THE NUMBER OF RATIONAL NUMBERS DETERMINED BY LARGE SETS OF INTEGERS

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ABSTRACT. When A and B are subsets of the integers in $[1, X]$ and $[1, Y]$ respectively, with $|A| \geq \alpha X$ and $|B| \geq \beta X$, we show that the number of rational numbers expressible as a/b with (a, b) in $A \times B$ is $\gg (\alpha\beta)^{1+\epsilon} XY$ for any $\epsilon > 0$, where the implied constant depends on ϵ alone. We then construct examples that show that this bound cannot in general be improved to $\gg \alpha\beta XY$. We also resolve the natural generalisation of our problem to arbitrary subsets C of the integer points in $[1, X] \times [1, Y]$. Finally, we apply our results to answer a question of Sárközy concerning the differences of consecutive terms of the product sequence of a given integer sequence.

1. INTRODUCTION

When A and B are intervals in the integers in $[1, X]$ and $[1, Y]$ respectively, satisfying $|A| \geq \alpha X$ and $|B| \geq \beta Y$, where X, Y real numbers ≥ 1, α, β are real numbers in $(0, 1]$, a standard application of the Möbius inversion formula shows that the number of rational numbers a/b with (a, b) in $A \times B$ is $\gg \alpha\beta XY$.

Our purpose is to investigate what might be deduced when in place of *intervals* we consider *arbitrary* subsets A and B of the integers in $[1, X]$ and $[1, Y]$ respectively with $|A| \geq \alpha X$ and $|B| \geq \beta Y$. When A and B are not intervals, it may happen that an abnormally large number of elements of these sets are multiples of certain integers, determining which in general is not easy. Nevertheless, since the sets under consideration are large, popular heuristics suggest that a non-trivial conclusion should still be accessible. What is pleasing is that we in fact have the following theorem, which is our principal conclusion. In the statement of this theorem and thereafter we write A/B to denote the subset of $\mathbf{Q}$ consisting of all rational numbers expressible as a/b with (a, b) in $A \times B$ for any A and B subsets of the integers ≥ 1.

THEOREM 1.1. — *Let α and β be real numbers in $(0, 1]$ and X and Y real numbers ≥ 1. When A and B are subsets of the integers in $[1, X]$ and $[1, Y]$ respectively, with $|A| \geq \alpha X$ and $|B| \geq \beta Y$ we have $|A/B| \gg (\alpha\beta)^{1+\epsilon} XY$ for any $\epsilon > 0$, where the implied constant depends on ϵ alone.*

Deferring the detailed proof of Theorem 1.1 to Section 2, let us summarize our argument with the aid of the following notation. For any integer $d \geq 1$, A and B subsets of the integers ≥ 1, we write $\mathcal{M}(A, B, d)$ to denote the subset of $A \times B$ consisting of all (a, b) in $A \times B$ with $\gcd(a, b) = d$. We show in Proposition 2.1 that for A and B as in Theorem 1.1 we have $\sup_{d \geq 1} |\mathcal{M}(A, B, d)| \geq \frac{1}{8}(\alpha\beta)^2 XY$. Starting from this initial bound we then obtain $\sup_{d \geq 1} |\mathcal{M}(A, B, d)| \gg (\alpha\beta)^{1+\epsilon} XY$ by a bootstrapping argument. Theorem 1.1 follows immediately from this last bound, since for any integer $d \geq 1$ we have $a/b \neq a_1/b_1$ for any two points (a, b) and (a_1, b_1) of $\mathcal{M}(A, B, d)$, and therefore $|A/B| \geq \sup_{d \geq 1} |\mathcal{M}(A, B, d)|$.

We supplement Theorem 1.1 with the following result which shows that the bound provided by Theorem 1.1 cannot be replaced with $|A/B| \gg \alpha\beta XY$. This bound, as we have already remarked, holds when A and B are intervals.

THEOREM 1.2. — *For any $\epsilon > 0$, there exists $\alpha > 0$ such that for all sufficiently large X there exists a subset A of the integers in $[1, X]$ satisfying $|A| \geq \alpha X$ and $|A/A| < \epsilon\alpha^2 X^2$.*

We prove Theorem 1.2 in Section 3. Our method depends on the observation that for any $\epsilon > 0$ and any set of prime numbers $\mathcal{P}$ with $|\mathcal{P}|$ sufficiently large, we have $|S(\mathcal{P})/S(\mathcal{P})| \leq \epsilon|S(\mathcal{P})|^2$, where $S(\mathcal{P})$ is the set of squarefree integers d formed from the primes in the subsets of $\mathcal{P}$ containing about half the primes in $\mathcal{P}$. By means of this observation we deduce that, for a suitable $\mathcal{P}$, the set of multiples of the elements of $S(\mathcal{P})$ in $[1, X]$, meets the conditions of Theorem 1.2.

The questions answered by the above theorems may be viewed as particular cases of a more general problem namely, for X and Y real numbers ≥ 1 and γ in $(0, 1]$, given a subset C of the integer points in $[1, X] \times [1, Y]$ satisfying $|C| \geq \gamma XY$, to determine in terms of γ, X and Y an optimal lower bound for $\mathrm{Frac}(C)$, the number of rational numbers a/b with (a, b) in C. Plainly, the above theorems take up the special case when C is of the form $A \times B$, that is, when C is equal to the product of its projections onto the co-ordinate axes.

It turns out, however, that aforementioned general problem is somewhat easily resolved. In effect, the method of Proposition 2.1 generalizes without additional effort to give the bound $|\mathrm{Frac}(C)| \geq \frac{1}{8}\gamma^2 XY$ and, interestingly, this bound is in fact optimal upto the constant $\frac{1}{8}$. More precisely, we have the following theorem.

THEOREM 1.3. — *For any γ in $(0, 1]$ and all sufficiently large X and Y there exists a subset C of the integer points in $[1, X] \times [1, Y]$ satisfying $|C| \geq \frac{\gamma}{8} XY$ and $|\mathrm{Frac}(C)| \leq \frac{\gamma^2}{2} XY$.*

We prove Theorem 1.3 at the end of Section 3 by explicitly describing sets C that satisfy the conditions of this theorem. Such sets are in general far from being of the form $A \times B$, which is only natural on account of Theorem 1.1. Indeed, our bootstrapping

argument for Theorem 1.1 depends crucially on the fact that this theorem is, from the more general point view, about sets C which are of the form $A \times B$.

We conclude this note with Section 4 where we apply Theorem 1.1 to obtain a near-optimal answer to the following question of Sárközy. When $\mathcal{A}$, $\mathcal{B}$ are sequences of integers, let $\mathcal{A}.\mathcal{B}$ be the sequence whose terms are the integers of the form ab, for some $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then Sárközy [4] asks if it is true that for any $\alpha > 0$ and $\mathcal{A}$ such that the lower asymptotic density $\underline{d}(\mathcal{A}) > \alpha$ there is a $c(\alpha)$ such that there are infinitely many pairs of consecutive terms of $\mathcal{A}.\mathcal{A}$ the difference between which is bounded by $c(\alpha)$.

Berczi [1] responded to the aforementioned question of Sárközy by showing that the minimum of the differences between consecutive terms of $\mathcal{A}.\mathcal{A}$ is $\ll \frac{1}{\alpha^4}$, where $\alpha = \underline{d}(\mathcal{A})$. Sandor [3] subsequently improved this by showing that this minimum is in fact $\ll \frac{1}{\alpha^3}$, with α now the upper asymptotic density $\overline{d}(\mathcal{A})$ of $\mathcal{A}$. Cilleruelo and Le [2] obtained the same bound when α is the upper Banach density of $\mathcal{A}$ and showed that this is the best possible bound for this density. The following result improves upon and generalizes Sandor's conclusion.

THEOREM 1.4. — *Let α and β be real numbers in $(0, 1]$ and let ϵ be > 0. When $\mathcal{A}$ and $\mathcal{B}$ are infinite sequences of integers with upper asymptotic densities α and β respectively, there are infinitely many pairs of consecutive terms of the product sequence $\mathcal{A}.\mathcal{B}$ the difference between which is $\ll \frac{1}{(\alpha\beta)^{1+\epsilon}}$, where the implied constant depends on ϵ alone.*

When $\mathcal{A}$ and $\mathcal{B}$ are the sequences of multiples of the integers h and k respectively, the difference between any two consecutive terms of the sequence $\mathcal{A}.\mathcal{B}$ is $\geq hk$. Since we have $\overline{d}(\mathcal{A}) = \frac{1}{h}$ and $\overline{d}(\mathcal{B}) = \frac{1}{k}$, we see that the conclusion of Theorem 1.4 is optimal up to a factor $\frac{1}{(\alpha\beta)^{\epsilon}}$.

Throughout this note, X, Y shall denote real numbers ≥ 1 and α, β, γ real numbers in $(0, 1]$. Also, the letter p shall denote a prime number. When I and J are subsets of a given set, $I \setminus J$ shall denote the set of elements of I that are not in J. In addition to the notation introduced so far, we shall write A_d to denote the subset of a set of integers A consisting of all multiples of d in A for any integer d. Finally, if $B = \{b\}$ with $b \geq 1$, we simply write A/b in place of A/B, by an abuse of notation.

2. PROOF OF THE BOUND

Let A and B be finite subsets of the integers ≥ 1. Then the family of subsets $\mathcal{M}(A, B, d)$ of $A \times B$, with d varying over the integers ≥ 1, is a partition of $A \times B$. Consequently, we have

$$(1) \qquad |A \times B| = \sum_{d \geq 1} |\mathcal{M}(A, B, d)| .$$

When A and B are contained in $[1, X]$ and $[1, Y]$ respectively, we have $|A_d| \leq X/d$ and $|B_d| \leq Y/d$, for any $d \geq 1$. Since $\mathcal{M}(A, B, d)$ is contained in $A_d \times B_d$, we then obtain $|\mathcal{M}(A, B, d)| \leq |A_d||B_d| \leq \frac{XY}{d^2}$, for all $d \geq 1$.

PROPOSITION 2.1. — *When A and B are subsets of the integers in the intervals $[1, X]$ and $[1, Y]$ respectively, with $|A| \geq \alpha X$ and $|B| \geq \beta Y$, we have $\sup_{d \geq 1} |\mathcal{M}(A, B, d)| \geq \frac{(\alpha\beta)^2 XY}{8}$.*

PROOF.— We adapt an argument from [2]. From (1) we have for any integer $T \geq 1$ that

$$(2) \quad |A \times B| = \sum_{1 \leq d \leq T} |\mathcal{M}(A, B, d)| + \sum_{T < d} |\mathcal{M}(A, B, d)| \leq \sum_{1 \leq d \leq T} |\mathcal{M}(A, B, d)| + \frac{XY}{T},$$

where the last inequality follows from $\sum_{T < d} |\mathcal{M}(A, B, d)| \leq \sum_{T < d} \frac{XY}{d^2} \leq \frac{XY}{T}$. Since $|A \times B| \geq \alpha\beta XY$ we conclude from (2) that

$$(3) \qquad \sup_{d \geq 1} |\mathcal{M}(A, B, d)| \geq \frac{1}{T} \sum_{1 \leq d \leq T} |\mathcal{M}(A, B, d)| \geq \left(\frac{\alpha\beta - \frac{1}{T}}{T} \right) XY$$

for any integer $T \geq 1$. Since $2 > \alpha\beta$, the interval $[\frac{2}{\alpha\beta}, \frac{4}{\alpha\beta}]$ contains an integer ≥ 1. The proposition now follows on setting T in (3) to be any such integer.

DEFINITION 2.1— We call a real number δ an *admissible exponent* if there exists a real number $C > 0$ such that for any α, β real numbers in $(0, 1]$, any X, Y real numbers ≥ 1 and any subsets A and B of the integers in $[1, X]$ and $[1, Y]$ with $|A| \geq \alpha X$ and $|B| \geq \beta Y$, we have $\sup_{d \geq 1} |\mathcal{M}(A, B, d)| \geq C(\alpha\beta)^\delta XY$. We call a C satisfying these conditions a *constant associated to* the admissible exponent δ.

Proposition 2.1 says that $\delta = 2$ is an admissible exponent. Proposition 2.2 will allow us to conclude that every $\delta > 1$ is an admissible exponent. The following lemma prepares us for an application of Hölder's inequality within the proof of Proposition 2.2.

For any integer $n \geq 1$ let $\tau(n)$ denote, as usual, the number of integers ≥ 1 that divide n. When D is an integer ≥ 1 we write $\tau_D(n)$ to denote the number of divisors d of n satisfying the condition $p|d \implies p \leq D$ for any prime number p.

LEMMA 2.1. — *When q is an integer ≥ 0 there is a real number $c(q) > 0$ such that for all real numbers $X \geq 1$ and integers $D \geq 1$ we have*

$$(4) \qquad \sum_{1 \leq n \leq X} \tau_D(n)^q \leq c(q)DX,$$

PROOF.— In effect, we have

$$(5) \qquad \sum_{1 \leq n \leq X} \tau_D(n)^q \ll X (\log 2D)^{2^q} \ll (2^q!) \, DX \, ,$$

where the implied constants are absolute. Plainly, the second inequality results from the elementary inequality $(\log t)^n \leq n! \, t$ for $t \geq 1$. We now prove the first inequality in (5). Let us write $\mathcal{D}$ for the set of integers m satisfying the condition $p|m \implies p \leq D$. For any integer $n \geq 1$, let $k(n)$ be the largest of the divisors of n lying in $\mathcal{D}$. We then have that

$$(6) \qquad \sum_{1 \leq n \leq X} \tau_D(n)^q = \sum_{\substack{1 \leq m \leq X, \\ m \in \mathcal{D}}} \tau(m)^q \sum_{\substack{1 \leq n \leq X, \\ k(n)=m}} 1 \leq X \sum_{m \in \mathcal{D}} \frac{\tau(m)^q}{m} \, ,$$

where we have used the upper bound X/m for the number of integers n in $[1, X]$ with $k(n) = m$. Let us write $S(q)$ for any integer $q \geq 0$ to denote the last sum in (6). Since Merten's formula gives $\prod_{1 \leq p \leq D}(1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log D}$, with γ here being Euler's constant, we have

$$(7) \qquad S(0) = \sum_{m \in D} \frac{1}{m} = \prod_{1 \leq p \leq D} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots \right) = \prod_{1 \leq p \leq D} \left(1 - \frac{1}{p}\right)^{-1} \ll \log 2D \, ,$$

where the implied constant is absolute. On noting that every divisor of an integer in $\mathcal{D}$ is again in $\mathcal{D}$ and using $\tau(dk) \leq \tau(d)\tau(k)$, valid for any integers d and $k \geq 1$, we obtain

$$(8) \qquad \sum_{m \in D} \frac{\tau(m)^q}{m} = \sum_{m \in D} \frac{\tau(m)^{q-1}}{m} \sum_{d|m} 1 = \sum_{(d,k) \in D \times D} \frac{\tau(dk)^{q-1}}{dk} \leq \left(\sum_{d \in D} \frac{\tau(d)^{q-1}}{d}\right)^2 .$$

In other words, $S(q) \leq S(q-1)^2$, for any $q \geq 1$. An induction on q then shows that for any integer $q \geq 0$ we have $S(q) \leq S(0)^{2^q} \ll (\log D)^{2^q}$, where the implied constant is absolute. On combining this bound with (6) we obtain the first inequality in (5).

PROPOSITION 2.2. — *If $\delta > 1$ is an admissible exponent then so is $\frac{3\delta(1+1/q)-2}{2\delta-1}$ for every integer $q \geq 1$.*

PROOF.— Let q be a given integer ≥ 1 and, for the sake of conciseness, let us write δ' to denote $\frac{3\delta(1+1/q)-2}{2\delta-1}$, which is > 1 since $\delta > 1$.

When C is a constant associated to δ, let us set C' to be the unique real number > 0 satisfying

$$(9) \qquad \frac{1}{8C'} = \left(\frac{C'}{C}\right)^{\frac{1}{2(\delta-1)}} 8^{\frac{\delta}{\delta-1}} (4c(q))^{\frac{\delta}{q(\delta-1)}} \, ,$$

where $c(q)$ is the implied constant in (4) of Lemma 2.1. It is easily seen from (9) that by replacing C with a smaller constant associated to δ if necessary we may assume that $\frac{1}{4} \geq C'$.

We shall show that δ' is an admissible exponent with C' a constant associated to δ'. Thus let α, β be real numbers in $(0, 1]$ and X, Y real numbers ≥ 1. Also, let A and B be any subsets of the integers in $[1, X]$ and $[1, Y]$ satisfying $|A| \geq \alpha X$ and $|B| \geq \beta Y$. We shall show that

$$(10) \qquad \sup_{d \geq 1} |\mathcal{M}(A, B, d)| \geq C'(\alpha\beta)^{\delta'} XY .$$

Replacing α and β with $\alpha' \geq \alpha$ and $\beta' \geq \beta$ such that $\alpha' \leq |A| \leq 2\alpha'$ and $\beta' \leq |B| \leq 2\beta'$ if necessary, we reduce to the case when $|A| \leq 2\alpha X$ and $|B| \leq 2\beta Y$.

Let us first dispose of the possibility that an abnormally large number of the integers in A and B are multiples of a given integer. Thus let $\alpha_d = |A_d|/X$ and $\beta_d = |B_d|/Y$, for any integer $d \geq 1$. Suppose that there exists an integer $d \geq 1$ such that

$$(11) \qquad \alpha_d \beta_d \geq \left(\frac{C'}{C}\right)^{\frac{1}{\delta}} (\alpha\beta)^{\frac{\delta'}{\delta}} d^{\frac{2}{\delta}-2} .$$

Then A_d and B_d are both non-empty and therefore X and Y are both $\geq d$. Further, the sets A_d/d and B_d/d are subsets of the integers in $[1, X/d]$ and $[1, Y/d]$. Since δ is an admissible exponent, C a constant associated to δ, and we have $|A_d/d| = (d\alpha_d)|X/d|$, $|B_d/d| = (d\beta_d)|X/d|$, there exists an integer $d' \geq 1$ such that

$$(12) \qquad |\mathcal{M}(A_d/d, B_d/d, d')| \geq C(d^2\alpha_d\beta_d)^{\delta}\frac{XY}{d^2} \geq C'(\alpha\beta)^{\delta'} XY ,$$

where the last inequality follows from (11). Since $|\mathcal{M}(A_d/d, B_d/d, d')|$ does not exceed $|\mathcal{M}(A, B, dd')|$, we obtain (10) from (12). We may therefore verify (10) assuming that for every integer $d \geq 1$ we have

$$(13) \qquad \alpha_d \beta_d < \left(\frac{C'}{C}\right)^{\frac{1}{\delta}} (\alpha\beta)^{\frac{\delta'}{\delta}} d^{\frac{2}{\delta}-2} .$$

With the aid of (13) we shall in fact obtain a more precise conclusion than (10). Let us set $K = \frac{(\alpha\beta)^{1-\delta'}}{8C'}$ and $L = 1 + [K]$. We shall show that

$$(14) \qquad \frac{1}{L}\sum_{1 \leq d \leq L} |\mathcal{M}(A, B, d)| \geq C'(\alpha\beta)^{\delta'} XY ,$$

so that we have $|\mathcal{M}(A, B, d)| \geq C'(\alpha\beta)^{\delta'} XY$ for some integer $d \leq L$, which of course implies (10). Note that since L is roughly about $\frac{(\alpha\beta)^{1-\delta'}}{C'}$, (14) is what one might expect from (1).

Let D be an integer in $[\frac{2}{\alpha\beta}, \frac{4}{\alpha\beta}]$. Thus in particular $D > 1$. When $L \geq D$ we obtain (14) even without (13). In effect, we then have $K \geq 1$ and hence that $L < 2K$ or, what is the same thing, that $L < \frac{(\alpha\beta)^{1-\delta'}}{4C'}$ from which (14) follows on noting that for any integer $T \geq D$, and in particular for $T = L$, we have from (3) that

$$(15) \qquad \frac{1}{T} \sum_{1 \leq d \leq T} |\mathcal{M}(A, B, d)| \geq \left(\frac{\alpha\beta - \frac{1}{T}}{T} \right) XY \geq \left(\frac{\alpha\beta - \frac{1}{D}}{T} \right) XY \geq \frac{\alpha\beta XY}{2T} \, .$$

Suppose now that $1 \leq L < D$. Let us first verify that for any integer T such that $1 \leq T < D$ we have the following inequality on account of (13).

$$(16)$$
$$\sum_{T < d \leq D} |\mathcal{M}(A, B, d)| \leq \left(\frac{C'}{C} \right)^{\frac{1}{2\delta}} (\alpha\beta)^{\frac{\delta'}{2\delta}} T^{\frac{1}{\delta} - 1} (XY)^{\frac{1}{2}} \left(\sum_{T < d \leq D} |A_d| \right)^{\frac{1}{2}} \left(\sum_{T < d \leq D} |B_d| \right)^{\frac{1}{2}} \, .$$

Indeed, for any integer d satisfying $T < d \leq D$ we have that

$$(17) \qquad |A_d||B_d| = (\alpha_d X \beta_d Y)^{\frac{1}{2}} |A_d|^{\frac{1}{2}} |B_d|^{\frac{1}{2}} \leq \left(\frac{C'}{C} \right)^{\frac{1}{\delta}} (\alpha\beta)^{\frac{\delta'}{2\delta}} T^{\frac{1}{\delta} - 1} (XY)^{\frac{1}{2}} |A_d|^{\frac{1}{2}} |B_d|^{\frac{1}{2}} \, ,$$

where the last inequality follows from (13) on noting that $d^{\frac{1}{\delta} - 1} \leq T^{\frac{1}{\delta} - 1}$ for d satisfying $T < d \leq D$, since $\delta \geq 1$. On combining the bound $|\mathcal{M}(A, B, d)| \leq |A_d||B_d|$ with (17) and an application of the Cauchy-Schwarz inequality we obtain (16).

We now estimate the sums on the right hand side of (16). An application of Hölder's inequality gives

$$(18) \qquad \sum_{T < d \leq D} |A_d| = \sum_{T < d \leq D} \sum_{\substack{n \in A, \\ d | n}} 1 \leq \sum_{n \in A} \tau_D(n) \leq |A|^{1 - \frac{1}{q}} \left(\sum_{1 \leq n \leq X} \tau_D(n)^q \right)^{\frac{1}{q}} \, .$$

From Lemma 2.1 we have the upper bound $c(q)DX$ for the last sum in (18). Since $|A| \leq 2\alpha X$ and $D \leq \frac{4}{\alpha\beta}$, we deduce from (18) that

$$(19) \qquad \sum_{T < d \leq D} |A_d| \leq (2\alpha)^{1 - \frac{2}{q}} \beta^{-\frac{1}{q}} (4c(q))^{\frac{1}{q}} X.$$

Arguing similarly, we obtain the bound

$$(20) \qquad \sum_{T < d \leq D} |B_d| \leq (2\beta)^{1 - \frac{2}{q}} \alpha^{-\frac{1}{q}} (4c(q))^{\frac{1}{q}} Y.$$

With these estimates we conclude from (16) that for any integer T satisfying $1 \leq T < D$ we have

$$(21) \qquad \sum_{T < d \le D} |\mathcal{M}(A,B,d)| \ \le \ 2 \left(\frac{C'}{C} \right)^{\frac{1}{2\delta}} (\alpha\beta)^{\frac{\delta'}{2\delta} + \frac{1}{2} - \frac{3}{2q}} T^{\frac{1}{\delta}-1} (4c(q))^{\frac{1}{q}} XY \ ,$$

We now reveal that our choices for C' and δ' were made so that K satisfies the relation

$$(22) \qquad 2 \left(\frac{C'}{C} \right)^{\frac{1}{2\delta}} (\alpha\beta)^{\frac{\delta'}{2\delta} + \frac{1}{2} - \frac{3}{2q}} K^{\frac{1}{\delta}-1} (4c(q))^{\frac{1}{q}} \ = \ \frac{\alpha\beta}{4} \ ,$$

as may be confirmed by a modest calculation using the expressions defining C' and δ' in terms of C and δ.

We see that $\sum_{L < d \le D} |\mathcal{M}(A,B,d)| \le \frac{\alpha\beta}{4} XY$ using (21) for $T = L$ together with (22) and noting that $K < L$. Since (15) applied with $T = D$ gives us $\sum_{1 \le d \le D} |\mathcal{M}(A,B,d)| \ge \frac{\alpha\beta}{2} XY$, we conclude that when $1 \le L < D$ we have

$$(23) \qquad \frac{1}{L} \sum_{1 \le d \le L} |\mathcal{M}(A,B,d)| \ \ge \ \frac{\alpha\beta}{4L} XY \ .$$

If $L = 1$ we obtain (14) from (23) on noting that $\frac{\alpha\beta}{4} \ge C'(\alpha\beta)^{\delta'}$, since $\frac{1}{4} \ge C'$ and $1 \le \delta'$. When $1 < L < D$ we have $1 \le K$ and hence $L < \frac{(\alpha\beta)^{1-\delta'}}{4C'}$ so that (14) results from (23) in this final case as well.

COROLLARY 2.1. — *Every $\delta > 1$ is an admissible exponent.*

PROOF. — Let q be any integer ≥ 4 and let $\{\delta_n(q)\}_{n \ge 1}$ the sequence of real numbers determined by the relations $\delta_1(q) = 2$ and

$$(24) \qquad \delta_{n+1}(q) = \frac{3\delta_n(q)\left(1 + \frac{1}{q}\right) - 2}{2\delta_n(q) - 1}$$

for $n \ge 1$. Then each $\delta_n(q)$ is an admissible exponent by Propositions 2.1 and 2.2. It is easily verified that the sequence $\delta_n(q)$ is decreasing and has a limit $\delta(q)$ given by the relation

$$(25) \qquad \delta(q) = 1 + \frac{3}{4q} + \frac{1}{2}\sqrt{\frac{6}{q} + \frac{9}{4q^2}}.$$

Plainly, any $\delta > \delta(q)$ is an admissible exponent. The corollary now follows on taking q arbitrarily large in (25).

Theorem 1.1 follows from the above corollary and the definition of admissible exponents on recalling that $|A/B| \ge \sup_{d \ge 1} |\mathcal{M}(A,B,d)|$.

3. COUNTEREXAMPLES

Let us first prove Theorem 1.2. To this end, given an integer $m \geq 1$ let $\mathcal{P}$ denote any set of $2m$ prime numbers and, for any subset I of $\mathcal{P}$, let $d(I) = \prod_{p \in I} p$. If $S(\mathcal{P})$ denotes the set of all $d(I)$ with $|I| = m$, we have the following lemma.

LEMMA 3.1. — *For any $\epsilon > 0$, we have $|S(\mathcal{P})/S(\mathcal{P})| \leq \epsilon|S(\mathcal{P})|^2$ for all sufficiently large m.*

PROOF. — Plainly, we have $|S(\mathcal{P})| = \binom{2m}{m}$. Let $\mathcal{Q}$ be the set of ordered pairs of disjoint subsets of $\mathcal{P}$. Then, for any I and J subsets of $\mathcal{P}$, we have

$$\text{(1)} \qquad \frac{d(I)}{d(J)} = \frac{d(I \setminus J)}{d(J \setminus I)} \, ,$$

and since $I \setminus J$ and $J \setminus I$ are disjoint, $(I \setminus J, J \setminus I)$ is in $\mathcal{Q}$. Thus $|S(\mathcal{P})/S(\mathcal{P})| \leq |\mathcal{Q}|$. Let us associate any (U, V) in $\mathcal{Q}$ to the map from $\mathcal{P}$ to the three element set $\{1, 2, 3\}$ that takes U to 1, V to 2 and the complement of $U \cup V$ in $\mathcal{P}$ to 3. It is easily seen that this association in fact gives a bijection from $\mathcal{Q}$ onto the set of maps from $\mathcal{P}$ to $\{1, 2, 3\}$ and hence that $|\mathcal{Q}| = 3^{2m}$. In summary, we deduce that

$$\text{(2)} \quad |S(\mathcal{P})/S(\mathcal{P})| \ \leq \ |\mathcal{Q}| \ = \ 3^{2m} \ = \ \frac{3^{2m}}{\binom{2m}{m}^2} \, |S(\mathcal{P})|^2 \ \leq \ (2m+1)^2 \left(\frac{3}{4}\right)^{2m} |S(\mathcal{P})|^2 \, ,$$

where we have used the inequality $\binom{2m}{m} \geq \frac{2^{2m}}{2m+1}$. The lemma follows from (2) on noting that $(2m+1)^2 \left(\frac{3}{4}\right)^{2m} \to 0$ as $m \to +\infty$.

PROOF OF THEOREM 1.2. — Given an integer $m \geq 1$, it is easily deduced from the prime number theorem that the interval $[T, T + T/m]$ contains at least $2m$ prime numbers when T is sufficiently large. For such a T, let $\mathcal{P}$ be a subset of $2m$ prime numbers in $[T, T + T/m]$. If $\mathcal{A}(\mathcal{P})$ is the sequence of integers ≥ 1 that are divisible by at least one of the integers $d(I)$ in $S(\mathcal{P})$ then a simple application of the principle of inclusion and exclusion implies that $\mathcal{A}(\mathcal{P})$ has an asymptotic density $\alpha(\mathcal{P})$ that is given by the relation

$$\text{(3)} \qquad \alpha(\mathcal{P}) \ = \ \sum_{1 \leq r \leq \binom{2m}{m}} (-1)^{r+1} \sum_{1 \leq i_1 < i_2 \ldots < i_r \leq \binom{2m}{m}} \frac{1}{d(I_{i_1} \cup I_{i_2} \ldots \cup I_{i_r})} \, ,$$

where $I_1, I_2, \ldots, I_{\binom{2m}{m}}$ are the subsets of cardinality m in $\mathcal{P}$.

For any i we have $T^m \leq d(I_i) \leq (1 + \frac{1}{m})^m T^m < eT^m$. Consequently, for the term $r = 1$ in (3) we obtain

$$\text{(4)} \qquad \sum_{1 \leq i \leq \binom{2m}{m}} \frac{1}{d(I_i)} \ \geq \ \frac{\binom{2m}{m}}{eT^m}.$$

When $r \geq 2$, we have that $d(I_{i_1} \cup I_{i_2} \ldots \cup I_{i_r})$, for any distinct indices $i_1, i_2 \ldots, i_r$, has at least $k+1$ prime factors in $\mathcal{P}$ and hence is $\geq T^{m+1}$. It follows from (3) and these bounds that we have

$$\alpha(\mathcal{P}) \geq \frac{\binom{2m}{m}}{eT^m} - \frac{2\binom{2m}{m}}{T^{m+1}} \geq \frac{\binom{2m}{m}}{3T^m} \tag{5}$$

when T is sufficiently large. In particular, on recalling that $|S(\mathcal{P})| = \binom{2m}{m}$, we obtain that for any integer $m \geq 1$, we have

$$\alpha(\mathcal{P}) \geq \frac{|S(\mathcal{P})|}{3T^m} \tag{6}$$

for all sufficiently large T and $\mathcal{P}$ any set of $2m$ prime numbers in $[T, T+T/m]$.

Finally, for $\mathcal{P}$ as above and any $X \geq 1$, let us set $A = \mathcal{A}(\mathcal{P}) \cap [1, X]$. Since $\alpha(\mathcal{P})$ is the asymptotic density of $\mathcal{A}(\mathcal{P})$, we have from (6) that $|A| \geq \frac{|S(\mathcal{P})|}{4T^m} X$, for all large enough X and T. Clearly, each integer in A is of the form $d(I)n$, for some $d(I)$ in $S(\mathcal{P})$ and an integer n, which must necessarily be $\leq \frac{X}{T^m}$, since A is in $[1, X]$ and $d(I) \geq T^m$. Consequently, we have we have $|A/A| \leq \frac{|S(\mathcal{P})/S(\mathcal{P})|}{T^{2m}} X^2$, for all large enough X and T. On comparing $|A|$ and $|A/A|$ by means of Lemma 3.1, we see that A meets the conditions of Theorem 1.2 when m, T and X are all sufficiently large.

PROOF OF THEOREM 1.3. — The number of primitive integer points, that is, integer points with coprime co-ordinates, in $[1, \gamma X] \times [1, \gamma Y]$ is $\sim \frac{6}{\pi^2} \gamma^2 XY$ as $X, Y \to \infty$. Thus for any γ in $(0, 1]$ and all sufficiently large X and Y, there is a subset S of the primitive integer points in $[1, \gamma X] \times [1, \gamma Y]$ satisfying $\frac{\gamma^2}{4} XY \leq |S| \leq \frac{\gamma^2}{2} XY$. Let us take for C the union of the sets $d.S$ with d varying over the interval $[1, \frac{1}{\gamma}]$, where each $d.S$ is the set of (da, db) with (a, b) varying over S. Then C is contained in $[1, X] \times [1, Y]$. Moreover, the sets $d.S$ are disjoint but $\mathrm{Frac}(d.S) = \mathrm{Frac}(S)$, for each d, and $|\mathrm{Frac}(S)| = |S|$. We therefore have $|C| = [\frac{1}{\gamma}]|S| \geq \frac{\gamma}{8} XY$ but $|\mathrm{Frac}(C)| = |\mathrm{Frac}(S)| = |S| \leq \frac{\gamma^2}{2} XY$.

4. GAPS IN PRODUCT SEQUENCES

We now deduce Theorem 1.4 from Theorem 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be sequences with upper asymptotic densities α and β. Then there exist infinitely many real numbers X and $Y \geq 1$ such that $|\mathcal{A} \cap (\frac{X}{2}, X]| \geq \frac{\alpha X}{4}$ and $|\mathcal{B} \cap (\frac{Y}{2}, Y]| \geq \frac{\beta Y}{4}$. For such X and Y let us apply Theorem 1.1 to the sets $A = \mathcal{A} \cap (\frac{X}{2}, X]$ and $B = \mathcal{B} \cap (\frac{Y}{2}, Y]$. We then have that $|A/B| \gg (\alpha\beta)^{1+\epsilon} XY$, where the implied constant depends on ϵ alone. Since A/B is a subset of the interval $[\frac{X}{2Y}, \frac{2X}{Y}]$, which is of length $\frac{X}{Y}$, we deduce that there are distinct a/b and a'/b' in A/B such that

$$0 < \left| \frac{a}{b} - \frac{a'}{b'} \right| \ll \frac{X/Y}{(\alpha\beta)^{1+\epsilon}XY} = \frac{1}{(\alpha\beta)^{1+\epsilon}Y^2} \ . \tag{1}$$

Since $|bb'| \leq Y^2$, it follows from (1) that difference between the distinct terms ba' and $b'a$ of the product sequence $\mathcal{A}.\mathcal{B}$ is $\ll \frac{1}{(\alpha\beta)^{1+\epsilon}}$. Since there are infinitely many distinct X and Y satisfying the required conditions, there are infinitely many such pairs of terms in the product sequence $\mathcal{A}.\mathcal{B}$.

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