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The least common multiple of a quadratic sequence

Javier Cilleruelo

Abstract

For any irreducible quadratic polynomial $f(x)$ in $\mathbb{Z}[x]$, we obtain the estimate $\log \text{l.c.m.}(f(1),\dots,f(n)) = n\log n + Bn + o(n)$, where B is a constant depending on f.

1. Introduction

The problem of estimating the least common multiple of the first n positive integers was first investigated by Chebyshev [Che52] when he introduced the function $\Psi(n) = \sum_{p^m \leqslant n} \log p = \log \text{l.c.m.}(1,\dots,n)$ in his study of the distribution of prime numbers. The prime number theorem asserts that $\Psi(n) \sim n$, so the asymptotic estimate $\log \text{l.c.m.}(1,\dots,n) \sim n$ is equivalent to the prime number theorem. The analogous asymptotic estimate for any linear polynomial $f(x) = ax + b$ is also known [Bat02] and it is a consequence of the prime number theorem for arithmetic progressions:

$$\log \text{l.c.m.}(f(1),\dots,f(n)) \sim n\frac{q}{\phi(q)} \sum_{\substack{1\leqslant k\leqslant q\\(k,q)=1}} \frac{1}{k}, \tag{1}$$

where $q = a/(a,b)$.

We address here the problem of estimating $\log \text{l.c.m.}(f(1),\dots,f(n))$ when f is an irreducible quadratic polynomial in $\mathbb{Z}[x]$. When f is a reducible quadratic polynomial, the asymptotic estimate is similar to that we obtain for linear polynomials. This case is studied in § 4 with considerably less effort than the irreducible case. We state our main theorem.

Theorem 1. *For any irreducible quadratic polynomial $f(x) = ax^2 + bx + c$ in $\mathbb{Z}[x]$, we have*

$$\log \text{l.c.m.}(f(1),\dots,f(n)) = n\log n + Bn + o(n),$$

where $B = B_f$ is defined by the formula

$$B_f = \gamma - 1 - 2\log 2 - \sum_p \frac{(d/p)\log p}{p-1} + \frac{1}{\phi(q)} \sum_{\substack{1\leqslant r\leqslant q\\(r,q)=1}} \log\left(1+\frac{r}{q}\right)$$
$$+ \log a + \sum_{p\mid 2aD} \log p\left(\frac{1+(d/p)}{p-1} - \sum_{k\geqslant 1} \frac{s(f,p^k)}{p^k}\right). \tag{2}$$

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In this formula γ is the Euler constant, $D = b^2 - 4ac = dl^2$, where d is a fundamental discriminant, (d/p) is the Kronecker symbol, $q = a/(a,b)$ and $s(f,p^k)$ is the number of solutions of $f(x) \equiv 0 \pmod{p^k}$ which can be easily calculated using Lemma 2.

For the simplest case, $f(x) = x^2 + 1$, the constant B_f in Theorem 1 can be written as

$$B_f = \gamma - 1 - \frac{\log 2}{2} - \sum_{p \neq 2} \frac{(-1/p)\log p}{p-1}, \tag{3}$$

where $(-1/p)$ is the Kronecker symbol (or Legendre symbol, since p is odd) defined by $(-1/p) = (-1)^{(p-1)/2}$ when p is odd.

In § 3, we give an alternative expression for the constant B_f which is more convenient for numerical computations. As an example, we will see that the constant B_f in (3) can be written as

$$\begin{aligned} B_f &= \gamma - 1 - \frac{\log 2}{2} + \sum_{k=1}^{\infty} \frac{\zeta'(2^k)}{\zeta(2^k)} + \sum_{k=0}^{\infty} \frac{L'(2^k,\chi_{-4})}{L(2^k,\chi_{-4})} - \sum_{k=1}^{\infty} \frac{\log 2}{2^{2^k}-1} \\ &= -0.066\,275\,634\,213\,060\,706\,383\,563\,177\,025\ldots. \end{aligned}$$

It would be interesting to extend our estimates to irreducible polynomials of higher degree, but we have found a serious obstruction in our argument. Some heuristic arguments and computations allow us to conjecture that the asymptotic estimate

$$\log \text{l.c.m.}(f(1),\ldots,f(n)) \sim (\deg(f)-1)n\log n \tag{4}$$

holds for any irreducible polynomial f in $\mathbb{Z}[x]$ of degree $\deg(f) \geqslant 3$. In § 2.4, we explain the obstruction to prove this conjecture. There we also prove that

$$\log \text{l.c.m.}(f(1),\ldots,f(n)) \sim n\log n \tag{5}$$

holds for any irreducible quadratic polynomial $f(x)$. Although this estimate is weaker than Theorem 1, the proof is easier.

To obtain the linear term in Theorem 1, we need a more involved argument. An important ingredient in this part of the proof is a deep result about the distribution of the solutions of the quadratic congruences $f(x) \equiv 0 \pmod{p}$ when p runs over all the primes. It was proved by Duke *et al.* [DFI95] (for $D < 0$) and by Toth [Tot00] (for $D > 0$). Actually we need a more general statement of this result, due to Toth.

Theorem 2 [Tot00]. *For any irreducible quadratic polynomial f in $\mathbb{Z}[x]$, the sequence*

$$\{\nu/p,\ 0 \leqslant \nu < p \leqslant x,\ p \in S,\ f(\nu) \equiv 0 \pmod{p}\}$$

is well distributed in $[0,1)$ as x tends to infinity for any arithmetic progression S containing infinitely many primes p for which the congruence $f(x) \equiv 0 \pmod{p}$ has solutions.

2. Proof of Theorem 1

2.1 Preliminaries

For $f(x) = ax^2 + bx + c$, we define $D = b^2 - 4ac$ and

$$L_n(f) = \text{l.c.m.}(f(1),\ldots,f(n)).$$

Since $L_n(f) = L_n(-f)$, we can assume that $a > 0$. Also, we can assume that b and c are non-negative integers. If this is not the case, we consider a polynomial $f_k(x) = f(k+x)$ for a k such that $f_k(x)$ has non-negative coefficients. Then we observe that $\log L_n(f) = \log L_n(f_k) + O_k(\log n)$ and that this error term is negligible for the statement of Theorem 1.

We define the numbers $\beta_p(n)$ by the formula

$$L_n(f) = \prod_p p^{\beta_p(n)}, \tag{6}$$

where the product runs over all the primes p. The primes involved in this product are those for which the congruence $f(x) \equiv 0 \pmod p$ has some solution. Except for some special primes (those such that $p|2aD$), the congruence $f(x) \equiv 0 \pmod p$ has zero or two solutions. We will discus this in detail in Lemma 2.

We denote by $\mathcal{P}_f$ the set of non-special primes for which the congruence $f(x) \equiv 0 \pmod p$ has exactly two solutions. More concretely,

$$\mathcal{P}_f = \{p : p \nmid 2aD, (D/p) = 1\},$$

where (D/p) is the Kronecker symbol. This symbol is just the Legendre symbol when p is an odd prime.

The quadratic reciprocity law shows that the set $\mathcal{P}_f$ is the set of the primes lying in exactly $\varphi(4D)/2$ of the $\varphi(4D)$ arithmetic progressions modulo $4D$, coprime with $4D$. As a consequence of the prime number theorem for arithmetic progressions, we have

$$\#\{p \leqslant x : p \in \mathcal{P}_f\} \sim \frac{x}{2\log x}$$

or, equivalently,

$$\sum_{\substack{0 \leqslant \nu < p \leqslant x \\ f(\nu) \equiv 0 \pmod p}} 1 \sim \frac{x}{\log x}.$$

Let $C = 2a + b$. We classify the primes involved in (6) into:

- *special* primes: those such that $p|2aD$;
- $p \in \mathcal{P}_f$: *small* primes: $p < n^{2/3}$,
  medium primes: $n^{2/3} \leqslant p < Cn$: *bad* primes: $p^2 | f(i)$ for some $i \leqslant n$, *good* primes: $p^2 \nmid f(i)$ for any $i \leqslant n$,
  large primes: $Cn \leqslant p \leqslant f(n)$.

We will use different strategies to deal with each class.

2.2 Large primes

We consider $P_n(f)$ and the numbers $\alpha_p(n)$ defined by

$$P_n(f) = \prod_{i=1}^{n} f(i) = \prod_p p^{\alpha_p(n)}. \tag{7}$$

The next lemma allow us to analyze the large primes involved in (6).

Lemma 1. *If $p \geqslant 2an + b$, then $\alpha_p(n) = \beta_p(n)$.*

Proof. If $\beta_p(n)=0$, then $\alpha_p(n)=0$. If $\alpha_p(n)>\beta_p(n)\geqslant 1$, then there exist $i<j\leqslant n$ such that $p|f(i)$ and $p|f(j)$. It implies that $p|f(j)-f(i)=(j-i)(a(j+i)+b)$. Thus, $p|(j-i)$ or $p|a(j+i)+b$, which is not possible because $p\geqslant 2an+b$. □

Since $C=2a+b$, we can write

$$\log L_n(f)=\log P_n(f)+\sum_{p<Cn}(\beta_p(n)-\alpha_p(n))\log p. \tag{8}$$

Indeed, we can take C to be any constant greater than $2a+b$. As we will see, the final estimate of $\log L_n(f)$ will not depend on C.

The estimate of $\log P_n(f)$ is easy:

$$\begin{aligned}\log P_n(f)&=\log\prod_{k=1}^{n}f(k)=\log\prod_{k=1}^{n}ak^2\left(1+\frac{b}{ka}+\frac{c}{k^2a}\right)\\&=n\log a+\log(n!)^2+\sum_{k=1}^{n}\log\left(1+\frac{b}{ka}+\frac{c}{k^2a}\right)\\&=2n\log n+n(\log a-2)+O(\log n)\end{aligned} \tag{9}$$

and we obtain

$$\log L_n(f)=2n\log n+n(\log a-2)+\sum_{p<Cn}(\beta_p(n)-\alpha_p(n))\log p+O(\log n). \tag{10}$$

2.3 The number of solutions of $f(x)\equiv 0 \pmod{p^k}$ and the special primes

The number of solutions of the congruence $f(x)\equiv 0 \pmod{p^k}$ will play an important role in the proof of Theorem 1. We write $s(f,p^k)$ to denote this quantity.

The lemma below summarizes all the cases for $s(f,p^k)$. We observe that, except for a finite number of primes, those dividing $2aD$, we have that $s(f;p^k)=2$ or 0 depending on $(D/p)=1$ or -1.

Lemma 2. *Let $f(x)=ax^2+bx+c$ be an irreducible polynomial and $D=b^2-4ac$.*

(i) *If $p\nmid 2a$, $D=p^lD_p$ and $(D_p,p)=1$, then*

$$s(f,p^k)=\begin{cases}p^{\lfloor k/2\rfloor}, & k\leqslant l,\\ 0, & k>l,\ l \text{ odd or } (D_p/p)=-1,\\ 2p^{l/2}, & k>l,\ l \text{ even } (D_p/p)=1.\end{cases}$$

(ii) *If $p|a$, $p\neq 2$, then* $s(f,p^k)=\begin{cases}0, & \text{if } p|b,\\ 1, & \text{if } p\nmid b.\end{cases}$

(iii) *If b is odd, then, for all $k\geqslant 2$,* $s(f,2^k)=s(f,2)=\begin{cases}1, & \text{if } a \text{ is even},\\ 0, & \text{if } a \text{ is odd and } c \text{ is odd},\\ 2, & \text{if } a \text{ is odd and } c \text{ is even}.\end{cases}$

(iv) *If b is even and a is even, then $s(f,2^k)=0$ for any $k\geqslant 1$.*

(v) *If b is even and a is odd, let $D=4^lD'$, $D'\not\equiv 0 \pmod 4$.*

(a) *If $k\leqslant 2l-1$, $s(f;2^k)=2^{\lfloor k/2\rfloor}$.*

(b) *If* $k=2l$, $s(f;2^k)=\begin{cases}2^l, & D'\equiv 1 \pmod 4,\\ 0, & D'\not\equiv 1 \pmod 4.\end{cases}$

(c) *If* $k\geqslant 2l+1$, $s(f;2^k)=\begin{cases}2^{l+1}, & D'\equiv 1 \pmod 8,\\ 0, & D'\not\equiv 1 \pmod 8.\end{cases}$

Proof. The proof is a consequence of elementary manipulations and Hensel's lemma. When the modulo is an odd prime p and $p\nmid a$, the congruence $ax^2+bx+c\equiv 0 \pmod p$ is equivalent to the congruence $y^2\equiv D \pmod p$. Hensel's lemma (see for example [HW08, Theorem 123]) provides a method to obtain all the solutions of the congruence $y^2\equiv D \pmod{p^{k+1}}$ from the solutions of $y^2\equiv D \pmod{p^k}$. In this way, we obtain all the distinct cases contained in part (i) of the lemma. Part (ii) is trivial and parts (iii)–(v) correspond to the case $p=2$, which can be analyzed easily. □

Corollary 1. *If* $p\nmid 2aD$, *then* $s(f,p^k)=1+(D/p)$.

Proof. In this case, $l=0$ and $D_p=D$ in Lemma 2. Thus, $s(f,p^k)=0=1+(D/p)$ if $(D/p)=-1$ and $s(f,p^k)=2=1+(D/p)$ if $(D/p)=1$. □

Lemma 3. *For any irreducible quadratic polynomial* f *and for* $\alpha_p(n)$ *defined as in (7) we have*

$$\alpha_p(n)=n\sum_{k\geqslant 1}\frac{s(f,p^k)}{p^k}+O\left(\frac{\log n}{\log p}\right), \tag{11}$$

where $s(f;p^k)$ *denotes the number of solutions of* $f(x)\equiv 0 \pmod{p^k}$, $0\leqslant x<p^k$.

Proof. We observe that the maximum exponent $\alpha_{p,i}$ such that $p^{\alpha_{p,i}}|f(i)$ can be written as $\alpha_{p,i}=\sum_{k\geqslant 1,p^k|f(i)}1$. Thus,

$$\alpha_p(n)=\sum_{i\leqslant n}\alpha_{p,i}=\sum_{i\leqslant n}\sum_{\substack{k\geqslant 1\\ p^k|f(i)}}1=\sum_{k\geqslant 1}\sum_{\substack{i\leqslant n\\ p^k|f(i)}}1. \tag{12}$$

The trivial estimate

$$s(f;p^k)\left[\frac{n}{p^k}\right]\leqslant\sum_{i\leqslant n,p^k|f(i)}1\leqslant s(f;p^k)\left(\left[\frac{n}{p^k}\right]+1\right)$$

gives

$$\sum_{\substack{i\leqslant n\\ p^k|f(i)}}1=n\frac{s(f;p^k)}{p^k}+O(s(f;p^k)). \tag{13}$$

Putting (13) in (12) and observing that $k\leqslant \log f(n)/\log p$ and that $s(f,p^k)\ll 1$, we get

$$\alpha_p(n)=n\sum_{k\geqslant 1}\frac{s(f,p^k)}{p^k}+O\left(\frac{\log n}{\log p}\right).$$ □

Since $p^{\beta_p(n)}\leqslant f(n)$, we have always the trivial estimate

$$\beta_p(n)\leqslant \log f(n)/\log p\ll \log n/\log p. \tag{14}$$

When we substitute (14) and (11) in (10), for the special primes we obtain

$$\log L_n(f) = 2n\log n + n\left(\log a - 2 - \sum_{p\mid 2aD}\sum_{k\geqslant 1}\frac{s(f,p^k)\log p}{p^k}\right) + \sum_{p<Cn,p\nmid 2aD}(\beta_p(n)-\alpha_p(n))\log p + O(\log n). \tag{15}$$

Lemma 3 has an easier formulation for non-special primes.

Lemma 4. *For any $p\nmid 2aD$, we have*

$$\alpha_p(n) = n\frac{1+(D/p)}{p-1} + O\left(\frac{\log n}{\log p}\right). \tag{16}$$

Proof. It is a consequence of Lemma 3 and Corollary 1. □

2.4 The asymptotic estimate

This subsection is a break in the proof of Theorem 1 to prove, in an easy way, that the weaker estimate

$$\log \text{l.c.m.}(f(1),\dots,f(n)) \sim n\log n \tag{17}$$

holds for any irreducible quadratic polynomial f.

We substitute (14) and (16) in (15) to obtain

$$\begin{aligned}\log L_n(f) &= 2n\log n + \sum_{p<Cn,p\nmid 2aD}(\beta_p(n)-\alpha_p(n))\log p + O(n)\\ &= 2n\log n - n\sum_{p<Cn,p\nmid 2aD}\frac{\log p}{p-1} - n\sum_{p<Cn,p\nmid 2aD}\frac{(D/p)\log p}{p-1}\\ &\quad + O\left(\sum_{p<Cn}\log n\right) + O(n).\end{aligned} \tag{18}$$

Now we get (17) using that $\sum_{p\leqslant x}\log p/(p-1)\sim\log x$ and that the sum $\sum_p (D/p)\log p/(p-1)$ is a convergent sum.

This is the moment to explain the main obstruction to obtain the analogous estimate for polynomials of degree $\deg(f)\geqslant 3$. For example, we consider the polynomial $f(x)=x^3+2$. Using the same approach used in the quadratic case, we get

$$\log L_n(f) = 3n\log n + \sum_{p<3n^2}(\beta_p(n)-\alpha_p(n))\log p + O(n). \tag{19}$$

We observe that the primes involved in the sum have the quadratic bound $3n^2$ instead of the linear bound we have in the case of quadratic polynomials. The reason is that if $p\mid k^3+2$ and $p\mid j^3+2$ with $j<k\leqslant n$, we only can say that $p\leqslant 3n^2$.

It is easy to check that $\beta_p(n)\ll \log n/\log p$ and $\alpha_p(n) = ns_p/(p-1)+O(\log n/\log p)$, where s_p is the number of solutions of $x^3+2\equiv 0 \pmod p$. Then we obtain

$$\log L_n(f) = 3n\log n - n\sum_{p<n}\frac{s_p\log p}{p-1} + O(n) + \sum_{n<p<3n^2}(\beta_p(n)-\alpha_p(n))\log p. \tag{20}$$

The Frobenius density theorem [LS96] implies that $s_p = 1$ on average, so $\sum_{p\leqslant x} s_p \log p/(p-1) \sim \log x$. Then, in the case $f(x) = x^3 + 2$, we have

$$\log L_n(f) = 2n \log n(1 + o(1)) + \sum_{n<p<3n^2} (\beta_p(n) - \alpha_p(n)) \log p. \tag{21}$$

We observe that $\beta_p(n) = \alpha_p(n)$ unless there exist $j < k \leqslant n$ such that $p|k^3 + 2$, $p|j^3 + 2$. In that case, we have the trivial bound $|\beta_p(n) - \alpha_p(n)| \ll 1$. Thus, to obtain the asymptotic $\log L_n(f) \sim 2n \log n$ for $f(x) = x^3 + 2$, we should prove that

$$|\{p : n \leqslant p \leqslant 3n^2, p|k^3 + 2, p|j^3 + 2 \text{ for some } 1 \leqslant j < k \leqslant n\}| = o(n).$$

In general, when $f(x)$ is an irreducible polynomial, the asymptotic estimate $\log \text{l.c.m.}(f(1), \dots, f(n)) \sim (\deg(f) - 1)n \log n$ would follow from the estimate

$$|\{p : n \leqslant p \ll n^{\deg(f)-1}, p|f(k), p|f(j) \text{ for some } 1 \leqslant j < k \leqslant n\}| = o(n). \tag{22}$$

This is obviously true when $\deg(f) = 2$, but we do not know how to prove it when $\deg(f) \geqslant 3$.

We come back to the proof of Theorem 1.

2.5 Medium primes

These primes can also be classified into bad and good primes. Bad primes are those p such that $p^2|f(i)$ for some $i \leqslant n$. Good primes are those are not bad primes.

As we have seen in the previous section, for any prime $p \in \mathcal{P}_f$, the congruence $f(x) \equiv 0 \pmod p$ has exactly two solutions, say $0 \leqslant \nu_{p,1}, \nu_{p,2} < p$.

If p is a good prime, we have that $\alpha_p(n)$ is just the number of integers $i \leqslant n$ such that $p|f(i)$. All these integers have the following form:

$$\nu_{p,1} + kp, \quad 0 \leqslant k \leqslant \left[\frac{n - \nu_{p,1}}{p}\right], \tag{23}$$

$$\nu_{p,2} + kp, \quad 0 \leqslant k \leqslant \left[\frac{n - \nu_{p,2}}{p}\right]. \tag{24}$$

Also, it is clear that if p is a good prime, then $\beta_p(n) \leqslant 1$. These observations motivate the following definition.

Definition 1. For any $p \in \mathcal{P}_f$, we define

$$\alpha_p^*(n) = \left[\frac{n - \nu_{p,1}}{p}\right] + \left[\frac{n - \nu_{p,2}}{p}\right] + 2, \tag{25}$$

$$\beta_p^*(n) = \begin{cases} 1, & \text{if } \beta_p(n) \geqslant 1, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Lemma 5. *For any $p \in \mathcal{P}_f$, we have:*

(i) $\alpha_p(n) - \alpha_p^*(n) = 2n/p(p-1) + O(\log n/\log p)$;

(ii) $\alpha_p(n) = \alpha_p^*(n)$ *and* $\beta_p(n) = \beta_p^*(n)$ *if* $p^2 \nmid f(i)$ *for any* $i \leqslant n$.

Proof. (i) Lemma 4 implies that $\alpha_p(n) = 2n/(p-1) + O(\log n/\log p)$ when $p \in \mathcal{P}_f$. On the other hand, we have that $\alpha_p^*(n) = 2n/p + O(1)$. Thus, $\alpha_p(n) - \alpha_p^*(n) = 2n/p(p-1) + O(\log n/\log p)$.

(ii) The first assertion has been explained at the beginning of the subsection. For the second, if $p \nmid f(i)$ for any $i \leqslant n$, then $\beta_p(n) = \beta_p^*(n) = 0$. And, if $p | f(i)$ for some $i \leqslant n$, we have that $\beta_p^*(n) = \beta_p(n) = 1$, since $p^2 \nmid f(i)$. □

By substituting (16) and (14) in (15) for small primes, we obtain

$$\log L_n(f) = 2n \log n + n\left(\log a - 2 - \sum_{p|2aD} \sum_{k\geqslant 1} \frac{s(f,p^k)\log p}{p^k}\right) - \sum_{\substack{p<n^{2/3} \\ p\nmid 2aD}} \frac{(1+(D/p))\log p}{p-1} + \sum_{\substack{n^{2/3}\leqslant p<Cn \\ p\in\mathcal{P}_f}} (\beta_p(n)-\alpha_p(n))\log p + O(n^{2/3}). \quad (27)$$

Now we split the last sum in (27) into

$$\begin{aligned}\sum_{\substack{n^{2/3}\leqslant p<Cn \\ p\in\mathcal{P}_f}} (\beta_p(n)-\alpha_p(n))\log p &= \sum_{\substack{n^{2/3}\leqslant p<Cn \\ p\in\mathcal{P}_f}} (\beta_p(n)-\beta_p^*(n)-\alpha_p(n)+\alpha_p^*(n))\log p \\ &\quad + \sum_{\substack{p<Cn \\ p\in\mathcal{P}_f}} \beta_p^*(n)\log p - \sum_{\substack{n^{2/3}\leqslant p<Cn \\ p\in\mathcal{P}_f}} \alpha_p^*(n)\log p + O(n^{2/3}) \\ &= S_1(n) + S_2(n) - S_3(n) + O(n^{2/3}). \end{aligned} \quad (28)$$

To estimate $S_1(n)$, we observe that Lemma 5(ii) implies that $\beta_p(n) - \beta_p^*(n) - \alpha_p(n) + \alpha_p^*(n) = 0$ for any good prime p. On the other hand, Lemma 5(i) and (14) imply that $|\beta_p(n) - \beta_p^*(n) - \alpha_p(n) + \alpha_p^*(n)| \ll \log n/\log p$. Thus,

$$|S_1(n)| \ll \log n \ |\{p : n^{2/3} < p < Cn, p \text{ bad}\}|. \quad (29)$$

Lemma 6. *The number of bad primes $p \nmid D$, $Q \leqslant p < 2Q$ is $\ll n^2/Q^2$.*

Proof. Let P_r be the set of all primes p such that $f(i) = ai^2 + bi + c = rp^2$ for some $i \leqslant n$. For $p \in P_r$, we have $(2ai+b)^2 - 4arp^2 = D$ and, then, $|(2ai+b)/p - 2\sqrt{ra}| \ll 1/p^2 \ll 1/Q^2$. We observe that all the fractions $(2ai+b)/p$, $1 \leqslant i \leqslant n$, $Q \leqslant p < 2Q$ are pairwise different. Otherwise, $(2ai+b)p' = (2ai'+b)p$ and then $p|2ai+b$. But it would imply that $p|(2ai+b)^2 - 4arp^2 = D$, which is not possible. On the other hand, $|(2ai+b)/p - (2ai'+b)/p'| \geqslant 1/pp' \gg 1/Q^2$. Thus, the number of primes $p \in P_r$ lying in $[Q, 2Q]$ is $\ll 1$. We finish the proof by observing that $r \leqslant f(n)/Q^2 \ll n^2/Q^2$. □

Now, if we split the interval $[n^{2/3}, Cn]$ into dyadic intervals and apply the lemma above to each interval, we obtain $|S_1(n)| \ll n^{2/3}\log n$.

To estimate $S_3(n) = \sum_{n^{2/3}<p<Cn, p\in\mathcal{P}_f} \alpha_p^*(n)$, we start by writing

$$\begin{aligned}\alpha_p^*(n) &= \left[\frac{n-\nu_{p,1}}{p}\right] + \left[\frac{n-\nu_{p,2}}{p}\right] + 2 \\ &= \frac{2n}{p} + \left(\frac{1}{2} - \frac{\nu_{p,1}}{p}\right) + \left(\frac{1}{2} - \frac{\nu_{p,2}}{p}\right) + \frac{1}{2} - \left\{\frac{n-\nu_{p,1}}{p}\right\} + \frac{1}{2} - \left\{\frac{n-\nu_{p,2}}{p}\right\}.\end{aligned}$$

Thus,

$$S_3(n) = n \sum_{n^{2/3}<p<Cn} \frac{(1+(D/p))\log p}{p} \tag{30}$$

$$+ \sum_{\substack{n^{2/3}<p<Cn \\ 0\leqslant\nu<p \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \frac{\nu}{p}\right)\log p + \sum_{\substack{n^{2/3}<p<Cn \\ 0\leqslant\nu<p \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \left\{\frac{n-\nu}{p}\right\}\right)\log p \tag{31}$$

$$= n \sum_{n^{2/3}<p<Cn} \frac{(1+(D/p))\log p}{p-1} + O(n^{2/3}) \tag{32}$$

$$+ \sum_{\substack{0\leqslant\nu<p<Cn \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \frac{\nu}{p}\right)\log p + \sum_{\substack{0\leqslant\nu<p<Cn \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \left\{\frac{n-\nu}{p}\right\}\right)\log p. \tag{33}$$

Substituting this in (28) and then in (27), we obtain

$$\log L_n(f) = 2n\log n + n\left(\log a - 2 - \sum_{p\mid 2aD}\sum_{k\geqslant 1}\frac{s(f,p^k)\log p}{p^k}\right)$$

$$- \sum_{\substack{p<Cn \\ p\nmid 2aD}} \frac{(1+(D/p))\log p}{p-1} + S_2(n) - T_1(n) - T_2(n) + O(n^{2/3}\log n), \tag{34}$$

where

$$S_2(n) = \sum_{\substack{p<Cn \\ p\in\mathcal{P}_f}} \beta_p^*(n)\log p \tag{35}$$

$$T_1(n) = \sum_{\substack{0\leqslant\nu<p<Cn \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \frac{\nu}{p}\right)\log p \tag{36}$$

$$T_2(n) = \sum_{\substack{0\leqslant\nu<p<Cn \\ f(\nu)\equiv 0 \pmod p}} \left(\frac{1}{2} - \left\{\frac{n-\nu}{p}\right\}\right)\log p. \tag{37}$$

The sums $T_1(n)$ and $T_2(n)$ will be $o(n)$ as a consequence of Theorem 2. But this is not completely obvious and we will provide a detailed proof in the next subsection.

First we will obtain in the next lemma a simplified expression for (34).

Lemma 7. *For any irreducible quadratic polynomial we have*

$$\log L_n(f) = n\log n + cn + S_2(n) - T_1(n) - T_2(n) + O(n^{2/3}\log n), \tag{38}$$

where

$$c = \log a - \log C - 2 + \gamma - \sum_{p\nmid 2aD}\frac{(d/p)\log p}{p-1} + \sum_{p\mid 2aD}\log p\left(\frac{1}{p-1} - \sum_{k\geqslant 1}\frac{s(f,p^k)}{p^k}\right)$$

and $S_2(n)$, $T_1(n)$ and $T_2(n)$ are as in (35)–(37).

Proof. Let $D = l^2 d$, where d is a fundamental discriminant. First we observe that $(D/p) = (l/p)^2(d/p)$ and that if $p\nmid D$, then $(D/p) = (d/p)$. As a consequence of the prime number theorem

on arithmetic progressions, we know that the sum $\sum_p (d/p)\log p/(p-1)$ is convergent. On the other hand, the well-known estimate $\sum_{p\leqslant x}\log p/(p-1)=\log x-\gamma+o(1)$, where γ is the Euler constant, implies that

$$\sum_{\substack{p<Cn\\ p\nmid 2aD}}\frac{(1+(D/p))\log p}{p-1}=\log n+\log C-\gamma-\sum_{p\mid 2aD}\frac{\log p}{p-1}+\sum_{p\nmid 2aD}\frac{(d/p)\log p}{p-1}+o(1). \tag{39}$$

Finally, we substitute (39) in (34). □

2.6 Equidistribution of the roots (mod p) of a quadratic polynomial

Now we develop a method to prove that $T_1(n)$, $T_2(n)$ and other similar sums which will appear in the estimate of $S_2(n)$ are all $o(n)$. These sums are all of the form

$$\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x)\log p \tag{40}$$

for some function $a(\nu,p,x)\ll 1$. By partial summation, we also get easily that

$$\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x)\log p=\log x\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x)-\int_1^x\frac{1}{t}\sum_{\substack{0\leqslant\nu<p\leqslant t,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x) \tag{41}$$

$$=\log x\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x)+o(x/\log x). \tag{42}$$

Hence, to prove that the sums (40) are $o(x)$, we must show that

$$\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} a(\nu,p,x)=o(x/\log x).$$

Theorem 2 implies, in particular, that for any arithmetic progression S and for any piecewise-continuous function g in $[0,1]$ such that $\int_0^1 g=0$, we have that

$$\sum_{\substack{0\leqslant\nu<p\leqslant x,p\in S\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}} g(\nu/p)=o(x/\log x). \tag{43}$$

Lemma 8. *Let f be an irreducible polynomial in $\mathbb{Z}[x]$. We have that the sums $T_1(n)$ and $T_2(n)$ defined in (36) and (37) are both $o(n)$.*

Proof. To prove that $T_1(n)=o(n)$, we apply (43) to the function $g(x)=x-1/2$.

To prove that $T_2(n)=o(n)$, the strategy is splitting the range of the primes into small intervals such that the n/p are almost constant in each interval. We take H a large, but fixed, number and we divide the interval $[1,Cn]$ into H intervals $L_h=(((h-1)/H)Cn,(h/H)Cn]$, $h=1,\dots,H$. Now we write

$$\sum_{\substack{0\leqslant\nu<p<n\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)}}\left(\left\{\frac{n-\nu}{p}\right\}-\frac{1}{2}\right)=\Sigma_{31}+\Sigma_{32}+\Sigma_{33}+O(n/(H^{1/3}\log n)), \tag{44}$$

where

$$\Sigma_{31} = \sum_{H^{2/3}\leqslant h\leqslant H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p}} \left(\left\{\frac{H}{h}-\frac{\nu}{p}\right\}-\frac{1}{2}\right),$$

$$\Sigma_{32} = \sum_{H^{2/3}\leqslant h\leqslant H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p\\ \nu/p\notin[H/h,H/(h-1)]}} \left(\left\{\frac{n}{p}-\frac{\nu}{p}\right\}-\left\{\frac{H}{h}-\frac{\nu}{p}\right\}\right),$$

$$\Sigma_{33} = \sum_{H^{2/3}\leqslant h\leqslant H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p\\ \nu/p\in[H/h,H/(h-1)]}} \left(\left\{\frac{n}{p}-\frac{\nu}{p}\right\}-\left\{\frac{H}{h}-\frac{\nu}{p}\right\}\right).$$

To estimate Σ_{31}, we apply (43) with the function $\{H/h - x\} - 1/2$ in each L_h and we obtain

$$\Sigma_{31} = o(Hn/\log n) = o(n\log n), \tag{45}$$

since H is a constant.

To bound Σ_{32}, we observe that if $p\in L_h$ and $\nu/p\notin[H/h, H/(h-1)]$, then

$$0\leqslant \left\{\frac{n}{p}-\frac{\nu}{p}\right\}-\left\{\frac{H}{h}-\frac{\nu}{p}\right\} = \frac{n}{p}-\frac{H}{h}\leqslant \frac{H}{h(h-1)}.$$

Thus,

$$|\Sigma_{32}| \ll \sum_{H^{2/3}\leqslant h<H}\sum_{p\in L_h}\frac{H}{h^2} \ll \sum_{H^{2/3}\leqslant h<H}\sum_{p\in L_h}\frac{1}{H^{1/3}} \ll \frac{\pi(n)}{H^{1/3}} \ll \frac{n}{H^{1/3}\log n}. \tag{46}$$

To bound Σ_{33}, first we observe that

$$\begin{aligned}\Sigma_{33} &\ll \sum_{H^{2/3}\leqslant h<H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p\\ \nu/p\in[H/h,H/(h-1)]}} 1\\ &= \sum_{H^{2/3}\leqslant h<H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p}} \left(\chi_{[H/h,H/(h-1)]}(\nu/p)-\frac{H}{h(h-1)}\right)\\ &\quad + \sum_{H^{2/3}\leqslant h<H} \sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p}} \frac{H}{h(h-1)},\end{aligned}$$

where, here and later, $\chi_{[a,b]}(x)$ denotes the characteristic function of the interval $[a,b]$.

Theorem 2 implies that

$$\sum_{\substack{0\leqslant \nu<p\in L_h\\ f(\nu)\equiv 0 \pmod p}} \left(\chi_{[H/h,H/(h-1)]}(\nu/p)-\frac{H}{h(h-1)}\right) = o(n/\log n).$$

Thus,

$$\Sigma_{33} \ll \sum_{H^{2/3}\leqslant h<H} o\left(\frac{n}{\log n}\right) + \sum_{H^{2/3}\leqslant h<H} \sum_{\substack{0\leqslant \nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p}} \frac{1}{H^{1/3}}$$

$$\ll o(n/\log n) + \frac{\pi(n)}{H^{1/3}} \ll o(n/\log n) + O(n/(H^{1/3}\log n)). \tag{47}$$

Estimates (45), (46) and (47) imply that $\Sigma_3 \ll o(n/\log n) + n/(H^{1/3}\log n)$. Since H can be chosen arbitrarily large, we have that $\Sigma_3 = o(n/\log n)$, which finishes the proof. □

To present Lemma 10, we need some previous considerations.

For primes $p \in \mathcal{P}_f$, the congruence $f(x) \equiv 0 \pmod p$ has exactly two solutions, say $0 \leqslant \nu_{p,1}, \nu_{p,2} < p$.

In some parts of the proof of Theorem 1, we will need to estimate some quantities depending on $\min(\nu_{p,1}, \nu_{p,2})$. For this reason it is convenient to know how they are related.

If $f(x) = ax^2 + bx + c$ and $p \in \mathcal{P}_f$, then $\nu_{p,1} + \nu_{p,2} \equiv -b/a \pmod p$. The next lemma will give more information when the prime p belongs to some particular arithmetic progression.

Lemma 9. *Let $q = a/(a,b)$, $l = b/(a,b)$. For any r, $(r,q) = 1$ and for any prime $p \equiv lr^{-1} \pmod q$ and $p \in \mathcal{P}_f$, we have*

$$\frac{\nu_{p,1}}{p} + \frac{\nu_{p,2}}{p} \equiv \frac{r}{q} - \frac{l}{pq} \pmod 1. \tag{48}$$

Proof. To avoid confusion, we denote by $\overline{q}_p$ and $\overline{p}_q$ the inverses of $q \pmod p$ and $p \pmod q$, respectively. From the obvious congruence $q\overline{q}_p + p\overline{p}_q \equiv 1 \pmod{pq}$, we deduce that $\overline{q}_p/p + \overline{p}_q/q - 1/pq \in \mathbb{Z}$. Since $p \equiv l\overline{r}_q \pmod q$, we obtain $\overline{q}_p/p \equiv 1/pq - r\overline{l}_q/q \pmod 1$. Thus,

$$\frac{\nu_{p,1}}{p} + \frac{\nu_{p,2}}{p} \equiv \frac{-l\overline{q}_p}{p} \equiv -l\left(\frac{1}{pq} - \frac{r\overline{l}_q}{q}\right) \equiv \frac{r}{q} - \frac{l}{pq} \pmod 1. \qquad \square$$

Since the two roots are symmetric with respect to $r/2q - l/2pq$, necessarily one of them lies in $[r/2q - l/2pq, 1/2 + r/2q - l/2pq) \pmod 1$ and the other in the complementary set.

Definition 2. For $(r,q) = 1, 1 \leqslant r \leqslant q, p \equiv lr^{-1} \pmod q$ and $p \in \mathcal{P}_f$, we define $\nu_{p,1}$ to be the root of $f(x) \equiv \pmod p$ such that

$$\frac{\nu_{p,1}}{p} \in T_{rp} = \left[\frac{r}{2q} - \frac{l}{2pq}, \frac{1}{2} + \frac{r}{2q} - \frac{l}{2pq}\right) \pmod 1,$$

and we define $\nu_{p,2}$ to be the root of $f(x) \equiv 0 \pmod p$ such that $\nu_{p,2}/p \in [0,1)\backslash T_{rp}$.

Lemma 10. *Assume the notation above. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, c_1, c_2$ be constants and $g_1(x), g_2(x)$ two linear functions satisfying*

$$J_n(p) = \left[g_1\left(\frac{n}{p}\right) + \frac{c_1}{p}, g_2\left(\frac{n}{p}\right) + \frac{c_2}{p}\right] \subset T_{rp}$$

for any prime $p \in K_n = [\alpha_1 n + \beta_1, \alpha_2 n + \beta_2]$. We have

$$\sum_{\substack{p\in K_n\cap\mathcal{P}_f \\ p\equiv lr^{-1} \pmod q}} \left(\chi_{J_n(p)}\left(\frac{\nu_{p,1}}{p}\right) - 2|J_n(p)|\right)\log p = o(n), \tag{49}$$

where χ_I is the characteristic function of the set I.

Proof. Since $J_n(p) \subset T_{rp}$, then $\nu_2/p \notin J_n(p)$ and we can write

$$\sum_{\substack{p\in K_n\cap\mathcal{P}_f\\ p\equiv lr^{-1}\,(\mathrm{mod}\ q)}} \chi_{J_n(p)}\left(\frac{\nu_{p,1}}{p}\right)\log p = \sum_{\substack{1\leqslant\nu\leqslant p\in K_n,\\ f(\nu)\equiv 0\,(\mathrm{mod}\ p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \chi_{J_n(p)}\left(\frac{\nu}{p}\right)\log p$$

and

$$\sum_{\substack{p\in K_n\cap\mathcal{P}_f\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} 2|J_n(p)|\log p = \sum_{\substack{1\leqslant\nu\leqslant p\in K_n,\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} |J_n(p)|\log p.$$

Thus,

$$\sum_{\substack{p\in K_n\cap\mathcal{P}_f\\ p\equiv lr^{-1}(\mathrm{mod}\ q)}} \left(\chi_{J_n(p)}\left(\frac{\nu_{p,1}}{p}\right) - 2|J_n(p)|\right)\log p = \sum_{\substack{1\leqslant\nu\leqslant p\in K_n,\\ f(\nu)\equiv 0\ (\mathrm{mod}\ p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{J_n(p)}\left(\frac{\nu}{p}\right) - |J_n(p)|\right)\log p.$$

The proof will be accomplished by showing that

$$\sum_{\substack{1\leqslant\nu\leqslant p\in K_n,\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{J_n(p)}\left(\frac{\nu}{p}\right) - |J_n(p)|\right) = o(n/\log n). \tag{50}$$

We split K_n into intervals $L_h = (((h-1)/H)n, (h/H)n]$ of length n/H and two extra intervals I, F (the initial and the final intervals) of length $\leqslant n/H$. Here h runs over a suitable set of consecutive integers $\mathcal{H}$ of cardinality $\ll(\alpha_2-\alpha_1)H$.

Let I_h denote the interval $[g_1(H/h) + c_1H/(nh), g_2(H/h) + c_2H/(nh)]$.

We write

$$\sum_{\substack{1\leqslant\nu\leqslant p\in K_n,\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{J_n(p)}\left(\frac{\nu}{p}\right) - |J_n(p)|\right) = \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4, \tag{51}$$

where

$$\Sigma_1 = \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{I_h}\left(\frac{\nu}{p}\right) - |I_h|\right),$$

$$\Sigma_2 = \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} (|I_h| - |J_n(p)|),$$

$$\Sigma_3 = \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{J_n(p)}\left(\frac{\nu}{p}\right) - \chi_{I_h}\left(\frac{\nu}{p}\right)\right),$$

$$\Sigma_4 = \sum_{\substack{0\leqslant\nu<p\in I\cup F\\ f(\nu)\equiv 0\,(\mathrm{mod}\,p)\\ p\equiv lr^{-1}\,(\mathrm{mod}\,q)}} \left(\chi_{I_h}\left(\frac{\nu}{p}\right) - |J_n(p)|\right).$$

The inner sum in Σ_1 can be estimated as we did in Lemma 8 (with the function $g(x) = \chi_I(x) - |I|$ instead of $g(x) = x - 1/2$), and we get again that $\Sigma_1 = o(n/\log n)$.

To estimate Σ_2 and Σ_3, we observe that if $p \in L_h$, then $J_n(p)$ and I_h are almost equal. Actually, comparing the end points of both intervals and because g is a linear function, we have that $\chi_{J_n(p)}(x) = \chi_{I_h}(x)$ except for an interval (or union of two intervals) E_h of measure

$$|E_h| \ll \min(1, H/h^2).$$

In particular, the estimate $||J_n(p)| - |I_h|| \ll \min(1, H/h^2)$ holds.

Thus, we have

$$\begin{aligned}\Sigma_2 &\ll \sum_{h\in\mathcal{H}} \sum_{p\in L_n} \min(1, H/h^2) \ll \sum_{h\leqslant H^{2/3}} \sum_{p\in L_h} 1 + \sum_{H^{2/3}<h\in\mathcal{H}} \sum_{p\in L_h} \frac{1}{H^{1/3}} \\ &\ll \pi(n/H^{1/3}) + \frac{1}{H^{1/3}}\pi(\alpha_1 n + \alpha_2) \ll n/(H^{1/3}\log n).\end{aligned}$$

To bound Σ_3, first we observe that

$$\begin{aligned}\Sigma_3 &\ll \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p \\ p\equiv lr^{-1} \pmod q}} \chi_{E_h}(\nu/p) \\ &= \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p \\ p\equiv lr^{-1} \pmod q}} (\chi_{E_h}(\nu/p) - |E_h|) + \sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p \\ p\equiv lr^{-1} \pmod q}} |E_h|.\end{aligned}$$

Theorem 2 implies that

$$\sum_{\substack{0\leqslant\nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p \\ p\equiv lr^{-1} \pmod q}} (\chi_{E_h}(\nu/p) - |E_h|) = o(n/\log n).$$

On the other hand,

$$\begin{aligned}\sum_{h\in\mathcal{H}} \sum_{\substack{0\leqslant\nu<p\in L_h \\ f(\nu)\equiv 0 \pmod p \\ p\equiv lr^{-1} \pmod q}} |E_h| &\ll \sum_{h\leqslant H^{2/3}} \sum_{p\in L_h} 1 + \sum_{H^{2/3}<h\in\mathcal{H}} \sum_{p\in L_h} \frac{H}{h^2} \\ &\ll \pi(n/H^{1/3}) + \frac{1}{H^{1/3}}\pi(\alpha_1 n + \alpha_2) \\ &\ll \frac{n}{H^{1/3}\log n}.\end{aligned}$$

Thus, $\Sigma_3 \ll o(n/\log n) + n/(H^{1/3}\log n)$.

Finally, we estimate Σ_4. We observe that

$$|\Sigma_4| \leqslant \sum_{p\in I} 1 + \sum_{p\in F} 1 \ll n/(H\log n)$$

as a consequence of the prime number theorem. Then

$$\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 = O(n/(H^{1/3}\log n)) + O(n/(H\log n)) + o(n/\log n),$$

finishing the proof because we can take H arbitrarily large. □

2.7 Estimate of $S_2(n)$ and end of the proof

Lemma 11. *For $S_2(n)$ defined in (35) we have the estimate*

$$S_2(n) = n\left(1 + \log C - \log 4 + \frac{1}{\phi(q)} \sum_{(r,q)=1} \log\left(1 + \frac{r}{q}\right)\right) + o(n). \tag{52}$$

Proof. Following the notation of Lemma 9, we split

$$S_2(n) = \sum_{\substack{(r,q)=1\\ 1\leqslant r\leqslant q}} S_{2r}(n) + \sum_{p\leqslant l} \beta_p^*(n)\log p = \sum_{\substack{(r,q)=1\\ 1\leqslant r\leqslant q}} S_{2r}(n) + O(1),$$

where

$$S_{2r}(n) = \sum_{\substack{l<p\leqslant Cn\\ p\equiv lr^{-1} \pmod q}} \beta_p^*(n)\log p. \tag{53}$$

Since $p \equiv lr^{-1} \pmod q$, Lemma 9 implies that $\nu_{p,1}/p + \nu_{p,2}/p \equiv r/q - l/pq \pmod 1$. We also observe that, since $p > l$, we have that $0 < r/q - l/pq \leqslant 1$.

Now we will check that

$$\beta_p^*(n) = \begin{cases} 1, & \text{if } \dfrac{n}{p} \geqslant \dfrac{1}{2} + \dfrac{r}{2q} - \dfrac{l}{2pq},\\[2ex] \chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p), & \text{if } \dfrac{r}{q} - \dfrac{l}{pq} < \dfrac{n}{p} < \dfrac{1}{2} + \dfrac{r}{2q} - \dfrac{l}{2pq},\\[2ex] \chi_{[r/2q-l/2pq,r/q-l/pq]}(\nu_{p,1}/p), & \text{if } \dfrac{r}{2q} - \dfrac{l}{2pq} \leqslant \dfrac{n}{p} \leqslant \dfrac{r}{q} - \dfrac{l}{pq},\\[2ex] \chi_{[r/q-l/pq-n/p,r/q-l/pq]}(\nu_{p,1}/p), & \text{if } \dfrac{n}{p} < \dfrac{r}{2q} - \dfrac{l}{2pq}. \end{cases}$$

We observe that $\beta_p^*(n) = 1$ if and only if $\nu_{p,1}/p \leqslant n/p$ or $\nu_{p,2}/p \leqslant n/p$. We recall that

$$\frac{r}{2q} - \frac{l}{2pq} \leqslant \frac{\nu_{p,1}}{p} < \frac{1}{2} + \frac{r}{2q} - \frac{l}{2pq}. \tag{54}$$

Also, we observe that Lemma 9 implies that

$$\frac{\nu_{p,2}}{p} = \begin{cases} \dfrac{r}{q} - \dfrac{l}{pq} - \dfrac{\nu_{p,1}}{p}, & \text{if } \dfrac{\nu_{p,1}}{p} \leqslant \dfrac{r}{q} - \dfrac{l}{pq},\\[2ex] \dfrac{r}{q} - \dfrac{l}{pq} - \dfrac{\nu_{p,1}}{p} + 1, & \text{if } \dfrac{\nu_{p,1}}{p} > \dfrac{r}{q} - \dfrac{l}{pq}. \end{cases} \tag{55}$$

- Assume that $n/p \geqslant 1/2 + r/2q - l/2pq$. Then $\nu_{p,1} < p(1/2 + r/2q - l/2pq) < n$, so $\beta_p^*(n) = 1$.
- Assume that $r/q - l/pq < n/p < 1/2 + r/2q - l/2pq$.
  - If $\chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p) = 1$, then $\nu_{p,1} \leqslant n$, so $\beta_p^*(n) = 1$.
  - If $\chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p) = 0$, then $\nu_{p,1}/p > n/p > r/q - l/pq$. Relations (54) and (55) imply that $\nu_{p,2}/p = 1 + r/q - l/pq - \nu_{p,1}/p > 1/2 + r/2q - l/2pq > n/p$. Since $\nu_{p,1} > n$ and $\nu_{p,2} > n$, we get $\beta_p^*(n) = 0$.
- Assume that $r/2q - l/2pq \leqslant n/p \leqslant r/q - l/pq$.
  - If $\chi_{[r/2q-l/2pq,r/q-l/pq]}(\nu_{p,1}/p) = 1$, then (55) imply that $0 < \nu_{p,2}/p \leqslant r/2q - l/2pq$, which implies that $\nu_{p,2} \leqslant n$, so $\beta_p^*(n) = 1$.

* If $\chi_{[r/2q-l/2pq,r/q-l/pq]}(\nu_{p,1}/p)=0$, then $\nu_{p,1}/p>r/q-l/pq\geqslant n/p$ and (55) imply that $\nu_{p,2}/p=r/q-l/pq-\nu_{p,1}/p+1>r/q-l/pq\geqslant n/p$. Since $\nu_{p,1}>n$ and $\nu_{p,2}>n$, we get $\beta_p^*(n)=0$.

– Assume that $n/p<r/2q-l/2pq$.

* If $\chi_{[r/q-l/pq-n/p,r/q-l/pq]}(\nu_{p,1}/p)=1$, then $\nu_{p,1}/p\leqslant r/q-l/pq$ and (55) imply that $\nu_{p,2}/p=r/q-l/pq-\nu_{p,1}/p\leqslant r/q-l/pq-(r/q-l/pq-n/p)=n/p$, so $\beta_p^*(n)=1$.
* If $\chi_{[r/q-l/pq-n/p,r/q-l/pq]}(\nu_{p,1}/p)=0$, we distinguish two cases:
  · if $r/2q-l/2q\leqslant\nu_{p,1}/p<r/q-l/pq-n/p$, then $\nu_1,p/p\geqslant r/2q-l/2q>n/p$, and also we have that $\nu_{p,2}/p=r/q-l/pq-\nu_{p,1}/p>r/q-l/pq-(r/q-l/pq-n/p)=n/p$. Thus, $\beta_p^*(n)=0$;
  · if $r/q-l/pq<\nu_{p,1}/p<1/2+r/2q-l/2pq$, then $\nu_{p,1}/p>(1/2)(r/q-l/pq)>n/p$. On the other hand, $\nu_{p,2}/p=r/q-l/pq-\nu_{p,1}/p+1>r/q-l/pq-(1/2+r/2q-l/2pq)+1=1/2+r/2q-l/2pq>n/p$. Thus, again we have that $\beta_p^*(n)=0$.

Now we split $S_{2r}(n)=\sum_{i=1}^4 S_{2ri}(n)$ according to the ranges of the primes involved in the lemma above.

$$S_{2r1}(n)=\sum_{\substack{l<p\leqslant(n+l/(2q))/(1/2+r/(2q))\\ p\equiv lr^{-1}\pmod q\\ p\in\mathcal{P}_f}}\log p,$$

$$S_{2r2}(n)=\sum_{\substack{(n+l/(2q))/(1/2+r/(2q))<p<(n+l/q)/(r/q)\\ p\equiv lr^{-1}\pmod q\\ p\in\mathcal{P}_f}}\chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p)\log p,$$

$$S_{2r3}(n)=\sum_{\substack{(q/r)(n+l/q)\leqslant p\leqslant(2q/r)(n+l/q)\\ p\equiv lr^{-1}\pmod q\\ p\in\mathcal{P}_f}}\chi_{[r/2q-l/2pq,r/q-l/pq]}(\nu_{p,1}/p)\log p,$$

$$S_{2r4}(n)=\sum_{\substack{(2q/r)(n+l/2q)<p<Cn\\ p\equiv lr^{-1}\pmod q\\ p\in\mathcal{P}_f}}\chi_{[r/q-l/pq-n/p,r/q-l/pq]}(\nu_{p,1}/p)\log p.$$

Since $(q,D)=1$ and the primes are odd numbers, the primes $p\equiv lr^{-1}\pmod q$, $p\in\mathcal{P}_f$ lie in a set of $\phi(4qD)/(2\phi(q))$ arithmetic progressions modulo $4qD$. The prime number theorem for arithmetic progressions implies that

$$\sum_{\substack{p\leqslant x\\ p\equiv lr^{-1}\pmod q, p\in P_f}}\log p\sim\frac{x}{2\phi(q)}\tag{56}$$

and

$$\sum_{\substack{ax<p\leqslant bx\\ p\equiv lr^{-1}\pmod q, p\in P_f}}\frac{\log p}{p}=\frac{\log(b/a)}{2\phi(q)}+o(1).\tag{57}$$

We will use these estimates and Lemma 10 to estimate $S_{2ri}(n)$, $i=1,2,3,4$.

By (56), we have

$$S_{2r1}(n)=\frac{n}{\phi(q)}\frac{q}{q+r}+o(n).\tag{58}$$

To estimate S_{2r2}, we write

$$\begin{aligned}
S_{2r2}(n) &= \sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q}} \chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p)\log p\\
&= \sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q}} \left(\frac{2n}{p}-\frac{r}{q}+\frac{l}{pq}\right)\log p\\
&\quad+\sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q}} \left(\chi_{[r/2q-l/2pq,n/p]}(\nu_{p,1}/p)\right.\\
&\quad\left.-2\left(\frac{n}{p}-\frac{r}{2q}+\frac{l}{2pq}\right)\right)\log p.
\end{aligned}$$

Lemma 10 implies that the last sum is $o(n)$. Thus,

$$\begin{aligned}
S_{2r2} &= \sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q\\ p\in\mathcal{P}_f}} \left(\frac{2n}{p}-\frac{r}{q}\right)\log p+o(n)\\
&= 2n \sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q\\ p\in\mathcal{P}_f}} \frac{\log p}{p}\\
&\quad-\frac{r}{q}\sum_{\substack{((n+l)/(2q))/(1/2+r/(2q))<p<((n+l)/q)/(r/q)\\ p\equiv lr^{-1} \pmod q\\ p\in\mathcal{P}_f}} \log p+o(n)\\
&= \frac{n}{\phi(q)}\log\left(\frac{1}{2}+\frac{q}{2r}\right)-\frac{n}{\phi(q)}\left(\frac{1}{2}-\frac{r}{q+r}\right)+o(n)
\end{aligned}$$

by (56) and (57).

To estimate $S_{2r3}(n)$, we write

$$\begin{aligned}
S_{2r3}(n) &= \sum_{\substack{(q/r)(n+l/q)\leqslant p\leqslant(2q/r)(n+l/q)\\ p\equiv lr^{-1} \pmod q\\ p\in\mathcal{P}_f}} \left(\frac{r}{q}-\frac{l}{pq}\right)\log p\\
&\quad+\sum_{\substack{(q/r)(n+l/q)\leqslant p\leqslant(2q/r)(n+l/q)\\ p\equiv lr^{-1} \pmod q\\ p\in\mathcal{P}_f}} \left(\chi_{[r/2q-l/2pq,r/q-l/pq]}(\nu_{p,1}/p)-\left(\frac{r}{q}-\frac{l}{pq}\right)\right)\log p\\
&= \frac{n}{2\phi(q)}+o(n)
\end{aligned}$$

by (56) and Lemma 10.

To estimate $S_{2r4}(n)$, we write

$$\begin{aligned} S_{2r4}(n) &= \sum_{\substack{(2q/r)(n+l/2q)<p<Cn \\ p\equiv lr^{-1} \pmod q \\ p\in\mathcal{P}_f}} \left(\frac{2n}{p}+\frac{2l}{pq}\right)\log p \\ &= \sum_{\substack{(2q/r)(n+l/2q)<p<Cn \\ p\equiv lr^{-1} \pmod q \\ p\in\mathcal{P}_f}} \left(\chi_{[r/q-l/pq-n/p,r/q-l/pq]}(\nu_{p,1}/p) - \left(\frac{2n}{p}+\frac{2l}{pq}\right)\right)\log p \\ &= \frac{n}{\phi(q)}(\log C - \log(2q/r)) + o(n) \end{aligned}$$

by (57) and Lemma 10.

Thus,

$$\begin{aligned} S_{2r}(n) &= S_{2r1}(n) + S_{2r2}(n) + S_{2r3}(n) + S_{2r4}(n) + O(1) \\ &= \frac{n}{\phi(q)}\frac{q}{q+r} + o(n) \\ &\quad + \frac{n}{\phi(q)}\log\left(\frac{1}{2}+\frac{q}{2r}\right) - \frac{n}{\phi(q)}\left(\frac{1}{2}-\frac{r}{q+r}\right) + o(n) \\ &\quad + \frac{n}{2\phi(q)} + o(n) \\ &\quad + \frac{n}{\phi(q)}(\log C - \log(2q/r)) + o(n) \\ &= \frac{n}{\phi(q)}(1 + \log C - \log 4 + \log(1+r/q)) + o(n). \end{aligned}$$

Now sum over all $r \leqslant q$, $(r,q)=1$ to finish the estimate of $S_2(n)$. □

Finally, we substitute (52) in (38) to conclude the proof of Theorem 1.

3. Computation of the constant B_f

The sum $\sum_p (d/p)\log p/(p-1)$, appearing in the formula of the constant B_f, converges very slowly. The next lemma gives an alternative expression for this sum, more convenient in order to obtain a fast computation.

Lemma 12. *We have the identity*

$$\sum_p \frac{(d/p)\log p}{p-1} = \sum_{k=1}^{\infty}\frac{\zeta'(2^k)}{\zeta(2^k)} - \sum_{k=0}^{\infty}\frac{L'(2^k,\chi_d)}{L(2^k,\chi_d)} + \sum_{p\mid d} s_p, \tag{59}$$

where $s_p = \sum_{k=1}^{\infty}\log p/(p^{2^k}-1)$.

Proof. For $s>1$, we consider the function $G_d(s) = \prod_p (1-1/p^s)^{(d/p)}$. Taking the derivative of the logarithm of $G_d(s)$, we obtain that

$$\frac{G_d'(s)}{G_d(s)} = \sum_p \frac{(d/p)\log p}{p^s-1}. \tag{60}$$

Since $L(s,\chi_d)=\prod_p(1-(d/p)^s/p)^{-1}$, we have

$$G_d(s)L(s,\chi_d)=\prod_p\Big(1-\frac{1}{p^s}\Big)^{(d/p)}\Big(1-\frac{(d/p)}{p^s}\Big)^{-1} \tag{61}$$

$$=\prod_{(d/p)=-1}\Big(1-\frac{1}{p^{2s}}\Big)^{-1} \tag{62}$$

$$=\prod_p\Big(1-\frac{1}{p^{2s}}\Big)^{((d/p)-1)/2}\prod_{p|d}\Big(1-\frac{1}{p^{2s}}\Big)^{1/2} \tag{63}$$

$$=G_d^{1/2}(2s)\zeta^{1/2}(2s)T^{1/2}(2s), \tag{64}$$

where $T(s)=\prod_{p|d}(1-1/p^s)$.

The derivative of the logarithm gives

$$\frac{G_d'(s)}{G_d(s)}-\frac{G_d'(2s)}{G_d(2s)}=\frac{\zeta'(2s)}{\zeta(2s)}+\frac{T_d'(2s)}{T_d(2s)}-\frac{L'(s,\chi_d)}{L(s,\chi_d)}.$$

Thus,

$$\frac{G_d'(s)}{G_d(s)}-\frac{G_d'(2^ms)}{G_d(2^ms)}=\sum_{k=0}^{m-1}\Big(\frac{G_d'(2^ks)}{G_d(2^ks)}-\frac{G_d'(2^{k+1}s)}{G_d(2^{k+1}s)}\Big) \tag{65}$$

$$=\sum_{k=1}^{m}\frac{\zeta'(2^ks)}{\zeta(2^ks)}+\sum_{k=1}^{m}\frac{T_d'(2^ks)}{T_d(2^ks)}-\sum_{k=0}^{m-1}\frac{L'(2^ks,\chi_d)}{L(2^ks,\chi_d)}. \tag{66}$$

By (60), we have that, for $s\geqslant 2$,

$$\begin{aligned}\Big|\frac{\zeta'(s)}{\zeta(s)}\Big|&\leqslant\sum_{n\geqslant 2}\frac{\Lambda(n)}{n^s-1}\leqslant\frac{\log 2}{2^s-1}+\sum_{n\geqslant 3}\frac{\log n}{n^s-1}\\&\leqslant\frac{4}{3}\frac{\log 2}{2^s}+\frac{9}{8}\sum_{n\geqslant 3}\frac{\log n}{n^s}\leqslant\frac{4}{3}\frac{\log 2}{2^s}+\frac{9}{8}\int_2^\infty\frac{\log x}{x^s}\,dx\\&=\frac{4}{3}\frac{\log 2}{2^s}+\frac{9}{8}\Big(\frac{\log 2}{2^{s-1}(s-1)}+\frac{1}{2^{s-1}(s-1)^2}\Big)\\&\leqslant\frac{1}{2^s(s-1)}\Big(\frac{20\log 2+8}{9}\Big)\leqslant\frac{5}{2}\cdot\frac{2^{-s}}{s-1}.\end{aligned}$$

Thus, $|\zeta'(2^k)/\zeta(2^k)|\leqslant(5/2)\cdot(2^{-2^k}/(2^k-1))$. The same estimate holds for $|G_d'(2^k)/G_d(2^k)|$, $|T_d'(2^k)/T_d(2^k)|$ and $|L'(2^k,\chi_d)/L(2^k,\chi_d)|$. When $m\to\infty$ and then $s\to 1$, we get

$$\sum_p\frac{(d/p)\log p}{p-1}=\sum_{k=1}^{\infty}\frac{\zeta'(2^k)}{\zeta(2^k)}-\sum_{k=0}^{\infty}\frac{L'(2^k,\chi_d)}{L(2^k,\chi_d)}+\sum_{k=1}^{\infty}\frac{T_d'(2^k)}{T_d(2^k)}. \tag{67}$$

Finally, we observe that $T_d'(2^k)/T_d(2^k)=\sum_{p|d}\log p/(p^{2^k}-1)$, so $\sum_{k=1}^{\infty}T_d'(2^k)/T_d(2^k)=\sum_{p|d}s_p$. □

The advantage of the lemma above is that the series involved converge very fast. For example,

$$\sum_{k=0}^{\infty}\frac{L'(2^k,\chi_d)}{L(2^k,\chi_d)}=\sum_{k=0}^{6}\frac{L'(2^k,\chi_d)}{L(2^k,\chi_d)}+\text{Error}$$

with $|\text{Error}|\leqslant 10^{-40}$.

Hence, we can write $B_f = C_0 + C_d + C(f)$, where C_0 is an universal constant, C_d depends only on d and $C(f)$ depends on f. More precisely,

$$C_0 = \gamma - 1 - 2\log 2 - \sum_{k=1}^{\infty} \frac{\zeta'(2^k)}{\zeta(2^k)} = -1.172\,547\,167\,419\,014\,850\,858\,752\,152\,8364\ldots,$$

$$C_d = \sum_{k=0}^{\infty} \frac{L'(2^k,\chi_d)}{L(2^k,\chi_d)} - \sum_{p|d} s_p,$$

$$C(f) = \frac{1}{\phi(q)} \sum_{\substack{1\leqslant r\leqslant q\\(r,q)=1}} \log\left(1+\frac{r}{q}\right) + \log a + \sum_{p|2aD} \log p\left(\frac{1+(d/p)}{p-1} - \sum_{k\geqslant 1} \frac{s(f,p^k)}{p^k}\right).$$

The values of s_p and $\sum_{k\geqslant 0} L'(2^k,\chi_d)/L(2^k,\chi_d)$, can be calculated with MAGMA with high precision. We include some of the values of C_d and $C(f)$:

C_{-4}	$= +0.346\,538\,435\,736\,895\,987\,549 - s_2$	$= +0.066\,550\,762\,366\,036\,180\,349\ldots,$
C_{-8}	$= -0.076\,694\,093\,066\,485\,311\,184 - s_2$	$= -0.356\,681\,766\,437\,345\,118\,384\ldots,$
C_{-3}	$= +0.586\,272\,400\,297\,149\,523\,649 - s_3$	$= +0.435\,045\,713\,698\,422\,447\,292\ldots,$
C_{-7}	$= -0.070\,022\,837\,990\,444\,988\,815 - s_7$	$= -0.111\,373\,766\,208\,260\,107\,471\ldots,$
C_{-15}	$= -0.486\,320\,692\,903\,261\,758\,405 - s_3 - s_5$	$= -0.707\,190\,640\,126\,000\,030\,028\ldots,$
$C(x^2+1)$	$= (3\log 2)/2$	$= 1.039\,720\,770\,839\,917\,964\,125\ldots,$
$C(x^2+2)$	$= (3\log 2)/2$	$= 1.039\,720\,770\,839\,917\,964\,125\ldots,$
$C(x^2+x+1)$	$= \log 2 + (\log 3)/6$	$= 0.876\,249\,228\,671\,296\,924\,649\ldots,$
$C(x^2+x+2)$	$= \log 2 + (\log 7)/(42)$	$= 0.739\,478\,374\,585\,071\,816\,681\ldots,$
$C(2x^2+1)$	$= 3\log 2$	$= 2.079\,441\,541\,679\,835\,928\,251\ldots,$
$C(2x^2+x+1)$	$= 2\log 2 + \log 3 + (\log 7)/(42)$	$= 1.838\,090\,663\,253\,181\,508\,076\ldots,$
$C(2x^2+x+2)$	$= \log 2 + (7\log 3)6 + (\log 5)/(20)$	$= 2.055\,333\,412\,961\,111\,634\,775\ldots,$
$C(2x^2+2x+1)$	$= 3\log 2$	$= 2.079\,441\,541\,679\,835\,928\,251\ldots.$

The table below contains the constant $B = B_f$ for all irreducible quadratic polynomials $f(x) = ax^2 + bx + c$ with $0 \leqslant a, b, c \leqslant 2$. When f_1, f_2 are irreducible quadratic polynomials such that $f_1(x) = f_2(x+k)$ for some k, we only include one of them since $L_n(f_1) = L_n(f_2) + O(\log n)$.

$f(x)$	d	q	B_f
x^2+1	-4	1	$-0.066\,275\,634\,213\,060\,706\,38\ldots$
x^2+2	-8	1	$-0.489\,508\,163\,016\,442\,005\,11\ldots$
x^2+x+1	-3	1	$+0.138\,747\,774\,950\,704\,521\,08\ldots$
x^2+x+2	-7	1	$-0.544\,442\,559\,042\,203\,141\,64\ldots$
$2x^2+1$	-8	1	$+0.550\,212\,607\,823\,475\,959\,00\ldots$
$2x^2+x+1$	-7	2	$+0.554\,169\,729\,625\,906\,549\,74\ldots$
$2x^2+x+2$	-15	2	$+0.175\,595\,605\,416\,096\,753\,88\ldots$
$2x^2+2x+1$	-4	1	$+0.973\,445\,136\,626\,857\,257\,74\ldots$

The table below shows the error term $E_f(n) = \log L_n(f) - n\log n - B_f n$ for the polynomials above and some values of n.

$f(x)$	$E_f(10^2)$	$E_f(10^3)$	$E_f(10^4)$	$E_f(10^5)$	$E_f(10^6)$	$E_f(10^7)$
x^2+1	-18	$+6$	-111	$+34$	-2634	-1557
x^2+2	-36	-11	-263	-761	-1462	-8457
x^2+x+1	-6	-9	$+17$	-654	-2528	-1685
x^2+x+2	$+9$	-20	-218	-2120	$+687$	-686
$2x^2+1$	-15	-1	-301	-251	$+1084$	$-14\,821$
$2x^2+x+1$	-1	$+6$	$+18$	-1289	$+235$	-2553
$2x^2+x+2$	-34	$+4$	-295	$+27$	$+1169$	$+1958$
$2x^2+2x+1$	-9	-89	$+9$	-232	-2876	$-10\,624$

4. Quadratic reducible polynomials

To complete the problem of estimating the least common multiple of quadratic polynomials, we will study here the case of reducible quadratic polynomials. As this case is much easier than the irreducible case, we will give a complete description for the sake of completeness.

If $f(x) = ax^2+bx+c$ with $g = (a,b,c) > 1$, it is easy to check that $\log L_n(f) = \log L_n(f') + O(1)$, where $f'(x) = a'x^2+b'x+c'$ with $a' = a/g$, $b' = b/g$, $c' = c/g$.

If $f(x) = (ax+b)^2$ with $(a,b) = 1$, then, since $(m^2,n^2) = (m,n)^2$, we have that $L_n((ax+b)^2) = L_n^2(ax+b)$ and we can apply (1) to get

$$\log \text{l.c.m.}\{(a+b)^2,\dots,(an+b)^2\} \sim 2n\frac{a}{\phi(a)}\sum_{\substack{1\leqslant k\leqslant a\\(k,a)=1}}\frac{1}{k}. \tag{68}$$

Now we consider the more general case $f(x) = (ax+b)(cx+d)$, $(a,b) = (c,d) = 1$.

Theorem 3. *Let $f(x) = (ax+b)(cx+d)$ with $(a,b) = (c,d) = 1$ and $ad \neq bd$. Let $q = ac/(a,c)$. We have*

$$\log \text{l.c.m.}(f(1),\dots,f(n)) \sim \frac{n}{\varphi(q)}\sum_{1\leqslant r\leqslant q,(r,q)=1}\max\left(\frac{a}{(br)_a},\frac{c}{(dr)_c}\right). \tag{69}$$

Proof. Suppose that $p^2|L_n(f)$. It implies that $p^2|(ai+b)(ci+d)$ for some i. If $p|ai+b$ and $p|ci+d$, then $p|(ad-bc)i$. If $p\nmid(ad-bc)$, then $p|i$ and consequently $p|b$ and $p|d$. Thus, if $p\nmid(ad-bc)bd$ and $p^2|(ai+b)(ci+d)$, then $p^2|(ai+b)$ or $p^2|(ci+d)$. In these cases, $p\leqslant M_n = \max(\sqrt{an+b},\sqrt{cn+d},|(ad-bd)bd|)$.

Thus, we write

$$L_n(f) = \prod_{p\leqslant M_n}p^{\beta_p(n)}\prod_{p>M_n}p^{\epsilon_p(n)} = \prod_{p\leqslant M_n}p^{\beta_p(n)-\epsilon_p(n)}\prod_p p^{\epsilon_p(n)}, \tag{70}$$

where $\epsilon_p(n) = 1$ if $p|f(i)$ for some $i\leqslant n$ and $\epsilon_p(n) = 0$ otherwise. Since $p^{\beta_p(n)}\leqslant f(n)$, we have that $\beta_p(n)\ll \log n/\log p$ and then

$$\sum_{p\leqslant M_n}(\beta_p(n)-\epsilon_p(n))\log p \ll (\log n)\pi(M_n) \ll \sqrt{n}. \tag{71}$$

Thus,

$$\log L_n(f) = \sum_{\substack{p|f(i) \\ \text{for some } i\leqslant n}} \log p + O(\sqrt{n}). \tag{72}$$

Let $q = ac/(a,c)$. Suppose that $p \equiv r^{-1} \pmod q$, $(r,q)=1$. Let $k=(br)_a$ be the least positive integer such that $k \equiv br \pmod a$. Then $p|(ai+b)$ for some $i \leqslant n$ if and only if $kp \leqslant an+b$. Similarly, let $j=(dr)_c$ be the least positive integer such that $j \equiv dr \pmod c$. Again, $p|(ci+d)$ for some $\leqslant i \leqslant n$ if $jp \leqslant cn+d$. Thus, the primes $p \equiv r^{-1} \pmod{ac}$ counted in the sum above are those such that $p \leqslant \max((an+b)/k, (cn+d)/j)$. The prime number theorem for arithmetic progressions implies that there are $\sim (n/\varphi(q)) \max(a/k, c/j)$ of such primes.

We finish the proof by summing over all $1 \leqslant r \leqslant q$, $(r,q)=1$. □

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Javier Cilleruelo franciscojavier.cilleruelo@uam.es
Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM) and
Departamento de Matemáticas, Universidad Autónoma de Madrid,
28049 Madrid, Spain