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EXACT SOLUTIONS FOR NONLOCAL NONLINEAR FIELD EQUATIONS IN COSMOLOGY

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Abstract

A method for the search of exact solutions for equation of a nonlocal scalar field in a non-flat metric is considered. In the Friedmann–Robertson–Walker metric the proposed method can be used in the case of an arbitrary potential, with the exception of linear and quadratic potentials, and allows to get in quadratures solutions, which depend on two arbitrary parameters. Exact solutions have been found for an arbitrary cubic potential, which consideration is motivated by the string field theory, as well as for exponential, logarithmic and power potentials. It has been shown that one can add the k-essence field to the model to get exact solutions for all Einstein equations.

1 GRAVITATIONAL MODELS WITH A NONLOCAL SCALAR FIELD

Nonlocal cosmological models, based on the string field theory (SFT) [1, 2] and p-adic string theory [3] are actively developed [4]–[22]. Characteristic properties of nonlocal models are the null energy condition violation and arising of phantom fields, which is connected with high derivative terms. Local models with phantom fields are considered as physically unacceptable ones because of a problem quantum instability [23, 24]. In papers [25, 26] the instability problem is reduced to such choosing of effective theory parameters that the instability turns out to be essential only at times that are not described in the framework of the effective theory approximation. It is mathematically expressed in that the terms, which result to instability, are treated as corrections essential only at small energies below the physical cutoff. The given approach allows considering such effective theories as physically acceptable with the presumption that an effective theory admits immersion in a fundamental theory, for example, the string field theory.

The interest in cosmological models related to open string field theory [4] is provoked by the possibility of obtaining solutions describing transitions from a perturbed vacuum to the true vacuum (the so-called *rolling solutions*). After all massive fields (or some of the lower massive fields) are integrated out using the equations of motion, the open string tachyon gets a potential with a nontrivial vacuum corresponding to the minimal value of energy. In the dark energy

model [4] (see also [27, 28, 29, 30]) it is implied that the Universe is a slowly decomposing $D3$–brane, whose decay is described by an open string tachyon mode. According to the Sen conjugation [1], the tachyon motion from an unstable vacuum to the stable vacuum describes the D-brane transition to the true vacuum. In fact one obtains a nonlocal potential with a nonlocality parameter determined by the string scale. Using a suitable redefinition of the fields, one can make the potential local, at that the kinetic term becomes nonlocal. This nonstandard kinetic term leads to a behavior similar to the behavior of a phantom field, and it can be approximated with a phantom kinetic term. Hence, the behavior of an open string tachyon can be effectively simulated by a scalar field with a negative kinetic term [31].

The string field theory also gives asymptotic conditions on the solutions [27, 28, 29, 30]. Special interest is represented with solutions, which are bounded on the whole of real axis and have a nonzero asymptotic at $t \to +\infty$. In this paper such solutions of the field equation in the Friedmann–Robertson–Walker metric have been found for cubic and logarithmic potentials.

Let us consider the gravitational model with a nonlocal scalar field ϕ, which is described by the following action:

$$S = \int d^4x \sqrt{-g}\, \alpha' \left(\frac{R}{16\pi G_{\mathrm{N}}} + \frac{1}{g_{\mathrm{o}}^2} \left(\frac{1}{2}\phi \mathcal{F}(\Box_g)\phi - V(\phi) \right) - \Lambda \right), \tag{1}$$

where G_{N} is the Newtonian gravitational constant ($8\pi G_{\mathrm{N}} = 1/M_{\mathrm{P}}^2$, M_{P} is the Planck mass), α' is the string length squared, g_{o} is the open string coupling constant. We use the signature $(-, +, +, +)$, the matrix $g_{\mu\nu}$ is the metric tensor, g is the determinant of $g_{\mu\nu}$, R is the scalar curvature. The potential $V(\phi)$ is a twice continuously differentiable function, the cosmological constant is considered as a part of the potential $V(\phi)$. The d'Alembert operator $\Box_g$ is applied to scalar functions and can be written as follows:

$$\Box_g = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}\, g^{\mu\nu} \partial_\nu \,. \tag{2}$$

The scalar field ϕ is dimensionless, while $[\alpha'] = \mathrm{cm}^2$ and $[g_{\mathrm{o}}] = \mathrm{cm}$. It is convenient to introduce dimensionless coordinates $\bar{x}_\mu = x_\mu/\sqrt{\alpha'}$, the dimensionless gravitational constant $\overline{G}_{\mathrm{N}} = G_{\mathrm{N}}/\alpha'$, and the dimensionless coupling constant $\bar{g}_{\mathrm{o}} = g_{\mathrm{o}}/\sqrt{\alpha'}$. The curvature scalar calculated in dimensionless coordinates is denoted as $\bar{R}$, the corresponding d'Alembert operator is marked as $\overline{\Box}_g$. We get action (1) in the following form:

$$S = \int d^4\bar{x}\, \sqrt{-g} \left(\frac{\bar{R}}{16\pi \overline{G}_{\mathrm{N}}} + \frac{1}{\bar{g}_{\mathrm{o}}^2} \left(\frac{1}{2}\phi\, \mathcal{F}(\overline{\Box}_g)\phi - V(\phi) \right) \right),$$

In the following formulae we always use dimensionless coordinates and parameters and omit bars over them.

The function $\mathcal{F}$ is assumed to be an analytic function on whole complex plane (i.e. an entire function), therefore, one can represent it by the convergent series expansion:

$$\mathcal{F}(\Box_g) = \sum_{n=0}^{\infty} f_n \Box_g^{\,n}. \tag{3}$$

From action (1) we obtain the following equations

$$G_{\mu\nu} = 8\pi G_{\mathrm{N}} T_{\mu\nu} - 8\pi G_{\mathrm{N}} \Lambda g_{\mu\nu}, \tag{4}$$

$$\mathcal{F}(\Box_g)\phi = \frac{dV}{d\phi}, \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor.

The energy–momentum (stress) tensor $T_{\mu\nu}$ is calculated by the standard formula

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{g_{\mathrm{o}}^2}\Big(E_{\mu\nu} + E_{\nu\mu} - g_{\mu\nu}\left(g^{\rho\sigma} E_{\rho\sigma} + W\right)\Big), \tag{6}$$

where

$$E_{\mu\nu} \equiv \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \partial_\mu \Box_g^l \phi \partial_\nu \Box_g^{n-1-l}\phi, \qquad W \equiv \frac{1}{2}\sum_{n=2}^{\infty} f_n \sum_{l=1}^{n-1} \Box_g^l \phi \Box_g^{n-l}\phi - \frac{f_0}{2}\phi^2 + V(\phi). \tag{7}$$

The case of a quadratic potential has been studied in papers [14, 15, 17, 19]. In paper [14] the algorithm of localization of the Einstein equations has been proposed and exact solutions in the spatially flat Friedmann–Robertson–Walker metric have been found. This algorithm can be used only in the cases of linear and quadratic potentials. At the same time from the string theory one gets cubic and the fourth degree potentials, so, a quadratic potential can be considered only as an approximation [17]. In this connection the search of solutions for equation (5) with a cubic potential in the Friedmann–Robertson–Walker metric is actively conducted [16, 18].

In this paper we propose the method of the search of solutions for (5) in the Friedmann–Robertson–Walker metric, which gives exact two-parameter solutions in the close form or in quadratures. This method allows to find solutions for an arbitrary potential $V(\phi)$, with the exception of the cases of linear and quadratic potentials. In the paper we consider the case of cubic potential, which is connected with the string theory (Section 3), as well as cases of logarithmic, exponential and power potentials (Section 4).

Note that in distinguish to the localization method [19], which allows to localize all Einstein equations, this method solves only equation (5), whereas equations (4), after the substitution of the obtained solution, are inconsistent in the general case. In Section 5 we show that the supplement of a scalar k-essence field gives an exact solution of the system of all nonlocal Einstein equations. The question about possible types of the additional matter and ability to obtain an exact solution for all Einstein equations in modified gravitation models, for example, in $f(R)$ gravitational model, without additional matter requires distinct investigations.

For cubic and exponential potentials approximate solutions for equation (5) with the string field theory inspired form of $\mathcal{F}(\Box)$ have been found by G. Calcagni and G. Nardelli [18] as a generalization of their solutions in the Minkowski space [32]. In distinguish from [18] in this paper we obtain the exact solutions for equation (5), in addition, the Hubble parameter $H(t)$ is a solution of equation, it is not given a priori.

2 SOLUTIONS FOR EQUATIONS OF MOTION

Let us consider nonlocal Klein–Gordon equation in the case of an arbitrary potential:

$$\mathcal{F}(\Box_g)\phi = V'(\phi), \tag{8}$$

where a prime denotes the derivative with respect to ϕ. A particular solution of (8) is a solution of the following system of local equations:

$$\sum_{n=0}^{N-1} f_n \Box_g^n \phi = V'(\phi) - C, \qquad f_N \Box_g^N \phi = C, \tag{9}$$

where $N - 1$ is a natural number, C is an arbitrary constant.

In the case $f_1 \neq 0$ we can choose $N = 2$. In the spatially flat Friedmann–Robertson–Walker metric with the interval:

$$ds^2 = - dt^2 + a^2(t) \left(dx_1^2 + dx_2^2 + dx_3^2 \right), \tag{10}$$

where $a(t)$ is the scale factor, we obtain from (9) the following system:

$$f_1 \Box_g \phi = - f_1 \left(\ddot{\phi} + 3H\dot{\phi} \right) = V'(\phi) - f_0 \phi - C, \qquad f_2 \Box_g^2 \phi = C. \tag{11}$$

The function $H(t)$ is the Hubble parameter: $H \equiv \dot{a}(t)/a(t)$, and a dot denotes the time derivative. It is easy to see that at $f_2 = 0$ solutions can exist only for $C = 0$. Let us consider the case $f_2 \neq 0$. The first equation of (11) can be rewritten in the following form:

$$H = - \frac{1}{3\dot{\phi}} \left(\ddot{\phi} + \tilde{V}'(\phi) - \frac{C}{f_1} \right), \tag{12}$$

where

$$\tilde{V}'(\phi) \equiv \frac{1}{f_1} \left(V'(\phi) - f_0 \phi \right). \tag{13}$$

Equation

$$(\partial_t^2 + 3H\partial_t) \left(\ddot{\phi} + 3H\dot{\phi} \right) = \frac{C}{f_2}, \tag{14}$$

transfigures to the following form

$$(\partial_t^2 + 3H\partial_t)\tilde{V}' = \tilde{V}'''\dot{\phi}^2 + \tilde{V}''(\ddot{\phi} + 3H\dot{\phi}) = - \frac{C}{f_2}. \tag{15}$$

We eliminate H and obtain

$$\dot{\phi}^2 = \frac{1}{\tilde{V}'''} \left(\tilde{V}''\tilde{V}' - \frac{C}{f_1}\tilde{V}'' - \frac{C}{f_2} \right). \tag{16}$$

Equation (16) can be solved in quadratures. The obtained solution depends on two arbitrary parameters C and t_0, the latter papameter corresponds to the time shift.

At $C = 0$ we get the following equation

$$\dot{\phi}^2 = \frac{\tilde{V}'\tilde{V}''}{\tilde{V}'''} \equiv \frac{(V' - f_0\phi)(V'' - f_0)}{f_1 V'''}, \tag{17}$$

using which one can get solutions at $f_2 = 0$ as well.

Remark that, the proposed method can not be used in the cases of linear and quadratic potentials, since for them $\widetilde{V}''' \equiv 0$. Consequently this way of the search of solutions is suited only for nonlinear in ϕ equation (8). In the case of linear in ϕ equation (8) the search of solutions is possible due to the localization method [14, 15, 19].

3 CUBIC POTENTIAL

The case of cubic potential is actively studied, since it is connected with the bosonic string field theory [16, 18]. Let us find solutions for equation (8) at

$$V(\phi) = B_3\phi^3 + B_2\phi^2 + B_1\phi + B_0, \tag{18}$$

where B_0, B_1, B_2, and B_3 are arbitrary constants, but $B_3 \neq 0$. We get (16) in the following form

$$\dot{\phi}^2 = 4C_3\phi^3 + 6C_2\phi^2 + 4C_1\phi + C_0, \tag{19}$$

where

$$C_0 = \frac{(B_1 - C)(2B_2 - f_0)}{6f_1 B_3} - \frac{Cf_1^2}{6f_1 f_2 B_3}, \qquad C_2 = \frac{2B_2 - f_0}{4f_1}, \tag{20}$$

$$C_1 = \frac{6B_3(B_1 - C) + (2B_2 - f_0)^2}{24f_1 B_3}, \qquad C_3 = \frac{3B_3}{4f_1}. \tag{21}$$

Note, that $C_3 \neq 0$ since $B_3 \neq 0$. Constants B_2 and f_0 appear in equation (19) only in the combination $2B_2 - f_0$. Using the transformation

$$\phi = \frac{1}{2C_3}(2\xi - C_2), \tag{22}$$

we get the following equation

$$\dot{\xi}^2 = 4\xi^3 - g_2\xi - g_3, \tag{23}$$

where

$$g_2 = 3C_2^2 - 4C_1 C_3 = \frac{(2B_2 - f_0)^2 - 12B_3(B_1 - C)}{16f_1^2}, \tag{24}$$

$$g_3 = 2C_1 C_2 C_3 - C_2^3 - C_0 C_3^2 = -\frac{3B_3 C}{32f_2 f_1}. \tag{25}$$

A solution of equation (23) is either the Weierstrass elliptic function

$$\xi(t) = \wp(t - t_0, g_2, g_3), \tag{26}$$

where t_0 is an arbitrary number, or a degenerate elliptic function. As known [33], the Weierstrass elliptic function is a double periodic meromorphic function, which has one double pole in the fundamental parallelogram of periods. If $\phi(t)$ is an elliptic function, then the Hubble parameter $H(t)$ is an elliptic function as well.

Let us consider degenerated cases. At $g_2 = 0$ and $g_3 = 0$ the general solution for equation (23) is

$$\xi = \frac{1}{(t - t_0)^2}, \tag{27}$$

therefore,

$$\phi_1 = \frac{1}{C_3(t - t_0)^2} - \frac{C_2}{2C_3} = \frac{4f_1}{3B_3(t - t_0)^2} - \frac{2B_2 - f_0}{6B_3}. \tag{28}$$

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Substituting ϕ_1 into (12), we get

$$H_1 = \frac{5}{3(t - t_0)}. \tag{29}$$

From conditions $g_2 = 0$ and $g_3 = 0$ it follows that

$$C = 0, \qquad B_1 = \frac{(2B_2 - f_0)^2}{12B_3}. \tag{30}$$

Solutions, which are bounded on the whole real axis and tends to a finite limit at $t \to \infty$, attract the special interest. Such a solution is the following function

$$\phi_2 = D_2 \tanh(\beta(t - t_0))^2 + D_0, \tag{31}$$

$$D_2 = \frac{4}{3B_3} f_1 \beta^2, \qquad D_0 = \frac{1}{18B_3} \left(3(f_0 - 2B_2) - 16 f_1 \beta^2 \right), \tag{32}$$

where β is a root of the following equation

$$1024 f_2 f_1 \beta^6 + 576 f_1^2 \beta^4 + 324 B_3 B_1 - 27(2B_2 - f_0)^2 = 0. \tag{33}$$

Bounded real solutions for equation (19) correspond to real roots of equation (33). Pure imaginary roots of equation (33) correspond to unbounded real solutions for equation (19), because $\tanh(\beta t)^2 = -\tan(i\beta t)^2$. The solution ϕ_2 exists at

$$C = \frac{1}{36B_3} \left(64 f_1^2 \beta^4 - 3(2B_2 - f_0)^2 + 36 B_3 B_1 \right). \tag{34}$$

The Hubble parameter has the following form:

$$H_2 = \frac{\beta(2\cosh(\beta t)^2 - 3)}{3\cosh(\beta t)\sinh(\beta t)} -$$

$$- \frac{3B_3(D_2 \tanh(\beta t)^2 + D_0)^2 + (2B_2 - f_0)(D_2 \tanh(\beta t)^2 + D_0) + B_1}{6 f_1 D_2 \beta \tanh(\beta t)(1 - \tanh(\beta t)^2)}.$$

The parameter t_0 is an arbitrary complex number, so, using the equality $\tanh(t + i\pi/2) = \coth(t)$, one gets the following real solutions

$$\tilde{\phi}_2 = D_2 \coth(\beta(t - t_0))^2 + D_0. \tag{35}$$

Note that solutions in terms of hyperbolic functions exist only at $C \neq 0$, since $g_3 = 0$ at $C = 0$, and solution (28) is the unique solution in terms of elementary functions.

4 EXACT SOLUTIONS FOR OTHER TYPES OF PO-TENTIAL

4.1 Logarithmic Potential

Note that one can get equation (16) in the form (19), staring from a nonpolynomial potential as well. Indeed, let

$$V(\phi) = C_1 \ln(\alpha\phi), \tag{36}$$

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where C_1 and α are arbitrary constants. The parameter α does not enter into equation (16):

$$\dot\phi^2 = \frac{f_0^2}{2f_1 C_1}\phi^4 + \frac{C(f_2 f_0 - f_1^2)}{2f_1 f_2 C_1}\phi^3 + \frac{C}{2f_1}\phi - \frac{C_1}{2f_1}. \tag{37}$$

At $f_0 \neq 0$ in the general case the Jacobi elliptic functions are solutions for (37). At $f_0 = 0$ and $C \neq 0$ equation (37) is a particular case of equation (19), which solutions are the Weierstrass elliptic functions.

Let us analyse real solutions in terms of the elementary functions. Such solutions have been found only at $f_0 = 0$.

At $C = 0$ one gets the following solutions

$$\phi_0(t) = -\frac{\sqrt{-2C_1 f_1}}{2f_1}(t - t_0), \qquad H_0 = \frac{2}{3(t - t_0)} + \frac{f_0}{3f_1}(t - t_0). \tag{38}$$

At $C \neq 0$ the following solution exists

$$\phi_{ln} = \tilde D_2 \tanh^2(A(t - t_0)) + \tilde D_0, \tag{39}$$

where A is an arbitrary number,

$$\tilde D_2 = -\frac{C_1(32 f_2 A^2 - 9 f_1)}{18 A f_1 (16 f_2 A^2 - 3 f_1)\varsigma}, \quad \tilde D_0 = \frac{C_1}{12 A f_1 \varsigma}, \quad \varsigma = \pm\sqrt{\frac{C1}{144 f_2 A^2 - 27 f_1}},$$

at that $C = 64 A^3 f_2 \varsigma$. The Hubble parameter

$$H_{ln} = \frac{A(2\cosh(At)^2 - 3)}{3\cosh(At)\sinh(At)} - \frac{C_1 \cosh(At)^2}{6 A f_1 \tilde D_2((\tilde D_2 \tanh(At)^2 + \tilde D_0)\tanh(At)}. \tag{40}$$

corresponds to this solution.

4.2   Exponential Potential

In paper [18] the exponential potential has been consider in addition to the cubic potential and approximate solutions for equation (8) have been obtained.

Let $V(\phi) = C_1 e^{\alpha\phi}$. At $f_0 = 0$ and $C = 0$ solutions for (9) are elementary functions and have the following form:

$$\phi_{exp}(t) = \frac{1}{\alpha}\ln\left(\frac{4 f_1}{C_1 \alpha^2 (t - t_0)^2}\right), \qquad H_{exp}(t) = \frac{1}{t - t_0}. \tag{41}$$

Note that the obtained Hubble parameter $H_{exp}(t)$ is proportional to the Hubble parameter, which has been used in paper [18], and to the Hubble parameter, obtained in the case of cubic potential (formula (29)).

4.3 Power Potential

Let us consider solutions in the case of the potential $V(\phi) = C_1\phi^n$. At $f_0 = 0$ equation (16) is as follows:

$$\dot{\phi}^2 = \frac{C_1^2 f_2 n^2(n-1)\phi^n - C_1 C f_2 n(n-1)\phi - C f_1^2 \phi^{3-n}}{f_1 f_2 C_1 n(n-1)(n-2)} \tag{42}$$

At $C = 0$ this equation is equivalent to

$$\dot{\phi}^2 = \frac{C_1 n\phi^n}{f_1(n-2)} \tag{43}$$

and has a solution in the form of the elementary function:

$$\phi_n(t) = 2^{2/(n-2)} \left(\frac{f_1}{C_1 n(n-2)(t-t_0)^2}\right)^{1/(n-2)}. \tag{44}$$

The corresponding Hubble parameter is equal to

$$H_n(t) = \frac{3n-4}{3(n-2)(t-t_0)}. \tag{45}$$

At $n = 4/3$ we get a particular solution to equation (8) in the Minkowski space:

$$\phi_m(t) = \pm\frac{2C_1\sqrt{-2f_1C_1}}{27f_1^2}(t-t_0)^3. \tag{46}$$

Note that in the Minkowski space exact bounded at all values of t solutions for nonlocal equations with power potentials have been found in paper [28].

5 COSMOLOGICAL MODEL WITH A NONLOCAL SCALAR FIELD AND A k-ESSENCE FIELD

The goal of this Section is to show, that the supplement of the k-essence scalar field Ψ allows to obtain a system of the Einstein equations, which has an exact solution, at that the Hubble parameter and the nonlocal field are given by formulae (12) and (16) respectively. The k-essence models are considered in cosmology both as inflation models [34, 35, 36], and as dark energy models [37, 38, 39, 40, 26], (see also [41] and references therein).

Let us consider the following action

$$S_2 = \int d^4x \sqrt{-g}\left(\frac{R}{16\pi G_{\mathrm{N}}} + \frac{1}{g_o^2}\left(\frac{1}{2}\phi\mathcal{F}(\Box_g)\phi - V(\phi)\right) - \mathcal{P}(\Psi, X)\right), \tag{47}$$

where $X \equiv -g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi$. In the Friedmann–Robertson–Walker metric the function Ψ depends only on time, so $X = \dot{\Psi}^2$.

Following paper [26], we select the pressure in the following form

$$\mathcal{P}(\Psi, X) = \frac{1}{2}(p_q(\Psi) - \varrho_q(\Psi)) + \frac{1}{2}(p_q(\Psi) + \varrho_q(\Psi))X + \frac{1}{2}M^4(\Psi)(X-1)^2. \tag{48}$$

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We consider functions $p_q(\Psi)$, $\varrho_q(\Psi)$, and $M^4(\Psi)$ as arbitrary differentiable functions. The energy density of the k-essence field is equal to

$$\mathcal{E}(\Psi, X) = (p_q(\Psi) + \varrho_q(\Psi))X + 2M^4(\Psi)(X^2 - X) - \mathcal{P}(\Psi, X). \tag{49}$$

The Einstein equations, which have been obtained using the variation of S_2, have the following form:

$$3H^2 = 8\pi G_{\mathrm{N}}(\varrho + \mathcal{E}),$$
$$2\dot{H} + 3H^2 = -8\pi G_{\mathrm{N}}(p + \mathcal{P}). \tag{50}$$

Varying the action S_2, we also get equation (8) and equation for the k-essence scalar field Ψ

$$\dot{\mathcal{E}} = -3H(\mathcal{E} + \mathcal{P}). \tag{51}$$

In the Friedmann–Robertson–Walker metric the energy–momentum tensor (6) has the following form

$$T_{\mu\nu} = g_{\mu\nu}\mathrm{diag}(-\varrho, p, p, p), \tag{52}$$

where

$$\varrho = \frac{1}{g_{\mathrm{o}}^2}\left(\sum_{n=1}^{\infty}\frac{f_n}{2}\left(\sum_{l=0}^{n-1}\partial_t\Box^l\phi\partial_t\Box^{n-1-l}\phi + \sum_{l=1}^{n-1}\Box^l\phi\Box^{n-l}\phi\right) - \frac{f_0}{2}\phi^2 + V(\phi)\right),$$

$$p = \frac{1}{g_{\mathrm{o}}^2}\left(\sum_{n=1}^{\infty}\frac{f_n}{2}\left(\sum_{l=0}^{n-1}\partial_t\Box^l\phi\partial_t\Box^{n-1-l}\phi - \sum_{l=1}^{n-1}\Box^l\phi\Box^{n-l}\phi\right) + \frac{f_0}{2}\phi^2 - V(\phi)\right).$$

Let ϕ_2 is a solution to system (9) at $N = 2$. Using $\Box_g^2\phi_2 = C/f_2$, we get

$$\varrho(\phi_2) = E_{00}(\phi_2) + W(\phi_2), \qquad p(\phi_2) = E_{00}(\phi_2) - W(\phi_2), \tag{53}$$

where

$$E_{00}(\phi_2) = \frac{1}{2g_{\mathrm{o}}^2}\left(f_1(\partial_t\phi_2)^2 + 2f_2\partial_t\phi_2\partial_t\Box_g\phi_2 + f_3(\partial_t\Box_g\phi_2)^2\right),$$

$$W(\phi_2) = \frac{1}{g_{\mathrm{o}}^2}\left(\frac{f_2}{2}(\Box_g\phi_2)^2 + \frac{f_3 C}{f_2}\Box_g\phi_2 + \frac{f_4 C^2}{2f_2^2} - \frac{f_0}{2}\phi_2^2 + V(\phi_2)\right).$$

The k-essence models (without additional fields) have one useful property. For any real differentiable function $H_0(t)$ there exist such differentiable functions $\varrho_q(\Psi)$ and $p_q(\Psi)$, that functions $H_0(t - t_0)$ and $\Psi(t) = t - t_0$ are a particular solution to system (50)–(51). This property can be generalized on the case of the models with an additional nonlocal scalar field, which are described by action (47). Indeed, at $\Psi(t) = t - t_0$ we obtain

$$\mathcal{E} = \varrho_q(\Psi) = \varrho_q(t - t_0), \qquad \mathcal{P} = p_q(\Psi) = p_q(t - t_0). \tag{54}$$

Substituting into (50) expression of ϱ_q and p_q, we get

$$\varrho_q(\Psi) = \varrho_q(t - t_0) = \frac{3}{8\pi G_{\mathrm{N}}}H^2(t - t_0) - \varrho(t - t_0),$$

$$p_q(\Psi) = p_q(t - t_0) = -\varrho_q(t - t_0) - \varrho(t - t_0) - p(t - t_0) - \frac{1}{4\pi G_{\mathrm{N}}}\dot{H}(t - t_0). \tag{55}$$

It is easy to see that system (50)–(51) and equation (8) have the exact particular solution, at that functions $H(t - t_0)$ and $\phi(t - t_0)$ are the obtained solution of equation (8) and $\Psi(t) = t - t_0$.

So, the algorithm to obtain exact solutions is as follows: for the given potential $V(\phi)$ we find $H(t)$ and $\phi(t)$, calculate the energy–momentum tensor and substitute the obtained values into (55). The obtained values of ϱ_q and p_q give the exact solvable model with a nonlocal scalar field and the k-essence field. The function $M(\Psi)$ can be selected arbitrarily.

Let us illustrate this scheme on the simplest example, connected with a cubic potential, and consider solution (28)–(29). Conditions (30) of existence of this solution leave constants B_3 and B_2 arbitrary. Using the arbitrariness of B_2, we can, without loss of generality, put $f_0 = 0$. Also, for compactness of the record we put $t_0 = 0$. For solution (28)–(29) we get

$$\Box_g \phi_1 = \frac{16 f_1}{3 B_3 t^4}, \tag{56}$$

therefore,

$$E_{00} = \frac{32 f_1^2 (f_1 t^4 + 16 f_2 t^2 + 64 f_3)}{9 B_3^2 t^{10}}, \qquad W = \frac{128 f_2 f_1^2}{9 g_o^2 B_3^2 t^8} + \frac{1}{g_o^2} V(\phi_1). \tag{57}$$

Consequently we get

$$\varrho_q(\Psi) = \frac{B_2^3 - 27 B_0 B_3^2}{27 g_o^2 B_3^2} + \frac{25}{24 \pi G_N \Psi^2} - \frac{160 f_1^3}{27 g_o^2 B_3^2 \Psi^6} - \frac{640 f_2 f_1^2}{9 g_o^2 B_3^2 \Psi^8} - \frac{2048 f_3 f_1^2}{9 g_o^2 B_3^2 \Psi^{10}},$$

$$p_q(\Psi) = -\frac{B_2^3 - 27 B_0 B_3^2}{27 g_o^2 B_3^2} - \frac{5}{8 \pi G_N \Psi^2} - \frac{32 f_1^3}{27 g_o^2 B_3^2 \Psi^6} - \frac{128 f_2 f_1^2}{3 g_o^2 B_3^2 \Psi^8} - \frac{2048 f_3 f_1^2}{9 g_o^2 B_3^2 \Psi^{10}}.$$

Thus, we can make a conclusion that the adding of the k-essence field allows to construct a self-consistent system of the Einstein equations for the obtained solution to equation (8). The choice of the k-essence scalar field as an additional field is reasoned by the goal to get a self-consistent system of the Einstein equations with no restriction on the type of the potential and solutions.

6 CONCLUSIONS

In this paper the method of the search of exact solutions to the nonlocal field equation (8) in the Friedmann–Robertson–Walker metric has been proposed. It has been demonstrated that for an arbitrary potential besides of linear or quadratic ones there exists a particular two-parameter solution, which can be found in quadratures.

The string field theory inspired case of a cubic potential has been consider in detail. It is shown that solutions are either Weierstrass elliptic functions or degenerate elliptic functions, for example, solutions in terms of hyperbolic tangent have been obtained. Exact solutions in terms of elementary functions have been found for logarithmic, exponential and power potentials as well.

Note that the proposed method allows to find a solution only for the field equation, but not for the whole system of the Einstein equations. It has been shown that the supplement of the k-essence scalar field allows to construct a self-consistent system of the Einstein equations for the obtained solution to equation (8). The localization of the Einstein equations in cosmological

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models with a nonlocal scalar field and an arbitrary potential is an actual problem, which studying requires distinct investigations[1].

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