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The Tensor Hierarchies of Pure $N = 2, d = 4, 5, 6$ Supergravities

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Abstract

We study the supersymmetric tensor hierarchy of pure (gauged) $N = 2, d = 4, 5, 6$ supergravity and compare them with those of the pure, ungauged, theories (worked out in Ref. [1] for $d = 5$) and the predictions of the Kač-Moody approach made in Ref. [2]. We find complete agreement in the ungauged case but we also find that, after gauging, new Stückelberg symmetries reduce the number of independent *physical* top-forms. The analysis has to be performed to all orders in fermion fields.

We discuss the construction of the worldvolume effective actions for the p -branes which are charged with respect to the $(p + 1)$ -form potentials and the relations between the tensor hierarchies and p -branes upon dimensional reduction.

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Introduction

The embedding-tensor formalism [3, 4, 5, 6] has proven to be a powerful method to determine, combining the requirements of duality and gauge invariance, (a large part of) the $(p + 1)$ -form content (*tensor hierarchy*) of field theories and, in particular, of supergravity theories. Thus, using the general results on 4-dimensional tensor hierarchies of Ref. [6], the supersymmetric tensor hierarchy of $N = 1, d = 4$ supergravity (and, implicitly, the most general form of the $N = 1, d = 4$ theories) was found in Ref. [7]. It turned out to contain more forms than expected according to the arguments of Ref. [6], based on (bosonic) gauge symmetry, but it is still possible to make sense of all of them since the new 3-forms could be associated to new deformations of the theory and the new 4-forms to new constraints involving the embedding tensor and other deformation parameters. The 3-forms transform into the gravitino under supersymmetry and they were shown to be associated to supersymmetric domain-wall solutions in Ref. [8].

At the same time, it has been conjectured that the $(p + 1)$ -form spectra different supergravity theories are determined by very extended symmetry groups related to their duality groups (see, e.g. Ref. [2] and references therein). It is not guaranteed that these two approaches (the embedding tensor approach and the *Kač-Moody* (KM) approach) will give the same $(p + 1)$ -form spectra and, in order to find all the $(p + 1)$ -forms of a given supergravity (and all the p -branes associated to them) it is important to compare the predictions of both methods. In this paper we will do this for the simplest $N = 2, d = 4, 5, 6$ theories.

Our first goal will be to extend the results obtained for the $N = 1, d = 4$ theories in [7] to the $N = 2, d = 4, 5, 6$ theories focusing, for simplicity, on the pure supergravity theories (i.e. in absence of matter couplings). We will, therefore, find the supersymmetric tensor hierarchies of these theories using the embedding-tensor formalism and using the results in Refs. [1, 2] and [9]. We will then compare the spectra of $(p + 1)$ -forms obtained with the predictions of the KM approach made in Ref. [2].

The absence of matter couplings limits the range of $(p + 1)$ -forms that we can find. However, those associated to the R-symmetry of the theory (not considered in Ref. [10]), which transform into gravitini (the only fermions available in the pure $N = 2$ theories), and which, therefore, may be associated to dynamical p -branes, should not be missed in our analysis. We will check this correspondence by constructing explicitly with the $(p + 1)$ -forms of the tensor hierarchies candidates to the bosonic part of κ -symmetric worldvolume actions⁴.

This paper is organized as follows: in Section 1 we give our conventions for the pure, ungauged, $N = 2, d = 4$ supergravity theory and in Section 2 we construct its tensor hierarchy as a particular case of the generic (bosonic) 4-dimensional hierarchy of Ref. [6], on the basis of the R-symmetry group of the theory. In Section 3 we consider the gauging of the R-symmetry group, constructing the supersymmetry transformation rules for all the $(p + 1)$ -form fields in the tensor hierarchy to lowest order in fermions and checking their closure up to duality relations. In Section 4 we briefly review the analogous results for the minimal $d = 5$ supergravity found in Ref. [1, 2], from the point of view of the 5-dimensional tensor hierarchy of Ref. [9] and in Section 5 we cover the 6-dimensional case, which is much simpler due to the absence of 1-forms. In

⁴Some previous partial results were also given in Ref. [11].

Section 6 we discuss the construction of the effective actions for the p -branes of these theories and in Section 7 we present our conclusions.

1 Pure, ungauged, $N = 2, d = 4$ supergravity

In order to pave the way for further generalizations, we are going to describe pure $N = 2, d = 4$ supergravity as a particular case of the general matter-coupled $N = 2, d = 4$ supergravity⁵. To make contact with the conventions used in the embedding-tensor formalism we will write symplectic products with symplectic indices M, N .

The supergravity multiplet of the $N = 2, d = 4$ theory consists of the graviton $e^a{}_\mu$, a pair of gravitini $\psi_{I\mu}$, ($I = 1, 2$) which we describe as Weyl spinors, and one graviphoton which we denote by $A^\Lambda{}_\mu$ even though the index Λ only takes one value to distinguish this fundamental (electric) field from its (magnetic) dual $A_{\Lambda\mu}$. The bosonic action is

$$S = - \int \left[\star R + 4 \Im \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge \star F^\Sigma + 4 \Re \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma \right], \quad (1.4)$$

where $\mathcal{N}_{\Lambda\Sigma}$ is the period ‘‘matrix’’ with only one component with negative imaginary part. In this case the choice of period ‘‘matrix’’ (or, equivalently, of constant ‘‘canonical symplectic section’’ \mathcal{V}^M) is arbitrary, as far as the constraints are satisfied.

We define G_Λ by

$$G_\Lambda{}^+ = \mathcal{N}_{\Lambda\Sigma}^* F^{\Lambda+}, \quad (1.5)$$

and define the 2-dimensional symplectic vector

$$(G^M) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}. \quad (1.6)$$

The supersymmetry transformations of the supergravity fields to all orders in fermions are

⁵See Refs. [12, 13] and the original references [14, 15]. Our conventions are given in Refs. [16, 17, 18] and follow closely those of Ref. [12]. In particular, our σ matrices satisfy

$$(\sigma^x \sigma^y)^I{}_J = \delta^{xy} \delta^I{}_J + i \varepsilon^{xyz} (\sigma^z)^I{}_J, \quad (1.1)$$

and we define

$$(\sigma^x)_I{}^J \equiv [(\sigma^z)^I{}_J]^*, \quad (1.2)$$

and one finds that

$$(\sigma^x)_I{}^J = \varepsilon_{IK} (\sigma^z)^K{}_L \varepsilon^{LJ}, \quad \sigma^x [{}^I{}_J \varepsilon^K]{}^J = \sigma^x [{}^K{}_I \varepsilon_J]{}^K = 0 \quad \sigma^x{}_I{}^J = \sigma^x J{}_I. \quad (1.3)$$

$$\delta_\epsilon e^a{}_\mu = -\frac{i}{4}\bar{\psi}_{I\mu}\gamma^a\epsilon^I + \text{c.c.}, \quad (1.7)$$

$$\delta_\epsilon\psi_{I\mu} = \tilde{\nabla}_\mu\epsilon_I + \varepsilon_{IJ}\tilde{T}^+{}_{\mu\nu}\gamma^\nu\epsilon^J, \quad (1.8)$$

$$\delta_\epsilon A^\Lambda{}_\mu = \frac{1}{4}\mathcal{L}^{\Lambda*}\varepsilon^{IJ}\bar{\psi}_{I\mu}\epsilon_J + \text{c.c.}, \quad (1.9)$$

where $\tilde{\nabla}_\mu$ is the Lorentz covariant derivative that uses the torsionful spin connection $\tilde{\omega}_\mu{}^{ab}$:

$$\begin{aligned} \tilde{\omega}_{abc} &= -\tilde{\Omega}_{abc} + \tilde{\Omega}_{bca} - \tilde{\Omega}_{cab}, \\ \tilde{\Omega}_{abc} &= \Omega_{abc} + \frac{1}{2}T_{abc}, \\ \Omega_{abc} &= e_a{}^\mu e_b{}^\nu \partial_{[\mu} e_{c|\nu]}, \\ T_{\mu\nu}{}^a &= -\frac{i}{2}\bar{\psi}_{[\mu} \gamma^a \psi_{\nu]}^I, \end{aligned} \quad (1.10)$$

\mathcal{L}^Λ is the upper component of the canonically-normalized symplectic section

$$(\mathcal{V}^M) = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix}, \quad \langle \mathcal{V} | \mathcal{V}^* \rangle \equiv \mathcal{V}^M{}^* \mathcal{V}_M = \mathcal{L}^{*\Lambda} \mathcal{M}_\Lambda - \mathcal{L}^\Lambda \mathcal{M}_\Lambda^* = -i, \quad (1.11)$$

and where \tilde{T} is the supercovariant graviphoton field strength which can be written in the form

$$\tilde{T}^+ = \langle \mathcal{V} | \tilde{F}^+ \rangle = \tilde{G}^{M+} \mathcal{V}_M, \quad (1.12)$$

\tilde{F}^Λ being given by

$$\tilde{F}^\Lambda{}_{\mu\nu} = F^\Lambda{}_{\mu\nu} + \frac{1}{4} [\mathcal{L}^\Lambda \varepsilon_{IJ} \bar{\psi}_{\mu}^I \psi_{\nu}^J + \text{c.c.}]. \quad (1.13)$$

A convenient choice of symplectic section and period matrix, satisfying the relations

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma, \quad \mathcal{L}^{*\Lambda} \mathcal{L}^\Sigma = -\frac{1}{2} \Im \mathcal{N}^{\Lambda\Sigma}, \quad (1.14)$$

(where the upper indices in the period matrix indicate that we are dealing with the inverse) is⁶

$$\mathcal{V} = (\mathcal{V}^M) = \begin{pmatrix} i \\ \frac{1}{2} \end{pmatrix}, \quad \mathcal{N}_{\Lambda\Sigma} = -\frac{i}{2}, \quad (1.15)$$

but we will leave them undetermined in what follows and we will only use their constancy and their general properties.

⁶It can be shown that the most general \mathcal{V}^M satisfying the normalization constraint can always be brought to this one by a symplectic transformation.

2 The tensor hierarchy of pure $N = 2, d = 4$ supergravity

In this section we are going to determine the tensor hierarchy of pure $N = 2, d = 4$ supergravity by adapting the generic result of Ref. [6] to the actual global symmetries and field content of the theory at hands.

2.1 Global symmetries of the ungauged theory

The bosonic global symmetry of this theory is the $Sp(2)$ group of electric-magnetic duality rotations of the vector fields that preserve the equations of motion but not the action:

$$\delta_{\alpha^a} G^M = \alpha^a T_{aN}{}^M G^N, \quad (2.1)$$

where the generators $T_{aN}{}^M$ satisfy the algebra

$$[T_a, T_b] = -f_{ab}{}^c T_c, \quad f_{12}{}^3 = -f_{23}{}^1 = -f_{31}{}^2 = -1. \quad (2.2)$$

This symmetry cannot be gauged using the electric and magnetic vector fields A^M because they are charged under it⁷. One can also see that the quadratic constraint of the embedding-tensor formalism cannot be satisfied for $\vartheta_M{}^a \neq 0$.

The other global symmetry of the theory is the R-symmetry group $U(2) = U(1) \times SU(2)$ which only acts on the spinors of the theory according to

$$\delta\psi_{\mu I} = \alpha^{\mathbf{x}} T_{\mathbf{x}I}{}^J \psi_{\mu J}, \quad (2.3)$$

where

$$T_{\mathbf{x}I}{}^J = -\frac{i}{2} \sigma^{\mathbf{x}}{}_I{}^J, \quad \mathbf{x} = 0, x, \quad x = 1, 2, 3, \quad (2.4)$$

where $\sigma^x{}_I{}^J$ s are the Pauli matrices and $\sigma^0{}_I{}^J \equiv \delta_I{}^J$. The non-vanishing commutators between these generators are⁸

$$[T_x, T_y]_I{}^J = -\varepsilon_{xyz} T_z{}_I{}^J, \quad (2.5)$$

so, in the conventions of Ref. [6], the structure constants are $f_{xy}{}^z = +\varepsilon_{xyz}$. The $\mathbf{x} = 0$ transformations are just multiplication by a (“Kähler”) $U(1)_{\text{Kähler}}$ phase to be distinguished from the $U(1)$ subgroups of $SU(2)$.

The standard quadratic constraint that expresses the invariance of the embedding tensor takes for this symmetry the form

$$\vartheta_M{}^x \vartheta_N{}^y \varepsilon_{xyz} = 0, \quad (2.6)$$

i.e. the two 3-component vectors $\vartheta^{\Lambda x}$ and $\vartheta_{\Lambda}{}^x$ are parallel and we can write

⁷One can only expect to be able to gauge one of the Abelian subgroups (there is no non-Abelian 2-dimensional subgroup) with just two gauge fields. It is easy to see that this is impossible.

⁸We will not distinguish between upper and lower $SU(2)$ indices x, y, z .

$$\vartheta_M^x = \alpha_M \vartheta^x, \quad (2.7)$$

where α_M is an arbitrary 2-component symplectic vector.

The quadratic constraint $\vartheta^M [x \vartheta_M^y] = 0$ is automatically satisfied. The quadratic constraint $\vartheta^M [x \vartheta_M^0] = 0$ is satisfied if $\vartheta_M^0 = \vartheta^0 \alpha_M$.

ϑ^x selects the $U(1) \subset SU(2)$ which is to be gauged combined with $U(1)_{\text{Kahler}}$. The generator of the Abelian symmetry which is gauged is, therefore, $\vartheta^x T_x$. The vector α_M selects a combination of the two vector fields $\alpha_M A^M$ that will act as a gauge vector.

It is well known that $U(1)_{\text{Kahler}}$ cannot be gauged in $N = 2, d = 4$ supergravity and, therefore, we will set $\vartheta^0 = 0$ from the onset. Furthermore, since the electric-magnetic duality group $Sp(2)$ cannot be gauged, we are also going to set $T_a = 0$ from the beginning⁹.

2.2 The tensor hierarchy of pure $N = 2, d = 4$ supergravity

In principle, we can naively substitute the indices of the symmetries, 1-forms and embedding tensor involved in our problem into the generic formulae of Ref. [6]. We only have to consider the R-symmetry with $f_{xy}^z = +\varepsilon_{xyz}$ and $\vartheta_M^x = \alpha_M \vartheta^x$, which leads to $X_{MNP} = 0$.

We should also take into account that both the 4-form D_x^{NM} and the associated 3-form gauge parameter Λ_x^{NM} can be taken to be antisymmetric in the upper indices. Then, we can define

$$D_x^{(1)} \equiv \frac{1}{2} \Omega_{PQ} D_x^{PQ}, \quad (2.8)$$

$$D_x^{(2)} \equiv -\frac{1}{8} \varepsilon_{xyz} D_{yz}, \quad (2.9)$$

$$\Lambda_x^{(1)} \equiv \frac{1}{2} \Omega_{PQ} \Lambda_x^{PQ}, \quad (2.10)$$

$$\Lambda_x^{(2)} \equiv -\frac{1}{8} \varepsilon_{xyz} \Lambda_{yz}, \quad (2.11)$$

Furthermore, the triplets $D_x^{(1)}, D_x^{(2)}$ only appear everywhere through their sum, so their Stückelberg shifts with 4-form parameters $\tilde{\Lambda}_x$ cancel each other. This indicates the existence of just one independent triplet of 4-forms and we define¹⁰

$$D_x \equiv D_x^{(1)} + D_x^{(2)}, \quad \Lambda_x \equiv \Lambda_x^{(1)} + \Lambda_x^{(2)}. \quad (2.12)$$

Finally, we find that D_x can have additional gauge transformations that leave invariant the 4-form field strength G_x^M :

$$\delta D_x = \vartheta^x \tilde{\Lambda}, \quad (2.13)$$

⁹There is no way to define consistent supersymmetry rules for the corresponding 2-forms B_a anyway.

¹⁰Alternatively, we may use the Stückelberg shift to eliminate one of these triplets.

where $\tilde{\Lambda}$ is a 4-form. This Stückelberg shift cannot be used to eliminate the complete triplet of 4-forms D_x , but only a combination of them.

We find that, out of all the possible fields present in the generic tensor hierarchy, A^M, B_x, C_x^M and D_x are interconnected by gauge transformations, while $B_0, C_0^M, D_{0x}, D_0^{NM}$ and D^{NPQ} are decoupled from them. We can advance that we have been able to construct consistent supersymmetry transformation rules for all the fields in the first group plus B_0 (as in Ref. [11]) but not for the rest of the fields in the second group. Thus we will ignore them from now on although in presence of matter they might be coupled consistently with the rest of the theory.

Taking all this information into account, we find the gauge transformations

$$\delta_h A^M = -d\Lambda^M + \frac{1}{2}\alpha^M \vartheta^x \Lambda_x, \quad (2.14)$$

$$\delta_h B_x = \mathfrak{D}\Lambda_x - \varepsilon_{xyz} \vartheta^y \Lambda_z^M \alpha_M, \quad (2.15)$$

$$\delta_h C_x^M = \mathfrak{D}\Lambda_x^M - F^M \wedge \Lambda_x - \delta_h A^M \wedge B_x + \Lambda^M H_x - 2\alpha^M \vartheta^y \varepsilon_{xyz} \Lambda_z, \quad (2.16)$$

$$\begin{aligned} \delta_h D_x &= \mathfrak{D}\Lambda_x + \frac{1}{2}[F_M + \frac{1}{4}\alpha_M \vartheta^y B_y] \wedge \Lambda_x^M + \frac{1}{2}\delta_h A_M \wedge C_x^M \\ &\quad - \frac{1}{2}\Lambda_M G_x^M - \frac{1}{8}\varepsilon_{xyz} \mathfrak{D}\Lambda_y \wedge B_z + \frac{1}{4}\varepsilon_{xyz} \Lambda_y \wedge H_z + \vartheta^x \tilde{\Lambda}. \end{aligned} \quad (2.17)$$

where the $SU(2)$ -covariant derivatives are given, e.g., by

$$\mathfrak{D}\Lambda_x = d\Lambda_x + \varepsilon_{xyz} \vartheta^y \alpha_M A^M \wedge \Lambda_z. \quad (2.18)$$

We also find the gauge-covariant field strengths

$$F^M = dA^M - \frac{1}{2}\alpha^M \vartheta^x B_x, \quad (2.19)$$

$$H_x = \mathfrak{D}B_x + \alpha_M \varepsilon_{xyz} \vartheta^y C_z^M, \quad (2.20)$$

$$G_x^M = \mathfrak{D}C_x^M + [F^M + \frac{1}{4}\alpha^M \vartheta^y B_y] \wedge B_x + 2\alpha^M \varepsilon_{xyz} \vartheta^y D_z, \quad (2.21)$$

and the hierarchical Bianchi identities

$$dF^M = -\frac{1}{2}\alpha^M \vartheta^x H_x, \quad (2.22)$$

$$\mathfrak{D}H_x = \alpha_M \varepsilon_{xyz} \vartheta^y G_z^M. \quad (2.23)$$

For the decoupled 2-form B_0 we find trivial results

$$\delta_h B_0 = d\Lambda_0, \quad H_0 = dB_0, \quad dH_0 = 0. \quad (2.24)$$

This is the tensor hierarchy that naively follows from the general one. However, we observe that by setting $\vartheta^0 = 0$ we have removed B_0 from F^M , for instance. This decoupling allows for more general gauge transformations for B_0 that cannot be determined by the embedding-tensor method and will be determined by supersymmetry.

3 General gauging of pure $N = 2, d = 4$ supergravity

We want to gauge the global symmetries of the theory discussed in the previous section using as gauge fields the electric and magnetic graviphoton, using the embedding-tensor formalism¹¹. We have seen that gauge invariance requires higher-rank fields but their presence must be compatible with supersymmetry.

We are going to proceed as in Ref. [7]: first of all, we make the following electric-magnetic invariant Ansatz for the gravitini supersymmetry transformations¹² to lowest order in fermions:

$$\delta_\epsilon \psi_{\mu I} = \tilde{\mathfrak{D}}_\mu \epsilon_I + \epsilon_{IJ} \mathcal{V}_M \tilde{F}^{M+}{}_{\mu\nu} \gamma^\nu \epsilon^J - \frac{1}{4} \mathcal{V}^M \alpha_M \vartheta^x \varepsilon_{IK} \sigma^x{}^K{}_J \gamma_\mu \epsilon^J, \quad (3.1)$$

where

$$\tilde{\mathfrak{D}}_\mu \epsilon_I = \tilde{\nabla}_\mu \epsilon_I + \frac{i}{2} A^M{}_\mu \alpha_M \vartheta^x \sigma^x{}_I{}^J \epsilon_J, \quad (3.2)$$

is the Lorentz and gauge-covariant derivative of the supersymmetry parameter ϵ_I that uses the torsionful spin connection in Eq. (1.10) and where \tilde{F}^M is the supercovariantization of the vector field strength of the tensor hierarchy of the tensor hierarchy defined in Eq. (2.19)

$$\tilde{F}^M{}_\mu \equiv F^M{}_{\mu\nu} + \frac{1}{4} [\mathcal{V}^M \varepsilon_{IJ} \vartheta^I{}_\mu \psi^J{}_\nu + \text{c.c.}] . \quad (3.3)$$

It should be distinguished from \tilde{G}^M , defined in Eq. (1.6), because the latter only depends on the electric vector field A^Λ .

This supersymmetry transformation reduces to the standard one for purely electric gaugings upon use of the duality relation $\tilde{F}_\Lambda = \tilde{G}_\Lambda$ which we can also write

$$\tilde{F}^M = \tilde{G}^M . \quad (3.4)$$

The supersymmetry variations of the bosonic fields (Vierbein and electric graviphoton) are not modified, but we have to add a supersymmetry transformation for the magnetic vector field A_Λ compatible with symplectic symmetry. A symplectic-covariant Ansatz that gives correctly the supersymmetry transformation rule for the electric graviphoton Eq. (1.9) when \mathcal{V}^M is given by Eq. (1.15) and also coincides with the uncoupled case of Ref. [11]) is

¹¹The electrically-gauged theory was constructed in [19, 20].

¹²Symplectic indices are raised and lowered with the symplectic metric according to $\mathcal{V}_M = \mathcal{V}^N \Omega_{MN}$ and $\mathcal{V}^M = \Omega^{NM} \mathcal{V}_N$, where $\Omega_{MN} = \Omega^{MN}$ and $\Omega^{MP} \Omega_{NP} = \delta^M{}_N$.

$$\delta_\epsilon A^M{}_\mu = -\frac{1}{4}\mathcal{V}^{M*}\epsilon^{IJ}\bar{\epsilon}_I\psi_{\mu J} + \text{c.c.} \quad (3.5)$$

The local supersymmetry algebra acting on $A^M{}_\mu$ closes into

$$[\delta_\eta, \delta_\epsilon]A^M{}_\mu = [\delta_{\text{g.c.t.}}(\xi) + \delta_h(\Lambda^M, \Lambda_x) + \delta_{\text{susy}}(\kappa)]A^M{}_\mu, \quad (3.6)$$

where $\delta_{\text{g.c.t.}}(\xi)$ is a general coordinate transformation with parameter ξ^μ , $\delta_h(\Lambda^M, \Lambda_x)$ is the gauge transformation predicted by the tensor hierarchy given in Eq. (2.14) with parameters Λ^M, Λ_x and $\delta_{\text{susy}}(\kappa)$ is the supersymmetry transformation of Eq. (3.5) with parameter κ if a term of the form

$$F^M{}_{\nu\mu} + 2\Im(\mathcal{V}^{*M}\mathcal{V}_N \star F^{N+}{}_{\nu\mu}) \quad (3.7)$$

vanishes. This term, upon use of Eq. (A.6), takes the form of the duality relation Eq. (3.4).

The parameters $\xi, \Lambda^M, \Lambda_x, \kappa$ are given by the spinor bilinears

$$\xi^{\mathbf{x}}{}_\mu \equiv \frac{i}{4}\sigma^{\mathbf{x}}{}_{J^I}(\bar{\epsilon}_I\gamma_\mu\eta^J - \bar{\eta}_I\gamma_\mu\epsilon^J) \in \mathbb{R}, \quad \xi^0{}_\mu \equiv \xi_\mu, \quad (3.8)$$

$$X \equiv \frac{1}{2}\epsilon_{IJ}\bar{\epsilon}^I\eta^J, \quad (3.9)$$

$$\Lambda^M = \Re(\mathcal{V}^M X) + \xi^\mu A^M{}_\mu, \quad (3.10)$$

$$\Lambda_{x\mu} = -\frac{1}{2}\xi^{\mathbf{x}}{}_\mu - \xi^\nu B_{x\nu\mu}. \quad (3.11)$$

$$\kappa^I = -\xi^\mu\psi^I{}_\mu. \quad (3.12)$$

The use of the vector field strengths F^M containing a triplet of 2-forms $B_{x\mu\nu}$ predicted by the tensor hierarchy in the gravitini supersymmetry transformations is clearly justified due to the presence of Stückelberg shifts of the vector fields. Now, the consistency of this construction requires the existence of a triplet of 2-forms $B_{x\mu\nu}$ with consistent supersymmetry transformations and with the gauge transformations predicted by the tensor hierarchy.

Observe that the more general Ansatz

$$\alpha\mathcal{V}^M\sigma^{\mathbf{x}}{}_{IJ}\bar{\epsilon}^I\psi_\mu{}^J + \text{c.c.}, \quad (3.13)$$

which would include for a possible triplet of 1-forms not predicted by the tensor hierarchy does not work for $\mathbf{x} \neq 0$.

3.1 2-forms

According to the tensor hierarchy predictions, we expect a set of 2-forms $B_{\mathbf{x}\mu\nu}$, 3 of which are required by the consistency of the supersymmetry transformations of A^M . Actually, we can only make Ansatzes for the supersymmetry transformation rules of as many 2-forms:

$$\delta_\epsilon B_{\mathbf{x}\mu\nu} = \beta i \sigma^{\mathbf{x}}_I{}^J \bar{\epsilon}^I \gamma_{[\mu} \psi_{\nu]J} + \text{c.c.}, \quad (3.14)$$

where the constant β has to be a real for the local supersymmetry algebra to close.

For $\mathbf{x} = x$ we find that the transformation

$$\delta_\epsilon B_{x\mu\nu} = \frac{i}{4} \sigma^x_I{}^J \bar{\epsilon}^I \gamma_{[\mu} \psi_{\nu]J} + \text{c.c.}, \quad (3.15)$$

leads to closure of the local supersymmetry algebra

$$[\delta_\eta, \delta_\epsilon] B_{x\mu\nu} = [\delta_{\text{g.c.t.}}(\xi) + \delta_h(\Lambda_x, \Lambda_x^M) + \delta_{\text{susy}}(\kappa)] B_{x\mu\nu}, \quad (3.16)$$

if we impose the duality (or on-shell) condition

$$\tilde{H}_x = 0, \quad (3.17)$$

where \tilde{H}_x is the 3-form supercovariant field strength

$$\tilde{H}_{x\mu\nu\rho} = H_{x\mu\nu\rho} - \frac{3i}{4} \sigma^x_I{}^J \bar{\psi}^I_{[\mu} \gamma_{\nu]} \psi_{J|\rho].} \quad (3.18)$$

The parameters ξ, Λ_x, κ are the spinor bilinears defined before (as they must) and Λ_x^M is given by

$$\Phi^x_{\mu\nu} \equiv \sigma^x_K{}^J \varepsilon_{IK} \bar{\epsilon}^I \gamma_{\mu\nu} \eta^J, \quad (3.19)$$

$$\Lambda_x^M{}_{\mu\nu} \equiv +\frac{1}{4} \Re(\mathcal{V}^M \Phi^x_{\mu\nu}) - a^M B_{x\mu\nu} - \xi^\rho C_x^M{}_{\rho\mu\nu}. \quad (3.20)$$

Observe that the three 2-forms $\Phi^x_{\mu\nu}$ are anti-selfdual.

Thus, we find complete agreement with the tensor hierarchy prediction and consistency with the supersymmetry transformations proposed for A^M . Furthermore, the on-shell condition $\tilde{H}_x = 0$ can be rewritten in the form

$$H_x = \star j_{N_x}, \quad (3.21)$$

where

$$j_{N_x}{}^\mu \equiv \frac{i}{8} \epsilon^{\mu\alpha\beta\gamma} \sigma^x_I{}^J \bar{\psi}^I_\alpha \gamma_{\beta\gamma} \psi_{J\gamma}, \quad (3.22)$$

is the triplet of Noether currents associated to the global $SU(2)$ invariance, in agreement with the general arguments of Ref. [6]. At lowest order in fermion fields these Noether currents vanish and instead of a duality condition the on-shell condition H_x resembles a gauge-triviality condition for the 2-forms.

For $\mathbf{x} = 0$ we get, to lowest order in fermions

$$[\delta_\eta, \delta_\epsilon] B_{0\mu\nu} = 2\partial_{[\mu}(-2\beta\xi_{\nu]}) + 8\beta\Im[X\mathcal{V}_M F^{M+}{}_{\mu\nu}] + \beta\alpha_M \vartheta^x \Im[\mathcal{V}^M \Phi^x_{\mu\nu}]. \quad (3.23)$$

Now, using the identities¹³

¹³All these identities can be derived from the constraint $\mathcal{V}^* M \mathcal{V}_M = -i$ and Eq. (A.5).

$$\Im(\mathcal{V}^M X) = -2\Re(\mathcal{V}^{*M}\mathcal{V}_N)\Re(\mathcal{V}^N X), \quad (3.24)$$

$$\Im(X\mathcal{V}^M F^{M+}) = 4\mathcal{M}_{MN}\Re(X\mathcal{V}^M)\Im(\mathcal{V}^{*N}\mathcal{V}_M F^{M+}), \quad (3.25)$$

$$\Im(\mathcal{V}^M \Phi^x) = -2\mathcal{M}^M{}_N \Re(\mathcal{V}^N \Phi^x), \quad (3.26)$$

where

$$\mathcal{M}_{MN} \equiv \Re(\mathcal{V}_M^* \mathcal{V}_N) \quad (3.27)$$

and using Eq. (A.6) and the on-shell conditions $\tilde{G}_\Lambda = \tilde{F}_\Lambda$ we can rewrite our previous result, again to lowest order in fermions, in the form

$$\begin{aligned} [\delta_\eta, \delta_\epsilon] B_{0\mu\nu} &= 2\partial_{[\mu}(-2\beta\xi_{\nu]}) - 16\beta\mathcal{M}_{MN}\Re(X\mathcal{V}^M)F^N{}_{\mu\nu} \\ &\quad + 2\beta\alpha^M \mathcal{M}_{MN}\vartheta^x \Re(\mathcal{V}^N \Phi^x{}_{\mu\nu}), \end{aligned} \quad (3.28)$$

which involves quantities that have appeared in other commutators.

The last term in the r.h.s. is a 2-form Stückelberg shift. Its presence confirms that $\vartheta^0 = 0$ because F^M could never be gauge invariant containing $\vartheta^0 B_0$. On the other hand, this modification of the generic tensor hierarchy prediction is possible because the generic case one does not take into account the possible vanishing of components of the embedding tensor.

To make progress we need to identify the field strength H_0 of B_0 , which cannot have the trivial form predicted by the tensor hierarchy. The r.h.s. of the commutator suggests that H_0 must contain a term of the form $\mathcal{M}_{MN}A^M \wedge dA^N$ and a coupling to the 3-forms C_x^M , which, as we will see, are the only 3-forms available. The only non-trivial possibility turns out to be

$$H_0 = dB_0 + \mathcal{M}_{MN} [A^M \wedge dA^N + \alpha^M \vartheta^x C_x^M], \quad (3.29)$$

where B_0 must transform according to

$$\delta B_{0\mu\nu} = 2\partial_{[\mu}\Lambda_{0|\nu]} + \mathcal{M}_{MN} [2A^M{}_{[\mu}\partial_{\nu]}\Lambda^N - \alpha^M \vartheta^x (\Lambda_x{}^N{}_{\mu\nu} + \Lambda^N B_{x\mu\nu} + \Lambda_{x[\mu} A^N{}_{\nu]})]. \quad (3.30)$$

Then, we see that ($\beta = -1/8$) the supersymmetry transformation

$$\delta_\epsilon B_{0\mu\nu} = -\frac{i}{8}\bar{\epsilon}^I \gamma_{[\mu}\psi_{\nu]I} + \text{c.c.} + 2\mathcal{M}_{MN}A^M{}_{[\mu}\delta_\epsilon A^N{}_{\nu]}, \quad (3.31)$$

closes to all order in fermion fields if we impose the duality (on-shell) condition

$$\tilde{H}_0 = 0, \quad (3.32)$$

where

$$\tilde{H}_{0\mu\nu\rho} = H_{0\mu\nu\rho} + \frac{3i}{8}\bar{\psi}^I{}_{[\mu}\gamma_{\nu]}\psi_{I|\rho]} . \quad (3.33)$$

This on-shell condition can also be rewritten as a duality between the bosonic 3-form field strength H_0 and the Noether current 1-form j_{N0} associated to the invariance under the global $U(1)_{\text{Kahler}}$.

The 1-form parameter Λ_0 is given by

$$\Lambda_{0\mu} \equiv -\frac{1}{4}\xi_\mu - b_{0\mu} + \mathcal{M}_{MN} [a^M + 2\Re(X\mathcal{V}^M)] A^N{}_\mu . \quad (3.34)$$

Again, the consistency of these results relies on the existence of the appropriate 3-forms, which we explore next.

3.2 3-forms

The only supersymmetry transformations for 3-forms that lead to closure of the local supersymmetry algebra are

$$\delta_\epsilon C_x{}^M{}_{\mu\nu\rho} = -\frac{3}{8}\mathcal{V}^{*M}\sigma^x{}_I{}^J\varepsilon^{IK}\bar{\epsilon}_K\gamma_{[\mu\nu}\psi_{\rho]J} + \text{c.c.} - 3\delta_\epsilon A^M{}_{[\mu}B_{x|\nu\rho]} . \quad (3.35)$$

In particular, it is easy to see that the supersymmetry algebra does not close with

$$\delta_\epsilon C_0{}^M{}_{\mu\nu\rho} = \lambda\mathcal{V}^{*M}\varepsilon^{IJ}\bar{\epsilon}_I\gamma_{[\mu\nu}\psi_{\rho]J} + \text{c.c.} , \quad (3.36)$$

for any values of λ , just as it did not close on the candidate A^x and we conclude that C_0^M cannot be introduced in this theory, which agrees with the fact that a ϑ^0 cannot be introduced, either.

The closure of the local supersymmetry algebra requires the use of the previously found on-shell conditions $\tilde{F}^M = \tilde{G}^M$ and $H_x = 0$ and a of the new condition

$$\tilde{G}_x{}^M = -\frac{3}{4}\star\mathcal{M}^{MN}\alpha_N\vartheta^x , \quad (3.37)$$

where the supercovariant 4-form field strength $\tilde{G}_x{}^M$ is given by

$$\tilde{G}_x{}^M = G_x{}^M - \frac{3}{4}[\mathcal{V}^M\sigma^x{}_I{}^K\varepsilon_{JK}\bar{\psi}^I{}_{[\mu}\gamma_{\nu\rho}\psi^J{}_{\sigma]} + \text{c.c.}] . \quad (3.38)$$

The on-shell condition is the supersymmetrization of the one proposed in Ref. [6]

$$G_x{}^M = \frac{1}{2}\star\frac{\partial V}{\partial\vartheta_M^x} , \quad (3.39)$$

for a manifestly symplectic-invariant (constant) potential V given by

$$V = -\frac{3}{4}\mathcal{M}^{MN}\alpha_M\alpha_N\vartheta^x\vartheta^x , \quad (3.40)$$

which generalizes the standard one.

Finally, the 3-form gauge parameter is given by the bilinear

$$\Lambda_{x\mu\nu\rho} = +\frac{3}{16}(\star\xi^x)_{\mu\nu\rho} + \frac{3}{8}\varepsilon_{xyz}B_{y[\mu\nu}b_{z|\rho]} + \frac{1}{2}a_P C_x{}^P{}_{\mu\nu\rho} - d_{x\mu\nu\rho} , \quad d_{x\mu\nu\rho} \equiv \xi^\sigma D_{x\sigma\mu\nu\rho} . \quad (3.41)$$

3.3 4-forms

There are three candidates to supersymmetry transformation rules of 4-forms¹⁴:

$$\delta_\epsilon D'_x{}_{\mu\nu\rho\sigma} = \sigma^x{}_I{}^J \bar{\epsilon}^I \gamma_{[\mu\nu\rho}\psi_{\sigma]} J + \text{c.c.}, \quad (3.42)$$

$$\delta_\epsilon D''_x{}_{\mu\nu\rho\sigma} = C_{xM}{}_{\mu\nu\rho} \delta_\epsilon A^M{}_{|\sigma]}, \quad (3.43)$$

$$\delta_\epsilon D'''_x{}_{\mu\nu\rho\sigma} = \varepsilon_{xyz} B_{y[\mu\nu]} \delta_\epsilon B_{z|\rho\sigma]}. \quad (3.44)$$

Let us first consider the ungauged case, for simplicity. In this case, the 4-forms decouple from the rest of the hierarchy and the gauge transformations may differ from those derived in the gauged case. The commutator of two supersymmetries closes for all three candidates at the lowest order on fermions, which would contradict the prediction made in Ref. [2] in the framework of the KM approach that there are only two triplets of 4-forms. Thus, we are lead to study the quartic terms in fermions in the r.h.s. of the commutators, as in Ref. [27]. These terms, which do not correspond to any of the gauge parameters found for the lower-rank forms, do not vanish for any of the three candidates and we can do two different things about it:

1. We can define as many 4-form gauge parameters as quartic terms we find. There are four different quartic terms and they appear in the commutator of the three candidates, but always in a fixed combination so there is, actually, only one independent 4-form gauge parameter. This gauge parameter may be used to gauge away one of the three candidates, which would leave us with the two independent triplets predicted by the KM approach.
2. We can construct linear combinations

$$aD'_x + bD''_x + cD'''_x, \quad (3.45)$$

of the three (triplets) of candidate 4-forms and choose the coefficients so that the quartic terms vanish. Since they always appear in the same combination we get only one constraint for the coefficients a, b, c

$$6a + \frac{3}{8}b + \frac{1}{2}c = 0, \quad (3.46)$$

which leaves us with two independent combinations on which the supersymmetry algebra, again in perfect agreement with the KM approach prediction.

Let us now consider the gauged case. The r.h.s. of the commutator contains new terms quadratic in fermions but the same terms quartic in fermions, so the above discussion still applies. All the quadratic terms but two correspond to gauge parameters already defined. The two

¹⁴The transformation $\bar{\epsilon}^I \gamma_{[\mu\nu\rho}\psi_{\sigma]} I + \text{c.c.}$ is the transformation of the the volume 4-form.

terms that do not are the total derivative of a 3-form and a 4-form shift. If the latter does not cancel, we can gauge away one more triplet, so only one would remain. To cancel it, we must have

$$c = \frac{3}{8}b, \quad (3.47)$$

which also leaves us with only one possible triplet of 4-forms, up to overall normalization. The overall normalization is fixed by the requirement that the remaining combination coincides with the triplet of 4-forms predicted by the tensor hierarchy $a = -3/16$.

Summarizing: in the gauged case new Stückelberg symmetries appear which leave use with only one triplet which is the one predicted by the tensor hierarchy and transforms under supersymmetry according to

$$\delta_\epsilon D_x{}_{\mu\nu\rho\sigma} = -\frac{3}{16}\sigma^x{}_I{}^J\bar{\epsilon}^I\gamma_{[\mu\nu\rho}\psi_{\sigma]J} + \text{c.c.} + 2C_x{}^M{}_{[\mu\nu\rho]}\delta_\epsilon A^M{}_{|\sigma]} + \frac{3}{4}\varepsilon_{xyz}B_y{}_{[\mu\nu]}\delta_\epsilon B_z{}_{|\rho\sigma]}. \quad (3.48)$$

The supersymmetry algebra closes with the gauge transformations Eq. (2.17) plus the Stückelberg shift

$$\tilde{\Lambda}_{\mu\nu\rho\sigma} = -\frac{9}{4}\alpha^M\mathcal{M}_{MN}\Re(X\mathcal{V}^N)\varepsilon_{\mu\nu\rho\sigma} + \alpha_M b_y{}_{[\mu}[C_y{}^M + 3A^M B_y]{}_{|\nu\rho\sigma]}, \quad (3.49)$$

which could be used to eliminate one component of the triplet, at the expense of breaking explicit $SU(2)$ -invariance.

We conclude that the tensor hierarchy of pure, gauged, $N = 2, d = 4$ supergravity contains two 1-forms which are $SU(2)$ singlets, A^M , four 2-forms (a triplet and a singlet) B_x and B_0 , six 3-forms (two $SU(2)$ triplets) $C_x{}^M$ and one triplet of 4-forms D_x (two in the ungauged case). It seems that the predictions of the KM approach have to be modified after gauging.

4 Pure $N = 2, d = 5$ supergravity

In this section we are going to study the case of pure minimal supergravity in 5 dimensions [21]. The possible $(p + 1)$ forms that can be coupled to the ungauged theory consistently with supersymmetry have been studied in Refs. [1] and [2]. Here we are going to revise those results taking into account the predictions of the generic 5-dimensional tensor hierarchy constructed in [9] using as global symmetry group the R-symmetry group $SU(2)$ (gauge invariance is clearly a pre-condition for supersymmetry invariance).

The supergravity multiplet of the $N = 2, d = 5$ theory consists of the graviton $e^a{}_\mu$, a symplectic-Majorana gravitino $\psi_{I\mu}$, ($I = 1, 2$) and one graviphoton A_μ .

The global symmetry group of this theory (or its equations of motion) reduces to the $SU(2)$ R-symmetry group and so the embedding tensor is ϑ^x , $x = 1, 2, 3$. In the standard formulations of the theory it appears as a Fayet-Iliopoulos term that selects the $U(1)$ subgroup of $SU(2)$ which is going to be gauged by the graviphoton. The formulation of the gauged theory in terms of the

fundamental fields is, therefore, well known¹⁵: the bosonic action of the fundamental fields is given by¹⁶

$$S = \int \left[\star R + \frac{1}{2} c^{2/3} F \wedge \star F + \frac{1}{3\sqrt{3}} c F \wedge F \wedge A - \star V \right], \quad (4.1)$$

where

$$F = dA, \quad V = -4c^{-2/3} \vartheta^x \vartheta^x, \quad (4.2)$$

and where the supersymmetry transformations of the fundamental fields to all order in fermions are

$$\delta_\epsilon e^a{}_\mu = \frac{i}{2} \bar{\epsilon}_i \gamma^a \psi_\mu^i, \quad (4.3)$$

$$\delta_\epsilon A_\mu = -\frac{i\sqrt{3}}{2} c^{-1/3} \bar{\epsilon}_i \psi_\mu^i, \quad (4.4)$$

$$\delta_\epsilon \psi_\mu^i = \tilde{\mathcal{D}}_\mu \epsilon^i - \frac{1}{8\sqrt{3}} c^{1/3} \tilde{F}^{\alpha\beta} (\gamma_{\mu\alpha\beta} - 4g_{\mu\alpha} \gamma_\beta) \epsilon^i + \frac{i}{2\sqrt{3}} c^{-1/3} \vartheta^x \sigma^{xi}{}_j \gamma_\mu \epsilon^j. \quad (4.5)$$

The covariant derivative is given by

$$\tilde{\mathcal{D}}_\mu \epsilon^i \equiv \tilde{\nabla}_\mu \epsilon^i + \frac{i}{2} A_\mu \vartheta^x \sigma^{xi}{}_j \epsilon^j, \quad (4.6)$$

where $\tilde{\nabla}_\mu$ is the Lorentz-covariant derivative with the torsionful connection $\tilde{\omega}_\mu{}^{ab}$ defined in Eq. (1.10) where the torsion is now given by

$$T_{\mu\nu}{}^a \equiv -\frac{i}{2} \bar{\psi}_{i[\mu} \gamma^a \psi_{\nu]}^i. \quad (4.7)$$

$\tilde{F}_{\mu\nu}$ is the supercovariant 2-form field strength:

$$\tilde{F}_{\mu\nu} \equiv F_{\mu\nu} + \frac{i\sqrt{3}}{2} c^{-1/3} \bar{\psi}_{i\mu} \psi_{\nu}^i. \quad (4.8)$$

Following [9] one finds that the tensor hierarchy contains, in addition to the graviphoton A , its dual 2-form B , a triplet of 3-forms C_x a triplet of 4-forms D_x , and, possibly, one 5-form E which may be consistently eliminated from the hierarchy if the 4-form gauge parameter $\Lambda^{(4)}$ vanishes:

¹⁵See [22], whose conventions we follow, and references therein.

¹⁶The constant c stands for the unique components of the totally-symmetric tensor C_{IJK} .

$$\delta_h A = -d\Lambda^{(0)}, \quad (4.9)$$

$$\delta_h B = d\Lambda^{(1)} + \frac{1}{\sqrt{3}}c\Lambda^{(0)}F - \vartheta^x \Lambda^{(2)}_x, \quad (4.10)$$

$$\delta_h C_x = \mathfrak{D}\Lambda^{(2)}_x + \varepsilon_{xyz}\vartheta^y(\Lambda^{(0)}C_z - \Lambda^{(3)}_z), \quad (4.11)$$

$$\delta_h D_x = \mathfrak{D}\Lambda^{(3)}_x - F \wedge \Lambda^{(2)}_x + \varepsilon_{xyz}\vartheta^y\Lambda^{(0)}D_z - \vartheta^x \Lambda^{(4)}, \quad (4.12)$$

$$\delta_h E = d\Lambda^{(4)}. \quad (4.13)$$

The corresponding gauge-covariant field strengths are

$$F = dA, \quad (4.14)$$

$$H = dB + \frac{1}{\sqrt{3}}cA \wedge F + \vartheta^x C_x, \quad (4.15)$$

$$G_x = \mathfrak{D}C_x + \varepsilon_{xyz}\vartheta^y D_z, \quad (4.16)$$

$$K_x = \mathfrak{D}D_x + F \wedge C_x + \vartheta^x E, \quad (4.17)$$

$$L = dE. \quad (4.18)$$

In Ref. [1] it was shown that there is no independent 5-form E that can be introduced in the supersymmetric theory (consistently with their find that $\Lambda^{(4)} = 0$) while in Ref. [2] it was shown that the ungauged supersymmetric theory may admit an independent triplet of 5-forms which should be decoupled from the rest of the tensor hierarchy even in the gauged case, according to the above results. In order to clarify these points we are going to construct the supersymmetric tensor hierarchy of the gauged theory. Closing the supersymmetry algebra on the different fields of the tensor hierarchy we will find the values of the gauge parameters and we will be able to determine the necessity or impossibility of adding 5-forms to it.

The supersymmetry algebra closes on the graviphoton giving

$$[\delta_\eta, \delta_\epsilon] A_\mu = \delta_h A_\mu + \delta_\xi A_\mu + \delta_\kappa A_\mu, \quad (4.19)$$

where $\delta_h A_\mu$ is the above gauge transformation for A_μ with gauge parameter

$$\Lambda^{(0)} = \lambda^{(0)} + \xi^\mu A_\mu, \quad \lambda^{(0)} \equiv \frac{\sqrt{3}i}{2}c^{-1/3}\bar{\epsilon}_i \eta^i, \quad (4.20)$$

and δ_ξ is a general coordinate transformation (\mathcal{L}_ξ) with parameter

$$\xi_\mu = \frac{i}{2} \bar{\epsilon}_i \gamma_\mu \eta^i, \quad (4.21)$$

and $\delta_\kappa A_\mu$ is a supersymmetry transformation with parameter

$$\kappa^i = -\xi^\mu \psi_\mu^i. \quad (4.22)$$

For the 2-form we find that the supersymmetry algebra closes for the supersymmetry transformation

$$\delta_\epsilon B_{\mu\nu} = -i\sqrt{3}c^{1/3} \bar{\epsilon}_i \gamma_{[\mu} \psi_{\nu]}^i + \frac{2}{\sqrt{3}} c A_{[\mu} \delta_\epsilon A_{\nu]}, \quad (4.23)$$

with the parameters of the tensor hierarchy gauge transformations being given by

$$\Lambda^{(1)}{}_\mu = \sqrt{3}c^{1/3} \xi_\mu - \xi^\rho B_{\rho\mu} + \frac{1}{\sqrt{3}} c \lambda^{(0)} A_\mu, \quad (4.24)$$

$$\Lambda^{(2)}{}_{x\mu\nu} = \lambda^{(2)}{}_{x\mu\nu} - \xi^\rho C_{x\rho\mu\nu}, \quad \lambda^{(2)}{}_{x\mu\nu} \equiv \frac{1}{2} \sigma^{xk}{}_j \varepsilon_{ik} \bar{\epsilon}^i \gamma_{\mu\nu} \eta^j, \quad (4.25)$$

if the supercovariant field strength 3-form is given by

$$\tilde{H}_{\mu\nu\rho} \equiv H_{\mu\nu\rho} + \frac{i3\sqrt{3}}{2} c^{1/3} \bar{\psi}_i \gamma_{[\mu} \psi_{\nu]}^i, \quad (4.26)$$

(H being as predicted by the tensor hierarchy) the duality relation

$$\tilde{H} = c^{2/3} \star \tilde{F}, \quad (4.27)$$

is satisfied.

The supersymmetry algebra also closes in the 3-form with supersymmetry transformations

$$\delta_\epsilon C_{x\mu\nu\rho} = -\frac{3i}{2} \sigma^x{}_i{}^k \varepsilon_{jk} \bar{\epsilon}^i \gamma_{[\mu\nu} \psi_{\rho]}^j, \quad (4.28)$$

with the parameters of the gauge transformations predicted by the tensor hierarchy being given by

$$\Lambda^{(3)}{}_{x\mu\nu\rho} = \lambda^{(3)}{}_{x\mu\nu\rho} - \xi^\sigma D_{x\sigma\mu\nu\rho} + \lambda^{(0)} C_{x\mu\nu\rho}, \quad \lambda^{(3)}{}_{x\mu\nu\rho} \equiv \frac{\sqrt{3}}{2} c^{-1/3} \sigma^x{}_j{}^k \varepsilon_{ik} \bar{\epsilon}^i \gamma_{\mu\nu\rho} \eta^j, \quad (4.29)$$

if the on-shell condition

$$\tilde{G}_{x\mu\nu\rho\sigma} \equiv G_{x\mu\nu\rho\sigma} + 3i \sigma^x{}_i{}^k \varepsilon_{jk} \bar{\psi}^i{}_{[\mu} \gamma_{\nu\rho} \psi_{\sigma]}^j = 0, \quad (4.30)$$

is satisfied (with the same interpretation as the on-shell condition $\tilde{H}_x = 0$ in the 4-dimensional case), and on the 4-forms D_x with

$$\delta_\epsilon D_{x\mu\nu\rho\sigma} = 2\sqrt{3}c^{-1/3} \sigma^x{}_i{}^k \varepsilon_{jk} \bar{\epsilon}^i \gamma_{[\mu\nu\rho} \psi_{\sigma]}^j + 4C_{x[\mu\nu\rho} \delta_\epsilon A_{\sigma]}, \quad (4.31)$$

with

$$\Lambda^{(4)}{}_{\mu\nu\rho\sigma} = -\xi^\delta E_{\delta\mu\nu\rho\sigma}, \quad (4.32)$$

if the (on-shell) duality relation

$$\star \tilde{K}_x = -4c^{-2/3} \vartheta^x = \frac{1}{2} \frac{\partial V}{\partial \vartheta^x}, \quad (4.33)$$

is also satisfied, with

$$\tilde{K}_{x\mu_1\cdots\mu_5} \equiv K_{x\mu_1\cdots\mu_5} - 5\sqrt{3}c^{-1/3} \sigma^x{}_i{}^k \epsilon_{jk} \bar{\psi}^i{}_{[\mu} \gamma_{\nu\rho\sigma} \psi^j{}_{\lambda]}. \quad (4.34)$$

The 4-form shift $\Lambda^{(4)}{}_x$ appears only because we have included a 5-form E in K_x . However, as discussed in Ref. [1], a non-trivial 5-form singlet cannot be introduced in the theory. Therefore, The supersymmetric tensor hierarchy seems to stop at the 4-form level and there is no 5-form singlet E . However, in Ref. [2] it was shown that, in the ungauged case, as predicted by the KM approach, the supersymmetry algebra closes for a triplet of 5-forms with supersymmetry transformations of the form

$$\delta_\epsilon E_{x\mu_1\cdots\mu_5} = B_{[\mu_1\mu_2} \delta_\epsilon C_{|x|\mu_3\mu_4\mu_5]} + \frac{1}{2\sqrt{3}} c A_{[\mu_1} \delta_\epsilon D_{|x|\mu_2\mu_3\mu_4\mu_5]}, \quad (4.35)$$

up to an overall normalization constant which cannot be fixed because this triplet does not couple to the rest of the fields of the tensor hierarchy. We have checked the closure of the supersymmetry algebra to all orders in fermions for this triplet in the ungauged case but in the gauged case the closure takes place only up to 5-form Stückelberg shifts proportional to $\vartheta^x \Lambda^{(5)}$ and $\varepsilon_{xyz} \vartheta^y \Lambda^{(5)}{}_z$ which, on the one hand, cannot be compensated by those of any other 5-forms and, on the other hand, can be used to gauge away the full E_x , recovering the spectrum of higher-rank forms predicted by the tensor hierarchy, as in the 4-dimensional case.

5 Pure $N = (2, 0)$, $d = 6$ supergravity

Let us now consider pure $N = (2, 0)$, $d = 6$ supergravity [23]. We shall be extremely brief. The supergravity multiplet consists of the graviton $e^a{}_\mu$, a positive-chirality symplectic-Majorana-Weyl gravitino $\psi_{\mu I}$ and a 2-form $B_{\mu\nu}$ with field strength $H = dB$ and with supercovariant field strength that is constrained to be self-dual (i.e. $\tilde{H}^- = 0$).

The bosonic equations of motion, which cannot be derived from a Lorentz-covariant Lagrangian are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} = 0, \quad (5.1)$$

$$H^- = 0. \quad (5.2)$$

The supersymmetry transformations for the supergravity fields are:

$$\delta_\epsilon e^a{}_\mu = \bar{\psi}^I{}_\mu \gamma^a \epsilon_I, \quad (5.3)$$

$$\delta_\epsilon \psi_{I\mu} = \tilde{\nabla}_\mu \epsilon_I + \frac{1}{4} \tilde{H}_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon_I, \quad (5.4)$$

$$\delta_\epsilon B_{\mu\nu} = \tilde{e}^I{}_{[\mu} \gamma_{\nu]I}. \quad (5.5)$$

The global symmetry group of this theory reduces to the $SU(2)$ R-symmetry group but there are no 1-forms available to gauge them and the embedding tensor vanishes. The 6-dimensional tensor hierarchy allows for deformations which are not gaugings and depend on deformation parameters which do not depend on the embedding tensor¹⁷ [7]. However, the absence of 1-forms implies the absence of 3-forms in the tensor hierarchy and the vanishing of any other deformation parameters and non-trivial constraints between them. This implies, according to the general arguments of Ref. [7] the absence of 5- and 6-forms in the tensor hierarchy. The only higher-rank forms allowed would be a triplet of 4-forms D_x related to the Noether currents of the $SU(2)$ R-symmetry group which are bilinear in gravitini. The tensor hierarchy would, then, consist of $\{B_{\mu\nu}, D_{x\mu\nu\rho\sigma}\}$ with trivial gauge transformations and field strengths

$$H \equiv dB, \quad K_x \equiv dD_x. \quad (5.6)$$

It is not difficult to see that supersymmetry confirms these results: it is not possible to construct consistent supersymmetry transformations for any other kind of higher-rank fields (except for the one corresponding to the volume 6-form).

In the ungauged, though, it is possible to construct consistent supersymmetry transformation for a triplet of 6-forms F_x :

$$\delta_\epsilon F_x = B_{[\mu_1\mu_2]} \delta_\epsilon D_{x|\mu_3\cdots\mu_6]}, \quad (5.7)$$

in agreement with the predictions of the KM approach [2].

6 Extended objects and their effective actions

Our goal in this section is to discuss the consistency of our previous results: we will study the relations between the fields of the $d = 4, 5, 6$ tensor hierarchies through dimensional reduction and the possible existence of supersymmetric branes charged with respect to them, constructing the bosonic parts of some of their worldvolume actions.

The 3 pure supergravity theories with 8 supercharges that we have considered are related by dimensional reduction and truncation of the extra vector multiplet that appears in each reduction¹⁸. Then, their higher-rank potentials should also be related by dimensional reduction if we

¹⁷Supersymmetry constraints may modify this last statement.

¹⁸See, e.g. Ref. [24].

ignore the additional 1-forms of the vector supermultiplets, that we truncate. That this is the case can immediately be seen in Figure 1. Observe that, as usual, electric-magnetic pairs in 4 dimensions are associated to rotations in 6 dimensions. Observe also that one additional form in any dimension would imply the existence of related forms in the other two which leads us to conclude that indeed we have obtained all the possible higher-rank forms.

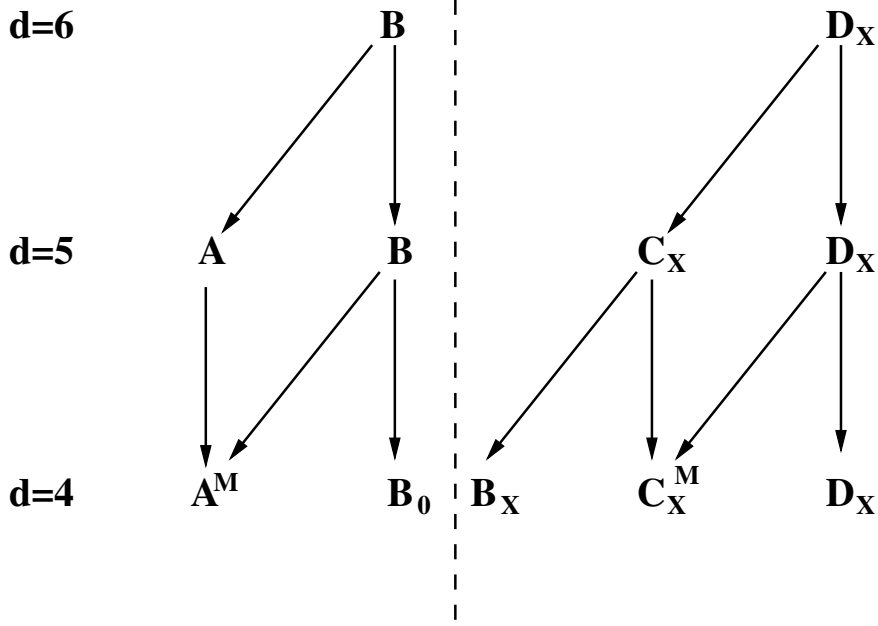


Figure 1: Relations between the fields of the tensor hierarchies of the $d = 4, 5, 6$ pure supergravities with 8 supercharges. The fields that descend from the 6-dimensional 4-forms D_x , to the right of the dashed line, are associated to worldvolume theories with 2 bosonic and 2 fermionic (8 divided by 2 (κ -symmetry) and by 2 (e.o.m.)) degrees of freedom. The fields that descend from the 6-dimensional 2-form B with selfdual 3-form field strength are associated to worldvolume theories with 1 bosonic (right- or left movers for $p = 1$) and 1 fermionic degrees of freedom.

If there are supersymmetric extended objects charged with respect to these forms, then they must also be related by simple and double dimensional reductions. On the other hand, all the dynamical p -branes of these theories must be charged with respect to the $(p + 1)$ -form potentials that we have found, since these are the only potentials of the theory transforming into the gravitini under supersymmetry¹⁹. It should, then, be possible to construct a κ -symmetric action for each of them with a Wess-Zumino term lead by the pullback corresponding $(p + 1)$ -form potential.

A necessary condition for the κ -symmetric actions to exist is Bose-Fermi matching of worldvolume degrees of freedom [25, 26]. The worldvolume theory of the 6-dimensional 3-brane (whose worldvolume has the simplest standard form) does not need any additional worldvolume bosonic fields: the 2 bosonic degrees of freedom associated to the transverse coordinates exactly

¹⁹We expect this fact to remain true even after coupling to matter.

match the 2 fermionic degrees of freedom that result from dividing the 8 of the minimal spinor by 2×2 (κ -symmetry and fermionic equations of motion). Reducing to $d = 5$ we get 2-branes and 3-branes. The theory of the latter contains an additional worldvolume scalar (or, equivalently, a worldvolume 2-form).

Simple dimensional reduction of the 5-dimensional 3-brane yields the worldvolume theory of another 3-brane (now spacetime-filling) with two non-geometrical scalars (or 2-forms). Double dimensional reduction gives a domain wall with one additional scalar which can be dualized into a vector which we expect to be of Born-Infeld type. The simple dimensional reduction of the 5-dimensional 2-brane gives essentially the same result (up to electric-magnetic rotation) while the double gives a string with no additional degrees of freedom coupling to B_x . All the theories obtained from the 6-dimensional 3-brane have the same number of bosonic and fermionic degrees of freedom.

Things are different if we start from the 6-dimensional (self-dual) string. Bose-Fermi matching can only be achieved between the left- or right-moving transverse scalars (which are 4) and the 2 fermionic degrees of freedom. This characteristic is inherited by the 5- and 4-dimensional strings which one obtains by simple dimensional reduction and which must contain additional, non-geometric, worldvolume scalars.

As we have seen, in each of the ungauged $d = 4, 5, 6$ cases there is one additional triplet of top forms. These triplets are obviously related by dimensional reduction. None of them seems to couple to supersymmetric spacetime-filling branes, at least within the framework of a conventional worldvolume action.

7 Conclusions

In this paper we have taken a step towards the democratic formulation of $d = 4, 5, 6$ supergravity theories with 8 supercharges (often called $N = 2$ theories), finding all the potentials that transform into the gravitini under supersymmetry. In the 4-dimensional case, our results complement those obtained by de Vroome and de Wit in Ref. [10] and in the 5-dimensional case those of Kleinschmidt and Roest in Ref. [2].

We have seen that the predictions of the bosonic tensor hierarchies are essentially satisfied in the supersymmetric case. In the 4-dimensional case we find some differences due to the impossibility of gauging the $U(1)$ factor of the R-symmetry group, which leads to an additional constraint of the embedding tensor. However, we have also found that in the ungauged case there are more fields (only top forms) than predicted by this approach. This is, on the other hand, in complete agreement with the predictions of the KM approach. In the 4- and 5-dimensional cases we have shown that, after gauging, the new top forms can either be completely gauged away or must be combined with other fields due to the appearance of new Stückelberg shifts that depend on the embedding tensor. To determine completely the number of independent top forms it has been crucial to study the closure of the supersymmetry algebra to all orders in fermion fields, as in Ref. [27].

The determination and study of the top form fields of these theories was precisely one of our goals. We have determined them but their physical meaning is, though, not completely clear: on

general grounds the top forms should be associated to the constraints imposed on the embedding tensor. However, in the 4-dimensional case we have solved all those constraints and a Lagrange-multiplier D_x is not really needed. Furthermore, we find one extra triplet in each dimension that does not fit into the tensor hierarchy.

On the other hand, the 4-dimensional top forms in the tensor hierarchy seem to be associated to the possible supersymmetric truncations from $N = 2$ to $N = 1$ in $d = 4$. Supersymmetric truncations are not possible in $d = 5, 6$, in agreement with the absence of top forms in the tensor hierarchy in those dimensions. The additional triplets of top forms do not seem to be related to truncations and their existence, albeit predicted by the KM approach, remains mysterious.

It would be interesting to construct explicitly the worldvolume actions that contain the potentials we have found and check their relations via dimensional reduction. This would shed more light on the relations between the corresponding supersymmetric objects in different dimensions and also on the possible intersections between supersymmetric objects of the same theory. It would also be very interesting to construct the complete democratic formulations of the $N = 2, d = 4, 5, 6$ supergravities with matter couplings with the help of the tensor hierarchies. This is, at any rate a necessary step to find the most general $N = 2$ supergravity theories. However, it would also give us a deeper understanding of these theories and very useful tools to work with them.

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A Some formulae

Using Eq. (1.14) and the normalization of the canonical symplectic section Eq. (1.11) (taking into account that in this case \mathcal{L}^Λ and \mathcal{M}_Λ stand for a single number), we find that

$$\Im(\mathcal{V}^{*M}\mathcal{V}^N) = -\frac{1}{2}\Omega^{MN}, \quad (\text{A.1})$$

$$\Re(\mathcal{V}^{*M}\mathcal{V}^N) = -\frac{1}{2} \begin{pmatrix} \Im\mathcal{N}^{\Lambda\Sigma} & \Im\mathcal{N}^{\Lambda\Omega}\Re\mathcal{N}_{\Omega\Sigma} \\ \Re\mathcal{N}_{\Lambda\Omega}\Im\mathcal{N}^{\Omega\Sigma} & \Im\mathcal{N}_{\Lambda\Sigma} + \Re\mathcal{N}_{\Lambda\Omega}\Im\mathcal{N}^{\Omega\Gamma}\Re\mathcal{N}_{\Gamma\Sigma} \end{pmatrix}, (\text{A.2})$$

and

$$\Im(\mathcal{V}^{*M}\mathcal{V}_N) = -\frac{1}{2}\delta^M_N, \quad (\text{A.3})$$

$$\Re(\mathcal{V}^{*M}\mathcal{V}_N) = -\frac{1}{2} \begin{pmatrix} \Im\mathcal{N}^{\Lambda\Omega}\Re\mathcal{N}_{\Omega\Sigma} & -\Im\mathcal{N}^{\Lambda\Sigma} \\ \Im\mathcal{N}_{\Lambda\Sigma} + \Re\mathcal{N}_{\Lambda\Omega}\Im\mathcal{N}^{\Omega\Gamma}\Re\mathcal{N}_{\Gamma\Sigma} & -\Re\mathcal{N}_{\Lambda\Omega}\Im\mathcal{N}^{\Omega\Sigma} \end{pmatrix} \quad (\text{A.4})$$

so, in particular,

$$\Re(\mathcal{V}^{*M}\mathcal{V}_N)\Re(\mathcal{V}^{*N}\mathcal{V}_P) = -\frac{1}{4}\delta^M_P, \quad (\text{A.5})$$

and

$$\Im(\mathcal{V}^{*M}\mathcal{V}_N F^{N+}) = -\frac{1}{2}F^M - \frac{1}{4} \begin{pmatrix} \Im\mathcal{N}^{\Lambda\Omega} \star (G_\Omega - F_\Omega) \\ (G_\Lambda - F_\Lambda) + \Re\mathcal{N}_{\Lambda\Sigma}\Im\mathcal{N}^{\Sigma\Omega}(G_\Omega - F_\Omega) \end{pmatrix}. \quad (\text{A.6})$$

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