Synthesis of Spatial RPRP Loops for a Given Screw System

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Abstract. The dimensional synthesis of spatial chains for a prescribed set of positions can be used for the design of parallel robots by joining the solutions of each serial chain at the end effector. In some cases, this may yield a system with negative mobility. The synthesis of some overconstrained but movable linkages can be done by comparing the known screw system associated to the motion of the linkage to that generated by the tasks positions defined by the synthesis. This paper present the simplest case, that of the spatial RPRP closed chain, for which one solution exists.

Key words: Dimensional synthesis, overconstrained linkages, finite screw systems.

1 Introduction

Synthesis of parallel robots has focused mainly on type or structural synthesis, using group theory, screw theory, or geometric methods, see for instance [1]. Dimensional synthesis examples exist, which focus on optimizing performance indices [2], [3] or on reachable workspace sizing [4], [5], [6]; see also [7].

The dimensional synthesis of spatial serial chains for a prescribed set of positions can be used for the design of parallel robots by synthesizing all supporting legs for the same set of positions. There are a few examples of finite-position dimensional synthesis of parallel robots in the literature, most of them doing partial synthesis. Wolbrecht et al. [8] perform synthesis of 3-RRS, 4-RRS and 5-RRS symmetric parallel manipulators; Kim and Tsai [9] and Rao [10] solve the partial kinematic synthesis of a 3-RPS parallel manipulator. This method yields, in some cases, a system with negative mobility.

One interesting question is whether the finite screw surfaces generated by the task positions can give any information for the synthesis of the overconstrained closed linkages. Using Parkin's definition for pitch [11], the screws corresponding to finite displacements can form screw systems. Huang [12] showed that the single RR chain forms a finite screw system of third order; however, the set of finite displacements of the coupler of the Bennett linkage form a cylindroid, which is a general 2-system of screws [13]. Baker [14] has also studied the motion of the Bennett linkage. Perez and McCarthy [15] used two arbitrary displacements to generate the cylindroid associated to the Bennett linkage in order to perform dimensional synthesis. Husty et al. [16] use the geometry of the Study quadric to obtain simpler equations for the synthesis and analysis. Following this approach, Brunnthaler [17] presents a new solution for the spatial 2-RR closed chain. Pfurner and Husty [18] present the constraint manifold of overconstrained 2-3R parallel robots as 6R closed chains.

In this paper, the focus is on the simplest of the overconstrained linkages, the closed spatial RPRP linkage. Recently, Huang [19] has shown that the set of screws corresponding to displacements of this linkage forms a 2-screw system. We use this result in order to synthesize RPRP linkages with positive mobility and for a given shape of the screw system of the relative displacements. In order to do so, we state the design equations using the Clifford algebra of dual quaternions [20], which has a direct relation to the screw system. The design yields a single RPRP linkage.

2 Clifford algebra equations for the synthesis

2.1 Forward Kinematics

The approach used in this paper for stating design equations is based on the method of Lee and Mavroidis [21]. They equate the forward kinematics of a serial chain to a set of goal displacements and consider the Denavit-Hartenberg parameters as variables. A more efficient formulation consists of stating the forward kinematics of relative displacements using the even Clifford subalgebra $C^+(P^3)$, also known as dual quaternions. In this section, we follow the approach presented in [20].

The Plücker coordinates $\mathsf{S} = (\mathbf{S}, \mathbf{C}\times\mathbf{S})$ of a line can be identified with the Clifford algebra element $\mathsf{S} = \mathbf{S} + \varepsilon\mathbf{C}\times\mathbf{S}$. Similarly, the screw $\mathsf{J} = (\mathbf{S}, \mathbf{V})$ becomes the element $\mathsf{J} = \mathbf{S} + \varepsilon\mathbf{V}$. Using the Clifford product we can compute the exponential of the screw $\frac{\theta}{2}\mathsf{J}$,

$$e^{\frac{\theta}{2}\mathsf{J}} = (\cos\frac{\theta}{2} - \frac{d}{2}\sin\frac{\theta}{2}\varepsilon) + (\sin\frac{\theta}{2} + \frac{d}{2}\cos\frac{\theta}{2}\varepsilon)\mathsf{S} = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\mathsf{S}. \quad (1)$$

The exponential of a screw defines a unit dual quaternion, which can be identified with a relative displacement from an initial position to a final position in terms of a rotation around and slide along an axis.

For a serial chain with n joints, in which each joint can rotate an angle θ_i around, and slide the distance d_i along, the axis S_i, for $i = 1,\dots,n$, the forward kinematics of relative displacements (with respect to a reference position) can be expressed as the composition of Clifford algebra elements. Let θ_0 and $\mathbf{d}_0$ be the joint parameters of this chain when in the reference configuration, so we have $\Delta\hat{\theta} = (\theta - \theta_{\mathbf{0}} + (\mathbf{d} - \mathbf{d}_{\mathbf{0}})\varepsilon)$. Then, the movement from this reference configuration is defined by

$$\hat{Q}(\Delta\hat{\theta}) = e^{\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n},$$
$$= (\mathrm{c}\frac{\Delta\hat{\theta}_1}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1)(\mathrm{c}\frac{\Delta\hat{\theta}_2}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2)\cdots(\mathrm{c}\frac{\Delta\hat{\theta}_n}{2} + \mathrm{s}\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n). \quad (2)$$

Note that s and c denote the sine and cosine functions, respectively.

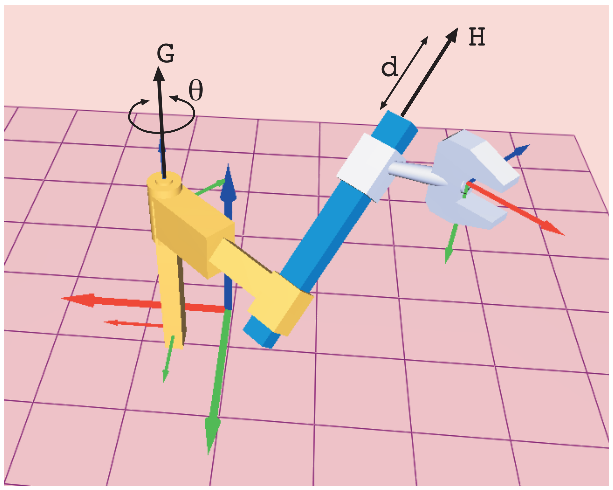

Fig. 1 The RP serial chain.

The RPRP linkage has a mobility $M = -2$ using the Kutzbach-Groebler formula; however, for a certain dimensions of the links, it moves with one degree of freedom. The RPRP linkage can be seen as a serial RP chain and a serial PR chain joined at their end-effectors.

The RP serial chain consists of a revolute joint followed by a prismatic joint, see Figure 1. In the PR serial chain, the order of the joints in the chain is switched. For both the RP and PR serial chains, let $\mathsf{G} = \mathbf{g} + \varepsilon\mathbf{g}^0$ be the revolute joint axis, with rotation θ, and $\mathbf{h}$ the prismatic joint direction with a slide d. The Clifford algebra forward kinematics equations of the RP chain are

$$\hat{Q}_{RP}(\theta, d) = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathsf{G})(1 + \varepsilon\frac{d}{2}\mathbf{h})$$
$$= (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{g}) + \varepsilon(-\frac{d}{2}\sin\frac{\theta}{2}\mathbf{g}\cdot\mathbf{h} + \frac{d}{2}\cos\frac{\theta}{2}\mathbf{h} + \sin\frac{\theta}{2}\mathbf{g}^0 + \frac{d}{2}\sin\frac{\theta}{2}\mathbf{g}\times\mathbf{h}). \quad (3)$$

For the *PR* chain, the only difference is a negative sign in the cross product. In Eq. (3), the angle and slide are measured from a certain reference configuration.

2.2 Design Equations and Counting

The design equations are created when a set of task positions are defined. The design variables that determine the dimensions of the chain are the position of the joint axes in the reference configuration.

Given a set of task positions expressed as relative displacements, $\hat{P}_{1j} = \cos\frac{\Delta\hat{\phi}_{1j}}{2} + \sin\frac{\Delta\hat{\phi}_{1j}}{2}\mathsf{P}_{1j}$, $j = 2, \ldots, m$, we equate them to the forward kinematics equations in Eq. (2),

$$\hat{P}_{1j} = e^{\frac{\Delta\hat{\theta}_{1j}}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_{2j}}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_{nj}}{2}\mathsf{S}_n}, \quad j = 2, \ldots, m. \tag{4}$$

The result is $8(m-1)$ design equations. The unknowns are the n joint axes S_i, $i = 1, \ldots, n$, and the $n(m-1)$ pairs of joint parameters $\Delta\hat{\theta}_{ij} = \Delta\theta_{ij} + \Delta d_{ij}\varepsilon$.

For the RP and PR serial chains, the design equations are

$$\hat{Q}_{RP}(\theta^j, d^j) = \hat{P}_{1j}, \quad j = 1, \ldots, m. \tag{5}$$

The counting of independent equations and unknowns allows to define the maximum number of arbitrary positions m that can be reached, based only on the type and number of joints of the serial chain, see [22] for details. Consider a serial chain with r revolute and p prismatic joints. The maximum number of task rotations is given by

$$m = \frac{3r + p + 6}{6 - (r + p)}. \tag{6}$$

For serial chains with less than three revolute joints, the structure of semi-direct product of the composition of displacements needs to be considered, and the maximum number of rotations n_R needs to be calculated too. Assuming that the orientations are given and that both the directions of the revolute joints and the angles to reach the task orientations are known, we can count, in a similar fashion, the number of translations n_T that the chain can be defined for.

$$m_R = \frac{3 + r}{3 - r}, \qquad m_T = \frac{2r + t + 3 - c}{3 - t}. \tag{7}$$

In order to determine the maximum number of task positions for the RP and PR chains, we apply Eq. (6) and Eq. (7), to obtain $m = 2.5$ task positions, $m_R = 2$ task rotations, and $m_T = 3$ task translations. Hence, we can define one arbitrary relative displacement and a second relative displacement whose orientation is not general.

3 Screw system for the RPRP Linkage

The linear combination of two arbitrary screws representing relative displacements form a 2-system known as the cylindroid, which turns out to be the manifold for the relative displacements of the closed 4R linkage. Huang [19], by intersecting the

3-systems associated with the finite displacements of the RP and PR dyads, shows that the screw surface of the closed RPRP linkage forms a 2-system of a special type, the fourth special type according to Hunt [23], also known as 2-IB [24]. The screws of this system are parallel, coplanar screws whose pitches vary linearly with their distance.

This screw system can be generated by two screws with same direction and finite pitches. Notice that this coincides with the results of the counting in previous section. The task positions defined for the synthesis of the RP (or PR) chain are two relative displacements with same direction and, in general, finite pitches. We can use those to generate the screw system.

For doing so, we define a first relative displacement, $\hat{S}_{12}$. The rotation axis of the displacement, $\mathbf{s}_{12}$ is common to both $\hat{S}_{12}$ and the second relative displacement. We set $\mathbf{s}_{12} = \mathbf{s}_{13}$ and select a rotation angle to define the relative rotation $\hat{s}_{13}$.

We can then set the slope of the pitch distribution in order to shape the screw system. The pitch for the finite displacement screws is [11]

$$p_{1i} = \frac{\frac{t_{1i}}{2}}{\tan\frac{\theta_{1i}}{2}}, \tag{8}$$

directly calculated from the dual quaternion. Similarly, a point on the screw axis is calculated as

$$\mathbf{c}_{1i} = \mathbf{s}_{1i} \times \mathbf{s}^0_{1i}. \tag{9}$$

Define the slope of the distribution as

$$K = \frac{p_{13} - p_{12}}{\mathbf{c}_{13} - \mathbf{c}_{12}} \tag{10}$$

If we set the value of K, we can solve for t_{13} in order to define the pitch of the second relative displacement, the location of its screw axis being defined.

4 Dimensional Synthesis of the RPRP Linkage for a Prescribed Screw System

The solution of the RP, and similarly, PR chains is simple and yields one solution. Given an arbitrary relative displacement $\hat{S}_{12} = (s^w_{12} + \mathbf{b}_{12}) + \varepsilon(s^{w0}_{12} + \mathbf{b}^0_{12})$ and a second displacement $\hat{S}_{13}$ such that both have same direction and a given pitch distribution, as explained in previous section, we equate them to the forward kinematics in Eq.(3). We can solve for the direction of the revolute joint $\mathbf{g}$ and the rotation angles,

$$\mathbf{g} = \frac{\mathbf{b}_{12}}{||\mathbf{b}_{12}||}, \qquad \tan\theta_{1i} = \frac{||\mathbf{b}_{1i}||}{s^w_{1i}}, \quad i = 2,3. \tag{11}$$

The equations corresponding to the dual part are linear in the moment of the revolute joint, $\mathbf{g}^0$,

$$\mathbf{g}^0 = \mathbf{b}^0_{1i} - \frac{d_{1i}}{2}(\cos\frac{\theta_{1i}}{2}\mathbf{h} + \sin\frac{\theta_{1i}}{2}\mathbf{g}\times\mathbf{h}), \quad i = 1,2. \tag{12}$$

Equating the solution of $\mathbf{g}^0$ for both relative displacements, we can solve linearly for $\mathbf{h}$ as a function of the slides d_{12}, d_{13}. The relation between the slides is given by the pitch condition,

$$\frac{s^{w0}_{12}}{\frac{d_{12}}{2}\sin\frac{\theta_{12}}{2}} = \frac{s^{w0}_{13}}{\frac{d_{13}}{2}\sin\frac{\theta_{13}}{2}} \tag{13}$$

Imposing $||h|| = 1$, we can solve for the slides to obtain one solution.

Using the same process, we can solve for the PR serial chain.

5 Example

The dual quaternions in Table 1 have been generated as explained. $\hat{S}_{12}$ has been randomly generated, while the rotation in $\hat{S}_{13}$ is such that it belongs to the workspace of the chain.

Table 1 Goal relative displacements for the RP and PR chains

$(0.660,-0.082,0.447,-0.596)+\varepsilon(-0.810,-1.611,0.375,-0.394)$
$(0.338,-0.042,0.229,0.911)+\varepsilon(-1.823,-1.189,1.169,0.328)$

We obtain one solution for the RPRP linkage, specified in Table 2 as the Plucker coordinates of the axes and the joint variables to reach the positions.

Table 2 Joint axes for the RPRP linkage at the reference configuration

Chain	Revolute joint G	Prismatic joint $\mathbf{h}$	Rotations $(\theta_{12},\theta_{13})$	Slides (d_{12},d_{13})
RP	$(-0.823,0.102,-0.558)$ $\varepsilon(0.704,3.313,-0.4315)$	$+$ $(-0.766,0.461,0.448)$	(-253.2,-48.6)	(-2.29,3.73)
PR	$(0.823,-0.102,0.558)$ $\varepsilon(-0.864,-0.844,1.121)$	$+$ $(-0.689,-0.714,0.119)$	(253.2,48.6)	(-2.29,3.73)

Comparing these results to the joint variable conditions in [19] we can see that our solution corresponds to the unfolded RPRP linkage. Figure 2 shows the chain reaching the three displacements, considering the reference displacement as the identity.

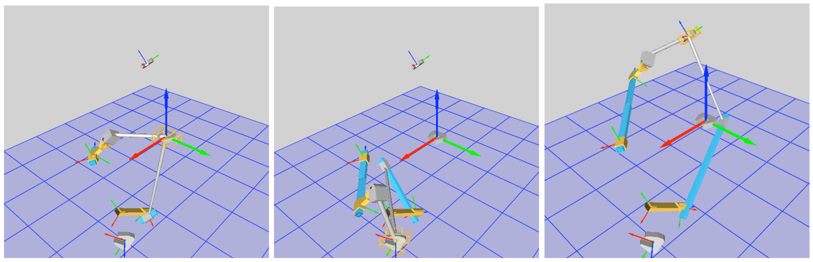

Fig. 2 The RPRP linkage reaching three positions

6 Conclusions

This papers presents the synthesis of an overconstrained closed linkage, the RPRP. The knowledge of the screw system that corresponds to the finite displacements of the linkage is used to ensure that the solutions of the synthesis of the RP and PR serial chains can be assembled to create a movable system. The counting of the maximum positions for the synthesis suffices to define the positions that generate the screw system. The synthesis yields a single RPRP linkage. Future work will further study the relation between the task positions and the screw systems of other spatial closed linkages.

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