

# Vector form factor of the pion : A model-independent approach \*

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We study a model-independent parameterization of the vector pion form factor that arises from the constraints of analyticity and unitarity. Our description should be suitable up to  $\sqrt{s} \simeq 1.2 \text{ GeV}$  and allows a model-independent determination of the mass of the  $\rho(770)$  resonance. We analyse the experimental data on  $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$  and  $e^+ e^- \rightarrow \pi^+ \pi^-$  in this framework, and its consequences on the low-energy observables worked out by chiral perturbation theory. An evaluation of the two pion contribution to the anomalous magnetic moment of the muon,  $a_\mu$ , and to the fine structure constant,  $\alpha(M_Z^2)$ , is also performed.

## 1. Introduction

Matrix elements of QCD hadron currents in exclusive processes provide, from a phenomenological point of view, a detailed knowledge on the hadronization mechanisms. Their evaluation, however, is a long-standing problem due to the fact that it involves strong interactions in an energy region driven by non-perturbative QCD. Within this framework semileptonic processes, as exclusive hadronic  $\tau$  decays ( $\tau^- \rightarrow H^- \nu_\tau$ ) or hadronic cross sections out of electron-positron annihilation ( $e^+ e^- \rightarrow H^0$ ), furnish an excellent dynamical system to explore. In the Standard Model their amplitudes are generically given by

$$M = C \mathcal{L}^\mu \mathcal{H}_\mu, \quad (1)$$

where  $C$  is a factor containing the relevant couplings,  $\mathcal{L}_\mu$  is the leptonic matrix element, easily calculable within the theory, and

$$\mathcal{H}_\mu = \langle H | J_\mu e^{iL_{strong}} | 0 \rangle, \quad (2)$$

with  $J_\mu$  the vector  $V_\mu$  ( $e^+ e^- \rightarrow H^0$ ) or left  $V_\mu - A_\mu$  ( $\tau^- \rightarrow H^- \nu_\tau$ ) hadron current. Symmetries help us to define a decomposition of  $\mathcal{H}_\mu$  in terms of the allowed Lorentz structure of implied momenta and a set of functions of Lorentz

invariants, the *form factors*  $F_i^H$ ,

$$\mathcal{H}_\mu = \sum_i \underbrace{(\dots)_\mu^i}_{\text{Lorentz struc.}} F_i^H(q^2, \dots). \quad (3)$$

Form factors are the goal of the hadronic matrix elements evaluation and, as can be noticed from the definition of  $\mathcal{H}_\mu$  in Eq. (2), are a strong interaction related problem in a non-perturbative regime.

In the last years experiments like ALEPH, CLEO-II, DELPHI, OPAL and CMD-2 [1,2,3,4] have provided an important amount and quality of experimental data on exclusive channels which phenomenological analysis is now mandatory. However most of these analyses are carried out within modelizations (including simplifying assumptions which may be not well controlled from QCD itself [5]) that, while of importance to get an understanding of the involved dynamics, could give a delusive interpretation of data.

The use of effective actions from QCD supplies a powerful model-independent procedure to work with. At very low energies [ $E \ll M_\rho$ , with  $M_\rho$  the mass of the  $\rho(770)$  resonance] the most important QCD feature is its chiral symmetry that is realized in chiral perturbation theory ( $\chi$ PT) [6] with a long and successful set of predictions both in strong and electroweak processes [7]. At higher energies [ $E \sim M_\rho$ ] resonance chiral theory is the analogous effective theory [8], where the lightest resonance fields are kept as explicit degrees of

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freedom. With the addition of dynamical constraints coming from short-distance QCD, resonance chiral theory becomes a predictive model-independent approach. This framework can be combined with S-matrix theory properties. On general grounds local causality of the interaction translates into the analyticity properties of amplitudes and, correspondingly, of form factors. Being analytic functions in complex variables the behaviour of form factors at different energy scales is related and, moreover, they are completely determined by their singularities. Dispersion relations embody rigorously these properties and are the appropriate tool to enforce them.

In this note we recall our work [9] on the vector form factor of the pion in the model-independent approach we have just sketched. We perform a numerical analysis of the recent  $e^+e^-$  CMD-2 data [1] and we reanalyse the  $\tau$  ALEPH data [2] when corrected by isospin breaking effects [10]. The output of these analyses is a determination of the  $\rho(770)^{\pm,0}$  masses, the low-energy parameters of the vector pion form factor, data on the  $\omega(782)$  resonance, particularly the  $\rho - \omega$  mixing, and a new evaluation of the two-pion contribution to the anomalous magnetic moment of the muon  $a_\mu$  and to  $\Delta\alpha(M_Z^2)$ .

## 2. Vector form factor of the pion

The pion vector form factor,  $F_V(s)$  is defined through

$$\langle \pi^+(p') \pi^-(p) | V_\mu^3 | 0 \rangle = (p - p')_\mu F_V(s), \quad (4)$$

where  $s = q^2 = (p + p')^2$  and  $V_\mu^3$  is the third component of the vector current associated to the  $SU(3)$  flavour symmetry of the QCD Lagrangian. This form factor drives the isovector hadronic part of  $e^+e^- \rightarrow \pi^+\pi^-$  and, in the isospin limit, of  $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ . At very low energies,  $F_V(s)$  has been studied in the  $\chi$ PT framework up to  $\mathcal{O}(p^6)$  [11,12]. A successful study at the  $\rho(770)$  energy scale has been carried out in the resonance chiral theory in Ref. [13].

Analyticity and unitarity properties of  $F_V(s)$  tightly constrain, on general grounds, the structure of the form factor [9,13]. Elastic unitarity and Watson final-state theorem relate the imagi-

nary part of  $F_V(s)$  to the partial wave amplitude  $t_1^1$  for  $\pi\pi$  elastic scattering, with angular momentum and isospin equal to one, as

$$\text{Im} F_V(s + i\varepsilon) = e^{i\delta_1^1} \sin(\delta_1^1) F_V(s)^*, \quad (5)$$

that shows that the phase of  $F_V(s)$  must be  $\delta_1^1$ . Thus analyticity and unitarity properties of the form factor are accomplished by demanding that it should satisfy a  $n$ -subtracted dispersion relation with the Omnès solution [13]

$$F_V(s) = \exp \left\{ \sum_{k=0}^{n-1} \frac{s^k}{k!} \frac{d^k}{ds^k} \ln F_V(s) \Big|_{s=0} + \frac{s^n}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dz}{z^n} \frac{\delta_1^1(z)}{z - s - i\varepsilon} \right\}. \quad (6)$$

This solution is strictly valid only below the inelastic threshold ( $s < 16m_\pi^2$ ), however higher multiplicity intermediate states are suppressed by phase space and ordinary chiral counting. The  $\delta_1^1(s)$  phase-shift, in Eq. (6), is rather well known, experimentally, up to  $E \sim 2$  GeV [14].

With an appropriate number of subtractions we can parameterize  $F_V(s)$  with the subtraction constants appearing in the first term of the exponential in Eq. (6). In Ref. [9] we have used three subtractions :

$$F_V(s) = \exp \left\{ \alpha_1 s + \frac{1}{2} \alpha_2 s^2 + \frac{s^3}{\pi} \int_{4m_\pi^2}^{\Lambda^2} \frac{dz}{z^3} \frac{\delta_1^1(z)}{z - s - i\varepsilon} \right\}, \quad (7)$$

where we have introduced an upper cut in the integration,  $\Lambda$ . This cut-off has to be taken high enough not to spoil the, a priori, infinite interval of integration, but low enough that the integrand is well known in the interval. The two subtraction constants  $\alpha_1$  and  $\alpha_2$  (a third one is fixed by the normalization  $F_V(0) = 1$ ) are related with the squared charge radius of the pion  $\langle r^2 \rangle_\pi^V$  and the quadratic term  $c_V^\pi$  in the low-energy expansion

$$F_V(s) = 1 + \frac{1}{6} \langle r^2 \rangle_\pi^V s + c_V^\pi s^2 + \mathcal{O}(s^3). \quad (8)$$

The input of the  $\delta_1^1(s)$  phase-shift is included as follows. Resonance chiral theory and vector

| Source       | $M_{\rho^\pm}$ (MeV) | $M_{\rho^0}$ (MeV) |
|--------------|----------------------|--------------------|
| Our fit      | $775.9 \pm 0.5$      | $777.8 \pm 0.7$    |
| Ref. [17]    | $773.8 \pm 0.6$      | $772.6 \pm 0.5$    |
| Average [18] | $775.9 \pm 0.5$      |                    |

Table 1

Comparison of our results for  $M_{\rho^\pm}$  and  $M_{\rho^0}$  with other recent figures. The average value [18] corresponds to  $e^+e^-$  and  $\tau$  data analyses only.

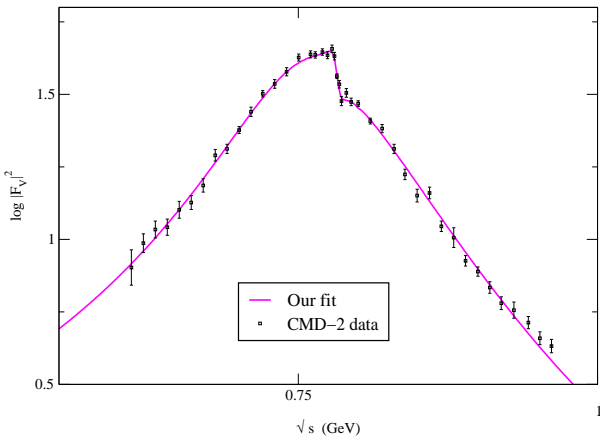


Figure 1. Comparison of experimental data from  $e^+e^- \rightarrow \pi^+\pi^-$  by CMD-2 [1] with the results of our fit.

meson dominance provide a model-independent analytic expression that describes properly the  $\rho(770)$  contribution [13]

$$\delta_1^1(s) = \arctan \left\{ \frac{M_\rho \Gamma_\rho(s)}{M_\rho^2 - s} \right\}, \quad (9)$$

with  $\Gamma_\rho(s)$  the off-shell  $\rho(770)$  width as computed in Ref. [15]. This phase-shift is accurate up to  $E \sim 1$  GeV. At higher energies heavier resonances with the same quantum numbers pop up and we use the available experimental data from Ochs [14].

$F_V(s)$  endows the hadronic dynamics in the  $e^+e^- \rightarrow \pi^+\pi^-$  process and, in the isospin limit in the  $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$  decay. In Ref. [9] we analysed the  $\tau$  ALEPH [2] data where radiative corrections were not taken into account. In this note we reanalyse these data when corrected for isospin

| Source                            | $\langle r^2 \rangle_V^\pi$ (GeV $^{-2}$ ) | $c_V^\pi$ (GeV $^{-4}$ ) |
|-----------------------------------|--|--------------------------|
| Our fit ( $\tau$ )                | $11.0 \pm 0.3$                             | $3.84 \pm 0.03$          |
| Our fit ( $e^+e^-$ )              | $11.5 \pm 0.2$                             | $3.73 \pm 0.02$          |
| $\mathcal{O}(p^6)$ $\chi$ PT [12] | $11.22 \pm 0.41$                           | $3.85 \pm 0.60$          |
| Ref. [17]                         | $11.17 \pm 0.05$                           | $3.60 \pm 0.03$          |

Table 2

Comparison of our results for the low-energy parameters of the pion form factor with other recent figures.

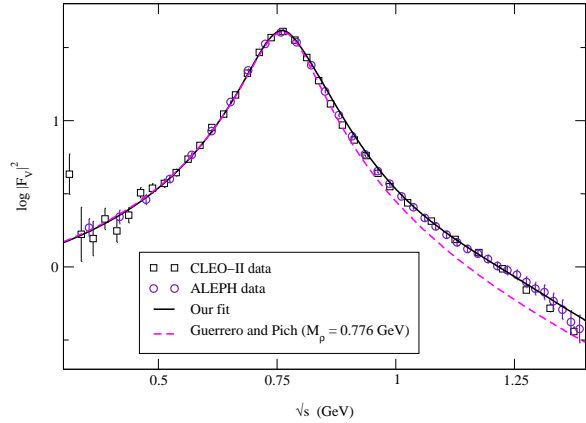


Figure 2. Comparison of experimental data from  $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$  decays by ALEPH [2] and CLEO-II [3] with the prediction of Ref. [13] and our fit to the ALEPH data.

breaking effects recently computed [10]. Analogously we study the recent experimental data on  $e^+e^- \rightarrow \pi^+\pi^-$  in the  $\rho(770)$  energy region by CMD-2 [1]. To analyse the  $e^+e^-$  data we need to include the effect of the  $\omega(782) \rightarrow \pi^+\pi^-$  process in our form factor. This we do through a  $\rho - \omega$  mixing term defined as in Ref. [16].

The results of our fit to CMD-2 data, with  $\chi^2/dof = 45.2/37$ , are shown in Fig. 1 while the  $M_{\rho^0}$  mass and low-energy parameters are given in Tables 1 and 2. In addition we get  $M_\omega = (781.8 \pm 0.3)$  MeV,  $\Gamma_\omega = (9.3 \pm 1.6)$  MeV and  $\Theta_{\rho\omega} = (-3.3 \pm 0.5) \times 10^{-3}$  GeV $^2$ . The analysis of the  $\tau$  ALEPH data gives the results shown in Fig. 2. The fit has a  $\chi^2/dof = 30.2/21$  and the values of  $M_{\rho^\pm}$  and the low-energy parameters can also be read in Tables 1 and 2. We also get

| $\sqrt{s_{max}}$ (GeV) | $a_{\mu}^{\pi\pi} _{\tau} \times 10^{10}$ | $a_{\mu}^{\pi\pi} _{e^+e^-} \times 10^{10}$ |
|------------------------|---|---|
| 0.5                    | $55.9 \pm 0.5$                            | $56.7 \pm 0.6$                              |
| 0.9                    | $488 \pm 7$                               | $475 \pm 5$                                 |
| 1.0                    | $507 \pm 7$                               | $490 \pm 6$                                 |
| 1.1                    | $513 \pm 8$                               | $494 \pm 6$                                 |

Table 3  
Results for the two-pion contribution to  $a_{\mu}$  from the analyses of  $\tau$  and  $e^+e^-$  data and for different values of the  $\sqrt{s_{max}}$  cut-off.

| $\sqrt{s_{max}}$ (GeV) | $10^4 \Delta\alpha(M_Z^2) _{\tau}$ | $10^4 \Delta\alpha(M_Z^2) _{e^+e^-}$ |
|------------------------|------------------------------------|--------------------------------------|
| 0.9                    | $31.9 \pm 0.5$                     | $30.7 \pm 0.4$                       |
| 1.0                    | $34.0 \pm 0.5$                     | $32.4 \pm 0.4$                       |
| 1.1                    | $34.8 \pm 0.6$                     | $33.0 \pm 0.5$                       |

Table 4  
Results for the two-pion contribution to  $\Delta\alpha(M_Z^2)$  from the analyses of  $\tau$  and  $e^+e^-$  data and for different values of the  $\sqrt{s_{max}}$  cut-off.

$\Delta M_{\rho^{\pm}-\rho^0} = (-1.9 \pm 0.9)$  MeV and  $\Delta\Gamma_{\rho^{\pm}-\rho^0} = (-0.2 \pm 0.6)$  MeV, to be compared with the figures in Ref. [18] :  $\Delta M_{\rho^{\pm}-\rho^0} = (-0.4 \pm 0.8)$  MeV and  $\Delta\Gamma_{\rho^{\pm}-\rho^0} = (0.1 \pm 1.9)$  MeV.

Finally, in Tables 3 and 4 we show the results for the two-pion vacuum polarization contribution to the anomalous magnetic moment of the muon,  $a_{\mu}^{\pi\pi}$  and to the shift in the fine-structure constant  $\Delta\alpha(M_Z^2)^{\pi\pi}$ , for different values of  $\sqrt{s_{max}}$  (the upper limit of the hadronic invariant mass in the dispersion integral that provides the hadron vacuum polarization contribution to both observables) and for the form factors coming from the analyses of  $e^+e^-$  and  $\tau$  data. Our results compare well with the recent computation in Ref. [19].

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