2.7}$$

Consequently, the system could be stable and there is no matter instability.

We remark that $-\mathcal{U}(R_b)$ can be regarded as the square of the effective mass m_{eff}^2 for the scalar mode R_p, given by $m_{\text{eff}}^2 \approx F''(R)/3$ [17]. This means that if $F''(R) > 0$, one has $m_{\text{eff}}^2 > 0$ and hence the scalar mode R_p is stable, whereas it is unstable for $F''(R) < 0$ because $m_{\text{eff}}^2 < 0$. If $F''(R) < 0$, this instability occurs inside matter when the deviation of R from $R_b \sim \kappa^2 \rho_{\text{matter}}$ emerges. Thus, it can be interpreted as a matter instability.

B. Time-dependent matter instability and curvature singularity

In Ref. [6], it has been shown that even for the viable models such as those in Refs. [7, 11], the increasing matter density could generate the curvature singularity, which we will describe in this subsection. We write $F(R)$ as[1]

$$F(R) = R + f(R). \tag{2.8}$$

The trace equation of the gravitational field is described as

$$3\Box f'(R) - R + Rf'(R) - 2f(R) = \kappa^2 T_{\text{matter}}. \tag{2.9}$$

We now consider a small region inside the star, where the energy density can be regarded as homogeneous and isotropic as assumed in Ref. [2]. Since we study the gravitational field of the star such as the Sun, whose mass density is $\rho_\odot = 1.4 \, \text{g cm}^{-3}$, we also suppose that the gravitational field is weak and the curvature is small because $R_\text{b} \sim \kappa^2 \rho_{\text{matter}}$. (If a neutron star has the solar mass $M_\odot = 2.0 \times 10^{33} \, \text{g}$ and the radius $1.0 \times 10^6 \text{cm}$, the mass density is $4.7 \times 10^{14} \, \text{g cm}^{-3}$, which is much larger than the solar mass density. The gravitational field of a neutron star is regarded as strong because a general relativistic treatment is necessary to explore a neutron star.) Then, we may replace $\Box$ in Eq. (2.9) with $-\partial_t^2$. By defining φ as $\varphi \equiv -f'(R)$, we can (at least locally) solve R with respect to φ: $R = R(\varphi)$. Using this solution, Eq. (2.9) can be rewritten as

$$\ddot{\varphi} = \mathcal{F}(\varphi, t) \equiv \frac{1}{3} \left\{ R(\varphi) + R(\varphi)\varphi + 2f(R(\varphi)) + \kappa^2 T_{\text{matter}}(t) \right\}, \tag{2.10}$$

where the dot denotes the time derivative of $\partial/\partial t$. Compared with Newton's equation of motion $m\ddot{\boldsymbol{x}} = \boldsymbol{F}$, we can regard $\mathcal{F}(\varphi, t)$ as a "force". From Eq. (2.10), we also obtain

$$0 = \frac{1}{2}\dot{\varphi}^2 - \int dt\dot{\varphi}(t)\mathcal{F}(\varphi(t), t). \tag{2.11}$$

We explore the case in which T_{matter} increases with time:

$$T_{\text{matter}} = -T_{\text{matter}\,0}\left(1 + \frac{t}{t_0}\right), \tag{2.12}$$

where $T_{\text{matter}\,0}$ is constant and t_0 is a time.

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[1] Note that the definition of $F(R)$ and $f(R)$ are different from those in Ref. [6].

In case of the viable models, e.g., those in Refs. [7, 11], $f(R)$ is given by

$$f(R) \sim -f_0 + \frac{f_1}{nR^n} , \tag{2.13}$$

for large R. Here, f_0 and f_1 are constants. If $f(R) = F(R) - R$ is written by Eq. (2.6), one finds $f_0 = (c_1/c_2) m^2$ and $f_1/n = (c_1/c_2^2) m^{2(n+1)}$. In this case, we have

$$\varphi = \frac{f_1}{R^{n+1}} , \tag{2.14}$$

and therefore $\mathcal{F}(\varphi, t)$ in Eq. (2.10) is given by

$$\mathcal{F}(\varphi, t) = \frac{1}{3} \left\{ \left(\frac{\varphi}{f_1} \right)^{-\frac{1}{n+1}} + f_1 \left(\frac{\varphi}{f_1} \right)^{\frac{n}{n+1}} - 2f_0 + \frac{2f_1}{n} \left(\frac{\varphi}{f_1} \right)^{\frac{n}{n+1}} - \kappa^2 T_{\mathrm{matter}\,0} \left(1 + \frac{t}{t_0} \right) \right\} . \tag{2.15}$$

Combining Eqs. (2.11) and (2.15), we obtain

$$\mathcal{E} = \frac{1}{2}\dot{\varphi}^2 + U(\varphi, t) ,$$

$$U(\varphi, t) = \frac{1}{3} \left\{ -\frac{(n+1)f_1}{n} \left(\frac{\varphi}{f_1} \right)^{\frac{n}{n+1}} - \frac{f_1^2(n+1)}{2n+1} \left(1 + \frac{2}{n} \right) \left(\frac{\varphi}{f_1} \right)^{\frac{2n+1}{n+1}} \right. $$
$$\left. + \left[\kappa^2 T_{\mathrm{matter}\,0} \left(1 + \frac{t}{t_0} \right) + 2f_0 \right] \varphi - \frac{\kappa^2 T_{\mathrm{matter}\,0}}{t_0} \int dt \varphi(t) \right\} , \tag{2.16}$$

where $\mathcal{E}$ is a constant corresponding to "energy" in the classical mechanics and $U(\varphi)$ to the potential. The case that R is large corresponds to the case that φ is small. Hence, we neglect the second term on the right-hand side of the expression for $U(\varphi, t)$ in (2.16), i.e., $-(1/3)\left[f_1^2(n+1)/(2n+1)\right](1 + 2/n)(\varphi/f_1)^{(2n+1)/(n+1)}$. First, we consider the case that $T_{\mathrm{matter}\,0} = 0$. It follows from Eq. (2.16) that the "potential" $U(\varphi)$ is given by

$$U(\varphi) = \frac{1}{3} \left\{ -\frac{n+1}{n} \left(\frac{\varphi}{f_1} \right)^{\frac{n}{n+1}} + 2f_0\varphi \right\} , \tag{2.17}$$

which vanishes at

$$\varphi = 0 , \quad \varphi_0 \equiv f_1^{-n} \left(\frac{n+1}{2nf_0} \right)^{n+1} , \tag{2.18}$$

and has a minimum at

$$\varphi = \varphi_{\mathrm{min}} \equiv f_1^{-n} (2f_0)^{-(n+1)} . \tag{2.19}$$

The conceptual form of $U(\varphi)$ is shown in Fig. 1. It is clear from Fig. 1 that if we start with an initial condition, for example, $\varphi > \varphi_0$ and $\dot{\varphi} \leq 0$, φ reaches $\varphi = 0$, which corresponds to the curvature singularity $R = \infty$.
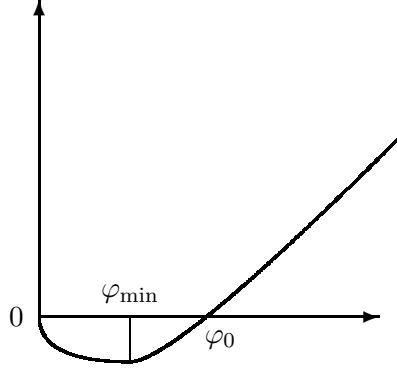
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FIG. 1: The "potential" $U(\varphi)$ in Eq. (2.17). The vertical and horizontal axes show $U(\varphi)$ and φ, respectively.
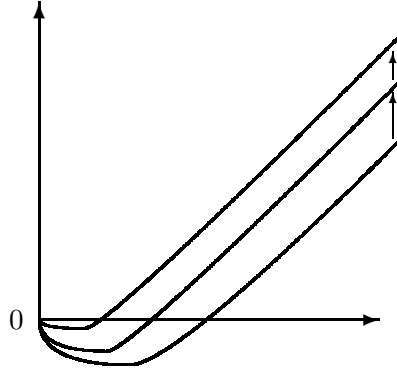

FIG. 2: The change of the "potential" $U(\varphi)$ in Eq. (2.16) with time t. Legend is the same as Fig. 1.

When $T_{\mathrm{matter}\,0} \neq 0$, the potential shifts up with time as in Fig. 2. By growing the energy density, the singularity $R = \infty$ can be realized easily. Even if $T_{\mathrm{matter}} = 0$, there could appear the curvature singularity but the increase of T_{matter}, which may occur by the star collapse, becomes a trigger to generate the singularity.

A prescription to avoid the singularity is to grow the potential at $R = \infty$. In order to realize the growing up of the potential, we add the following term, say, to $f(R)$ in Eq. (2.13)

$$f(R) \to f(R) - f_2 R^\alpha \,. \tag{2.20}$$

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Here, we assume f_2 is a positive constant: $f_2 > 0$ and α is a constant. If we choose $\alpha > 1$, the added term dominates when the curvature R is large and we find

$$\varphi \sim \alpha f_2 R^{\alpha - 1} \,. \tag{2.21}$$

Hence, the case $R \to \infty$ corresponds to the case $\varphi \to +\infty$. In this case, $\mathcal{F}(\varphi, t)$ in Eq. (2.10) is given by

$$\mathcal{F}(\varphi, t) \sim f_2 (\alpha - 2) \left(\frac{\varphi}{\alpha f_2} \right)^{\frac{\alpha}{\alpha - 1}} \,. \tag{2.22}$$

Since $\alpha > 0$, the magnitude of the "force" $\mathcal{F}(\varphi, t)$ becomes infinitely large when $R \to +\infty$. When $\alpha < 2$, the force is negative and works to decrease φ or the scalar curvature, and therefore the curvature singularity is not realized. In fact, from Eq. (2.11) we have the following form:

$$\mathcal{E} = \frac{1}{2}\dot{\varphi}^2 + U(\varphi, t) \,, \quad U(\varphi) \sim \frac{f_2^2 \alpha (\alpha - 1)(2 - \alpha)}{2\alpha - 1} \left(\frac{\varphi}{\alpha f_2} \right)^{\frac{2\alpha - 1}{\alpha - 1}} \,. \tag{2.23}$$

Thus, the "potential" $U(\varphi)$ is positive as long as $1 < \alpha < 2$ and becomes infinite when $\varphi \to \infty$, which means that we need infinite "work" to arrive at $\varphi \to \infty$. On the other hand, when $\alpha > 2$, the force is positive and therefore the force makes the curvature infinite and the curvature singularity is easily realized. Note that even if we choose f_2 to be negative, the result is not changed because the curvature singularity corresponds to $\varphi \to -\infty$ in this case although the sign of the "force" $\mathcal{F}(\varphi, t)$ is changed.

It is also important to remark the case of adding R^2 term, i.e., $\alpha = 2$ as $f(R) \to f(R) + f_3 R^2$, where f_3 is assumed to be a positive constant. For the "realistic" model in Eq. (2.13), since the inverse power term $f_1/(nR^n)$ can be negligible when the curvature is large, we may consider the model $F(R) \sim R + f_3 R^2$. In this case, we may regard $\varphi \sim -2f_3 R$. In the expression of the "force" $\mathcal{F}(\varphi, t)$ in Eq. (2.10), the second term $R(\varphi)\varphi$ is cancelled by the third term $2f(R(\varphi))$, which is consistent with Eq. (2.22), and the fourth term $\kappa^2 T_{\mathrm{matter}}(t)$ could be neglected. Hence, only the first term $R(\varphi)$ contributes to the dynamics and it gives the "force" linear to φ as $\mathcal{F}(\varphi, t) \sim -\varphi/(6f_3)$. Thus, from Eq. (2.11) we obtain quadratic "potential" $U(\varphi) \sim \varphi^2/(12f_3)$, which becomes positive infinity when R goes to positive infinity (φ goes to negative infinity). As a result, the curvature singularity is prevented.

Instead of the model in Eq. (2.13), we may investigate the exponential gravity model [8–10],

$$f(R) = -f_{\rm c} \left(1 - {\rm e}^{-\frac{R}{R_{\rm c}}} \right) , \tag{2.24}$$

where $f_{\rm c}$ and $R_{\rm c}$ are positive constants. In the model in Eq. (2.24), we have

$$\varphi = \frac{f_{\rm c}}{R_{\rm c}} {\rm e}^{-\frac{R}{R_{\rm c}}} . \tag{2.25}$$

Therefore, the curvature singularity $R \to +\infty$ corresponds to $\varphi = +0$. For the model in Eq. (2.24), we find

$$\mathcal{F}(\varphi, t) \sim -\frac{R_{\rm c}}{3} \ln \frac{R_{\rm c}\varphi}{f_{\rm c}} . \tag{2.26}$$

When $\varphi \to +0$, the "force" $\mathcal{F}(\varphi, t)$ becomes positive and infinite, but now it follows from Eq. (2.11) that we have the following form:

$$\mathcal{E} = \frac{1}{2}\dot{\varphi}^2 + U(\varphi, t) , \quad U(\varphi) \sim \frac{R_{\rm c}}{3}\varphi \left(\ln \frac{R_{\rm c}\varphi}{f_{\rm c}} - 1 \right) , \tag{2.27}$$

which gives the finite value of "potential" when $\varphi \to +0$. This implies that we only need finite "work" to arrive at $\varphi = +0$. Therefore, the curvature singularity $R \to \infty$ can be realized. This situation is almost the same as that in the model in Eq. (2.13) because when $\varphi \to +0$, the "force" in Eq. (2.15) diverges but the "potential" in Eq. (2.16) is finite.

We mention that there exists the following model with two exponential terms to realize inflation as well as the late-time cosmic acceleration [10]

$$f(R) = -f_{\rm c} \left(1 - {\rm e}^{-\frac{R}{R_{\rm c}}} \right) - f_{\rm i} \left[1 - {\rm e}^{-\left(\frac{R}{R_{\rm i}}\right)^q} \right] + \gamma R^{\beta} , \tag{2.28}$$

where $f_{\rm i}$, $R_{\rm i}(\gg R_{\rm c})$ and γ are positive constants, β is a constant, and $q(> 1)$ is a natural number. Here, we take $R_{\rm c}$ and $R_{\rm i}$ as the current curvature and the value of R during inflation. If (i) $R \lesssim R_{\rm c}$, (ii) $R_{\rm c} \ll R \ll R_{\rm i}$, and (iii) $R \gg R_{\rm i}$, the first, second, and third terms on the right-hand side of Eq. (2.28) becomes dominant, respectively. For the case (i), the model in Eq. (2.28) corresponds to the one in Eq. (2.24), whereas for the case (iii) the behavior of the model in Eq. (2.28) is similar to that in the case of Eq. (2.20) with the additional term domination. In terms of the case (ii), we examine the behavior for the corresponding form of $f(R)$ as $f(R) = -f_{\rm i} \left[1 - {\rm e}^{-(R/R_{\rm i})^q} \right]$. In this case, we obtain

$$\varphi \sim \frac{qf_{\rm i}}{R_{\rm i}} \left(\frac{R}{R_{\rm i}} \right)^{q-1} , \tag{2.29}$$

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where we have used $R \ll R_{\mathrm{i}}$. For $q > 1$, φ increases as R becomes large. Moreover, we find

$$\mathcal{F}(\varphi, t) \sim -\frac{q f_{\mathrm{i}}}{3} \left(\frac{R_{\mathrm{i}} \varphi}{q f_{\mathrm{i}}} \right)^{2q/(q-1)} . \tag{2.30}$$

The sign of the "force" $\mathcal{F}(\varphi, t)$ is negative and therefore the force works to decrease φ. Furthermore, Eq. (2.11) becomes

$$\mathcal{E} = \frac{1}{2} \dot{\varphi}^2 + U(\varphi, t) , \quad U(\varphi) \sim \frac{q-1}{3(3q-1)} \frac{(q f_{\mathrm{i}})^2}{R_{\mathrm{i}}} \left(\frac{R_{\mathrm{i}} \varphi}{q f_{\mathrm{i}}} \right)^{(3q-1)/(q-1)} . \tag{2.31}$$

Since $q > 1$, $U(\varphi)$ increases as φ becomes large. Consequently, the behavior of the model in Eq. (2.28) in the intermediate regime $R_{\mathrm{c}} \ll R \ll R_{\mathrm{i}}$ is stable. Note that this does not mean that the curvature singularity could not be generated because we are assuming the scalar curvature is finite as $R \ll R_{\mathrm{i}}$ although the curvature has a tendency to decrease by the "force" $\mathcal{F}(\varphi, t)$. Without the last term in Eq. (2.28), there could appear the curvature singularity when $R \gg R_{\mathrm{i}}$.

In Ref. [6], the emergence of the curvature singularity in the future has been investigated by the analytical studies, which have also been supplemented with numerical solutions. The equation for the evolution of R was reduced to the form of an oscillator equation $\ddot{x} + dV(x)/dx = 0$, where x and V corresponds to φ and U in this paper. In case of a time dependent potential $V = V(x, t)$, since it is impossible to obtain an analytical solution, the qualitative behavior of the solution has been analyzed.

III. REALIZATION OF THE CURVATURE SINGULARITY

A. Realization process of the curvature singularity

We discuss how the curvature singularity is realized. As an example, we investigate the viable model in Eq. (2.13). We consider a small region inside the star, which can be regarded as homogeneous and isotropic. In this case, the space-time is locally described by the flat Friedmann-Lemaître-Robertson-Walker (FLRW):

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1,2,3} \left(dx^i \right)^2 , \tag{3.1}$$

where $a(t)$ is the scale factor. Note that since we are examining the star collapse, the Hubble rate $H \equiv \dot{a}/a$ is negative, that is, the space-time is shrinking. We also mention that the

energy densities of the matters automatically increase because the region is shrinking. Thus, we do not give an explicit time dependence as in Eq. (2.12).

In case of cosmology, it is well-known that for the model in Eq. (2.13), Type II singularity can be realized, where the scale factor a and the effective energy density are finite but the effective pressure and the scalar curvature R diverge [18]. We apply this fact to the star collapse. Since a is finite, the energy density and the pressure from the matter are finite and can be neglected near the singularity. In such a situation, we find the Hubble rate H is given by

$$H \sim -\frac{h_{\mathrm{st}}}{(t_{\mathrm{st}} - t)^{\frac{n}{n+2}}} \, , \tag{3.2}$$

where h_{st} is a positive constant and t_{st} is the time when the curvature singularity in the star appears. When $t \to t_{\mathrm{st}}$, H is finite but $\dot{H}$ and therefore the scalar curvature $R = 6\dot{H} + 12H^2$ diverge. This implies that in the model in Eq. (2.13), the naked curvature singularity is generated in the finite future. In the above analysis, we have assumed that the region we are considering is almost homogeneous and isotropic. If the assumptions are valid near the singularity, the singularity appears simultaneously anywhere in the region. Even if the assumptions of the homogeneity and isotropy are broken, the naked singularities appear densely in the region. Furthermore, we remark that in general, the matter density becomes larger in the region deeper from the surface of the star. Hence, we expect that first, the naked curvature singularity could be generated near the center of the star. If the singularity generates the attractive force, the star shrinks more, but if the generated force is repulsive, there might occur the explosion. If the explosion could occur, however, H must change its sign from negative to positive, which seems to be difficult to be realized.

On the other hand, the model in Eq. (2.13) could generate the cosmological singularity, where the Hubble rate behaves similar to Eq. (3.2) as

$$H \sim \frac{h_{\mathrm{co}}}{(t_{\mathrm{co}} - t)^{\frac{n}{n+2}}} \, . \tag{3.3}$$

In the expanding cosmology, H should be positive and hence the constant h_{co} is positive. Moreover, the singularity in the universe occurs at $t = t_{\mathrm{co}}$. The values of t_{co} (and h_{co}) in Eq. (3.3) and t_{st} (and h_{st}) in Eq. (3.2) could be determined by the initial condition for H and $\dot{H}$. Since we are considering the collapsing star, the absolute values of H and $\dot{H}$ inside the star are expected to be large compared with the values in the expanding universe.

Therefore, we expect $t_{\mathrm{co}} > t_{\mathrm{st}}$, that is, the curvature singularity in the star occurs before the cosmological singularity.

We note that in order to remove the finite-time future singularity, adding a R^2 term to the form of $f(R)$ works [the first reference in Ref. [1], [18]]. Similarly, the R^2 term can cure the curvature singularity under consideration in this paper, as analyzed in Sec. II B. The R^2 term shifts the value of the potential near the curvature singularity in the scalar tensor description. This fact seems to show a close relation between the curvature singularity in the star under discussion and the finite-time future singularity in the context of cosmology.

B. Time scale for the realization of the curvature singularity

We demonstrate the estimation of the time scale for the realization of the curvature singularity by exploring the exponential gravity model in Eq. (2.24). In the flat FLRW background (3.1), the gravitational field equations derived from the action in Eq. (2.1) read

$$0 = -\frac{1}{2}F(R) + 3\left(H^2 + \dot{H}\right)F'(R) - 18\left(4H^2\dot{H} + H\ddot{H}\right)F''(R) + \kappa^2\rho_{\mathrm{matter}}, \quad (3.4)$$

$$0 = \frac{1}{2}F(R) - \left(\dot{H} + 3H^2\right)F'(R) + 6\left(8H^2\dot{H} + 6H\ddot{H} + 4\dot{H}^2 + \dddot{H}\right)F''(R)$$
$$+ 36\left(4H\dot{H} + \ddot{H}\right)^2 F'''(R) + \kappa^2 P_{\mathrm{matter}}, \quad (3.5)$$

where P_{matter} is the pressure of matter. Substituting Eq. (2.8) with Eq. (2.24) into Eq. (3.4), we find

$$0 = -3H^2 + f_{\mathrm{c}}\left\{\frac{1}{2} + \left[-\frac{1}{2} - 3\left(H^2 + \dot{H}\right)\frac{1}{R_{\mathrm{c}}} - 18\left(4H^2\dot{H} + H\ddot{H}\right)\frac{1}{R_{\mathrm{c}}^2}\right]\mathrm{e}^{-\frac{R}{R_{\mathrm{c}}}}\right\} + \kappa^2\rho_{\mathrm{matter}}.$$
$$(3.6)$$

We examine the case $R \to \infty$. In this case, it follows from Eq. (3.6) that H is expressed as

$$H \sim A\left(t_{\mathrm{s}} - t\right)\ln\frac{t_{\mathrm{s}} - t}{t_1} + H_{\mathrm{s}}, \quad (3.7)$$

where A is a constant, t_{s} is the time when the curvature singularity appears, t_1 is a time, and H_{s} is the value of H at $t = t_{\mathrm{s}}$. When $R \to \infty$, $t \to t_{\mathrm{s}}$ and $H \to H_{\mathrm{s}}$. In the limit of $t \to t_{\mathrm{s}}$, by combining Eq. (3.7) with Eq. (3.6), we obtain

$$0 = -3H_{\mathrm{s}}^2 + f_{\mathrm{c}}\left(\frac{1}{2} - \frac{3H_{\mathrm{s}}}{R_{\mathrm{c}}t_1}\mathrm{e}^{-\frac{12H_{\mathrm{s}}^2}{R_{\mathrm{c}}}+1}\right) + \kappa^2\rho_{\mathrm{matter}} + O\left(t_{\mathrm{s}} - t\right), \quad (3.8)$$

where we have taken $A = R_{\mathrm{c}}/6$ and used the fact that the term proportional to $\ddot{H}$ is dominant over other terms in Eq. (3.6) because $\ddot{H} \sim A/\left(t_{\mathrm{s}} - t\right)$ and $\dot{H} \sim -A\left\{\ln\left[\left(t_{\mathrm{s}} - t\right)/t_1\right] + 1\right\}$.

As an initial condition, we set the initial time $t = 0$ and $\dot{H} = 0$ at $t = 0$. By deriving the solution of $\dot{H} = 0$ at $t = 0$ in terms of t_1, we have $t_1 = e t_{\mathrm{s}}$. Moreover, by solving Eq. (3.8) in terms of t_1, we acquire

$$t_1 = \frac{6 H_{\mathrm{s}}}{R_{\mathrm{c}}} \mathcal{I} \mathrm{e}^{-\frac{12 H_{\mathrm{s}}^2}{R_{\mathrm{c}}} + 1}, \tag{3.9}$$

where

$$\mathcal{I} \equiv \frac{f_{\mathrm{c}}}{f_{\mathrm{c}} - 6 H_{\mathrm{s}}^2 + 2 \kappa^2 \rho_{\mathrm{matter}}}. \tag{3.10}$$

If we regard R_{c} as the current curvature, we get $R_{\mathrm{c}} \approx 12 H_{\mathrm{c}}^2$, where $H_{\mathrm{c}} = 2.1 h \times 10^{-42} \mathrm{GeV}$ [19] with $h = 0.7$ [20, 21] is the current value of the Hubble rate. From $t_1 = e t_{\mathrm{s}}$ and Eq. (3.9), we find

$$\frac{t_{\mathrm{s}}}{t_{\mathrm{a}}} = \frac{1}{2} \frac{H_{\mathrm{c}}^{-1}}{t_{\mathrm{a}}} \frac{H_{\mathrm{s}}}{H_{\mathrm{c}}} \mathrm{e}^{-\left(\frac{H_{\mathrm{s}}}{H_{\mathrm{c}}}\right)^2} \mathcal{I}, \tag{3.11}$$

where $t_{\mathrm{a}} = 2.9 \times 10^{17} \sec$ [19] is the age of the flat universe and $H_{\mathrm{c}}^{-1} / t_{\mathrm{a}} = 1.5$.

For the star collapse with the solar mass density $\rho_{\odot} = 1.4 \, \mathrm{g \, cm^{-3}}$, we find $H_{\mathrm{s}} / H_{\mathrm{c}} = 3.9 \times 10^{14}$, where we have used $3 H_{\mathrm{s}}^2 / \kappa^2 = \rho_{\odot}$ and the critical density $\rho_{\mathrm{crit}} \equiv 3 H_{\mathrm{c}}^2 / \kappa^2 = 9.2 \times 10^{-28} \, \mathrm{g \, cm^{-3}}$ [19]. In this case, from Eq. (3.11) we obtain $t_{\mathrm{s}} / t_{\mathrm{a}} = 2.9 \times 10^{14} \mathrm{e}^{-1.5 \times 10^{29}} \mathcal{I}$. If $\mathcal{I} \sim O(1)$, $t_{\mathrm{s}} / t_{\mathrm{a}} \ll 1$, i.e., t_{s} is much smaller than the age of the universe. The reason why the time scale for the appearance of the curvature singularity in the exponential model is so small originates from the exponential factor $\mathrm{e}^{-R/R_{\mathrm{c}}}$. We remark that if we apply the above analysis to cosmological circumstances, in which $H_{\mathrm{s}} \sim H_{\mathrm{c}}$, it follows from Eq. (3.11) that $t_{\mathrm{s}} / t_{\mathrm{a}} \sim 0.27 \mathcal{I}$. As a consequnce, if $\mathcal{I} \sim O(1)$, the time scale for the appearance of the curvature singularity in the universe is as long as the age of the universe.

IV. CONCLUSION

In the present paper, we have studied a curvature singularity appearing in the star collapse for some models of $F(R)$ gravity. We have explored the mechanics to generate the curvature singularity. Furthermore, we have explicitly illustrated that the higher derivative term R^{α} ($1 < \alpha \leq 2$) could cure the curvature singularity and viable $F(R)$ gravity models could become free of such a singularity. It is quite remarkable that same scenario works to cure star singularity as well as finite-time future singularity. In addition, we have discussed the realization process of the curvature singularity in the viable $F(R)$ gravity model and estimated the time scale of its appearance in exponential gravity. It has been shown that in

case of the star collapse, the time scale is much shorter than the age of the universe, whereas in cosmological circumstances, it is as long as the cosmological time.

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