

n -ARY ALGEBRAS: A REVIEW WITH APPLICATIONS

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ABSTRACT. This paper reviews the properties and applications of certain n -ary generalizations of Lie algebras in a self-contained and unified way. These generalizations are algebraic structures in which the two entries Lie bracket has been replaced by a bracket with n entries. Each type of n -ary bracket satisfies a specific *characteristic identity* which plays the rôle of the Jacobi identity for Lie algebras. Particular attention will be paid to *generalized Lie algebras*, which are defined by even multibrackets obtained by antisymmetrizing the associative products of its n components and that satisfy the *generalized Jacobi identity*, and to *Filippov (or n -Lie) algebras*, which are defined by fully antisymmetric n -brackets that satisfy the *Filippov identity*. Three-Lie algebras have surfaced recently in multi-brane theory in the context of the Bagger-Lambert-Gustavsson model. Because of this, Filippov algebras will be discussed at length, including the cohomology complexes that govern their central extensions and their deformations (it turns out that Whitehead's lemma extends to all semisimple n -Lie algebras). When the skewsymmetry of the Lie or n -Lie algebra bracket is relaxed, one is led to a more general type of n -algebras, the n -Leibniz algebras. These will be discussed as well, since they underlie the cohomological properties of n -Lie algebras.

The standard Poisson structure may also be extended to the n -ary case. We shall review here the even *generalized Poisson structures*, whose generalized Jacobi identity reproduces the pattern of the generalized Lie algebras, and the *Nambu-Poisson structures*, which satisfy the Filippov identity and determine Filippov algebras. Finally, the recent work of Bagger-Lambert and Gustavsson on superconformal Chern-Simons theory will be briefly discussed. Emphasis will be made on the appearance of the 3-Lie algebra structure and on why the A_4 model may be formulated in terms of an ordinary Lie algebra, and on its Nambu bracket generalization.

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1. INTRODUCTION AND OVERVIEW

In the last decades there has been an increasing interest in the applications of various n -ary generalizations of the ordinary Lie algebra structure to theoretical physics problems, which has peaked in the last three years. n -ary algebraic operations are, however, very old: ternary operations appeared for the first time associated with the cubic matrices that had been introduced by A. Cayley in the middle of the XIXth century and that were also considered by J.J. Sylvester some forty years later, still in that century. In spite of this, the modern mathematical work on general multioperator rings and algebras (not necessarily associative) begins much later with a series of papers by Kurosh (see [1] and earlier refs. therein). In particular, the *linear Ω -algebras* are given by a vector space on which certain multilinear operations are defined; they are reviewed in [2]. This general class of algebras is, however, larger than the two main generalizations of Lie algebras to be discussed in this review, which will be denoted generically as n -ary algebras. In these, the standard Lie bracket is replaced by a linear n -ary bracket with $n > 2$ entries, the algebra structure being defined by the characteristic identity satisfied by the n -ary bracket. There are two (main) ways of achieving this, depending on how the Jacobi identity (JI) of the ordinary Lie algebras is looked at. The JI can be viewed as the

statement that (a) a nested double Lie bracket gives zero when antisymmetrized with respect to its three entries or that (b) the Lie bracket is a derivation of itself. Both (a) and (b) are equivalent for ordinary Lie algebras where the JI is indeed a necessary identity that follows from the associativity of the composition of the Lie algebra elements in the Lie bracket.

Let now \mathfrak{H} be a generic n -ary algebra, $n > 2$, endowed with a skewsymmetric n -linear bracket $\wedge^n \mathfrak{H} \rightarrow \mathfrak{H}$, $(X_1, \dots, X_n) \mapsto [X_1, \dots, X_n]$. When \mathfrak{H} is defined using the *characteristic identity* that extends property (a) to the n -ary bracket, one is led to the *higher order Lie algebra* or *generalized Lie algebra* structure (GLA) [3–5], [6–8] (see also [9, 10]), denoted $\mathfrak{H} = \mathfrak{G}$, and the characteristic identity satisfied by its multibracket is called *generalized Jacobi identity* (GJI); GLAs will be discussed in Sec. 6. These algebras are a case of the more general antisymmetric linear Ω -algebras of Kurosh [1], to whom the earlier generalizations of Lie algebras may be traced; see [2] for a review of the russian school contributions to the subject and further references. The GLA generalization is natural for n even (for n odd, the *r.h.s.* of the GJI, rather than being zero, is a larger bracket with $(2n - 1)$ entries). GLAs may also be considered as a particular case (in which there is no violation of the GJI [11]) of the strongly homotopy algebras of Stasheff [12–15]. When possibility (b) is used as the guiding principle, then one obtains the *Filippov identity* [16] (FI) as characteristic identity and, correspondingly, the n -Lie or *Filippov algebras* (FA) [16]; both terms, Filippov and n -Lie, will be used indistinctly. FAs will be denoted by $\mathfrak{H} = \mathfrak{G}$ and reviewed in Sec. 7. The characteristic GJI and FI still admit further generalizations that essentially preserve their original structure, but these (see [17]) will not be considered here.

As with metric Lie algebras (Sec. 2.3), the n -bracket alone is often insufficient for applications and the existence of an inner product on the underlying algebra vector space is required. This leads to notion of metric Filippov n -algebras to be discussed in Sec. 7.6. When $n = 2$, both algebra structures \mathfrak{G} and \mathfrak{G} coincide and determine ordinary Lie algebras \mathfrak{g} . For $n \geq 3$, the GJI (n even) and the FI become different characteristic identities and define, respectively, GLAs \mathfrak{G} and n -Lie or FAs \mathfrak{G} . There is also the possibility of relaxing the antisymmetry of the n -bracket: this leads to ordinary ($n = 2$) Loday's (or Leibniz) algebras [18–20] and to their n -Leibniz algebra generalizations [21, 22], which differ from their Lie and n -Lie counterparts by having brackets that do not require anticommutativity. These algebras will be denoted by \mathcal{L} ($n = 2$) and \mathfrak{L} ($n \geq 3$) respectively, and will be considered in Secs. 4, 9.

Filippov algebras [16, 23–25] have recently been found useful in the search for an effective action describing the low energy dynamics of coincident M2-branes or, more specifically, in the Bagger-Lambert-Gustavsson (BLG) type models [26–32]. The fact that there is a unique simple¹ euclidean 3-Lie algebra (A_4) was actually rediscovered in the context of the first BLG model, where it follows [34, 35] by assuming that the metric needed for the BLG action has to be positive definite, a condition that may be relaxed. We shall discuss the BLG and related models in Secs. 14, 15; we just mention now that the original BLG action was subsequently reformulated [36, 29] without using a 3-Lie algebra, and that other models for low energy multiple M2-brane dynamics have appeared (albeit with $\mathcal{N}=6$ rather than $\mathcal{N}=8$ manifest supersymmetries) that do not use a FA structure [37]. Some modifications of the original BLG model based on A_4 have been considered in the literature using non-fully skewsymmetric 3-brackets. These define, in fact, various 3-Leibniz algebras, and some of these will be discussed (along with Lie-triple systems \mathfrak{T} , a particular case of 3-Leibniz algebras) in Sec. 10.1.

¹That there is only a simple n -Lie algebra for $n > 2$ had been found in [25, 16] (see Th. 53). For $n = 3$ this was also rediscovered in [33] (unaware of all earlier work), plus other results. This reference contains a new proposal for the use of FAs in the context of orbifold singularities in M-theory compactifications.

Nambu algebras are a particular, infinite-dimensional case of n -Lie algebras. Their n -bracket is provided by the Jacobian determinant of n functions or Nambu bracket [38], although Nambu did not write the characteristic identity satisfied by his bracket, which is none other than the FI. This was discussed in [16, 39–42], and *Nambu-Poisson structures* (N-P) have been much studied since Nambu’s original paper [38] (mostly devoted to $n = 3$) and Takhtajan general study [41] for arbitrary n , see [43, 40, 44–47, 21, 48–51] (ref. [21] also considers Nambu superalgebras). In fact, since the earlier considerations of p -branes as gauge theories of volume preserving diffeomorphisms [52, 53] (see also [54]), the infinite-dimensional FAs given by Nambu brackets have also appeared in applications to brane theory [55] and, in particular, in the Nambu three-bracket realization of the mentioned BLG model as a gauge theory associated with volume preserving diffeomorphisms in a three-dimensional space; see, in particular, refs. [56–58] (see also [59]). The BLG-Nambu bracket model in [58] will be considered in Sec. 14.

Much in the same way the Nambu-Poisson structures follow the pattern of FAs, it is also possible to introduce *generalized Poisson structures* (GPS) [4, 5, 60] (see further [50, 61, 62]), the n -even *generalized Poisson brackets* (GPB) of which satisfy the GJI and correspond to the GLAs earlier mentioned. Note, however, that besides the two properties that each n -ary Poisson generalization share respectively with the GLAs and FAs (skewsymmetry of both n -ary brackets plus the GJI (FI) for the GP (N-P) structures, respectively), the n -ary brackets of both GPS and of N-PS satisfy an additional condition, Leibniz’s rule. Adding the appropriate grading factors, it is also possible to define the *graded generalized Poisson structures* (graded GPS) [63]; of course, setting aside the Leibniz rule, the remaining structural properties of the graded GPS define the graded multibracket and the graded GJI of the *graded generalized Lie algebras* (graded GLAs). We conclude this paragraph by noting that there has been an extensive discussion since the papers by Nambu [38] and Takhtajan [41] about the difficulties of quantizing the N-P structures, a question to which we shall come back in Sec. 13.5; here we just refer in this connection to the papers above and, *e.g.*, to [55, 60, 64–66] and further references therein.

Although the GLA, FA and n -Leibniz algebra structures will be the main subject of this review (besides the triple systems to be briefly discussed in Sec. 10), these do not exhaust, of course, the wide class of algebra and Lie algebra generalizations. Without considering a characteristic identity, other n -algebraic structures have been studied *e.g.* in [67, 68]. Other Lie algebra generalizations have to do with introducing a grading; this includes the well known case of superalgebras (see *e.g.* [69] for a useful collection of definitions, results and basic references and [70] for a very early -perhaps the first- review). There is, besides, a full array of other algebraic structures. These include non-associative algebras in general [71, 72], Malčev (or Moufang-Lie in Malčev’s terminology) algebras [73, 74] and their ternary [74] and n -ary generalizations [75], some specific types of ternary structures [76–78]), F -Lie algebras [79–81], etc. Some of these algebras were motivated in part by certain aspects of fractional supersymmetry and/or the problem of parastatistics (see *e.g.* [82–87]) but they were also studied by their own interest (see [88] and [89] for a review of higher order generalizations of the Poincaré algebra and possible applications). None of these algebraic structures, however, will be considered here; we refer to the quoted papers for further information and references. Other types of structures and (anti)brackets/algebras, as the Batalin-Vilkovisky algebras, the master equation, etc. (see *e.g.* [90] and references therein), which are relevant *e.g.* in the B-V approach to quantization, will also be omitted. Nevertheless, a few important constructions such as Koszul’s [91], strongly homotopy algebras [12] (see also [92] in the context of closed string theory) and Gerstenhaber

algebras [93] will be briefly mentioned in Sec. 6.4 or in connection with Sec. 13; see also [94] for a unified review and [95] for other aspects.

Since both GLAs and FAs reduce to Lie algebras for $n = 2$, and Lie algebras are often a guide for any of their $n > 2$ generalizations, we start by summarizing some Lie algebra properties in Sec. 2.

1.1. Notation and conventions.

We will use \mathfrak{g} to denote standard Lie algebras and the larger case \mathcal{G} and \mathfrak{G} , respectively, for the n -ary higher order or *Generalized Lie algebras* (GLAs, \mathcal{G}) and the *Filippov* or *n -Lie algebras* (FAs, \mathfrak{G}). In general, we will use the same symbol for the different n -ary algebras and their underlying vector spaces. Ordinary $n = 2$ and ($n > 2$)-Leibniz algebras (LAs) will be denoted by \mathcal{L} and \mathfrak{L} respectively; triple systems will be denoted by \mathfrak{T} . The infinite-dimensional FAs generated by the Jacobian of functions, also referred to as Nambu algebras, will be denoted by \mathfrak{N} .

For the sake of distinguishing clearly among the different brackets considered, we shall refer to the n -ary brackets of the GLAs \mathcal{G} as *higher order brackets* or *multibrackets*, and to those of the FAs \mathfrak{G} or of the n -Leibniz ones \mathfrak{L} simply as *n -brackets*. The elements of the different algebras will be frequently denoted by capital letters X, Y etc (and, for FAs, occasionally by e_a). Chosen a basis, the structure constants of GLAs \mathcal{G} will be written as $C_{i_1 \dots i_{2s}}^j$ ($i = 1, \dots, \dim \mathcal{G}$) and those of the FAs \mathfrak{G} and LAs \mathfrak{L} as $f_{a_1 \dots a_n}^b$ ($a = 1, \dots, \dim(\mathfrak{G}, \mathfrak{L})$). For $n = 2s = 2$, $\mathcal{G} = \mathfrak{G} = \mathfrak{g}$ and \mathfrak{L} becomes \mathcal{L} .

All algebras in this review will be on \mathbb{R} or \mathbb{C} and, with the exception of Nambu and gauge algebras, finite-dimensional.

A terminological remark.

GLAs were called *Lie n -algebras* in [6]. Such a terminology may cause confusion with the very different n -Lie algebras (FAs) which were introduced by Filippov with this name a dozen of years earlier. Thus, rather than betting the distinction between n -Lie algebras (\equiv FAs) and GLAs on the precise location of a single letter (*n -Lie alg. vs. Lie n -alg.*), we shall use *n -Lie* and *FA* indistinctly for Filippov algebras and keep our *higher order* or *generalized Lie algebras* GLA terminology for the n -ary generalization of Lie algebras reviewed in Sec. 6.

A note on references.

Besides some references quoted by their historical value, most of them have been selected to give due credit to the relevant papers and, further, for their potential usefulness when they are directly related to the text. We do not consider helpful the recent practice in some (physics) e-papers of grouping twenty (or far more) references under a single number; this might result in quoting all possibly related work, but it is useless to the reader, who could perform such an indiscriminate search in the arXives if she or he wished so.

2. A SHORT SUMMARY OF LIE ALGEBRAS

We summarize in this section some ingredients of the theory of Lie algebras (see *e.g.* [96]) that may be useful when considering ($n > 2$)-ary generalizations.

2.1. General properties of Lie algebras.

A *Lie algebra structure* is a vector space \mathfrak{g} together with a bilinear operation $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, the Lie bracket $[\cdot, \cdot] : (X, Y) \mapsto [X, Y]$, that satisfies

$$[X, Y] = -[Y, X], \quad (1)$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (2)$$

A finite example is the associative algebra $\text{End } V$ of linear transformations of a finite vector space, which is the general linear Lie algebra $\mathfrak{gl}(\dim V)$; another are the Lie algebras \mathfrak{g} generated by the (left, say) invariant vector fields associated with the (then, right) action of a Lie group G on itself. An infinite-dimensional example is the Lie algebra of the vector fields $\mathfrak{X}(M)$ on a manifold M .

The Jacobi identity (JI) (2) may be looked at as a necessary consequence of the associativity of the composition of the bracket elements with $[X, Y] = XY - YX$. A second view of the JI is obtained by rewriting eq. (2) as

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] . \quad (3)$$

A linear transformation $D : \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra is said to be a *derivation* of the Lie algebra, $D \in \text{Der } \mathfrak{g}$, if

$$D[Y, Z] = [DY, Z] + [Y, DZ] \quad \forall Y, Z \in \mathfrak{g} . \quad (4)$$

Thus, eq. (3) states that, for all $X \in \mathfrak{g}$, $[X, \]$ is a derivation of \mathfrak{g} . This is the adjoint derivative acting from the left, $ad_X Y := [X, Y]$. In terms of ad_X , eq. (3) becomes

$$ad_X[Y, Z] = [ad_X Y, Z] + [Y, ad_X Z] . \quad (5)$$

Thus, the JI identity also expresses that $ad_X \in \text{Der } \mathfrak{g}$ is an inner derivation of $\mathfrak{g} \ \forall X \in \mathfrak{g}$.

Viewing the above as the result of a linear transformation on any $Z \in \mathfrak{g}$ and removing it, the previous equation yields

$$ad_X ad_Y - ad_Y ad_X = ad_{ad_X Y} \quad \text{or equivalently} \quad [ad_X, ad_Y] = ad_{[X, Y]} , \quad (6)$$

where the bracket on the *l.h.s.* of the second equation stands for the commutator in $\text{End } \mathfrak{g}$ and that of the *r.h.s.* is the original bracket in \mathfrak{g} . Thus, the map $ad : \mathfrak{g} \rightarrow \text{End } \mathfrak{g} = \mathfrak{gl}(\dim \mathfrak{g})$, $ad : X \mapsto ad_X$, is a Lie algebras homomorphism and defines the *adjoint representation* of \mathfrak{g} . Its image $ad \mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(\dim \mathfrak{g})$, the Lie algebra of *inner* derivations of \mathfrak{g} or $\text{InDer } \mathfrak{g}$, inner because they are defined by elements of \mathfrak{g} . The kernel of ad is the centre $Z(\mathfrak{g})$ of \mathfrak{g} and is an ideal of \mathfrak{g} ; thus, $\text{InDer } \mathfrak{g} = \mathfrak{g}/Z(\mathfrak{g})$. $\text{InDer } \mathfrak{g}$ is an ideal of the Lie algebra of all derivations $\text{Der } \mathfrak{g}$ of \mathfrak{g} since rearranging equation (4) it follows that $[D, ad_Y] = ad_{DY}$ for any $D \in \text{Der } \mathfrak{g}$, $Y \in \mathfrak{g}$. As a result, the quotient $\text{Der } \mathfrak{g}/\text{InDer } \mathfrak{g} = \text{OutDer } \mathfrak{g}$ is, by construction, the Lie algebra of the *outer derivations*. If \mathfrak{g} is semisimple, $\mathfrak{g} \rightarrow ad \mathfrak{g}$ is an isomorphism of Lie algebras and, further, all derivations are inner, $\text{Der } \mathfrak{g} = \text{InDer } \mathfrak{g}$. By *Ado's theorem*, every finite-dimensional Lie algebra over \mathbb{R} or \mathbb{C} is isomorphic to a matrix algebra (has a faithful finite representation). An important result towards the classification of arbitrary Lie algebras is the *Levi-Malčev decomposition theorem*: a Lie algebra is the semidirect sum $\mathfrak{g} = \text{Rad}(\mathfrak{g}) \ltimes \mathfrak{g}_L$ of a semisimple subalgebra (\mathfrak{g}_L , its *Levi factor*) and its radical $\text{Rad}(\mathfrak{g})$. The Lie algebra is called *reductive* when $\text{Rad}(\mathfrak{g}) = Z(\mathfrak{g})$, in which case $\mathfrak{g} = Z(\mathfrak{g}) \oplus \mathfrak{g}_L$ with $\mathfrak{g}_L = [\mathfrak{g}, \mathfrak{g}]$.

Once a basis $\{X_i\}$ of \mathfrak{g} is chosen, a Lie algebra may be described in terms of the corresponding *structure constants* C_{ij}^k of \mathfrak{g} , $[X_i, X_j] = C_{ij}^k X_k$. The defining conditions (1), (2) for a Lie algebra are then given by the expressions

$$C_{ij}^k = -C_{ji}^k \quad \text{and} \quad C_{[ij}^l C_{k]l}^s = 0 \quad (\text{JI}) \quad , \quad i, j, k = 1, \dots, r = \dim \mathfrak{g} \quad , \quad (7)$$

respectively, where the square brackets surrounding the indices denote total skewsymmetrisation. This will be defined throughout the paper by

$$[a_1 \dots a_n] := \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} a_{\sigma(1)} \dots a_{\sigma(n)} \quad , \quad (8)$$

where $\pi(\sigma) = 0, 1$ is the even or odd parity of the permutation σ in the group S_n of permutations of the indices $(1, \dots, n)$, without the ‘weight one’ factor $1/n!$. This means that

$$[a_1 \dots a_n] = \sum_{cycl \sigma \in S_n} (-1)^{\pi(\sigma)} a_{\sigma(1)} [a_{\sigma(2)} \dots a_{\sigma(n)}] \quad (9)$$

which, for n odd, does not produce any signs. In terms of the structure constants, the matrices $ad_X \in \text{End } \mathfrak{g}$ of the adjoint representation $ad_{X_i} X_j := [X_i, X_j]$ are given by $(ad_{X_i})^k_j = C_{ij}^k$.

2.2. A comment on associativity.

The *associator* of a triple product, which is given by

$$(X, Y, Z) := (XY)Z - X(YZ), \quad (10)$$

accounts for the lack of associativity in the same way that the commutator $[X, Y] = XY - YX$ measures the lack of commutativity. Non-associative algebras (see [71, 97, 98] for book discussions) under the composition have non-zero associators. It is possible to define an antisymmetric associator by

$$[X, Y, Z]_{\text{ant.assoc.}} := (X, Y, Z) + (Y, Z, X) + (Z, X, Y) - (Y, X, Z) - (X, Z, Y) - (Z, Y, X) \quad (11)$$

The above expression is equivalent to $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y]$. Clearly, if the associator is zero there is associativity and the ordinary JI is satisfied.

2.3. Metric Lie algebras.

A Lie algebra is called *metric* when it is endowed with an invariant, symmetric and non-degenerate bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ which defines the scalar product in the \mathfrak{g} vector space. This means that

$$X \cdot \langle Y, Z \rangle = \langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \quad \forall X, Y, Z \in \mathfrak{g} \quad (12)$$

i.e., the bilinear form $\langle \cdot, \cdot \rangle$ is ‘associative’ in the sense that $\langle [Y, X], Z \rangle = \langle Y, [X, Z] \rangle$.

If $\{\omega^i\}$ is a basis of the coalgebra \mathfrak{g}^* dual to $\{X_j\}$, the bilinear form and the invariance condition are expressed as

$$g = g_{ij} \omega^i \otimes \omega^j, \quad C_{li}^s g_{sj} + C_{lj}^s g_{is} = 0, \quad (13)$$

where the coordinates of g are $g_{ij} = g(X_i, X_j) \equiv \langle X_i, X_j \rangle$.

Semisimple Lie algebras are metric because their *Cartan-Killing metric* $k(X, Y) := \text{Tr}(ad_X ad_Y)$ is non-degenerate and invariant (trace forms are invariant), and hence defines an inner product; clearly, $k([Y, X], Z) = k(Y, [X, Z])$ follows from the associativity of $ad_X \in \text{End } \mathfrak{g}$. In terms of the structure constants, the coordinates of the Killing metric k are given by

$$k_{ij} := \text{Tr}(ad_{X_i} ad_{X_j}) = C_{il}^s C_{js}^l. \quad (14)$$

The Killing metric is negative-definite for the compact real form of a semisimple Lie algebra (when the generators are taken to be antihermitian). The metricity condition (12) is equivalent to the total skewsymmetry of the structure constants with all indices down, $C_{ijk} := C_{ij}^l g_{lk}$. Although all Lie algebras are endowed with a canonical *ad*-invariant inner product given by the Killing form, the inner product of metric nonsemisimple Lie algebras has to be imposed as an external structure. For instance, Abelian Lie algebras are metric relative to any inner product, since the adjoint action is trivial. For the structure of metric Lie algebras see [99–103].

2.4. Lie algebras and elementary differential geometry on Lie groups.

Let G be a r -dimensional Lie group with elements g parametrized by their coordinates $g^l, l = 1, \dots, r$ and let $L_{g'}g = g'g = R_{g'}g'$ ($g', g \in G$) be the left and right actions $G \times G \rightarrow G$ with obvious notation. The left (right) invariant vector fields LIVF $X_i^L(g)$ (RIVF, $X_i^R(g)$) on G reproduce the commutators of the Lie algebra \mathfrak{g} :

$$[X_i^L(g), X_j^L(g)] = C_{ij}^k X_k^L(g) \quad , \quad [X_i^R(g), X_j^R(g)] = -C_{ij}^k X_k^R(g) \quad , \quad (15)$$

$i, j, k = 1, \dots, r = \dim \mathfrak{g}$. In terms of the Lie derivative, the L - (R -) invariance conditions read²

$$L_{X_j^R(g)} X_i^L(g) = [X_j^R(g), X_i^L(g)] = 0 \quad , \quad L_{X_i^L(g)} X_j^R(g) = [X_i^L(g), X_j^R(g)] = 0 \quad , \quad (16)$$

which simply express that the left and right translations of G commute.

The Lie algebra \mathfrak{g} may be equally described in terms of invariant forms on the manifold of the associated Lie group G . Let $\omega^{Li}(g)$ be the basis of LI one-forms of G dual to a basis of \mathfrak{g} given by LIVF ($\omega^{Li}(g)(X_j^L(g)) = \delta_j^i$). The ω^i satisfy the Maurer-Cartan (MC) equations

$$d\omega^{Li}(g) = -\frac{1}{2} C_{jk}^i \omega^{Lj}(g) \wedge \omega^{Lk}(g) = -C_{jk}^i \omega^{Lj}(g) \otimes \omega^{Lk}(g) \quad , \quad (17)$$

and the JI $C_{[i_1 i_2] \rho}^\rho C_{i_3 \rho}^\sigma = 0$ is now implied by the nilpotency of d , $d^2 \equiv 0$.

The exterior derivative d of a p -form $\alpha \in \Lambda^p(M)$ on a manifold M is the $(p+1)$ -form defined by the functions in $\mathcal{F}(M)$ obtained by taking arguments on a set of $(p+1)$ arbitrary vector fields $X_1(x), \dots, X_{p+1}(x)$ on M . This expression is given by Palais formula,

$$(d\alpha)(X_1, \dots, X_p, X_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \alpha(X_1, \dots, \hat{X}_i, \dots, X_{p+1}) + \sum_{i < j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}) \quad . \quad (18)$$

Thus, if α is a LI form on G ,

$$d\alpha^L(X_{i_1}^L, \dots, X_{i_{p+1}}^L) = \sum_{s < t} (-1)^{s+t} \alpha^L([X_{i_s}^L, X_{i_t}^L], X_{i_1}^L, \dots, \hat{X}_{i_s}^L, \dots, \hat{X}_{i_t}^L, \dots, X_{i_{p+1}}^L) \quad , \quad (19)$$

since $\alpha^L(X_1^L, \dots, \hat{X}_i^L, \dots, X_{p+1}^L)$ is constant and the first term in (18) is then zero³.

The MC equations may be written in a more compact way by introducing the (canonical) \mathfrak{g} -valued LI one-form θ on G , $\theta(g) = \omega^i(g) \circ X_i(g)$, $\theta(X_i) = X_i$; then, the MC equations read

$$d\theta = -\theta \wedge \theta = -\frac{1}{2} [\theta, \theta] \quad , \quad (20)$$

since the \mathfrak{g} -valued bracket of two \mathfrak{g} -valued p, q -forms α, β is the \mathfrak{g} -valued $(p+q)$ -form

$$[\alpha, \beta] := \alpha^i \wedge \beta^j \circ [X_i, X_j] = C_{ij}^k \alpha^i \wedge \beta^j \circ X_k \quad , \quad [\alpha, \beta] = (-1)^{pq+1} [\beta, \alpha] \quad , \quad (21)$$

so that $[\theta, \theta]^i = 2 C_{jk}^i \omega^{Lj} \otimes \omega^{Lk}$.

The transformation properties of the MC forms $\omega^i(g)$ follow from the action of the Lie derivative on one-forms, $(L_Y \beta)(X) = Y \cdot \beta(X) - \beta([Y, X])$, which on the LI MC forms gives

$$L_{X_i(g)} \omega^j(g) = -C_{ik}^j \omega^k(g) \quad (22)$$

²The superindex L (R) in the fields refers to the left (right) invariance of them; LIVF (RIVF) generate right (left) translations.

³From now on we shall assume that the vector fields on G generating \mathfrak{g} and their dual one-forms are the left invariant ones (*i.e.*, $X \in \mathfrak{X}^L(G)$, etc.) and drop the superindex L . Superindices L, R will be used to avoid confusion when both LI and RI objects appear.

(if ω were RI, the *r.h.s* above and those in eq. (20) would have plus signs). A general LI p -form on G is a linear combination of exterior products of MC forms, $\alpha(g) = \frac{1}{p!} \alpha_{i_1 \dots i_p} \omega^{i_1}(g) \wedge \dots \wedge \omega^{i_p}(g)$, where $\alpha_{i_1 \dots i_p}$ are constants; for it

$$L_{X_i(g)} \alpha(g) = - \sum_{s=1}^p \frac{1}{p!} C_{ik}^{i_s} \alpha_{i_1 \dots i_p} \omega^{i_1}(g) \wedge \dots \wedge \widehat{\omega^{i_s}(g)} \wedge \omega^k(g) \wedge \dots \wedge \omega^{i_p}(g) \quad . \quad (23)$$

In terms of the Lie derivative L_X , the invariance of the scalar product is simply written as $L_X g = 0$ where now the $\omega^i(g)$ in eq. (13) are the MC forms.

3. LIE ALGEBRA COHOMOLOGY, CENTRAL EXTENSIONS AND DEFORMATIONS

We provide here a summary of the basic Lie algebra cohomology notions and expressions that will be useful later on when we consider the cohomology of n -Lie algebras (see [104, 105] and *e.g.*, [106] and references therein).

3.1. Main definitions.

Let \mathfrak{g} be a Lie algebra, \mathfrak{g}^* the dual of the \mathfrak{g} vector space and V a vector space which is a left $\rho(\mathfrak{g})$ -module *i.e.*, V carries a representation ρ of the Lie algebra \mathfrak{g} on V , $\rho(X_i)_C^A \rho(X_j)_B^C - \rho(X_j)_C^A \rho(X_i)_B^C = \rho([X_i, X_j])_B^A$, $A, B = 1, \dots, \dim V$.

Definition 1. (*V-valued p-dimensional cochains on a Lie algebra \mathfrak{g}*)

A V -valued p -cochain Ω^p on \mathfrak{g} is a skewsymmetric p -linear mapping

$$\Omega^p : \mathfrak{g} \times \dots \times \mathfrak{g} \rightarrow V \quad , \quad \Omega^A = \frac{1}{p!} \Omega_{i_1 \dots i_p}^A \omega^{i_1} \wedge \dots \wedge \omega^{i_p} \quad , \quad (24)$$

where $i_1, \dots, i_p = 1, \dots, r = \dim \mathfrak{g}$, $A = 1, \dots, \dim V$ and $\{\omega^i\}$ is a basis of \mathfrak{g}^* dual to the one $\{X_j\}$ of \mathfrak{g} ; the constants $\Omega_{i_1 \dots i_p}^A$ are the coordinates of the p -cochain in the (non-minimal) basis $\omega^{i_1} \wedge \dots \wedge \omega^{i_p}$. At this stage, the $\{\omega^i\}$ are simply elements of \mathfrak{g}^* *i.e.* linear maps, $\omega^i : \mathfrak{g} \rightarrow \mathbb{R}$, say.

Under the natural addition law of V -valued skewsymmetric covariant p -tensors on \mathfrak{g} , the cochains $\Omega^p \in C^p$ of a given order p form an abelian group which is denoted by $C^p(\mathfrak{g}, V)$. The action of the coboundary operator is given by

Definition 2. (*Coboundary operator (for the left action ρ of \mathfrak{g} on V)*)

The coboundary operator is the map $s : \Omega^p \in C^p(\mathfrak{g}, V) \mapsto (s\Omega^p) \in C^{p+1}(\mathfrak{g}, V)$ given by

$$\begin{aligned} (s\Omega^p)^A(X_1, \dots, X_{p+1}) &:= \sum_{i=1}^{p+1} (-)^{i+1} \rho(X_i)_B^A (\Omega^{pB}(X_1, \dots, \hat{X}_i, \dots, X_{p+1})) \\ &+ \sum_{\substack{j,k=1 \\ j < k}}^{p+1} (-)^{j+k} \Omega^{pA}([X_j, X_k], X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{p+1}) \quad ; \end{aligned} \quad (25)$$

$(s\Omega^p) = 0$ for $p \geq \dim \mathfrak{g}$. This defines the *Lie algebra cohomology complex* $(C^\bullet(\mathfrak{g}, V), s)$ for the representation ρ of \mathfrak{g} .

The structure of this formula is analogous to that for the exterior derivative d acting on forms $\alpha(x)$ on a finite-dimensional manifold M , eq. (18). The only difference is that in (25) the Ω is a p -antisymmetric covariant tensor defined on the vector space of \mathfrak{g} , the X_i are vectors and $\Omega(X_1, \dots, X_p)$ is a number, while in eq. (18) $\alpha(x)$ is a p -skewsymmetric covariant tensor *field*

on the manifold M and the $X_i(x)$ are vector *fields* on M . Of course, $s^2 \equiv 0$ on any Ω^p , as will be shown in Prop. 5.

We give below, for later use, a few examples of the action of the coboundary operator on cochains of the lower orders:

$$\begin{aligned}
 s\Omega^0(X_1) &= \rho(X_1)\Omega^0, \\
 s\Omega^1(X_1, X_2) &= \rho(X_1)\Omega^1(X_2) - \rho(X_2)\Omega^1(X_1) - \Omega^1([X_1, X_2]); \\
 s\Omega^2(X_1, X_2, X_3) &= \rho(X_1)\Omega^2(X_2, X_3) - \rho(X_2)\Omega^2(X_1, X_3) + \rho(X_3)\Omega^2(X_1, X_2) \\
 &\quad - \Omega^2([X_1, X_2], X_3) + \Omega^2([X_1, X_3], X_2) - \Omega^2([X_2, X_3], X_1) \\
 s\Omega^3(X_1, X_2, X_3, X_4) &= \rho(X_1)\Omega^3(X_2, X_3, X_4) - \rho(X_2)\Omega^3(X_1, X_3, X_4) \\
 &\quad + \rho(X_3)\Omega^3(X_1, X_2, X_4) - \rho(X_4)\Omega^3(X_1, X_2, X_3) \\
 &\quad - \Omega^3([X_1, X_2], X_3, X_4) + \Omega^3([X_1, X_3], X_2, X_4) - \Omega^3([X_1, X_4], X_2, X_3) \\
 &\quad - \Omega^3([X_2, X_3], X_1, X_4) + \Omega^3([X_2, X_4], X_1, X_3) - \Omega^3([X_3, X_4], X_1, X_2).
 \end{aligned} \tag{26}$$

For later generalization to n -ary algebras, it is convenient to rewrite the action of the coboundary operator for $\rho = 0$ using the adjoint derivative. In terms of ad_X , it obviously reads

$$\begin{aligned}
 (s\Omega^p)(X_1, \dots, X_{p+1}) &= \sum_{1 \leq j < k}^{p+1} (-1)^{j+k} \Omega^p(ad_{X_j} X_k, X_1, \dots, \hat{X}_j, \dots, \hat{X}_k, \dots, X_{p+1}) \\
 &= \sum_{1 \leq j < k}^{p+1} (-1)^j \Omega^p(X_1, \dots, \hat{X}_j, \dots, ad_{X_j} X_k, \dots, X_{p+1}) \quad .
 \end{aligned} \tag{27}$$

Clearly, $s^2 = 0$ follows from the fact that

$$ad_X ad_Y Z - ad_Y ad_X Z = ad_{[X, Y]} Z \quad , \tag{28}$$

eq. (5). At present, this is a mere change of notation, but it will prove useful later to define the n -Lie algebra cohomology since, as stated, FAs constitute a generalization of Lie algebras based on extending the adjoint derivative to the ($n > 2$)-bracket case.

Definition 3. (*Cocycles, coboundaries and the p -th cohomology group*)

A V -valued p -cochain Ω^p (or Ω^{pA} , making explicit the coordinate index of the $\rho(\mathfrak{g})$ -module V) is called a p -cocycle when $s\Omega^p = 0$. If a cocycle Ω^p may be written as $\Omega^p = s\Omega^{p-1}$ where Ω^{p-1} is a $(p-1)$ -cochain, the p -cocycle is trivial and Ω^p is called a p -coboundary. The spaces of p -cocycles and p -coboundaries are labelled, respectively, by $Z_\rho^p(\mathfrak{g}, V)$ and $B_\rho^p(\mathfrak{g}, V)$. The p -th Lie algebra cohomology group $H_\rho^p(\mathfrak{g}, V)$, with values in V for the representation ρ , is defined by the quotient group

$$H_\rho^p(\mathfrak{g}, V) = Z_\rho^p(\mathfrak{g}, V) / B_\rho^p(\mathfrak{g}, V) \quad . \tag{29}$$

Cohomology groups measure the lack of exactness of the sequence $\dots C^{p-1} \xrightarrow{s} C^p \xrightarrow{s} C^{p+1} \dots$ i.e., how much $sC^{p-1} \equiv B^p \subset C^p$ differs from the kernel Z^p of s acting on C^p ; if $H^p = 0$, $sC^{p-1} \equiv B^p = Z^p \equiv \ker s \subset C^p$.

Remark 4. (*Lie algebra cohomology and the MC equations*)

Let ω be the \mathfrak{g} -valued one-cochain on \mathfrak{g} defined by $\omega(X_i) = X_i$ (when defined on the group manifold, it is the canonical one-form θ on G). Then, the expression of the action of the coboundary operator in eq. (26) for the trivial representation gives $s\omega(X_i, X_j) = -\omega([X_i, X_j])$,

which may be written as $s\omega(X_i, X_j) = -(\omega \wedge \omega)(X_i, X_j)$ *i.e.*, as $s\omega = -\omega \wedge \omega$. This is the MC equation (20), which appears here as the expression of the action of the coboundary operator on the one-cochain ω . In Sec. 3.5, s will be identified with the exterior derivative, the ω^i with the left invariant MC forms on G and $d^2 \equiv 0$ will correspond to $s^2 \equiv 0$ above, which holds as a consequence of the JI.

3.2. Central extensions of a Lie algebra.

It is easy to see that the possible central extensions of a Lie algebra \mathfrak{g} (see *e.g.* [106]) are characterized by non-trivial two-cocycles for the trivial representation. Chosen a basis, the commutators of a central extension $\tilde{\mathfrak{g}}$ of a Lie algebra \mathfrak{g} are generically given by

$$[\tilde{X}_i, \tilde{X}_j] = C_{ij}^k \tilde{X}_k + \Omega^2(X_i, X_j)\Xi, \quad (30)$$

where $X_i \in \mathfrak{g}$, $\tilde{X}_i \in \tilde{\mathfrak{g}}$, Ξ is the central ($[\tilde{X}_i, \Xi] = 0$) generator in $\tilde{\mathfrak{g}}$, $\Omega^2(X_i, X_j) = -\Omega^2(X_j, X_i)$ and $[X_i, X_j] = C_{ij}^k X_k$ are the commutators of the original, unextended algebra \mathfrak{g} . If (30) has to be a Lie algebra, the Lie bracket in $\tilde{\mathfrak{g}}$ has to satisfy the JI, and this forces the antisymmetric bilinear map Ω^2 in (30) to be a two-cocycle, as it follows from the third equality for $s\Omega(X, Y, Z)$ in (26) for $\rho = 0$.

If Ω^2 is a trivial two-cocycle (a two-coboundary, $\Omega^2 = s\Omega^1$) there exists a basis of $\tilde{\mathfrak{g}}$ in which the Ξ can be removed from the *r.h.s* of (30). This means that $\tilde{\mathfrak{g}}$ is the direct sum $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ and that the central extension is actually trivial. Indeed, in this case (see the second equality in (26)) $\Omega^2(X_i, X_j) = (s\Omega^1)(X_i, X_j) = -\Omega^1([X_i, X_j]) = -C_{ij}^k \Omega(X_k)$. It is then sufficient to define new generators \tilde{X}' of $\tilde{\mathfrak{g}}$ by the linear combination $\tilde{X}'_k = \tilde{X}_k - \Omega^1(X_k)\Xi$ to obtain $[\tilde{X}'_i, \tilde{X}'_j] = C_{ij}^k (\tilde{X}_k - \Omega^1(X_k)) = C_{ij}^k \tilde{X}'_k$, $[\tilde{X}'_i, \Xi] = 0$, which shows that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ in an explicit manner. Of course, such a $\tilde{\mathfrak{g}}$ is a trivial central extension irrespective of the basis used to express its commutators.

Thus, the central extensions of a Lie algebra \mathfrak{g} are governed by the second cohomology group for the trivial representation, $H_0^2(\mathfrak{g})$. When $H_0^2(\mathfrak{g}) = 0$, all central extensions of \mathfrak{g} are trivial. This is the case for semisimple algebras by virtue of the Whitehead's Lemma 8 below.

3.3. Deformations of Lie algebras.

Deformations of algebras were studied long ago by Gerstenhaber [107] and by Nijenhuis and Richardson [108] specifically for the Lie algebra case (see also [109] for an overview and further early references). The idea is to find a new Lie algebra 'close' to the original one. This leads to a cohomology problem and to the notion of *stability* or *rigidity*; algebras that cannot be deformed are called rigid. The idea of stability has a clear physical meaning: it is associated with theories that are not deformable *i.e.*, that they are stable in the sense that they do not change in a qualitative (*i.e.*, structural) manner by smoothly changing a parameter. For instance, since the Poincaré algebra is a deformation of the Galilei one, Einsteinian mechanics may be looked as a stabilization of Newtonian mechanics (albeit a partial one, since the Poincaré algebra itself may still be deformed into either of the stable simple de Sitter and anti-de Sitter algebras, $so(1, 4)$ and $so(2, 3)$). The deformation process may also be applied to superalgebras (see [110]); for instance, $osp(1|4)$ is a deformation of the $N=1, D=4$ superPoincaré algebra.

Since we shall consider in Sec. 11.7 deformations of Filippov algebras, let us review briefly here the problem of deforming Lie algebras. This provides an example where the relevant cohomology is the Lie algebra cohomology for a representation, $\rho = ad$. The aim here is to obtain a deformation of the original Lie bracket $[X, Y]$, depending of a parameter t , in a way that still defines a Lie algebra. Thus, one looks for a new Lie bracket $[X, Y]_t$ depending on t

$([X, Y]_{t=0} = [X, Y])$, defined by

$$[X, Y]_t := [X, Y] + \sum_{n=1}^{\infty} t^n \alpha_n(X, Y) \quad , \quad (31)$$

where the α_n are necessarily bilinear and skewsymmetric \mathfrak{g} -valued maps, $\alpha_n : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$, $\alpha_n \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, that must satisfy the conditions that the JI imposes on the deformed bracket $[X, Y]_t$, which has to be a Lie algebra bracket. Thus, the dimension of the deformed Lie algebra is the same as that of the original one; only its structure is deformed. The first order deformation, $[X, Y]_t := [X, Y] + t\alpha(X, Y)$, is the *infinitesimal deformation*; not every infinitesimal deformation is the first-order term of a full deformation. When it is, the deformation is called *integrable*.

3.3.1. Infinitesimal deformations of a Lie algebra \mathfrak{g} .

To see how cohomology enters, consider an infinitesimal deformation *i.e.*, eq. (31) neglecting terms of order t^2 and higher. This means that, to find the conditions that the \mathfrak{g} -valued $\alpha_1 \equiv \alpha$ has to satisfy, only t -linear terms have to be kept in

$$[X, [Y, Z]_t]_t + [Y, [Z, X]_t]_t + [Z, [X, Y]_t]_t = 0 \quad , \quad (32)$$

which is the JI for the deformed Lie algebra. Using the JI for the original one \mathfrak{g} , the remaining (order t) terms give rise to the condition

$$ad_X \alpha(Y, Z) + ad_Y \alpha(Z, X) + ad_Z \alpha(X, Y) + \alpha(X, [Y, Z]) + \alpha(Y, [Z, X]) + \alpha(Z, [X, Y]) = 0 \quad . \quad (33)$$

Comparing with the third equality in eq. (26), we see that the skewsymmetric map α has to be a \mathfrak{g} -cohomology two-cocycle for the action $\rho = ad$, since eq. (33) simply reads $s\alpha(X, Y, Z) = 0$. Alternatively, we might take the *l.h.s.* of (33) to *define* $s\alpha(X, Y, Z)$ and thus the three-linear skewsymmetric map $s\alpha$; this would lead us to *derive* the cohomology relevant for the deformation problem (see below).

We may now ask ourselves whether this infinitesimal deformation is a true one *i.e.*, not removable by redefining the basis of the algebra, in which case the deformation would be trivial. It turns out that this will be so if the two-cocycle defining the infinitesimal deformation is a two-coboundary, hence obtained from a \mathfrak{g} -valued one-cochain β *i.e.*, if $\alpha(X, Y) = (\delta\beta)(X, Y) = ad_X \beta(Y) - ad_Y \beta(X) - \beta([X, Y])$ (see the second equality in eq. (26)). To check this, it is sufficient to redefine new generators $X' = X - t\beta(X)$ to find that, with $[X, Y] = Z$ and neglecting terms of order higher than t ,

$$\begin{aligned} [X', Y']_t &= [X, Y] + t\alpha(X, Y) - t ad_X \beta(Y) + ad_Y \beta(X) + (t\beta([X, Y]) - t\beta([X, Y])) \\ &= Z - t\beta(Z) + t(\alpha(X, Y) - (\delta\beta)(X, Y)) = Z' \quad , \end{aligned} \quad (34)$$

since the t -term in the last equality is zero for a two-coboundary $\alpha = \delta\beta$. Note that, again, enforcing that the redefinition manifestly exhibits the undeformed character of the algebra would show that the two-cocycle α is given by a specific expression in terms of a \mathfrak{g} -valued linear map (one-cochain) β , which then would determine the form of the two-coboundary generated by β . In this way, by studying the deformations of Lie algebras (and then by generalizing the action of the coboundary operator on higher order cochains, etc.) we would have been led naturally to the cohomology complex $(C^\bullet(\mathfrak{g}, \mathfrak{g}), s)$ of Def. 2 for the representation ad and, by extension, for an arbitrary one ρ . The JI is the key ingredient in the definition of the Lie algebra cohomology⁴.

⁴The Lie algebra cohomology complex relevant for the general (not necessarily central) extensions of \mathfrak{g} by an abelian Lie algebra kernel \mathscr{A} (see *e.g.* [106]) is also the cohomology for a representation ρ . We shall not look in this section at the cohomology of \mathfrak{g} from this point of view, which would lead us to the coboundary operator

The outcome of this discussion is that if $H_{ad}^2(\mathfrak{g}, \mathfrak{g}) = 0$ all two-cocycles are trivial, \mathfrak{g} cannot be deformed and hence the Lie algebra is rigid [107, 108]; therefore, by Witehead's Lemma 8 below, all semisimple algebras are stable. Note, however, that $H_{ad}^2(\mathfrak{g}, \mathfrak{g}) = 0$ is a *sufficient* condition for the rigidity of a Lie algebra, but not a necessary one; there are examples of Lie algebras which do not satisfy this condition *i.e.*, with $H_{ad}^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ -therefore, not semisimple- that nevertheless are rigid [111].

3.3.2. Higher order deformations.

Moving further one encounters an obstruction when $H^3 \neq 0$, which prevents the integrability of the infinitesimal deformation. To see it, let us consider (31) up to the t^2 term,

$$[X, Y]_t = [X, Y] + t\alpha_1(X, Y) + t^2\alpha_2(X, Y) \quad . \quad (35)$$

Imposing the condition that $[X, Y]_t$ now satisfies the JI to order t^2 , and taking into account that $\alpha_1(X, Y)$ gives an infinitesimal deformation and hence it is a two-cocycle, $s\alpha_1(X, Y, Z) = 0$, we obtain from eq. (35) that the t^2 terms have the form

$$\begin{aligned} & \alpha_1(X, \alpha_1(Y, Z)) + \alpha_1(Y, \alpha_1(Z, X)) + \alpha_1(Z, \alpha_1(X, Y)) \\ & + ad_X\alpha_2(Y, Z) + ad_Y\alpha_2(Z, X) + ad_Z\alpha_2(X, Y) \\ & + \alpha_2(X, [Y, Z]) + \alpha_2(Y, [Z, X]) + \alpha_2(Z, [X, Y]) \\ & \equiv \gamma(X, Y, Z) + s\alpha_2(X, Y, Z) \quad , \end{aligned} \quad (36)$$

where the first line above defines a \mathfrak{g} -valued three-linear map,

$$\gamma(X, Y, Z) := \alpha_1(X, \alpha_1(Y, Z)) + \text{cycl.}(X, Y, Z) \quad , \quad (37)$$

which is fully skewsymmetric and hence a three-cochain, $\gamma \in C^3(\mathfrak{g}, \mathfrak{g})$, and where the remaining (α_2) terms in eq. (36) have been identified as $s\alpha_2$ using eq. (26). It is then seen that $\gamma \in Z_{ad}^3(\mathfrak{g}, \mathfrak{g})$: indeed, since $\alpha_1 \in Z_{ad}^2(\mathfrak{g}, \mathfrak{g})$, $s\gamma(X_1, X_2, X_3, X_4) = 0$ with $s\gamma$ given by (26) for $\rho = ad$ and $\gamma = \Omega^3$.

Hence, when the cocycle γ is actually a three-coboundary, $\gamma = s\alpha'$ where α' is a two-cochain, it is sufficient to take $\alpha_2 = -\alpha'$ in (35) to see in (36) that the JI is fulfilled up to second order. We can now continue in the same way up to terms of order t^3 to find at this stage that again a three-cocycle appears potentially obstructing the deformation to that order, and so on. Thus, all the obstructions that prevent expanding an infinitesimal deformation to a one parameter family of deformations are elements of $H_{ad}^3(\mathfrak{g}, \mathfrak{g})$; as all three-cocycles have to be trivial, it follows that only if $H_{ad}^3(\mathfrak{g}, \mathfrak{g}) = 0$ all obstructions vanish and every infinitesimal deformation $\alpha_1 \in Z_{ad}^2(\mathfrak{g}, \mathfrak{g})$ is integrable.

3.4. Coordinates expression of the coboundary operator action for the trivial representation.

of Def. 2. This approach will be followed in Sec. 4.2, where the extension problem for Leibniz algebras \mathcal{L} is discussed. Sec. 4.2 can be readily translated to the Lie algebra case by adding the requirement of skewsymmetry where appropriate.

It is convenient to have the action of the coboundary operator of eq. (27) expressed in terms of the coordinates $\Omega_{i_1 \dots i_p} = \Omega(X_{i_1}, \dots, X_{i_p})$ of the p -cochain on which it acts. Eq. (27) gives

$$\begin{aligned} (s\Omega)_{i_1 \dots i_{p+1}} &= \sum_{\substack{s,t=1 \\ s < t}}^{p+1} (-1)^{s+t} C_{i_s i_t}^k \Omega_{k i_1 \dots \hat{i}_s \dots \hat{i}_t \dots i_{p+1}} \\ &= \frac{1}{2} \sum_{\substack{s,t=1 \\ s < t}}^{p+1} (-1)^{s+t} \epsilon_{i_s i_t}^{j_1 j_2} C_{j_1 j_2}^k \frac{1}{(p-1)!} \epsilon_{i_1 \dots \hat{i}_s \dots \hat{i}_t \dots i_{p+1}}^{j_3 \dots j_{p+1}} \Omega_{k j_3 \dots j_{p+1}} \\ &= -\frac{1}{2} \frac{1}{(p-1)!} C_{j_1 j_2}^k \Omega_{k j_3 \dots j_{p+1}} \sum_{\substack{s,t=1 \\ s < t}}^{p+1} (-1)^{s+t+1} \epsilon_{i_s i_t}^{j_1 j_2} \epsilon_{i_1 \dots \hat{i}_s \dots \hat{i}_t \dots i_{p+1}}^{j_3 \dots j_{p+1}} \end{aligned}$$

which, using

$$\sum_{\substack{s,t=1 \\ s < t}}^{p+1} (-1)^{s+t+1} \epsilon_{i_s i_t}^{j_1 j_2} \epsilon_{i_1 \dots \hat{i}_s \dots \hat{i}_t \dots i_{p+1}}^{j_3 \dots j_{p+1}} = \epsilon_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}} \quad , \quad (38)$$

which follows by developing the determinant that defines the antisymmetric Kronecker symbol

$$\epsilon_{j_1 \dots j_p}^{i_1 \dots i_p} = \det \begin{pmatrix} \delta_{j_1}^{i_1} & \dots & \delta_{j_p}^{i_1} \\ \vdots & & \vdots \\ \delta_{j_1}^{i_p} & \dots & \delta_{j_p}^{i_p} \end{pmatrix} \quad , \quad (39)$$

gives the coordinates of the $(p+1)$ -cochain (in fact, coboundary)

$$(s\Omega)_{i_1 \dots i_{p+1}} = -\frac{1}{2} \frac{1}{(p-1)!} \epsilon_{i_1 \dots i_{p+1}}^{j_1 \dots j_{p+1}} C_{j_1 j_2}^k \Omega_{k j_3 \dots j_{p+1}} \quad . \quad (40)$$

3.5. Chevalley-Eilenberg formulation of Lie algebra cohomology.

The Chevalley-Eilenberg (CE) formulation [104] makes use of the ‘localisation’ process which allows us to obtain invariant tensor *fields* on the group manifold G by left-translating the appropriate algebraic objects at the unit element $e \in G$ to an arbitrary group element g . In this way, the expression (25) for the Lie algebra coboundary operator and eq. (18) for the exterior derivative become equivalent if we take M as the manifold of the group G associated to \mathfrak{g} and convert the multilinear applications into invariant tensor fields on the group manifold. This is done by identifying \mathfrak{g} with $T_e(G)$, the vector tangent space at the unit element, and by moving from e to an arbitrary group element $g \in G$ by the left translation L_g generated by g . In this way, the basis elements X_i of the vector space \mathfrak{g} become LI vector fields $X_i(g)$ on the group manifold G satisfying the Lie algebra commutation relations, $[X_i(g), X_j(g)] = C^i_{jk} X^k(g)$, the $\omega^i \in \mathfrak{g}^*$ become the LI Maurer-Cartan (MC) one-forms $\omega^i(g)$ on G which characterize the Lie algebra from the MC equations dual point of view (eqs. (17)), and the p -cochains become LI p -forms on the group manifold G .

Let $V = \mathbb{R}$, so that ρ is trivial. Then, the first term in (25) is not present and, on LI one-forms, s and d act in the same manner. Since there is a one-to-one correspondence between p -antisymmetric maps on \mathfrak{g} and LI p -forms on G , a p -cochain in $C^p(\mathfrak{g}, \mathbb{R})$ is given by a LI p -form on G , which in terms of the MC forms may be written as

$$\Omega^p(g) = \frac{1}{p!} \Omega_{i_1 \dots i_p} \omega^{i_1}(g) \wedge \dots \wedge \omega^{i_p}(g) \quad , \quad (41)$$

with constant coordinates $\Omega_{i_1 \dots i_p}$. The Lie algebra cohomology coboundary operator s for the trivial representation $\rho = 0$ is thus given by the exterior differential d acting on LI p -forms on the group manifold G , which are the p -cochains (the explicit dependence of the forms $\Omega(g)$, $\omega^i(g)$ on g will be omitted henceforth).

Coordinates expression of the cocycle condition, $\rho = 0$.

A p -cochain Ω is a p -cocycle for the trivial representation of \mathfrak{g} if $(s\Omega)_{i_1 \dots i_{p+1}} = 0$ *i.e.*, when its coordinates satisfy (see eq. (40)),

$$C_{[j_1 j_2}^k \Omega_{i_1 \dots i_{p-1}]k} = 0 . \quad (42)$$

This condition also follows from eq. (41) by imposing $d\Omega = 0$ and using the MC eqs. (17). The nilpotency of s can be easily checked using the CE formulation of the Lie algebra cohomology:

Proposition 5. *(Nilpotency of the coboundary operator s)*

The Lie algebra cohomology operator s is nilpotent, $s^2 = 0$.

Proof. First, we notice that a V -valued p -cocycle may be written as $\Omega^{pA}(g) = \frac{1}{p!} \Omega_{i_1 \dots i_p}^A \omega^{i_1}(g) \wedge \dots \wedge \omega^{i_p}(g)$. Thus, looking at the definition of the coboundary operator in (25) and taking into account (19), we see that s may be written as

$$(s)_B^A = \rho(X_i)_B^A \omega^i + \delta_B^A d \quad , \quad (s = d + \rho(X_i) \omega^i) \quad . \quad (43)$$

Then, since $d^2=0$, the proposition follows from the fact that

$$\begin{aligned} s^2 &= (\rho(X_i) \omega^i + d)(\rho(X_j) \omega^j + d) = \rho(X_i) \rho(X_j) \omega^i \wedge \omega^j + \rho(X_i) \omega^i d + \rho(X_j) d(\omega^j) + d^2 \\ &= -\frac{1}{2} \rho(X_j) C_{ik}^j \omega^l \wedge \omega^k + \frac{1}{2} [\rho(X_i), \rho(X_j)] \omega^i \wedge \omega^j = 0 \quad , \end{aligned} \quad (44)$$

where use has been made of the fact that ρ is a representation of \mathfrak{g} , $[\rho(X_i), \rho(X_j)] = C_{ij}^k \rho(X_k)$. \square

In spite of s being given by d , the Lie algebra CE cohomology is in general different from the de Rham cohomology: a closed LI p -form α on G -*i.e.*, a p -cocycle- may be de Rham exact and hence de Rham trivial without being CE-trivial. This is because a de Rham exact form or de Rham coboundary, $\alpha = d\beta$, will not be a CE coboundary if the potential $(p-1)$ -form β of α is not a CE cochain *i.e.*, is not a LI form⁵. Nevertheless, for G compact the Lie and de Rham cohomologies coincide, $H_{DR}(G) = H_0(\mathfrak{g}, \mathbb{R})$, as stated by the following

Proposition 6. *(de Rham vs. CE cohomology) [104]*

Let G be a compact and connected Lie group. Every de Rham cohomology class on G contains one and only one bi-invariant form. The bi-invariant forms span a ring isomorphic to $H_{DR}(G)$.

Example 7. Let \mathfrak{g} be the abelian two-dimensional algebra. The corresponding Lie group is \mathbb{R}^2 , hence de Rham trivial. However, the translation algebra \mathbb{R}^2 has a non-trivial Lie algebra second cohomology group; it admits a one-parameter family of non-trivial central extensions, all isomorphic to the three-dimensional Heisenberg-Weyl algebra.

⁵This is the case *e.g.*, for certain forms on superspace group manifolds ('rigid' superspaces) which appear in M- and super- p -brane theory (see [112]). This is not surprising due to the absence of global considerations in the fermionic, Grassmann odd sector of supersymmetry. The Lie algebra cohomology notions may be, in fact, extended to superalgebras (see *e.g.* [113–115] and refs. therein).

3.6. Whitehead's lemma for vector valued cohomology.

Lemma 8. (*Whitehead's lemma [96]*)

Let \mathfrak{g} be a finite-dimensional semisimple Lie algebra over a field of characteristic zero and let V be a finite-dimensional irreducible $\rho(\mathfrak{g})$ -module such that $\rho(\mathfrak{g})V \neq 0$ (ρ non-trivial). Then,

$$H_\rho^q(\mathfrak{g}, V) = 0 \quad \forall q \geq 0 \quad . \quad (45)$$

If $q = 0$, the non-triviality of ρ and the irreducibility imply that $\rho(\mathfrak{g}) \cdot v = 0$ ($v \in V$) holds only for $v = 0$.

Proof. Since \mathfrak{g} is semi-simple, the Cartan-Killing metric k_{ij} is invertible, $k^{ij}k_{jk} = \delta_k^i$. Let τ be the operator on the space of p -cochains $\tau : C^p(\mathfrak{g}, V) \rightarrow C^{p-1}(\mathfrak{g}, V)$ defined by

$$(\tau\Omega)_{i_1 \dots i_{p-1}}^A = k^{ij} \rho(X_i)_{.B}^A \Omega_{j i_1 \dots i_{p-1}}^B \quad . \quad (46)$$

It is not difficult to check that on cochains the Laplacian-like operator $(s\tau + \tau s)$ gives⁶

$$[(s\tau + \tau s)\Omega]_{i_1 \dots i_p}^A = \Omega_{i_1 \dots i_p}^B I_2(\rho)_{.B}^A \quad , \quad (48)$$

where $I_2(\rho)_{.B}^A = k^{ij}(\rho(X_i)\rho(X_j))_{.B}^A$ is the quadratic Casimir in the representation ρ . By Schur's lemma it is proportional to the unit matrix. Hence, applying (48) to a cocycle $\Omega \in Z_\rho^p(\mathfrak{g}, V)$ we find

$$s\tau\Omega = \Omega I_2(\rho) \Rightarrow s(\tau\Omega I_2(\rho)^{-1}) = \Omega \quad . \quad (49)$$

Thus, Ω is the coboundary generated by the cochain $\tau\Omega I_2(\rho)^{-1} \in C_\rho^{p-1}(\mathfrak{g}, V)$. \square

For semisimple algebras and $\rho = 0$ we also have $H_0^1 = 0$ and $H_0^2 = 0$, but already $H_0^3 \neq 0$; the three-cocycle $\Omega_{i_2 i_2 i_3}$ is given by the fully antisymmetric structure constants $C_{i_1 i_2 i_3}$.

3.7. Simple Lie algebras, invariant polynomials, and cohomology.

Let \mathfrak{g} be simple. By virtue of Whitehead's Lemma, only the $\rho = 0$ case is interesting in the simple case since, if \mathfrak{g} is simple, $H_\rho^p(\mathfrak{g}, V) = 0$ for ρ non-trivial. The non-trivial cohomology groups are related to the primitive (*i.e.*, not reducible to products, see Def. 14) symmetric invariant tensors [116–123] on \mathfrak{g} , which in turn determine Casimir elements in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

Definition 9. (*Symmetric and invariant polynomials on \mathfrak{g}*)

A symmetric polynomial on \mathfrak{g} is given by a covariant symmetric LI tensor.

In terms of the MC forms on the group manifold G , a symmetric invariant polynomial is given by a LI covariant tensor field on G , $k = k_{i_1 \dots i_m} \omega^{i_1} \otimes \dots \otimes \omega^{i_m}$ with symmetric constant coordinates $k_{i_1 \dots i_m}$.

The polynomial k is said to be a *bi-invariant* (or *ad-invariant*) symmetric polynomial if it is also right-invariant, *i.e.* if $L_{X_l} k = 0 \quad \forall X_l \in \mathfrak{X}^L(G)$. Using (23) we find that

$$L_{X_l} k = 0 \Rightarrow C_{li_1}{}^s k_{si_2 \dots i_m} + C_{li_2}{}^s k_{i_1 s \dots i_m} + \dots + C_{li_m}{}^s k_{i_1 \dots i_{m-1} s} = 0 \quad . \quad (50)$$

⁶ For instance, for a two-cochain eq. (48) reads

$$\begin{aligned} [(s\tau + \tau s)\Omega]_{ij}^A &= g^{kl} \rho(X_i)_{.B}^A \rho(X_k)_{.C}^B \Omega_{lj}^C - g^{kl} \rho(X_j)_{.B}^A \rho(X_k)_{.C}^B \Omega_{li}^C - g^{kl} \rho(X_k)_{.B}^A C_{ij}^m \Omega_{lm}^B \\ &+ g^{kl} \rho(X_k)_{.B}^A \rho(X_l)_{.C}^B \Omega_{ij}^C + g^{kl} \rho(X_k)_{.B}^A \rho(X_i)_{.C}^B \Omega_{jl}^C + g^{kl} \rho(X_k)_{.B}^A \rho(X_j)_{.C}^B \Omega_{li}^C \\ &- g^{kl} \rho(X_k)_{.B}^A C_{ij}^m \Omega_{ml}^B - g^{kl} \rho(X_k)_{.B}^A C_{li}^m \Omega_{mj}^B - g^{kl} \rho(X_k)_{.B}^A C_{jl}^m \Omega_{mi}^B \\ &= g^{kl} [\rho(X_i), \rho(X_k)]_{.B}^A \Omega_{lj}^B - g^{kl} [\rho(X_j), \rho(X_k)]_{.B}^A \Omega_{li}^B + I_2(\rho)_{.B}^A \Omega_{ij}^B \\ &- g^{kl} \rho(X_k)_{.B}^A C_{li}^m \Omega_{mj}^B - g^{kl} \rho(X_k)_{.B}^A C_{jl}^m \Omega_{mi}^B = I_2(\rho)_{.B}^A \Omega_{ij}^B \quad . \end{aligned} \quad (47)$$

Since the coordinates of k are given by $k_{i_1 \dots i_m} = k(X_{i_1}, \dots, X_{i_m})$, eq. (50) is equivalent to stating that k is *ad*-invariant⁷, *i.e.*,

$$k([X_l, X_{i_1}], \dots, X_{i_m}) + k(X_{i_1}, [X_l, X_{i_2}], \dots, X_{i_m}) + \dots + k(X_{i_1}, \dots, [X_l, X_{i_m}]) = 0 \quad (51)$$

or, equivalently,

$$k(\text{Ad } g X_{i_1}, \dots, \text{Ad } g X_{i_m}) = k(X_{i_1}, \dots, X_{i_m}) \quad , \quad (52)$$

from which eq. (51) follows by taking the derivative $\partial/\partial g^l$ in $g = e \in G$.

The invariant symmetric polynomials just described can be used to construct Casimir elements of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ in the following way

Proposition 10. (*Higher order Casimirs*)

Let k be a symmetric invariant tensor. Then $k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}$ (coordinate indices of k raised using the Killing metric), is a Casimir of \mathfrak{g} of order m , $[k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}, Y] = 0 \forall Y \in \mathfrak{g}$.

Proof.

$$\begin{aligned} [k^{i_1 \dots i_m} X_{i_1} \dots X_{i_m}, X_s] &= \sum_{j=1}^m k^{i_1 \dots i_m} X_{i_1} \dots [X_{i_j}, X_s] \dots X_{i_m} \\ &= \sum_{j=1}^m k^{i_1 \dots i_m} X_{i_1} \dots C_{i_j s}^t X_t \dots X_{i_m} = 0 \end{aligned} \quad (53)$$

by (50). □

An easy way of obtaining symmetric (*ad*-)invariant polynomials (used *e.g.*, in the construction of characteristic classes) is given by

Proposition 11. Let X_i denote now a representation of \mathfrak{g} . Then, the symmetrized trace

$$k_{i_1 \dots i_m} = s\text{Tr}(X_{i_1} \dots X_{i_m}) \quad (54)$$

defines a symmetric invariant polynomial.

Proof. k is symmetric by construction and the *ad*-invariance is obvious since $\text{Ad } g X := g X g^{-1}$. □

The simplest illustration of a trace-invariant form is the non-singular Killing metric (eq. (14)) for a simple Lie algebra \mathfrak{g} ; its associated Casimir is the (second order) Casimir I_2 .

Example 12. Let $\mathfrak{g} = su(n)$, $n \geq 2$, and let X_i be (hermitian) matrices in the defining representation. Then

$$s\text{Tr}(X_i X_j X_k) \propto 2\text{Tr}(\{X_i, X_j\} X_k) = d_{ijk} \quad , \quad (55)$$

using that, for the $su(n)$ algebra, $\text{Tr}(X_k) = 0$ and with generators normalized to $\text{Tr}(X_i X_j) = \frac{1}{2} \delta_{ij}$, the anticommutator is given [124] by $\{X_i, X_j\} = c \delta_{ij} + d_{ijl} X_l$ with $c = 1/n$. The symmetric polynomial d_{ijk} leads to the third order Casimir I_3 ; for $su(2)$ only the Killing metric $k_{ij} = \delta_{ij}$ and the quadratic Casimir I_2 exist.

Example 13. In the case $\mathfrak{g} = su(n)$, $n \geq 4$, we have a fourth order polynomial

$$s\text{Tr}(X_{i_1} X_{i_2} X_{i_3} X_{i_4}) \propto d_{(i_1 i_2 l} d_{l i_3) i_4} + 2c \delta_{(i_1 i_2} \delta_{i_3) i_4} \quad , \quad (56)$$

where $()$ indicates symmetrization. The first term $d_{(i_1 i_2 l} d_{l i_3) i_4}$ gives the fourth order Casimir I_4 . It generalizes easily to higher n by nesting more d 's, leading to the Klein [118] form of the $su(n)$ Casimirs. The last part of (56) (see [125]) is clearly the symmetrized product of two copies of the order two Casimir I_2 and thus it is not *primitive*.

⁷For a LI p -form α , the equivalent to (51) and (19) show that a bi-invariant form on G is closed.

Definition 14. (*Primitive symmetric invariant polynomials*)

A symmetric invariant polynomial $k_{i_1 \dots i_m}$ on \mathfrak{g} is called primitive if it is not of the form

$$k_{i_1 \dots i_m} = k_{(i_1 \dots i_p)}^{(p)} k_{i_{p+1} \dots i_m}^{(q)}, \quad p + q = m, \quad (57)$$

where $k^{(p)}$ and $k^{(q)}$ are two lower order symmetric invariant polynomials.

Of course, we could also have considered eq. (56) for $su(3)$, but then it would not have led to a fourth-order primitive polynomial, since $su(3)$ is a rank 2 algebra and has only two primitive invariant polynomials (of ranks 2 and 3). Indeed, $d_{(i_1 i_2 l} d_{i_3) i_4}$ is not primitive for $su(3)$ and can be written in terms of $\delta_{i_1 i_2}$ as in (57) (see, *e.g.*, [126]; see also [125] and refs. therein). In general, for a compact simple algebra of rank l there are l invariant primitive polynomials and Casimirs [116–123] and, as shown below, l primitive Lie algebra cohomology cocycles (see Table in Sec. 3.9).

Lemma 15. (*p -cocycles as skewsymmetric invariant polynomials*)

Let Ω^p (eq. (41)) be a p -cocycle. Then, it defines a skewsymmetric invariant polynomial of rank p .

Proof. The statement follows since invariance means $L_{X_i} \Omega^p = 0$ *i.e.*,

$$\sum_{k=1}^p C_{ij_s}^k \Omega_{j_1 \dots j_{s-1} k j_{s+1} \dots j_p} = 0 \quad \text{or} \quad C_{i[j_1}^k \Omega_{j_2 \dots j_p]k} = 0, \quad (58)$$

which is satisfied by any p -cocycle. □

3.8. Cocycles from invariant polynomials.

To make explicit the connection between the invariant polynomials and the non-trivial cocycles of a simple Lie algebra \mathfrak{g} let us use the particular case $\mathfrak{g} = su(n)$ as a guide. On the manifold of the $SU(n)$ group one can construct the *odd* p -form

$$\Omega^p = \frac{1}{p!} \text{Tr}(\theta \wedge \overset{p}{\cdot} \wedge \theta), \quad (59)$$

where again $\theta = \omega^i X_i$ is the canonical form and we take $\{X_i\}$ in the defining representation; p has to be odd since otherwise Ω would be zero by virtue of the cyclic property of the trace and the anticommutativity of one-forms.

Proposition 16. *The odd LI form Ω^p on G in (59) is a non-trivial (CE) Lie algebra cohomology p -cocycle.*

Proof. Since Ω^p is LI by construction, it is sufficient to show that Ω^p is closed and that it is not the differential of a LI $(p-1)$ -form *i.e.*, that it is not a coboundary. By using (20) we get

$$d\Omega^p \sim \text{Tr}(\theta \wedge \overset{p+1}{\cdot} \wedge \theta) = 0, \quad (60)$$

since $p+1$ is even. Suppose now that $\Omega^p = d\Omega^{p-1}$, with Ω^{p-1} LI. Then Ω^{p-1} would be of the form (59) and hence zero because $(p-1)$ is also even. □

All non-trivial p -cocycles in $H_0^p(su(n), \mathbb{R})$ are of the form (59). The fact that these forms are closed and de Rham non-exact ($SU(n)$ is compact) allows us to use them to construct Wess-Zumino-Witten [127, 128] terms on the group manifold (see also [129]).

Let us set $p = 2m - 1$. Since $\theta = \omega^i \circ X_i$, the form Ω^p expressed in coordinates is

$$\Omega^p = \frac{1}{q!} \text{Tr}(X_{i_1} \dots X_{i_{2m-1}}) \omega^{i_1} \wedge \dots \wedge \omega^{i_{2m-1}}$$

$$\begin{aligned} &\propto \text{Tr}([X_{i_1}, X_{i_2}][X_{i_3}, X_{i_4}] \cdots [X_{i_{2m-3}}, X_{i_{2m-2}}]X_{i_{2m-1}})\omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-1}} \\ &= \text{Tr}(X_{l_1} \cdots X_{l_{m-1}}X_\sigma)C_{i_1 i_2}^{l_1} \cdots C_{i_{2m-3} i_{2m-2}}^{l_{m-1}} \omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-2}} \wedge \omega^\sigma \quad . \end{aligned} \quad (61)$$

We see here how the order m symmetric invariant polynomial $\text{Tr}(X_{l_1} \cdots X_{l_{m-1}}X_\sigma)$ appears in this context. There is symmetry in $l_1 \cdots l_{m-1}$ (and hence $\text{Tr}(X_{l_1} \cdots X_{l_{m-1}}X_\sigma)$ is fully symmetric) because there is antisymmetry on the i indices due to the ω^{i_i} 's.

Conversely, the following statement holds:

Proposition 17. (*Cocycles associated with invariant polynomials*)

Let $k_{i_1 \dots i_m}$ be a symmetric invariant polynomial. Then, the polynomial

$$\Omega_{\rho i_2 \dots i_{2m-2} \sigma} = C_{j_2 j_3}^{l_1} \cdots C_{j_{2m-2} \sigma}^{l_{m-1}} k_{\rho l_1 \dots l_{m-1}} \epsilon_{i_2 \dots i_{2m-2}}^{j_2 \dots j_{2m-2}} \quad (62)$$

is skewsymmetric and defines [125] the closed form $((2m-1)$ -cocycle)

$$\Omega^{2m-1} = \frac{1}{(2m-1)!} \Omega_{\rho i_2 \dots i_{2m-2} \sigma} \omega^\rho \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_{2m-2}} \wedge \omega^\sigma \quad . \quad (63)$$

Proof. To check the complete skew-symmetry of $\Omega_{\rho i_2 \dots i_{2m-2} \sigma}$ in (62) it is sufficient, due to the Levi-Civita symbol, to show the antisymmetry in ρ and σ . This is done by using the invariance of k (50) and the symmetry properties of k and ϵ to rewrite $\Omega_{\rho i_2 \dots i_{2m-2} \sigma}$ as the sum of two terms. The first one,

$$\begin{aligned} &\sum_{s=1}^{m-2} \epsilon_{i_2 \dots i_{2m-2}}^{j_2 \dots j_{2s} j_{2s+1} j_{2m-2} j_{2s+2} \dots j_{2m-3}} k_{\rho l_1 \dots l_{s-1} l_{m-1} l_s \dots l_{m-2} \sigma} \\ &\quad \times C_{j_2 j_3}^{l_1} \cdots C_{j_{2s} j_{2s+1}}^{l_s} C_{l_s j_{2m-2}}^{l_{m-1}} C_{j_{2s+2} j_{2s+3}}^{l_{s+1}} \cdots C_{j_{2m-4} j_{2m-3}}^{l_{m-2}} \end{aligned} \quad (64)$$

vanishes due to the standard JI, and the second one is

$$\Omega_{\rho i_2 \dots i_{2m-2} \sigma} = -\epsilon_{i_2 \dots i_{2m-2}}^{j_2 \dots j_{2m-2}} k_{\sigma l_1 \dots l_{m-1}} C_{j_2 j_3}^{l_1} \cdots C_{j_{2m-2} \rho}^{l_{m-1}} = -\Omega_{\sigma i_2 \dots i_{2m-2} \rho} \quad . \quad (65)$$

To show that $d\Omega = 0$ we make use of the fact that any bi-invariant form is closed. Since Ω is LI by construction, we only need to prove its right-invariance, but

$$\Omega \propto \text{Tr}(\theta \wedge \overset{2m-1}{\cdots} \wedge \theta) \quad (66)$$

is obviously RI since, under a right translation, the canonical one-form transforms by $R_g^* \theta = \text{Ad}_g^{-1} \theta$ (see Prop. 11). \square

Without discussing the origin of the invariant polynomials for the different compact simple Lie algebras [116–123, 125], we may conclude that to each symmetric primitive invariant polynomial of order m we can associate a non-trivial Lie algebra cohomology $(2m-1)$ -cocycle (see [125] for practical details). In fact, this one-to-one correspondence is a famous result of Chevalley [130, 131] after a conjecture of A. Weil⁸.

A question that might immediately arise (from the above explicit construction) is whether it could be extended further since, from the $l = \text{rank } \mathfrak{g}$ primitive invariant polynomials that exist for a simple \mathfrak{g} , we may obtain an arbitrary number of non-primitive ones (see eq. (57)) by taking symmetrized products of primitive polynomials and applying the above formulae. This question is answered negatively by Prop. 18 and Cor. 19 below [125].

⁸The correspondence between invariant polynomials and cocycles reflects its transgression character [132]. See further [133] in the context of the Hopf-Koszul-Samelson theorem; we thank G. Pinczon for pointing out this reference to us.

Proposition 18. *Let $k_{i_1 \dots i_m}$ be a symmetric G -invariant polynomial. Then,*

$$\epsilon_{i_1 \dots i_{2m}}^{j_1 \dots j_{2m}} C_{j_1 j_2}^{l_1} \dots C_{j_{2m-1} j_{2m}}^{l_m} k_{l_1 \dots l_m} = 0 \quad . \quad (67)$$

Proof. By replacing $C_{j_{2m-1} j_{2m}}^{l_m} k_{l_1 \dots l_m}$ in the *l.h.s* of (67) by the other terms in (50) we get

$$\epsilon_{i_1 \dots i_{2m}}^{j_1 \dots j_{2m}} C_{j_1 j_2}^{l_1} \dots C_{j_{2m-3} j_{2m-2}}^{l_{m-1}} \left(\sum_{s=1}^{m-1} C_{j_{2m-1} l_s}^k k_{l_1 \dots l_{s-1} k l_{s+1} \dots l_{m-1} j_{2m}} \right) \quad , \quad (68)$$

which is zero due to the JI (note that (67) also follows from (60)). □

Corollary 19. *(Primitive invariant polynomials vs. cocycles)*

Let k be a non-primitive symmetric invariant polynomial (57). Then the $(2m - 1)$ -cocycle Ω associated to it by eq. (62) is zero [125].

Thus, to a primitive symmetric m -polynomial it is possible to associate uniquely a Lie algebra $(2m - 1)$ -cocycle. Conversely, we also have the following

Proposition 20. *(Invariant polynomials from cocycles)*

Let $\Omega^{(2m-1)}$ be a primitive cocycle. The l polynomials $t^{(m)}$ given by

$$t^{i_1 \dots i_m} = [\Omega^{(2m-1)}]^{j_1 \dots j_{2m-2} i_m} C_{j_1 j_2}^{i_1} \dots C_{j_{2m-3} j_{2m-2}}^{i_{m-1}} \quad (69)$$

are invariant, symmetric and primitive (see [125, Lemma 3.2]).

This converse proposition relates the cocycles of the Lie algebra cohomology to Casimirs in the enveloping algebra $\mathcal{U}(\mathfrak{g})$. The polynomials in (69) have certain advantages (for instance, they have all traces equal to zero) [125] over other more conventional ones such as *e.g.*, those obtained in (54). A list of the ranks of the invariant polynomials and associated cocycles for the simple algebras is given for convenience in Table 1 below.

3.9. Simple compact algebras: cocycles and Casimir operators.

We have seen that the Lie algebra cocycles may be expressed in terms of LI forms on the group manifold G (Sec. 3.5). The equivalence of the Lie algebra (CE) cohomology and the de Rham cohomology in Prop. 6 in the simple compact case is specially interesting because, since all primitive cocycles are odd, compact groups behave as products of odd spheres from the point of view of real homology. This leads to a number of simple and elegant formulae concerning the Poincaré polynomials, Betti numbers, the primitive invariant tensors and the non-trivial Lie algebra cocycles, etc. The following table summarizes many of these results. Details on the topological properties of Lie groups may be found in [134–136, 130, 131, 137–141]; for book references see [142–144, 106].

\mathfrak{g}	$\dim \mathfrak{g}$	Orders m_i of invariants and Casimirs	Orders $p = 2m_i - 1$ of \mathfrak{g} -cocycles
A_l	$(l + 1)^2 - 1$ [$l \geq 1$]	$2, 3, \dots, l + 1$	$3, 5, \dots, 2l + 1$
B_l	$l(2l + 1)$ [$l \geq 2$]	$2, 4, \dots, 2l$	$3, 7, \dots, 4l - 1$
C_l	$l(2l + 1)$ [$l \geq 3$]	$2, 4, \dots, 2l$	$3, 7, \dots, 4l - 1$
D_l	$l(2l - 1)$ [$l \geq 4$]	$2, 4, \dots, 2l - 2, l$	$3, 7, \dots, 4l - 5, 2l - 1$
G_2	14	2, 6	3, 11
F_4	52	2, 6, 8, 12	3, 11, 15, 23
E_6	78	2, 5, 6, 8, 9, 12	3, 9, 11, 15, 17, 23
E_7	133	2, 6, 8, 10, 12, 14, 18	3, 11, 15, 19, 23, 27, 35
E_8	248	2, 8, 12, 14, 18, 20, 24, 30	3, 15, 23, 27, 35, 39, 47, 59

Table 1. *Orders (ranks) of the primitive invariant tensors and associated cocycles for the compact simple Lie algebras ($i = 1 \dots, l$)*

The structure constants always define a non-trivial three-cocycle, the one associated with the non-degenerate Cartan-Killing form; this is why in the table above there is always a 2 (3) in the third (last) column. It is worth noticing that the sum of the orders ($2m_i - 1$) of the different cocycles in each entry in the last column is the dimension of the corresponding algebra *i.e.*,

$$\sum_{i=1}^{\text{rank } \mathfrak{g}} (2m_i - 1) = \dim \mathfrak{g} \quad .$$

This is so because the sum of the dimensions of the odd spheres has to be equal to the dimension of the group manifold.

The cohomology ring of the compact simple \mathfrak{g} is generated by the l primitive cocycles of order ($2m_i - 1$).

4. LEIBNIZ (LODAY'S) ALGEBRAS AND COHOMOLOGY

4.1. Main definitions.

Loday's algebras, or Leibniz algebras [18–20] in Loday's terminology (see also [145, 146]), are a non-skewsymmetric ($[X, Y] \neq -[Y, X]$) version of Lie algebras. More specifically, a Leibniz algebra \mathcal{L} is given by the following

Definition 21. (*(Left) Leibniz algebra*)

A (left) Leibniz algebra (LA) is a vector space \mathcal{L} endowed with a bilinear operation $\mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ that satisfies the relation (*left Leibniz identity*)

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]] \quad \forall X, Y, Z \in \mathcal{L} \quad , \quad (70)$$

which is no longer equivalent to the $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ Lie algebra JI.

Although $[X, Y] \neq -[Y, X]$ for a Leibniz algebra, some anticommutativity is left in the double bracket, since eq. (70) implies

$$[[X, Y], Z] = -[[Y, X], Z] \quad . \quad (71)$$

A Leibniz algebra that satisfies $[X, X] = 0 \quad \forall X \in \mathcal{L}$ has an anticommutative bracket and therefore is a Lie algebra since then eq. (70) is the JI. Obviously, any Lie algebra \mathfrak{g} is also a Leibniz algebra. Many Lie algebra notions such as those of subalgebra, quotient by a two-sided ideal, etc. extend trivially to the Leibniz algebra case (other notions, such as that of representation, require more care since the Leibniz bracket is not skewsymmetric, see Sec. 4.2 below). For instance, a *homomorphism ϕ of Leibniz algebras* is a homomorphism of the underlying vector spaces such that $\phi([X, Y]) = [\phi(X), \phi(Y)]$. The elements of the form $[X, X]$ (and those of the form $[X, Y] + [Y, X]$) are central in \mathcal{L} . They generate a two-sided ideal \mathcal{I} of \mathcal{L} , and the quotient \mathcal{L}/\mathcal{I} is a Lie algebra.

Since the Leibniz bracket is not antisymmetric, one has to distinguish between left (above) and *right LAs*, for which the left derivation property of eq. (70) is replaced by the right one,

$$[[X, Y], Z] = [[X, Z], Y] + [X, [Y, Z]] \quad \forall X, Y, Z \in \mathcal{L}^r \quad (72)$$

or *right Leibniz identity* (which again becomes the JI in the anticommutative case), and eq. (71) by

$$[X, [Y, Z]] = -[X, [Z, Y]] \quad .$$

The images of the left and right *adjoint maps*, ad and ad^r , are derivations of the corresponding Leibniz algebras since, in terms of them, the above two defining equations read

$$ad_X[Y, Z] = [ad_X Y, Z] + [Y, ad_X Z] \quad , \quad [X, Y]ad_Z^r = [Xad_Z^r, Y] + [X, Yad_Z^r] \quad , \quad (73)$$

where here we have added the superscript r and further located ad^r at the right to emphasize its right action character. Thus, the left and the right adjoint derivatives, which give rise to the left and right Leibniz identities, are essentially different, as they are in general any left and right actions $\rho(X)$. In contrast with the Lie algebra case, taking the opposite $X \mapsto -X$ is not an antiautomorphism of \mathcal{L} . Nevertheless, left and right Leibniz algebras are still related in the following sense: if the bracket $[X, Y]$ satisfies (70) and hence defines a left Leibniz algebra, the bracket $[X, Y]' = [Y, X]$ satisfies eq. (72) and defines a right one. The centre $Z(\mathcal{L})$ of a Leibniz algebra may be defined as the kernel of ad .

An interesting question is whether there is some analogue of Lie's third theorem for Leibniz algebras *i.e.*, whether there is some kind of generalization of the notion of Lie group (some kind of 'Leibniz group') so that Leibniz algebras are its corresponding tangent structures. Such an object has been dubbed by Loday [18] as a *coquecigrue*⁹. However, the problem of integrating Leibniz algebras in general remains open, although progress has been made with the introduction of *Lie racks* [147]. There is, however, a geometrical interpretation of certain *three*-Leibniz algebras, the Lie-triple systems (see Sec. 10), as tangent spaces: see [148].

The notion of LA may be extended to include the Z_2 -graded or Leibniz superalgebra case [145], although this will not be treated here. Further, we shall only consider *left* LAs from now on.

4.2. Extensions of a Leibniz algebra \mathcal{L} by an abelian one \mathcal{A} .

By definition (see, *e.g.* [106] for the Lie algebra case), $\widetilde{\mathcal{L}}$ is said to be an extension a Leibniz algebra \mathcal{L} by an abelian one \mathcal{A} if \mathcal{A} is a two-sided ideal of $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}}/\mathcal{A} = \mathcal{L}$ *i.e.*, there is an exact homomorphisms sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \widetilde{\mathcal{L}} \longrightarrow \mathcal{L} \longrightarrow 0 \quad .$$

To determine a solution $\widetilde{\mathcal{L}}$, we need the data of the extension problem, namely \mathcal{L} and a \mathcal{L} -module \mathcal{A} which is the abelian LA, so that left *and* right actions $\rho : \mathcal{L} \mapsto \text{End}\mathcal{A}$ of \mathcal{L} on \mathcal{A} are given (we shall not write ρ, ρ^r hereafter to distinguish the left or right actions since they will be distinguished by their location). In contrast with the Lie algebra case, both left and right actions on \mathcal{A} are needed as the Leibniz bracket is not anticommutative in general.

Let us assume that $\widetilde{\mathcal{L}}$ is a solution to the extension problem and let us characterize its elements by $(A, \tau(X))$, where τ is a trivializing section injecting \mathcal{L} into $\widetilde{\mathcal{L}}$. Since \mathcal{A} is abelian, it is clear that the left and right actions given by

$$\rho(X)A := [\tau(X), A] = [\pi^{-1}(X), A] \quad , \quad A\rho(X) := [A, \tau(X)] = [A, \pi^{-1}(X)] \quad , \quad (74)$$

where $\pi^{-1}(X)$ is the fibre over X (π is the projection of $\widetilde{\mathcal{L}}$ onto \mathcal{L}), are well defined; indeed all the elements $\pi^{-1}(X) \in \widetilde{\mathcal{L}}$ (*i.e.* those in the class of the element $\tau(X)$ of $\widetilde{\mathcal{L}}$ that defines the element $X \in \mathcal{L}$) give rise to the same (left or right) action as $\tau(X)$. Let us write $\tau(X) = (0, X)$ and then denote the elements of $\widetilde{\mathcal{L}}$ by (A, X) . Then, the bracket in $\widetilde{\mathcal{L}}$ is defined by

$$[(A_1, X_1), (A_2, X_2)] = (\rho(X_1)A_2 + A_1\rho(X_2) + \omega^2(X_1, X_2), [X_1, X_2]) \quad (75)$$

⁹After Rabelais imaginary animal in his *Gargantua* [*coquecigrue* = coq (cock) + ciguë (hemlock) + grue (crane)], an embodiment of absolute absurdity.

where, in contrast with the Lie algebra case, the antisymmetry of the two-cochain $\omega^2(X_1, X_2)$ is not required. The presence of $\omega^2(X_1, X_2)$, $\omega^2 : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{A}$, indicates that \mathcal{L} is not necessarily a subalgebra of $\widetilde{\mathcal{L}}$ since, in general, τ is not a homomorphism of Leibniz algebras,

$$[\tau(X_2), \tau(X_2)] - \tau([X_1, X_2]) = \omega^2(X_1, X_2) \quad ; \quad (76)$$

in fact,

$$[\tau(X_1), \tau(X_2)] = [(0, X_1), (0, X_2)] = (\omega^2(X_1, X_2), [X_1, X_2]) \neq (0, [X_1, X_2]) = \tau([X_1, X_2]) \quad .$$

Our objective is to determine the conditions that the bracket in eq. (75) must satisfy for $\widetilde{\mathcal{L}}$ to be a LA. These follow by imposing the Leibniz identity (eq. (70)), namely

$$\begin{aligned} [(A_1, X_1), [(A_2, X_2), (A_3, X_3)]] &= [[(A_1, X_1), (A_2, X_2)], (A_3, X_3)] \\ &+ [(A_2, X_2), [(A_1, X_1), (A_3, X_3)]] \quad . \end{aligned} \quad (77)$$

A simple calculation shows that eq. (77) implies the relations

$$\begin{aligned} \rho(X_1)(\rho(X_2)A_3) &= \rho([X_1, X_2])A_3 + \rho(X_2)(\rho(X_1)A_3) \quad , \\ \rho(X_1)(A_2\rho(X_3)) &= (\rho(X_1)A_2)\rho(X_3) + A_2\rho([X_1, X_3]) \quad , \\ A_1\rho([X_2, X_3]) &= (A_1\rho(X_2))\rho(X_3) + \rho(X_2)(A_1\rho(X_3)) \quad , \end{aligned} \quad (78)$$

plus the condition

$$\begin{aligned} (s\omega^2)(X_1, X_2, X_3) &:= \rho(X_1)\omega^2(X_2, X_3) - \rho(X_2)\omega^2(X_1, X_3) - \omega^2(X_1, X_2)\rho(X_3) + \\ &- \omega^2([X_1, X_2], X_3) - \omega^2(X_2, [X_1, X_3]) + \omega^2(X_1, [X_2, X_3]) = 0 \quad , \end{aligned} \quad (79)$$

where the position of the ρ 's indicates the left or right action and the first equality defines $s\omega^2$; then, $s\omega^2 = 0$ characterizes ω^2 as a two-cocycle. Later we shall write expressions such as the above one in the form

$$\begin{aligned} (s\omega^2)(X_1, X_2, X_3) &:= X_1 \cdot \omega^2(X_2, X_3) - X_2 \cdot \omega^2(X_1, X_3) - \omega^2(X_1, X_2) \cdot X_3 + \\ &- \omega^2([X_1, X_2], X_3) - \omega^2(X_2, [X_1, X_3]) + \omega^2(X_1, [X_2, X_3]) = 0 \quad , \end{aligned} \quad (80)$$

where $X \cdot$ and $\cdot X$ stand, respectively, for the left and right action of $\rho(X)$.

The left and right actions in eqs. (78), viewed in $\widetilde{\mathcal{L}}$, correspond to adjoint ones. They read, in the same order,

$$\begin{aligned} [X, [Y, A]] &= [[X, Y], A] + [Y, [X, A]] \quad , \\ [X, [A, Y]] &= [[X, A], Y] + [A, [X, Y]] \quad , \\ [A, [X, Y]] &= [[A, X], Y] + [X, [A, Y]] \quad , \end{aligned} \quad (81)$$

where *e.g.*, we have written $[X, A]$ for the left action of $\tau(X)$ on the ideal A of $\widetilde{\mathcal{L}}$. The above equations are the statement that the actions $[X, A]$, $[A, X]$ define a Leibniz *representation* [19,20] of \mathcal{L} on \mathcal{A} ; note that both the left and right actions intervene in the definition. Eqs. (81) are the analogue of the left module for an associative algebra (there is a corresponding set of equations for the analogue of a right module [18]).

The sum of the last two equations in (81) gives (*cf.* (71))

$$[[X, A], Y] = -[[A, X], Y] \quad . \quad (82)$$

A representation such that

$$[X, A] = -[A, X] \quad \forall X \in \mathcal{L} \quad , \quad A \in \mathcal{A} \quad (83)$$

is satisfied is called a *symmetric representation* of \mathcal{L} . When the representation is symmetric, all eqs. (81) are equivalent among themselves. In particular, a representation of a Lie algebra is a symmetric representation in the Leibniz algebra sense¹⁰.

Different sections τ define images $\tau(X) \in \widetilde{\mathcal{L}}$ of $X \in \mathcal{L}$ that differ in an element of the abelian ideal \mathcal{A} . Thus, if τ' is another trivializing section it follows that $\tau'(X) = \tau(X) + \omega^1(X)$, where ω^1 is a linear map $\omega^1 : \mathcal{L} \rightarrow \mathcal{A}$. In analogy with (76), we may write $[\tau'(X_1), \tau'(X_2)] = \tau'([X_1, X_2]) + \omega'^2(X_1, X_2)$ and, comparing now with the result of computing $[\tau(X_1) + \omega^1(X_1), \tau(X_2) + \omega^1(X_2)]$, we immediately obtain that the two bilinear maps ω'^2, ω^2 are related by

$$\omega'^2(X_1, X_2) - \omega^2(X_1, X_2) = \rho(X_1)\omega^1(X_2) + \omega^1(X_2)\rho(X_1) - \omega^1(X_1, X_2) = (s\omega^1)(X_1, X_2), \quad (84)$$

where the last equality defines $(s\omega^1)(X_1, X_2)$. Thus, the extensions are characterized by actions that satisfy the *representation conditions* (eqs. (81)), which characterize \mathcal{A} as a (left) \mathcal{L} -module, and by the bilinear maps that satisfy eq. (79) *i.e.*, by the *two-cocycles* $\omega^2 \in Z_\rho^2(\mathcal{L}, \mathcal{A})$.

Using different trivializing sections to characterize the same extension $\widetilde{\mathcal{L}}$ corresponds to taking two-cocycles that are equivalent according to eq. (84) *i.e.*, that differ in the two-coboundary $\omega_{cob}^2 = s\omega^1 \in B_\rho^2(\mathcal{L}, \mathcal{A})$ generated by the one-cochain ω^1 .

Therefore, the inequivalent extensions $\widetilde{\mathcal{L}}$ of a LA \mathcal{L} by an abelian one \mathcal{A} which is a \mathcal{L} -module for the representation (81) are classified [19, 18] by the second cohomology group

$$H_\rho^2(\mathcal{L}, \mathcal{A}) = Z_\rho^2(\mathcal{L}, \mathcal{A})/B_\rho^2(\mathcal{L}, \mathcal{A}) \quad .$$

As with Lie algebras, this general situation has two important special subcases:

- When $\omega^2 = 0$ (or it is equivalent to zero, $\omega^2 = s\omega^1$), eq. (76) shows that $\tau : \mathcal{L} \rightarrow \widetilde{\mathcal{L}}$ is a homomorphism of Leibniz algebras; then, the extension *splits* (this is the case that for Lie algebras corresponds to the semidirect sum). This solution to the extension problem always exists, since only requires the definition of the actions ρ (*i.e.*, eqs. (81)), which are known as they are necessary data for the extension problem.
- When ω^2 is non-trivial but $\rho = 0$, all eqs. (81) are trivial and then the second cohomology group $H_0^2(\mathcal{L}, \mathcal{A}) = Z_0^2(\mathcal{L}, \mathcal{A})/B_0^2(\mathcal{L}, \mathcal{A})$ characterizes the possible central extensions of \mathcal{L} by \mathcal{A} .

Clearly, the case where both ρ and ω^2 are trivial corresponds to the direct sum $\widetilde{\mathcal{L}} = \mathcal{A} \oplus \mathcal{L}$ of Leibniz algebras, which does not contain any structure beyond that in the \mathcal{L} and \mathcal{A} LA summands themselves.

4.3. Leibniz algebra cohomology.

We are now in a position to generalize the previous results to define higher order cochains and the *Leibniz algebra cohomology complex* $(C^\bullet(\mathcal{L}, \mathcal{A}), s)$

Definition 22. (*Leibniz p -cochains*)

An \mathcal{A} -valued p -cochain is a p -linear map $\omega^p : \otimes^p \mathcal{L} \rightarrow \mathcal{A}$. The space of p -cochains will be denoted $C^p(\mathcal{L}, \mathcal{A})$.

Definition 23. (*Coboundary operator for Leibniz algebra cohomology*)

¹⁰Notice that if we had $A\rho(X) = -\rho(X)A$ and ω^2 were skewsymmetric, the first equality in eq. (79) would coincide with the third equation in (26) which defines the action of the Lie algebra coboundary operator on a two-cochain.

The coboundary operator s is the map $s : C^p(\mathcal{L}, \mathcal{A}) \rightarrow C^{p+1}(\mathcal{L}, \mathcal{A})$ defined by

$$\begin{aligned} (s\omega^p)(X_1, \dots, X_{p+1}) &:= \sum_{i=1}^p (-1)^{i+1} \rho(X_i) \omega^p(X_1, \dots, \widehat{X}_i, \dots, X_{p+1}) \\ &+ \sum_{\substack{i,j \\ 1 \leq i < j}}^{p+1} (-1)^i \omega^p(X_1, \dots, \widehat{X}_i, \dots, [X_i, X_j], \dots, X_{p+1}) \\ &+ (-1)^{p+1} \omega^p(X_1, \dots, X_p) \rho(X_{p+1}) \quad . \end{aligned} \quad (85)$$

We see that both the left and the right actions of \mathcal{L} on the \mathcal{A} -valued cochains intervene in the definition of s . When $\omega^p = \omega^2$, eq. (85) reproduces eq. (79). Eq. (85) is the expression of the coboundary operator for the Leibniz algebra cohomology [19, 18, 22] (there given for a right LA) and [21]. It is proved in [19] that $s^2 = 0$, so that $(C^\bullet(\mathcal{L}, \mathcal{A}), s)$ is indeed a LA cohomology complex.

When the representation is a *symmetric* one (eq. (83)), the last term in eq. (85) may be included in the first one as one more contribution to the sum. Then, eq. (85) adopts same form as the action of the Lie algebra coboundary operator, namely

$$\begin{aligned} (s\omega^p)(X_1, \dots, X_n) &:= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(X_i) \omega^p(X_1, \dots, \widehat{X}_i, \dots, X_{p+1}) \\ &+ \sum_{\substack{i,j \\ 1 \leq i < j}}^{p+1} (-1)^i \omega^p(X_1, \dots, \widehat{X}_i, \dots, [X_i, X_j], \dots, X_{p+1}) \quad , \end{aligned} \quad (86)$$

which formally coincides with eq. (25) (see eq. (27)); here, the proof that $s^2 = 0$ follows by analogy to the Lie algebra cohomology case.

The above two expressions also reproduce those in [149] (when particularized to the $n = 2$ case) and in [21].

4.4. Deformations of Leibniz algebras.

As for Lie algebras, one may ask the question of applying the cohomology complex in Def. 23 to the problem of deforming Leibniz algebras. It is clear that the first order deformations will be classified by $H^2(\mathcal{L}, \mathcal{L})$. We shall not discuss this here since it will be done later directly for the n -ary generalization of \mathcal{L} , the n -Leibniz algebras \mathfrak{L} (Secs. 12 and 11.7) from which the \mathcal{L} case follows for $n = 2$. It is also possible to consider Leibniz deformations of Lie and n -Lie algebras, since these are in particular Leibniz and n -Leibniz; see [150] and [151].

5. n -ARY ALGEBRAS

To extend the ordinary Lie algebra \mathfrak{g} structure to the case of brackets with $n > 2$ entries, we have to define first the n -ary brackets and then generalize the JI. The Lie bracket is naturally extended to an n -ary bracket by requiring it to be a multilinear application

$$[\ , \ , \ \overset{n}{\cdots} \ , \] : \mathfrak{H} \times \overset{n}{\cdots} \times \mathfrak{H} \rightarrow \mathfrak{H} \ ,$$

where \mathfrak{H} is a generic n -ary algebra. The next step is specifying the consistency condition to be satisfied by this n -ary bracket. As described in the Introduction, there are two natural interpretations of the JI; these lead to two main $n > 2$ different generalizations of the Lie algebra structure obtained, respectively, by extending to the n -ary brackets the antisymmetrization of

the nested Lie brackets $[, [,]]$ of the $n = 2$ case, or the derivation character of the adjoint map (intermediate possibilities between these two exist: see [152,153]).

6. HIGHER ORDER OR GENERALIZED LIE ALGEBRAS (GLAs)

GLAs make emphasis on the associativity of the composition of the elements in their multibracket. These algebras were introduced independently in [3–5] and [9,6–8]; we refer to Sec. 1.1 for the terminology. General linear antisymmetric ‘ Ω -algebras’ were considered earlier [1,2].

An obviously skewsymmetric higher order multilinear bracket is provided by the following

Definition 24. (*Higher order generalized Lie bracket or multibracket*)

Let X_i be arbitrary associative operators, $i = 1, \dots, r$. A *multibracket* of order n [3] is defined by the fully antisymmetrized product of its entries

$$[X_{i_1}, \dots, X_{i_n}] := \sum_{\sigma \in S_n} (-1)^{\pi(\sigma)} X_{i_{\sigma(1)}} \dots X_{i_{\sigma(n)}} , \quad (87)$$

which obviously reduces to the ordinary Lie bracket $[X_{i_1}, X_{i_2}] = \epsilon_{i_1 i_2}^{j_1 j_2} X_{j_1} X_{j_2}$ for $n = 2$. Using the Levi-Civita symbol (eq. (39)) , the above definition is obviously equivalent to

$$[X_{i_1}, \dots, X_{i_n}] = \epsilon_{i_1 \dots i_n}^{j_1 \dots j_n} X_{j_1} \dots X_{j_n} . \quad (88)$$

Since

$$\epsilon_{j_1 \dots j_n}^{i_1 \dots i_n} = \sum_{s=1}^n (-1)^{s+1} \delta_{j_s}^{i_1} \epsilon_{j_1 \dots \hat{j}_s \dots j_n}^{i_2 \dots i_n} = \sum_{s=1}^n (-1)^{s+n} \epsilon_{j_1 \dots \hat{j}_s \dots j_n}^{i_1 \dots i_{n-1}} \delta_{j_s}^{i_n} , \quad (89)$$

it is clear that a multibracket of order n may be expressed in terms of multibrackets of increasingly lower orders by using

$$\begin{aligned} [X_1, X_2, \dots, X_n] &= \sum_{s=1}^n (-1)^{s+1} X_s [X_2, X_3, \dots, \hat{X}_s, \dots, X_n] \\ &= \sum_{s=1}^n (-1)^{s+n} [X_1, X_2, \dots, \hat{X}_s, \dots, X_n] X_s . \end{aligned} \quad (90)$$

For instance, for the order three and four multibrackets we find

$$\begin{aligned} [X_1, X_2, X_3] &= X_1[X_2, X_3] - X_2[X_1, X_3] + X_3[X_1, X_2] \\ &= [X_2, X_3]X_1 - [X_1, X_3]X_2 + [X_1, X_2]X_3 , \end{aligned} \quad (91)$$

$$\begin{aligned} [X_1, X_2, X_3, X_4] &= X_1[X_2, X_3, X_4] - X_2[X_1, X_3, X_4] + X_3[X_1, X_2, X_4] - X_4[X_1, X_2, X_3] \\ &= -[X_2, X_3, X_4]X_1 + [X_1, X_3, X_4]X_2 - [X_1, X_2, X_4]X_3 + [X_1, X_2, X_3]X_4 . \end{aligned} \quad (92)$$

The associativity of the product of the entries in (87) implies that the multibracket necessarily satisfies an identity, which has a *different* structure depending on whether n is *even* or *odd*, according to the following

Lemma 25. (*Generalized Jacobi identity (GJI)*)

For n even, the higher order bracket (87) satisfies the following identity

$$\sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} [[X_{\sigma(1)}, \dots, X_{\sigma(n)}], X_{\sigma(n+1)}, \dots, X_{\sigma(2n-1)}] = 0 . \quad (93)$$

We shall refer to this identity (93) satisfied by the n even multibracket as the generalized Jacobi identity (GJI).

For n odd, the identity is structurally different: the r.h.s. of the above expression is proportional to the larger bracket $[X_1, \dots, X_{2n-1}]$ rather than zero.

Proof. In terms of the Levi-Civita symbol, the l.h.s. of (93) reads

$$\epsilon_{i_1 \dots i_{2n-1}}^{j_1 \dots j_{2n-1}} \epsilon_{j_1 \dots j_n}^{l_1 \dots l_n} [X_{l_1} \cdots X_{l_n}, X_{j_{n+1}}, \dots, X_{j_{2n-1}}] \quad . \quad (94)$$

Since the n entries in this bracket are also antisymmetrized, eq. (94) is equal to

$$\begin{aligned} & n! \epsilon_{i_1 \dots i_{2n-1}}^{l_1 \dots l_n j_{n+1} \dots j_{2n-1}} \epsilon_{j_{n+1} \dots j_{2n-1}}^{l_{n+1} \dots l_{2n-1}} \sum_{s=0}^{n-1} (-1)^s X_{l_{n+1}} \cdots X_{l_{n+s}} X_{l_1} \cdots X_{l_n} X_{l_{n+1+s}} \cdots X_{l_{2n-1}} \\ &= n!(n-1)! \epsilon_{i_1 \dots i_{2n-1}}^{l_1 \dots l_{2n-1}} X_{l_1} \cdots X_{l_{2n-1}} \sum_{s=0}^{n-1} (-1)^s (-1)^{ns} \\ &= n!(n-1)! [X_{i_1}, \dots, X_{i_{2n-1}}] \sum_{s=0}^{n-1} (-1)^{s(n+1)} \quad , \end{aligned} \quad (95)$$

where we have used the skewsymmetry of ϵ to relocate the block $X_{l_1} \cdots X_{l_n}$ in the second equality. Thus, the l.h.s. of (95) is proportional to a multibracket of order $(2n-1)$ times a sum, which is zero for *even* n and for n *odd* is equal to n [3, 11]. \square

The GJI (93) contains, in all, $\frac{(2n-1)!}{n!(n-1)!} = \binom{2n-1}{n}$ independent terms. For $n=2$ it reduces to the ordinary JI; for $n=4$, for instance, it has already 35 terms. It is easy to find (see [3, 51]) that a multibracket with n even entries is reducible to sums of products of $n/2$ ordinary two-brackets. All one has to do is to iterate the identity easily obtained from (89)

$$\epsilon_{j_1 \dots j_n}^{i_1 \dots i_n} = \sum_{t>s=1}^n (-1)^{s+t+1} \epsilon_{j_s j_t}^{i_1 i_2} \epsilon_{j_1 \dots \hat{j}_s \dots \hat{j}_t \dots j_n}^{i_3 \dots i_n} \quad . \quad (96)$$

For instance, for $n=4$ one immediately obtains

$$\begin{aligned} [X_1, X_2, X_3, X_4] &= [X_1, X_2][X_3, X_4] - [X_1, X_3][X_2, X_4] + [X_1, X_4][X_2, X_3] + \\ & \quad [X_2, X_3][X_1, X_4] - [X_2, X_4][X_1, X_3] + [X_3, X_4][X_1, X_2] \quad . \end{aligned} \quad (97)$$

For $n > 4$ (n even or odd), other decompositions of the multibracket in terms of lower order brackets are possible by using various resolutions of the Levi-Civita symbol into products of lower order ones.

For n even, the GJI makes it natural to define a higher order Lie algebra by means of

Definition 26. (*Higher order or generalized Lie algebra (GLA)*)

An order $n = 2p$ higher order generalized Lie algebra [3–5], [6, 7] is a vector space \mathcal{G} endowed with a fully skewsymmetric bracket $\mathcal{G} \times \cdots \times \mathcal{G} \rightarrow \mathcal{G}$, $(X_1, \dots, X_n) \mapsto [X_1, \dots, X_n] \in \mathcal{G}$, such that the GJI (93) is fulfilled.

Consequently, given a basis $\{X_i\}$ of \mathcal{G} ($i = 1, \dots, r = \dim \mathcal{G}$), a finite-dimensional GLA of order $n = 2p$ is defined by an expression of the form

$$[X_{i_1}, \dots, X_{i_{2p}}] = C_{i_1 \dots i_{2p}}^j X_j \quad , \quad (98)$$

where the multibracket is defined by eq. (87) and the constants $C_{i_1 \dots i_{2p}}^j$ are the *higher order algebra structure constants*. The defining properties of the higher order algebra, eqs. (87) and

(93), now translate into the antisymmetry of the $C_{i_1 \dots i_{2p}}^j$ in i_1, \dots, i_{2p} and in that these structure constants satisfy the GJI of the $n = 2p$ GLA,

$$\epsilon_{i_1 \dots i_{4p-1}}^{j_1 \dots j_{4p-1}} C_{j_1 \dots j_{2p}}^l C_{j_{2p+1} \dots j_{4p-1}}^s = 0 \quad \text{or} \quad C_{[j_1 \dots j_{2p}}^l C_{j_{2p+1} \dots j_{4p-1}]}^s = 0 \quad . \quad (99)$$

Clearly, for $n=2$ this reduces to the JI for a Lie algebra \mathfrak{g} , $C_{[ij}^l C_k]^s = 0$.

We now include, for the sake of completeness, the identities that are obtained when two multibrackets of different orders n, m are nested. These are given by the following

Proposition 27. (*Mixed order generalized Jacobi identities, MGJI*)

Let m, n be even. The mixed order generalized Jacobi identity for even order multibrackets reads

$$\epsilon^{j_1 \dots j_{n+m-1}} [[X_{j_1}, \dots, X_{j_n}], \dots, X_{j_{n+m-1}}] = 0 \quad . \quad (100)$$

Proof. Following the same reasoning as in Lemma 25,

$$\begin{aligned} & \epsilon_{i_1 \dots i_{n+m-1}}^{j_1 \dots j_{n+m-1}} \epsilon_{j_1 \dots j_n}^{l_1 \dots l_n} [X_{l_1} \dots X_{l_n}, X_{j_{n+1}}, \dots, X_{j_{n+m-1}}] \\ &= n! \epsilon_{i_1 \dots i_{n+m-1}}^{l_1 \dots l_n j_{n+1} \dots j_{n+m-1}} \epsilon_{j_{n+1} \dots j_{n+m-1}}^{l_{n+1} \dots l_{n+m-1}} \sum_{s=0}^{m-1} (-1)^s X_{l_{n+1}} \dots X_{l_{n+s}} X_{l_1} \dots X_{l_n} X_{l_{n+1+s}} \dots X_{l_{n+m-1}} \\ &= n!(m-1)! \epsilon_{i_1 \dots i_{n+m-1}}^{l_1 \dots l_{n+m-1}} X_{l_1} \dots X_{l_{n+m-1}} \sum_{s=0}^{m-1} (-1)^s (-1)^{ns} \\ &= n!(m-1)! [X_{i_1}, \dots, X_{i_{n+m-1}}] \sum_{s=0}^{m-1} (-1)^{(n+1)s} \quad , \end{aligned} \quad (101)$$

which is zero for n and m even as stated.

In contrast, if n and/or m are odd the sum $\sum_{s=0}^{m-1} (-1)^{(n+1)s}$ is different from zero (m if n is odd and 1 if n is even). In this case, the *l.h.s.* of (100) is proportional to the $(n+m-1)$ -commutator $[X_{i_1}, \dots, X_{i_{n+m-1}}]$. This MGJI, found independently in [11], had been defined as a multiplication of skewsymmetric multilinear maps in [154]. \square

It is simple to find the expression of the MGJI in terms of the structure constants. If n and m are the even orders of the nested brackets in eq. (100), and assuming that expressions corresponding to eq. (98) exist (this will be the case of Th. 28 below), the MGJI leads to

$$\epsilon^{i_1 \dots i_{n+m-1}} C_{i_1 \dots i_n}^l C_{i_{n+1} \dots i_{n+m-1}}^s = 0 \quad ; \quad (102)$$

eq. (99) corresponds to $n=m=2p$. We shall not discuss any further identities following from associativity, but the above are not the only ones if one allows for more nested brackets. This leads to the Bremner identities, initially proposed for the ternary commutator [155, 74] and later extended to more general cases (see [156] and [157, 158]).

Having defined n -GLAs, one faces the question of providing some examples. As we shall see, eq. (102) for $n = 2$ (eq. (42)) provides a hint for a wide class of them: a way of finding examples of these higher order algebras is to look at ordinary Lie algebra cohomology since, as we saw in Sec. 3.9, the primitive invariant polynomials of the simple compact algebras and their associated non-trivial cocycles are in one-to-one correspondence.

6.1. Higher order simple algebras associated with a compact simple Lie algebra \mathfrak{g} .

We present here a construction of GLAs for which the previous cohomology notions play a crucial rôle, namely the definition of the higher order Lie algebras associated with a compact simple algebra \mathfrak{g} . If the Lie algebra is simple $\Omega_{ij\rho} \equiv C_{ij\rho} = k_{\rho\sigma} C_{ij}^\sigma$ is, by (62), the non-trivial three-cocycle associated with the Cartan-Killing metric. Since $\Omega_{ij\rho}$ is given by the antisymmetric form of the structure constants of \mathfrak{g} , there always exists a three-cocycle. The question arises as to whether higher order cocycles (and therefore Casimirs of orders higher than two) can be used to define the structure constants of a higher order bracket. Given the odd dimensionality of the cocycles, these multibrackets will involve an even number of entries.

By way of an example, let us consider the case of $su(n)$, $n > 2$, and the four-bracket. Let X_i be matrices of the defining representation. The four-bracket is defined by

$$[X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] := \epsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} X_{j_1} X_{j_2} X_{j_3} X_{j_4} \quad . \quad (103)$$

Using the skew-symmetry in $j_1 \dots j_4$, we may rewrite (103) in terms of commutators as

$$\begin{aligned} [X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}] &= \frac{1}{2^2} \epsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} [X_{j_1}, X_{j_2}] [X_{j_3}, X_{j_4}] = \frac{1}{2^2} \epsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} X_{l_1} X_{l_2} \\ &= \frac{1}{2^2} \epsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} \frac{1}{2} (d_{l_1 l_2}^\sigma X_\sigma + c \delta_{l_1 l_2}) \\ &= \frac{1}{2^3} \epsilon_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} C_{j_1 j_2}^{l_1} C_{j_3 j_4}^{l_2} d_{l_1 l_2}^\sigma X_\sigma \equiv \Omega_{i_1 \dots i_4}^\sigma X_\sigma \quad , \end{aligned} \quad (104)$$

where in going from the first line to the second we have used that the factor multiplying $X_{l_1} X_{l_2}$ is symmetric in l_1, l_2 , so that we can replace $X_{l_1} X_{l_2}$ by $\frac{1}{2} \{X_{l_1}, X_{l_2}\}$ which is written in terms of the d 's. The contribution of the term proportional to c vanishes due to the JI. Thus, the structure constants $C_{i_1 \dots i_4}^\sigma$ of the four-bracket are given by the five-cocycle $\Omega_{i_1 \dots i_4}^\sigma$ associated with the primitive symmetric polynomial d_{ijk} .

The above result is, in fact, general. A $(2p+1)$ -cocycle Ω for the Lie algebra cohomology of \mathfrak{g} defines a higher order $2p$ algebra by

$$[X_{i_1}, \dots, X_{i_{2p}}] = \Omega_{i_1 \dots i_{2p}}^j X_j \quad , \quad (105)$$

where the structure constants satisfy the GJI (eqs. (93)). One may check directly that $\Omega_{i_1 \dots i_{2p}}^j$ defines a $(2p+1)$ -cocycle for \mathfrak{g} , since the MGJI for the ordinary Lie bracket of \mathfrak{g} and the above one give

$$\epsilon^{i_1 \dots i_{m+1}} C_{i_1 i_2}^l \Omega_{i_3 \dots i_{m+1} l}^j = 0 \quad (106)$$

by eq. (102). Lowering the index j with the invariant Killing metric we see that the fully antisymmetric structure constants $\Omega_{i_1 \dots i_{m+1}}$ above are, in fact, those of a cocycle for the Lie algebra cohomology of \mathfrak{g} since the above equation implies

$$\epsilon^{i_1 \dots i_{m+2}} C_{i_1 i_2}^l \Omega_{i_3 \dots i_{m+1} i_{m+2} l} = 0 \quad , \quad (107)$$

which is the $(2p+1)$ -cocycle condition (42). Thus, the following theorem follows [3]:

Theorem 28. (Higher order simple Lie algebras associated with a compact simple algebra \mathfrak{g})

Given a simple algebra \mathfrak{g} of rank l , there are $(l-1)$ $(2m_i-2)$ -higher order simple Lie algebras associated with \mathfrak{g} . They are given by the $(l-1)$ Lie algebra cocycles of order $(2m_i-1) > 3$ which are associated with the $(l-1)$ symmetric invariant polynomials on \mathfrak{g} of order $m_i > m_1 = 2$. The $m_1 = 2$ case (for which the invariant polynomial is the Killing metric) reproduces the original simple Lie algebra \mathfrak{g} ; for the remaining $(l-1)$ cases, the skewsymmetric $(2m_i-2)$ -commutators define an element of \mathfrak{g} by means of the $(2m_i-1)$ -cocycles, $(2m_i-1) > 3$. These higher order

structure constants (as the ordinary structure constants with all the indices written down) are fully antisymmetric cocycles and satisfy the GJI.

6.2. Multibrackets, higher order coderivatives and exterior derivatives.

Higher order brackets can be used to generalize the ordinary coderivation of multivectors.

Definition 29. (*Exterior coderivative*)

Let $\{X_i\}$ be a basis of \mathfrak{g} given in terms of LIVF on G and $\wedge^* \mathfrak{g}$ the exterior algebra of multivectors generated by them ($X_1 \wedge \cdots \wedge X_q \equiv \epsilon_{1\dots q}^{i_1\dots i_q} X_{i_1} \otimes \cdots \otimes X_{i_q}$). The exterior coderivation is the map of degree -1 , $\partial : \wedge^q \mathfrak{g} \rightarrow \wedge^{q-1} \mathfrak{g}$, defined by

$$\partial(X_1 \wedge \cdots \wedge X_q) = \sum_{\substack{l=1 \\ l < k}}^q (-1)^{l+k+1} [X_l, X_k] \wedge X_1 \wedge \cdots \wedge \hat{X}_l \wedge \cdots \wedge \hat{X}_k \wedge \cdots \wedge X_q \quad , \quad (108)$$

where $\partial(X_1, X_2) = [X_1, X_2]$.

This definition is analogous to that of the exterior derivative d , as given by (18) with its first term missing when one considers left-invariant forms (eq. (19)). As the exterior derivative d , ∂ is nilpotent, $\partial^2 = 0$, due to the JI for the commutator.

In order to generalize (108), let us note that $\partial(X_1 \wedge X_2) = [X_1, X_2]$, so that (108) can be interpreted as a formula that gives the action of ∂ on a q -vector in terms of that on a bivector. For this reason we may write ∂_2 for ∂ above. It is then natural to introduce an operator ∂_s that on a s -vector gives the multicommutator of order s . On an n -multivector its action is given by

Definition 30. (*Coderivation ∂_s*)

The general coderivation ∂_s of degree $-(s-1)$, s even, is the map $\partial_s : \wedge^q \mathfrak{g} \rightarrow \wedge^{q-(s-1)} \mathfrak{g}$ defined by

$$\begin{aligned} \partial_s(X_1 \wedge \cdots \wedge X_q) &:= \frac{1}{s!} \frac{1}{(q-s)!} \epsilon_{1\dots q}^{i_1\dots i_q} \partial_s(X_{i_1} \wedge \cdots \wedge X_{i_s}) \wedge X_{i_{s+1}} \wedge \cdots \wedge X_{i_q} \quad , \\ \partial_s \wedge^q \mathfrak{g} &= 0 \quad \text{for } s > q \quad , \\ \partial_s(X_1 \wedge \cdots \wedge X_s) &= [X_1, \dots, X_s] \quad . \end{aligned} \quad (109)$$

Note that, using (38), this expression reproduces (108) for $s=2$.

Proposition 31. *The coderivation (109) is nilpotent, i.e., $\partial_s^2 \equiv 0$.*

Proof. Let q and s be such that $q - (s-1) \geq s$ (otherwise the statement is trivial). Then,

$$\begin{aligned} \partial_s \partial_s(X_1 \wedge \cdots \wedge X_q) &= \\ \frac{1}{s!} \frac{1}{(q-s)!} \frac{1}{s!} \frac{1}{(q-2s+1)!} \epsilon_{1\dots q}^{i_1\dots i_q} \epsilon_{i_{s+1}\dots i_q}^{j_{s+1}\dots j_q} \{ &S [[X_{i_1}, \dots, X_{i_s}], X_{j_{s+1}}, \dots, X_{j_{2s-1}}] \wedge X_{j_{2s}} \wedge \dots \wedge X_{j_q} \\ -(q-2s+1)[X_{i_1}, \dots, X_{i_s}] \wedge [X_{j_{s+1}}, \dots, X_{j_{2s}}] \wedge X_{j_{2s+1}} \wedge \dots \wedge X_{j_q} \} &= 0 \quad . \end{aligned} \quad (110)$$

The sum over $i_{s+1} \dots i_q$ produces an overall antisymmetrization over the i, j indices. As a result, the first term above vanishes because, since s is even, the double bracket is the GJI. Similarly, the second one is also zero because the wedge product of the two s -brackets is antisymmetric while the resulting ϵ symbol is symmetric under the interchange $(i_1, \dots, i_s) \leftrightarrow (j_{s+1}, \dots, j_{2s})$. \square

Let us now see how the nilpotency condition (or equivalently the GJI) looks like in the simplest cases.

Example 32. (*The coderivation ∂_2*)

Consider $\partial \equiv \partial_2$. Then we have

$$\partial(X_1 \wedge X_2 \wedge X_3) = [X_1, X_2] \wedge X_3 - [X_1, X_3] \wedge X_2 + [X_2, X_3] \wedge X_1 \quad (111)$$

and

$$\partial^2(X_1 \wedge X_2 \wedge X_3) = [[X_1, X_2], X_3] - [[X_1, X_3], X_2] + [[X_2, X_3], X_1] = 0 \quad (112)$$

by the JI.

When we move to $\partial \equiv \partial_4$, the number of terms grows very rapidly. The explicit expression for $\partial^2(X_{i_1} \wedge \cdots \wedge X_{i_7}) = 0$ (which, as we know, is equivalent to the GJI) is given in full in [159] and contains $\binom{7}{3} = 35$ terms (note that the tenth term in eq. (32) there should read $[[X_{i_1}, X_{i_2}, X_{i_6}, X_{i_7}], X_{i_3}, X_{i_4}, X_{i_5}]$). In general, the GJI which follows from $\partial_{2m-2}^2(X_1 \wedge \cdots \wedge X_{4m-5}) = 0$ ($s = 2m - 2$) contains $\binom{4m-5}{2m-2}$ different terms, as for the GJI (93).

6.3. Higher order exterior derivative and generalized MC equations.

We now generalize the Lie algebra MC equations (eqs. 20) to the case of the GLAs of Th. 28 and write them in a BRST-like form. The result is a higher order BRST-type operator that contains all the information on the l possible GLAs \mathfrak{G} associated with a given simple Lie algebra \mathfrak{g} of rank l .

Let us first note that eq. (20) gives $d^2\theta = \frac{1}{2}[[\theta, \theta], \theta] = 0$ so that the JI reads

$$[[\theta, \theta], \theta] = 0 \quad . \quad (113)$$

In Sec. 6.2 we considered higher order coderivations which also had the property $\partial_s^2 = 0$ as a result of the GJI. We may now introduce the corresponding dual higher order exterior derivatives \tilde{d}_s to provide a generalization of the MC eq. (17). Since ∂_s was defined on multivectors that are product of left-invariant vector fields, the dual \tilde{d}_s will be given for left-invariant forms.

It is easy to introduce dual bases in $\wedge^q \mathfrak{g}^*$ and $\wedge^q \mathfrak{g}$. With $\omega^i(X_j) = \delta_j^i$, these are given by $\omega^{I_1} \wedge \cdots \wedge \omega^{I_q}$, $\frac{1}{q!} X_{I_1} \wedge \cdots \wedge X_{I_q}$, $I_1 < \cdots < I_q$, since $(\epsilon_{j_1 \dots j_q}^{i_1 \dots i_q} \omega^{j_1} \otimes \cdots \otimes \omega^{j_q}) (\frac{1}{q!} \epsilon_{l_1 \dots l_q}^{k_1 \dots k_q} X_{k_1} \otimes \cdots \otimes X_{k_q}) = \epsilon_{l_1 \dots l_q}^{i_1 \dots i_q}$ and $\epsilon_{L_1 \dots L_q}^{I_1 \dots I_q}$ is 1 if all indices coincide and 0 otherwise. Nevertheless it is customary to use the non-minimal ‘basis’ $\omega^{i_1} \wedge \cdots \wedge \omega^{i_q}$ to write $\alpha = \frac{1}{q!} \alpha_{i_1 \dots i_q} \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}$ with $\alpha_{i_1 \dots i_q} \equiv \alpha(X_{i_1}, \dots, X_{i_q}) = \frac{1}{q!} \alpha(X_{i_1} \wedge \cdots \wedge X_{i_q})$ since $(\omega^{i_1} \wedge \cdots \wedge \omega^{i_q})(X_{j_1}, \dots, X_{j_q}) = \epsilon_{j_1 \dots j_q}^{i_1 \dots i_q}$.

Definition 33. (*Higher order exterior derivative*)

The action of $\tilde{d}_m : \wedge^q \mathfrak{g}^* \rightarrow \wedge^{q+(2m-3)} \mathfrak{g}^*$ (recall that $s = 2m - 2$) is given by (cf. (115))

$$\begin{aligned} (\tilde{d}_m \alpha)(X_{i_1}, \dots, X_{i_{q+2m-3}}) &:= \\ & \frac{1}{(2m-2)!} \frac{1}{(q-1)!} \epsilon_{i_1 \dots i_{q+2m-3}}^{j_1 \dots j_{q+2m-3}} \alpha([X_{j_1}, \dots, X_{j_{2m-2}}], X_{j_{2m-1}}, \dots, X_{j_{q+2m-3}}) \quad , \quad (114) \\ (\tilde{d}_m \alpha)_{i_1 \dots i_{q+2m-3}} &= \frac{1}{(2m-2)!} \frac{1}{(q-1)!} \epsilon_{i_1 \dots i_{q+2m-3}}^{j_1 \dots j_{q+2m-3}} \Omega_{j_1 \dots j_{2m-2}} \cdot \rho \alpha_{\rho j_{2m-1} \dots j_{q+2m-3}} \quad , \end{aligned}$$

where the first (second) factorial in the denominator is the number of arguments inside (outside) the multibracket.

For $m = 2$, \tilde{d}_2 gives eq. (19) with $p = q$,

$$\tilde{d}_2 \alpha(X_{i_1}, \dots, X_{i_{q+1}}) = \frac{1}{(2 \cdot 2 - 2)!} \frac{1}{(q-1)!} \epsilon_{i_1 \dots i_{q+1}}^{j_1 \dots j_{q+1}} \alpha([X_{j_1}, X_{j_2}], X_{j_3}, \dots, X_{i_{q+1}}) \quad (115)$$

with the identification $d \equiv -\tilde{d}_2$.

Proposition 34. (*Higher order coderivative*)

$\tilde{d}_m : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+(2m-3)} \mathfrak{g}^*$ is dual to the coderivation $\partial_{2m-2} : \wedge^p \mathfrak{g} \rightarrow \wedge^{p-(2m-3)} \mathfrak{g}$ i.e., on a generic p -form,

$$\tilde{d}_m \alpha \propto \alpha \partial_{2m-2} \quad . \quad (116)$$

Proof. If α is a p -form, $\tilde{d}_m \alpha$ is a $(p + 2m - 3)$ -form and, since $\partial_{2m-2} : \wedge^{p+2m-3} \mathfrak{g} \rightarrow \wedge^p \mathfrak{g}$, eq. (109) tells us that

$$\begin{aligned} \alpha(\partial_{2m-2}(X_{i_1} \wedge \cdots \wedge X_{i_{p+2m-3}})) &= \frac{1}{(2m-2)!} \frac{1}{(p+2m-3-2m+2)!} \times \\ &\times \epsilon_{i_1 \dots i_{p+2m-3}}^{j_1 \dots j_{p+2m-3}} \alpha([X_{j_1}, \dots, X_{j_{2m-2}}] \wedge X_{j_{2m-1}} \wedge \cdots \wedge X_{j_{p+2m-3}}) \quad , \end{aligned} \quad (117)$$

which is proportional¹¹ to $(\tilde{d}_m \alpha)(X_{i_1} \wedge \cdots \wedge X_{i_{p+2m-3}})$. \square

Proposition 35. *The operator \tilde{d}_m satisfies Leibniz's rule,*

$$\tilde{d}_m(\alpha \wedge \beta) = \tilde{d}_m \alpha \wedge \beta + (-1)^p \alpha \wedge \tilde{d}_m \beta \quad . \quad (118)$$

Proof. If α and β are p and q forms, respectively, we get using (114)

$$\begin{aligned} \tilde{d}_m(\alpha \wedge \beta)_{i_1 \dots i_{p+q+2m-3}} &= \frac{1}{(2m-2)!} \frac{1}{(p+q-1)} \epsilon_{i_1 \dots i_{p+q+2m-3}}^{j_1 \dots j_{p+q+2m-3}} \Omega_{j_1 \dots j_{2m-2}}^\rho \cdot \\ &\cdot \left(\frac{1}{p!q!} \epsilon_{\rho j_{2m-1} \dots j_{p+q+2m-3}}^{k_1 \dots k_{p+q}} \alpha_{k_1 \dots k_p} \beta_{k_{p+1} \dots k_{p+q}} \right) \\ &= \frac{1}{(2m-2)!} \frac{1}{p!q!} \epsilon_{i_1 \dots i_{p+q+2m-3}}^{j_1 \dots j_{p+q+2m-3}} \Omega_{j_1 \dots j_{2m-2}}^\rho \left(p \alpha_{\rho j_{2m-1} \dots j_{p+2m-3}} \beta_{j_{p+2m-2} \dots j_{p+q+2m-3}} \right. \\ &\quad \left. + (-1)^p q \alpha_{j_{2m-1} \dots j_{p+2m-2}} \beta_{\rho j_{p+2m-1} \dots j_{p+q+2m-3}} \right) \\ &= \epsilon_{i_1 \dots i_{p+q+2m-3}}^{j_1 \dots j_{p+q+2m-3}} \left(\frac{1}{q!(p+2m-3)!} (\tilde{d}_m \alpha)_{j_1 \dots j_{p+2m-3}} \beta_{j_{p+2m-2} \dots j_{p+q+2m-3}} \right. \\ &\quad \left. + (-1)^p \frac{1}{p!(q+2m-3)!} \alpha_{j_{2m-1} \dots j_{p+2m-2}} (\tilde{d}_m \beta)_{j_1 \dots j_{2m-2} j_{p+2m-1} \dots j_{p+q+2m-3}} \right) \\ &= \left((\tilde{d}_m \alpha) \wedge \beta + (-1)^p \alpha \wedge (\tilde{d}_m \beta) \right)_{i_1 \dots i_{p+q+2m-3}} \quad . \end{aligned} \quad (119)$$

Thus, \tilde{d}_m is odd and satisfies Leibniz's rule. \square

The coordinates of $\tilde{d}_m \omega^\sigma$, where ω^σ is a MC form, are given by

$$\begin{aligned} (\tilde{d}_m \omega^\sigma)(X_{i_1}, \dots, X_{i_{2m-2}}) &= \frac{1}{(2m-2)!} \epsilon_{i_1 \dots i_{2m-2}}^{j_1 \dots j_{2m-2}} \omega^\sigma([X_{j_1}, \dots, X_{j_{2m-2}}]) \\ &= \omega^\sigma([X_{i_1}, \dots, X_{i_{2m-2}}]) = \omega^\sigma(\Omega_{i_1 \dots i_{2m-2}}^\rho X_\rho) = \Omega_{i_1 \dots i_{2m-2}}^\sigma \quad (120) \end{aligned}$$

from which we conclude that

$$\tilde{d}_m \omega^\sigma = \frac{1}{(2m-2)!} \Omega_{i_1 \dots i_{2m-2}}^\sigma \omega^{i_1} \wedge \cdots \wedge \omega^{i_{2m-2}} \quad . \quad (121)$$

For $m = 2$, $\tilde{d}_2 = -d$, eqs. (121) reproduce the MC eqs. (20). In the compact notation that uses the canonical one-form θ on G , this leads to

¹¹ One finds $\tilde{d}_m \alpha = \frac{(p+2m-3)!}{p!} \alpha \partial_{2m-2}$, where p is the order of the form α . The factor appears as a consequence of using the same definition (antisymmetrization with no weight factor) for the \wedge product of forms and vectors.

Proposition 36. (*Generalized Maurer-Cartan equations*)

The action of \tilde{d}_m on the canonical form θ is given by

$$\tilde{d}_m \theta = \frac{1}{(2m-2)!} \left[\theta, \overset{2m-2}{\dots}, \theta \right] , \quad (122)$$

where the multibracket of forms is defined by $\left[\theta, \overset{2m-2}{\dots}, \theta \right] = \omega^{i_1} \wedge \dots \wedge \omega^{i_{2m-2}} [X_{i_1}, \dots, X_{i_{2m-2}}]$.

Using Leibniz's rule for \tilde{d}_m we arrive at $\tilde{d}_m^2 \theta = -\frac{1}{(2m-2)!} \frac{1}{(2m-3)!} [\theta, \overset{2m-3}{\dots}, \theta, [\theta, \overset{2m-2}{\dots}, \theta]] = 0$, which expresses the GJI as

$$[\theta, \overset{2m-3}{\dots}, \theta, [\theta, \overset{2m-2}{\dots}, \theta]] = 0 , \quad (123)$$

which recovers the JI equation (113) for $m = 2$.

Each generalized Maurer-Cartan equation (123) can be expressed in terms of ghost (Grassmann odd) variables c^i , $c^i c^j = -c^j c^i$, $c^{i2} = 0$, by means of a 'generalized BRST operator',

$$s_{2m-2} = -\frac{1}{(2m-2)!} c^{i_1} \dots c^{i_{2m-2}} \Omega_{i_1 \dots i_{2m-2}} \cdot \frac{\partial}{\partial c^\sigma} . \quad (124)$$

By adding together all the l generalized BRST operators, the complete BRST operator is obtained. Then we have the following [3]

Theorem 37. (*Complete BRST operator*)

Let \mathfrak{g} be a simple Lie algebra. Then, there exists a nilpotent associated operator, the complete BRST operator associated with \mathfrak{g} , given by the odd vector field

$$\begin{aligned} s &= -\frac{1}{2} c^{j_1} c^{j_2} \Omega_{j_1 j_2} \cdot \frac{\partial}{\partial c^\sigma} - \dots - \frac{1}{(2m_i-2)!} c^{j_1} \dots c^{j_{2m_i-2}} \Omega_{j_1 \dots j_{2m_i-2}} \cdot \frac{\partial}{\partial c^\sigma} - \dots \\ &- \frac{1}{(2m_l-2)!} c^{j_1} \dots c^{j_{2m_l-2}} \Omega_{j_1 \dots j_{2m_l-2}} \cdot \frac{\partial}{\partial c^\sigma} \equiv s_2 + \dots + s_{2m_i-2} + \dots + s_{2m_l-2} , \end{aligned} \quad (125)$$

where $i = 1, \dots, l$, $\Omega_{j_1 j_2} \cdot \frac{\partial}{\partial c^\sigma} \equiv C_{j_1 j_2} \cdot \frac{\partial}{\partial c^\sigma}$ and $\Omega_{j_1 \dots j_{2m_i-2}} \cdot \frac{\partial}{\partial c^\sigma}$ are the corresponding l higher order cocycles, which encodes all the multialgebras \mathfrak{G} associated with a simple Lie algebra \mathfrak{g} .

Proof. We have to show that $\{s_{2m_i-2}, s_{2m_j-2}\} = 0 \ \forall i, j$. To prove it, let us write the anti-commutator explicitly:

$$\begin{aligned} \{s_{2m_i-2}, s_{2m_j-2}\} &= \frac{1}{(2m_i-2)!} \frac{1}{(2m_j-2)!} \times \\ &\times \left\{ (2m_j-2) c^{l_1} \dots c^{l_{2m_i-2}} \Omega_{l_1 \dots l_{2m_i-2}} \cdot \frac{\partial}{\partial c^\rho} c^{r_2} \dots c^{r_{2m_j-2}} \Omega_{r_2 \dots r_{2m_j-2}} \cdot \frac{\partial}{\partial c^\sigma} + i \leftrightarrow j \right. \\ &\left. + (c^{l_1} \dots c^{l_{2m_i-2}} c^{r_1} \dots c^{r_{2m_j-2}} \Omega_{l_1 \dots l_{2m_i-2}} \cdot \frac{\partial}{\partial c^\rho} \Omega_{r_1 \dots r_{2m_j-2}} \cdot \frac{\partial}{\partial c^\sigma} + i \leftrightarrow j) \frac{\partial}{\partial c^\rho} \frac{\partial}{\partial c^\sigma} \right\} \\ &= \frac{1}{(2m_i-2)!} \frac{1}{(2m_j-3)!} c^{l_1} \dots c^{l_{2m_i-2}} c^{r_2} \dots c^{r_{2m_j-2}} \Omega_{l_1 \dots l_{2m_i-2}} \cdot \frac{\partial}{\partial c^\rho} \Omega_{r_2 \dots r_{2m_j-2}} \cdot \frac{\partial}{\partial c^\sigma} \\ &+ i \leftrightarrow j , \end{aligned} \quad (126)$$

where we have used the fact that $\frac{\partial}{\partial c^\rho} \frac{\partial}{\partial c^\sigma}$ is antisymmetric in ρ, σ while the parenthesis multiplying it is symmetric. The term proportional to a single $\frac{\partial}{\partial c^\sigma}$ also vanishes as a consequence of equation (102). \square

The antisymmetric coefficients of $\partial/\partial c^\sigma$, etc. in s_{2m_i-2} can be viewed, in dual terms, as (even) multivectors of the type

$$\Lambda = \frac{1}{(2m-2)!} \Omega_{i_1 \dots i_{2m-2}}^\sigma x_\sigma \partial^{i_1} \wedge \dots \wedge \partial^{i_{2m-2}} \quad , \quad (127)$$

by replacing the inherent skewsymmetry associated with the odd character of the c^i by that introduced by the wedge product of the derivatives with respect to the even variables x_i . The resulting multivectors Λ have the property of having zero Schouten-Nijenhuis bracket [160,161] (see Appendix 2) among themselves by virtue of the GJI (99). As a result, they have precisely the property required to define the (*linear*) *generalized Poisson structures* that will be discussed in Sec. 13.3.

For other aspects of operators with similar structure see [162,3] and further references therein.

6.4. GLAs and strongly homotopy (SH) Lie algebras.

The above higher order Lie algebras turn out to be a special example of the strongly homotopy (SH) Lie algebras [12, 13, 163, 15] which we briefly mention below for completeness. These SH algebras allow for ‘controlled’ violations of the GJI, which are obviously absent for a GLA.

Definition 38. (*SH algebras* [12])

A *SH Lie structure* on a vector space V is a collection of skewsymmetric linear maps $l_s : V \otimes \dots \otimes V \rightarrow V$ such that

$$\sum_{i+j=s+1} \sum_{\sigma \in S_s} \frac{1}{(i-1)! j!} (-1)^{\pi(\sigma)} (-1)^{i(j-1)} l_i(l_j(v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(j)}) \otimes v_{\sigma(j+1)} \otimes \dots \otimes v_{\sigma(s)}) = 0 \quad . \quad (128)$$

For a general treatment of SH Lie algebras including v gradings see [12, 13, 163] and references therein. Note that $\frac{1}{(i-1)! j!} \sum_{\sigma \in S_s}$ is equivalent to the sum over the ‘unshuffles’, *i.e.*, over the permutations $\sigma \in S_s$ such that $\sigma(1) < \dots < \sigma(j)$ and $\sigma(j+1) < \dots < \sigma(s)$.

Example 39. For $s = 1$, eq. (128) just says that $l_1^2 = 0$ (l_1 is a differential). For $s = 2$, eq. (128) gives

$$-\frac{1}{2} l_1(l_2(v_1 \otimes v_2) - l_2(v_2 \otimes v_1)) + l_2(l_1(v_1) \otimes v_2 - l_1(v_2) \otimes v_1) = 0 \quad (129)$$

i.e., with $l_2(v_1 \otimes v_2) = [v_1, v_2]$,

$$l_1[v_1, v_2] = [l_1 v_1, v_2] + [v_1, l_1 v_2] \quad . \quad (130)$$

For $s = 3$, we have three maps l_1, l_2, l_3 , and eq. (128) reduces to

$$\begin{aligned} & [l_2(l_2(v_1 \otimes v_2) \otimes v_3) + l_2(l_2(v_2 \otimes v_3) \otimes v_1) + l_2(l_2(v_3 \otimes v_1) \otimes v_2)] + [l_1(l_3(v_1 \otimes v_2 \otimes v_3))] \\ & + [l_3(l_1(v_1) \otimes v_2 \otimes v_3) + l_3(l_1(v_2) \otimes v_3 \otimes v_1) + l_3(l_1(v_3) \otimes v_1 \otimes v_2)] = 0 \quad , \end{aligned} \quad (131)$$

i.e., adopting the convention that $l_s(v_1 \otimes \dots \otimes v_s) = [v_1, \dots, v_s]$,

$$\begin{aligned} & [[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2] \\ & = -l_1[v_1, v_2, v_3] - [l_1(v_1), v_2, v_3] - [v_1, l_1(v_2), v_3] - [v_1, v_2, l_1(v_3)] \quad . \end{aligned} \quad (132)$$

The *r.h.s* in (132) shows the violation of the (standard) Jacobi identity appearing in the *l.h.s.*

In the particular case in which a unique l_s (s even) is defined, we recover the GLA case since, for $i = j = s$, eq. (128) reproduces the GJI (93) in the form

$$\sum_{\sigma \in S_{2s-1}} \frac{1}{s!} \frac{1}{(s-1)!} (-1)^{\pi(\sigma)} l_s(l_s(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(s)} \otimes v_{\sigma(s+1)} \otimes \cdots \otimes v_{\sigma(2s-1)})) = 0 \quad . \quad (133)$$

Thus, the higher order Lie algebras correspond to a particular case of SH Lie algebras, the one that appears when the GJI is satisfied.

7. FILIPPOV OR n -LIE ALGEBRAS

This section reviews some basic properties of Filippov algebras (FAs) [16], [42, 23–25]; other questions, including the existence of the enveloping algebras of Filippov algebras are discussed in [164, 165]. We begin by discussing the crucial ingredient of n -Lie algebras, the derivation property that determines their characteristic identity, the *Filippov identity*, which distinguishes FAs from the higher order GLAs of the previous section. After discussing the general properties of the FAs and, in particular, their associated inner derivations Lie algebra, we look at examples of finite FAs (and, specially, the simple ones). The infinite dimensional case will be exemplified by the Nambu or Jacobian FAs (these are not the only examples of infinite-dimensional FAs: ternary Kac-Moody- and Virasoro-Witt-like algebras have been considered in [166] and [167], respectively).

7.1. Derivations of an n -bracket, the Filippov identity and n -Lie algebras.

Let \mathfrak{G} be a vector space endowed with a fully antisymmetric, multilinear application $[, , \dots , ,] : \mathfrak{G} \times \cdots \times \mathfrak{G} \rightarrow \mathfrak{G}$, $[X_1, X_2, \dots, X_n] \in \mathfrak{G}$, called *n -bracket*. Let ad be the map $ad : \wedge^{n-1} \mathfrak{G} \rightarrow \text{End } \mathfrak{G}$ that to each element of $\wedge^{n-1} \mathfrak{G}$ associates the *left multiplication* of \mathfrak{G} given by

$$ad_{X_1 X_2 \dots X_{n-1}} : Z \rightarrow [X_1, X_2, \dots, X_{n-1}, Z] \quad , \quad \forall X_i, Z \in \mathfrak{G} \quad , \quad (134)$$

which for $n=2$ reproduces the action of ad_X on a Lie algebra; note that the skewsymmetry of the n -bracket implies that of the $n-1$ arguments of ad . In similarity with the Lie algebra case, we now require that $ad_{X_1, X_2, \dots, X_{n-1}}$ is a *derivation* of the n -bracket *i.e.*, that the following property holds:

Definition 40. (*Inner derivations of the n -bracket*)

$ad_{X_1, X_2, \dots, X_{n-1}}$ is an inner (*left*) derivation of the n -bracket *i.e.*,

$$\begin{aligned} ad_{X_1, X_2, \dots, X_{n-1}}[Y_1, Y_2, \dots, Y_n] &= \sum_{i=1}^n [Y_1, \dots, ad_{X_1 \dots X_{n-1}} Y_i, \dots, Y_n] \\ &= \sum_{i=1}^n [Y_1, \dots, Y_{i-1}, [X_1, \dots, X_{n-1}, Y_i], Y_{i+1}, \dots, Y_n] . \end{aligned} \quad (135)$$

As for Lie algebras, the derivations $ad_{X_1, \dots, X_{n-1}}$ are called *inner* because they are characterized by $(n-1)$ elements of the n -Lie algebra \mathfrak{G} itself. The above reads, in full detail,

$$\begin{aligned} [X_1, X_2, \dots, X_{n-1}, [Y_1, Y_2, \dots, Y_n]] &= [[X_1, X_2, \dots, X_{n-1}, Y_1], Y_2, \dots, Y_n] + \\ [Y_1, [X_1, \dots, X_{n-1}, Y_2], Y_3, \dots, Y_n] &+ \cdots + [Y_1, \dots, Y_{n-1}, [X_1, X_2, \dots, X_{n-1}, Y_n]] \quad . \end{aligned} \quad (136)$$

Eq. (136), which contains $(n+1)$ terms, is the *Filippov identity* (FI). It reduces to the JI for $n=2$ and motivates the following

Definition 41. (*n -Lie or Filippov algebra (FA) [16]*)

An n -Lie algebra \mathfrak{G} is a vector space together with a multilinear fully skewsymmetric application $[\cdot, \cdot, \dots, \cdot] : \mathfrak{G} \times \dots \times \mathfrak{G} \rightarrow \mathfrak{G}$, the n -bracket, such that the Filippov identity (136) is satisfied. For $n=2$, the FA \mathfrak{G} reduces to an ordinary Lie algebra \mathfrak{g} .

Thus, the FI (136) that characterizes a FA just reflects that $ad_{X_1, \dots, X_{n-1}}$ is an inner derivation of the n -Lie algebra. Note that for $n > 2$ $ad_{X_1 X_2 \dots X_{n-1}}$ is a derivation of the n -bracket but not a representation of the elements of \mathfrak{G} themselves (but see Sec. 8.3 below), since the skewsymmetric map $ad : \mathfrak{G} \times \dots \times \mathfrak{G} \rightarrow \text{End } \mathfrak{G}$ is not defined on \mathfrak{G} itself unless $n=2$. It is only for $n = 2$ that ad is both a representation of \mathfrak{g} and a derivation of the Lie algebra.

The skewsymmetric sets (X_1, \dots, X_{n-1}) that determine inner derivations $ad_{X_1, \dots, X_{n-1}}$ of the FA \mathfrak{G} appear very frequently in the theory of FAs and it is convenient to denote them by a symbol, $\mathcal{X} \equiv (X_1, \dots, X_{n-1})$, and to give them a name. They will be called *fundamental objects* of the FA \mathfrak{G} , $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$; their properties will be discussed in Sec. 7.8. As for Lie algebras, the ad map is not injective in general, the extreme case being that of an abelian \mathfrak{G} , for which $ad_{\mathcal{X}} = 0 \ \forall \mathcal{X}$. The classes of fundamental objects \mathcal{X} obtained by taking the quotient by $\ker ad$ are the *inner derivations* of \mathfrak{G} . They define an ordinary Lie algebra $\text{InDer } \mathfrak{G} \equiv \text{Lie } \mathfrak{G}$ or *Lie algebra associated with* \mathfrak{G} , as will be discussed in Sec. 8.

Observation. Filippov uses [16] *right* multiplications, $R(X_1, \dots, X_{n-1}) : Y \mapsto [Y, X_1 \dots, X_{n-1}]$. Such a right inner derivation leads to

$$\begin{aligned} & [[Y_1, \dots, Y_n], X_1, \dots, X_{n-1}] = [[Y_1, X_1, \dots, X_{n-1}], Y_2, \dots, Y_n] + \\ & [Y_1, [Y_2, X_1, \dots, X_{n-1}], Y_3, \dots, Y_n] + \dots + [Y_1, \dots, Y_{n-1}, [Y_n, X_1, \dots, X_{n-1}]] . \end{aligned} \quad (137)$$

However, due to the full skewsymmetry of the Filippov n -bracket, all the terms in the ‘left’ (eq. (136)) and in the ‘right’ identity above differ in a $(-1)^{n-1}$ sign, and therefore both equations define one and the same expression (as it is of course the case of the JI, which may be written either as $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ or as $[[Y, Z], X] = [[Y, X], Z] + [Y, [Z, X]]$).

There may be derivations of a FA that are not defined through elements of \mathfrak{G} since, in general,

Definition 42. (*Derivations of a FA*)

A derivation of a FA is an element $D \in \text{End } \mathfrak{G}$ that satisfies the derivation property,

$$D[X_1, X_2, \dots, X_n] = \sum_{i=1}^n [X_1, \dots, D X_i, \dots, X_n] \quad . \quad (138)$$

As for $\text{InDer } \mathfrak{G}$ above, the space $\text{Der } \mathfrak{G}$ of all *derivations* D of \mathfrak{G} defined by eq. (138) generate a Lie algebra. It is checked that $\text{InDer } \mathfrak{g}$ is an ideal of $\text{Der } \mathfrak{G}$ (see, *e.g.* [25]). Therefore, the quotient $\text{Der}(\mathfrak{G})/\text{InDer } \mathfrak{G} = \text{OutDer } \mathfrak{G}$ is, by definition, the Lie algebra of *outer derivations* of the FA \mathfrak{G} , thus called because they cannot be realized in terms of fundamental elements of \mathfrak{G} .

7.1.1. Other forms of the Filippov identity.

The FI may be rewritten in various forms which is useful to have at hand. Using the skewsymmetry of the n -bracket, the FI may also be written as

$$[X_1, X_2, \dots, X_{n-1}, [Y_1, Y_2, \dots, Y_n]] = \sum_{i=1}^n (-1)^{n-i} [Y_1, \dots, \widehat{Y}_i, \dots, Y_n, [X_1, \dots, X_{n-1}, Y_i]] \quad (139)$$

or, equivalently,

$$[X_1, X_2, \dots, X_{n-1}, [Y_1, \dots, Y_n]] = \sum_{\text{cycl.perm.}} (-1)^{\pi(\sigma)} [[X_1, \dots, X_{n-1}, Y_{\sigma(1)}], Y_{\sigma(2)}, \dots, Y_{\sigma(n)}] \quad , \quad (140)$$

where the sum is extended to the n circular permutations σ of the n indices. For n odd, all circular permutations are even, and no signs appear; for n even, plus and minus signs alternate in (140).

Another useful form of the FI is provided by

$$[[X_{[a_1, \dots, X_{a_n}], X_{b_1}], \dots, X_{b_{n-1}}] = 0 \quad , \quad (141)$$

to be compared with eq. (93) for a GLA.

Finally, we give one more way of writing the FI. If we introduce a set of $(2n - 1)$ anticommuting, ‘ghost’ variables and set $B = b^a X_a$, $C = c^a X_a$, it is easy to rewrite eq. (136) in the compact form [58]

$$[B, \overset{n-1}{\cdot\cdot\cdot}, B, [C, \overset{n}{\cdot\cdot\cdot}, C]] = n[[B, \overset{n-1}{\cdot\cdot\cdot}, B, C], C, \overset{n-1}{\cdot\cdot\cdot}, C] \quad . \quad (142)$$

The proof is an immediate generalization of the $n = 2$ Lie algebra case, for which $[B, [C, C]] = 2[[B, C], C]$ reproduces the JI once the ghost variables are factored out.

The above anticommuting ghost variables can be used to prove the equivalence of eqs. (139) and (141), as we show below for the simplest $n = 3$ case. Assume that (141) is true. It may be written as

$$\begin{aligned} [X_{b_1}, X_{a_1}, [X_{a_2}, X_{a_3}, X_{a_4}]] &= [X_{b_1}, X_{a_2}, [X_{a_1}, X_{a_3}, X_{a_4}]] \\ &+ [X_{b_1}, X_{a_3}, [X_{a_2}, X_{a_1}, X_{a_4}]] + [X_{b_1}, X_{a_4}, [X_{a_2}, X_{a_3}, X_{a_1}]] \quad . \end{aligned} \quad (143)$$

Contracting with $b^{b_1} b^{a_1} c^{a_2} c^{a_3} c^{a_4}$ leads to

$$[B, B, [C, C, C]] = -3[B, C, [B, C, C]] \quad . \quad (144)$$

If we now contract with $c^{b_1} b^{a_1} b^{a_2} c^{a_3} c^{a_4}$ we obtain

$$- [B, C, [B, C, C]] = [[B, B, C], C, C] \quad , \quad (145)$$

and combining the last two equations the FI in the form (142) follows. Conversely, start from the conventional form (139). To show that this implies eq. (141), it is sufficient to prove the equivalent expression $[B, C, [C, C, C]] = 0$. This follows by applying the FI to $[C, C, [B, C, C]]$:

$$\begin{aligned} [C, C, [B, C, C]] &= [[C, C, B], C, C] + [B, [C, C, C], C] + [B, C, [C, C, C]] \\ &= [C, C, [B, C, C]] + 2[B, C, [C, C, C]] \quad , \end{aligned} \quad (146)$$

which gives $[B, C, [C, C, C]] = 0$. The proof may be extended to any n , and it is relegated to Appendix 1.

Proposition 43. *(($n - 1$)-Lie algebras from n -Lie algebras [16])*

Let \mathfrak{G} be an arbitrary n -Lie algebra and define, by fixing an element $A \in \mathfrak{G}$ in its n -bracket, the obviously $(n - 1)$ -linear and fully antisymmetric $(n - 1)$ -bracket by

$$[X_1, X_2, \dots, X_{n-1}]_{n-1} := [A, X_1, X_2, \dots, X_{n-1}]_n \quad . \quad (147)$$

Then, the $(n - 1)$ -bracket above satisfies the FI.

Proof. Clearly, with A fixed, the FI for the n -bracket implies the equality

$$\begin{aligned} [A, X_1, X_2, \dots, X_{n-2}, [A, Y_1, Y_2, \dots, Y_{n-1}]] &= [[A, X_1, X_2, \dots, X_{n-2}, A], Y_1, \dots, Y_{n-1}] + \\ [A, [A, X_1, X_2, \dots, X_{n-2}, Y_1], Y_2, \dots, Y_{n-1}] &+ \dots + [A, Y_1, \dots, Y_{n-2}, [A, X_1, X_2, \dots, X_{n-2}, Y_{n-1}]] \\ &= \sum_{i=1}^{n-1} [A, Y_1, \dots, Y_{i-1}, [A, X_1, X_2, \dots, X_{n-2}, Y_i], Y_{i+1}, \dots, Y_{n-1}] \end{aligned} \quad (148)$$

which, in turn, implies that the $(n - 1)$ -bracket defined by (147) satisfies the n terms FI and hence defines a $(n - 1)$ -Lie algebra on the same vector space of the n -Lie algebra. \square

Thus, given an n -Lie algebra, one may obtain by the above procedure a subordinated chain of m -Lie algebras of increasingly lower orders (see also [168] for further details and references). In particular, a 3-Lie algebra \mathfrak{G} defines a family of Lie algebras \mathfrak{g} characterized by the (fixed) elements of \mathfrak{G} , since $[X, Y] := [A, X, Y] \ \forall X, Y \in \mathfrak{G}$ satisfies the JI in \mathfrak{g} .

7.2. Structure constants of the n -Lie algebras and the FI.

Chosen a basis $\{X_a\}$ of \mathfrak{G} , $a = 1, \dots, \dim \mathfrak{G}$, the FA bracket may be defined by the n -Lie algebra structure constants,

$$[X_{a_1} \dots X_{a_n}] = f_{a_1 \dots a_n}{}^d X_d \quad . \quad (149)$$

The $f_{a_1 \dots a_n}{}^d$ are fully skewsymmetric in the a_i indices and satisfy the condition

$$f_{b_1 \dots b_n}{}^l f_{a_1 \dots a_{n-1}l}{}^s = \sum_{k=1}^n f_{a_1 \dots a_{n-1}b_k}{}^l f_{b_1 \dots b_{k-1}l b_{k+1} \dots b_n}{}^s \quad , \quad (150)$$

which expresses the FI

$$[X_{a_1}, \dots, X_{a_{n-1}}, [X_{b_1}, \dots, X_{b_n}]] = \sum_k [X_{b_1}, \dots, X_{b_{k-1}} [X_{a_1}, \dots, X_{a_{n-1}}, X_{b_k}], X_{b_{k+1}}, \dots, X_{b_n}]$$

in terms of the structure constants of \mathfrak{G} . For later convenience, we write the coordinates expression of the $n = 3$ FI explicitly:

$$f_{b_1 b_2 b_3}{}^l f_{a_1 a_2 l}{}^s = f_{a_1 a_2 b_1}{}^l f_{l b_2 b_3}{}^s + f_{a_1 a_2 b_2}{}^l f_{b_1 l b_3}{}^s + f_{a_1 a_2 b_3}{}^l f_{b_1 b_2 l}{}^s \quad . \quad (151)$$

Similarly, the form (141) of the FI leads in coordinates to

$$f_{[a_1 \dots a_n}{}^l f_{b_1] b_2 \dots b_{n-1} l}{}^s = 0 \quad , \quad (152)$$

to be compared in the n even case with the coordinates expression of the GJI for a GLA in eq. (99). Thus, every even FA defines a generalized Lie algebra, a fact that will find an analogue in Lemma 95 for the even n -ary generalizations of the Poisson structures to be discussed in Sec. 13.

To conclude, we give two more forms for the FI in coordinates. From expression (136) or (139) it follows that

$$f_{c_1 \dots c_n}{}^l f_{b_1 \dots b_{n-1} l}{}^s = \sum_{i=1}^n (-1)^{n-i} f_{b_1 \dots b_{n-1} c_i}{}^l f_{c_1 \dots \hat{c}_i \dots c_n l}{}^s \quad , \quad (153)$$

which is equivalent to

$$f_{c_1 \dots c_n}{}^l f_{b_1 \dots b_{n-1} l}{}^s = \frac{(-1)^{n-1}}{(n-1)!} f_{b_1 \dots b_{n-1} [c_1}{}^l f_{c_2 \dots c_n] l}{}^s \quad (154)$$

(the *r.h.s.* would become $(-1)^{n-1} n f_{b_1 \dots b_{n-1} [c_1} {}^l f_{c_2 \dots c_n] l} {}^s$ with unit weight antisymmetrization of the n indices c). Eq. (154) also follows by writing $B = b^a X_a$ etc and extracting the ghosts from eq. (142).

7.3. Structural properties of Filippov n -Lie algebras.

Let us go briefly through some basic properties of FAs following the pattern of the Lie algebras in Sec. 2 which, after all, constitute the $n=2$ FA case. Many Lie algebra results may be straightforwardly translated to general FA, although there are important differences, the main ones being that the rich variety of simple Lie algebras is drastically reduced when moving to $n > 2$ FAs and that *e.g.*, some Lie algebra concepts (as solvability) allow for more than one possible definition when extended to $n \geq 3$ n -Lie algebras. The structure of the FAs (for a review of $n = 3$ FAs, see [169]) was already developed in the original paper of Filippov [16] and in later work of Kasymov [23], where the notion of representation and the analogues of the Cartan subalgebra and Killing metric for Lie algebras were introduced for FAs. Further developments can be found in [24, 170] (see also [171, 172]) and in the very complete Ph. D. thesis of Ling [25]. This last paper proved the analogue of the Levi decomposition for finite-dimensional n -Lie algebras and showed that for $n > 2$ all $(n + 1)$ -dimensional simple n -Lie algebras are of *one* type up to isomorphisms, the one given by Filippov [16], thus classifying the simple FAs.

7.3.1. Basic definitions, properties and results.

Let \mathfrak{G} be a FA (Def. 41). Then,

A FA is *abelian* if $[X_1, \dots, X_n] = 0$ for all $X \in \mathfrak{G}$.

A subspace \mathfrak{h} of a FA \mathfrak{G} is a *Filippov subalgebra* when it is closed under under the n -bracket,

$$\mathfrak{h} \text{ subalgebra} \Leftrightarrow [Y_1, \dots, Y_n] \subset \mathfrak{h} \quad \forall Y \in \mathfrak{h} \quad .$$

A subspace $I \subset \mathfrak{G}$ is an *ideal* \mathfrak{G} if

$$[X_1, \dots, X_{n-1}, Y] \subset I \quad \forall X \in \mathfrak{G}, \forall Y \in I \quad .$$

A FA is *simple* if $[\mathfrak{G}, \dots, \mathfrak{G}] \neq \{0\}$ and has no ideals different from the trivial ones, $\{0\}$ and \mathfrak{G} .

It is easy to check that if I, J are ideals of \mathfrak{G} , $I + J = \{X + Y | X \in I, Y \in J\}$ and $I \cap J$ are also ideals of \mathfrak{G} . Thus, since the sum (resp. intersection) of ideals of \mathfrak{G} is an ideal of \mathfrak{G} , a FA \mathfrak{G} has maximal (resp. minimal) ideals. An ideal J is called *maximal* if the only ideals containing J are \mathfrak{G} and J . An ideal I of \mathfrak{G} is called *minimal* if the only ideals of \mathfrak{G} contained in I are 0 and I . Further, in analogy with the Lie algebras particular case (see *e.g.* [173]) the following Lemma [16, 25] holds:

Lemma 44.

Let I, J be ideals of \mathfrak{G} . Then, $(I + J)/I \sim J/(I \cap J)$. Futher, if $I \subset J$, J/I is an ideal of \mathfrak{G}/I and $(\mathfrak{G}/I)/(J/I) \sim \mathfrak{G}/J$ i.e., one can remove the ‘common factor’ I .

Definition 45. (Centre, centralizer, normalizer)

The *centre* $Z(\mathfrak{G})$ of a FA is given by $Z(\mathfrak{G}) = \{Z \in \mathfrak{G} | [X_1, \dots, X_{n-1}, Z] = 0 \forall X \in \mathfrak{G}\}$. It is an abelian ideal of \mathfrak{G} . More generally, the *centralizer* $C(\mathfrak{h})$ of a subset $\mathfrak{h} \subset \mathfrak{G}$ may be defined by the condition $[C(\mathfrak{h}), \mathfrak{h}, \mathfrak{G}, \dots, \mathfrak{G}] = 0$ (thus, and as for Lie algebras, $C(\mathfrak{G}) = Z(\mathfrak{G})$). Using the FI (136), it is seen that $C(\mathfrak{h})$ is a subalgebra of \mathfrak{G} . Similarly, the *normalizer* $N(\mathfrak{h})$ of a subalgebra \mathfrak{h} of \mathfrak{G} is defined by the condition $[N(\mathfrak{h}), \mathfrak{h}, \mathfrak{G}, \dots, \mathfrak{G}] \subset \mathfrak{h}$. Again, the FI shows that $N(\mathfrak{h})$ is a subalgebra of \mathfrak{G} ; clearly, \mathfrak{h} is an ideal of $N(\mathfrak{h})$.

In analogy with Lie algebras, we have the following (see further Th. 66 below)

Theorem 46.

All the derivations of a simple FA are inner [16].

Definition 47. (*Homomorphisms of FAs*)

A vector space homomorphism $\phi : \mathfrak{G} \rightarrow \mathfrak{G}'$, $\phi : X \in \mathfrak{G} \rightarrow \phi(X) \in \mathfrak{G}'$, is a *homomorphism of Filippov algebras* when the image of the n -bracket in \mathfrak{G} is the n -bracket of the images in \mathfrak{G}' ,

$$\phi([X_1, \dots, X_n]) = [\phi(X_1), \dots, \phi(X_n)] \quad .$$

The *kernel of an homomorphism* ϕ is the vector space $\ker \phi = \{Y \in \mathfrak{G} \mid \phi(Y) = 0\}$; $\ker \phi$ is an ideal of \mathfrak{G} since, if $\phi(Y) = 0$, $\phi([X_1, \dots, X_{n-1}, Y]) = [\phi(X_1), \dots, \phi(X_{n-1}), \phi(Y)] = 0$ and therefore $[X_1, \dots, X_{n-1}, Y] \in \ker \phi$. When $\ker \phi = 0$, ϕ is an *isomorphism* of FAs.

The quotient space of a FA \mathfrak{G} by an ideal I is also an n -Lie algebra, since the n -bracket of any n elements from each of the classes $X_1+I, \dots, X_{n+1}+I$, is an element in the class $[X_1, \dots, X_n]+I$ because I is an ideal. Therefore, given an homomorphism ϕ as above, there is an exact sequence of homomorphisms or *canonical decomposition of $\phi : \mathfrak{G} \rightarrow \mathfrak{G}'$* ,

$$0 \rightarrow \ker \phi \rightarrow \mathfrak{G} \rightarrow \mathfrak{G}/\ker \phi = \text{Im} \mathfrak{G} \rightarrow 0 \quad .$$

The first and second arrows are obviously *injective*, the third one is the *canonical projection* and the fourth homomorphism is also *surjective* (and trivially so). Clearly, if ϕ is surjective, $\mathfrak{G}/\ker \phi$ and \mathfrak{G}' are isomorphic FAs.

7.3.2. *Solvable Filippov algebras, radical. Semisimple FAs.*

Let \mathfrak{G} be a FA \mathfrak{G} , and define the *derived series of ideals* inductively by¹²

$$\mathfrak{G}^{(0)} = \mathfrak{G} \quad , \quad \mathfrak{G}^{(1)} = [\mathfrak{G}^{(0)}, \dots, \mathfrak{G}^{(0)}] \quad , \quad \dots \quad , \quad \mathfrak{G}^{(m)} = [\mathfrak{G}^{(m-1)}, \dots, \mathfrak{G}^{(m-1)}] \quad ;$$

if $\mathfrak{G}^{(1)} = 0$, \mathfrak{G} is abelian.

Definition 48. [16]

A FA \mathfrak{G} is *solvable* if $\mathfrak{G}^{(m)} = 0$ for some m ; then, $\mathfrak{G}^{(m-1)}$ is an abelian ideal.

The same definition of solvability applies to ideals $\mathfrak{h}, \mathfrak{h}'$ of \mathfrak{G} ; the sum $\mathfrak{h} + \mathfrak{h}'$ of two solvable ideals is also a solvable ideal of \mathfrak{G} . This means that any finite-dimensional \mathfrak{G} admits a maximal solvable ideal $\text{Rad}(\mathfrak{G})$ and, further, that $\mathfrak{G}/\text{Rad}(\mathfrak{G})$ does not contain non-zero solvable ideals.

The *radical* $\text{Rad}(\mathfrak{G})$ of a FA \mathfrak{G} is the maximal solvable ideal of \mathfrak{G} .

A FA is called *semisimple* when $\text{Rad}(\mathfrak{G}) = 0$.

The following three theorems [25] hold :

Theorem 49. (*Semisimple FAs*)

¹²The solvability notion for Lie algebras allows for various extensions when moving to FAs, $n > 2$, because the n -bracket has more than two entries. For an n -Lie algebra the notion of k -solvability was introduced by Kasymov [23] by taking $\mathfrak{G}^{(0,k)} = \mathfrak{G}$, $\mathfrak{G}^{(m,k)} = [\mathfrak{G}^{(m-1,k)}, \dots, \mathfrak{G}^{(m-1,k)}, \mathfrak{G}, \dots, \mathfrak{G}]$, where there are k entries $\mathfrak{G}^{(m-1,k)}$ at the beginning of the n -bracket. Filippov's solvability [16], used above, corresponds to k -solvability for $k = n$; k -solvability implies n -solvability for all k [25].

A finite-dimensional n -Lie algebra \mathfrak{G} is semisimple iff it is the direct sum of simple ideals,

$$\mathfrak{G} = \bigoplus_{s=1}^k \mathfrak{G}_{(s)} = \mathfrak{G}_{(1)} \oplus \cdots \oplus \mathfrak{G}_{(k)} . \quad (155)$$

where each ideal $\mathfrak{G}_{(s)}$ is simple as an n -Lie algebra. Then, $\text{Der } \mathfrak{G}$ is also semisimple and all derivations of \mathfrak{G} are inner, $\text{Der } \mathfrak{G} = \text{InDer } \mathfrak{G}$.

The above statements directly extend to FAs the familiar Lie algebra ones *e.g.*, that $\text{ad } \mathfrak{g} = \text{Der } \mathfrak{g}$ when \mathfrak{g} is semisimple.

Theorem 50. (*Levi decomposition of an n -Lie algebra*) [25]

Let \mathfrak{G} be a finite-dimensional n -Lie algebra. Then, \mathfrak{G} admits a Levi decomposition,

$$\mathfrak{G} = \text{Rad}(\mathfrak{G}) \ltimes \mathfrak{G}_L \quad , \quad \text{Rad}(\mathfrak{G}) \cap \mathfrak{G}_L = 0 \quad ,$$

where \mathfrak{G}_L is a semisimple n -Lie subalgebra called the Levi factor of \mathfrak{G} . Therefore, $\mathfrak{G}/\text{Rad}(\mathfrak{G})$ is semisimple.

Theorem 51. (*Reductive FAs and semisimplicity*) [16]

As for ordinary Lie algebras, an n -Lie algebra is reductive if its radical $\text{Rad } \mathfrak{G}$ is equal to its centre $Z(\mathfrak{G})$; then $\mathfrak{G} = Z(\mathfrak{G}) \oplus \mathfrak{G}_L$. It then follows from the previous theorems that \mathfrak{G} is reductive iff the Lie algebra $\text{InDer } \mathfrak{G}$ of inner derivations is semisimple.

7.4. Examples of Filippov algebras.

In its original paper [16], Filippov already provided many examples of n -Lie algebras, solvable and simple (in fact, all simple ones). The $n = 3$ FAs on \mathbb{R}^4 were given in [149]. We present explicitly here and in the next subsections a few additional FAs of interest.

Example 52. (*Matrix realizations of general FAs*)

In terms of matrices, an example of a 3-algebra is provided by [174] (see also [175])

$$[A, B, C] = \text{tr}(A)[B, C] + \text{tr}(B)[C, A] + \text{tr}(C)[A, B] . \quad (156)$$

This expression may be generalized to the arbitrary n -case by [174]

$$[A_1, A_2, \dots, A_n] = \sum_{i=1}^n (-1)^{i-1} \langle A_i \rangle [A_1, A_2, \dots, \hat{A}_i, \dots, A_n] \quad , \quad (157)$$

where the $\langle A_i \rangle$ are commuting numbers associated to the A_i ('traces') and \hat{A}_i is absent in the $(n-1)$ -bracket. If the $(n-1)$ -bracket is skewsymmetric and satisfies the corresponding $(n-1)$ -order FI, then the n -bracket is also skewsymmetric, satisfies the FI and thus defines a FA.

For other matrix realizations of three-brackets see *e.g.*, [56, 176, 177] and Th. 55 below.

7.5. The simple n -Lie algebras.

7.5.1. The euclidean A_{n+1} algebras. [16]

Let us first consider the simple case of the Filippov three-algebra A_4 , which is defined on a four-dimensional real euclidean vector space V . Let v_1^a, v_2^a, v_3^a ($a = 1, 2, 3, 4$) be the coordinates of three vectors $v_1, v_2, v_3 \in V$ in a basis $\{e_i\}$ of V . The 3-bracket of the three vectors is then defined by the 'vector product' of v_1, v_2, v_3 ,

$$[v_1, v_2, v_3] = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ v_1^1 & v_1^2 & v_1^3 & v_1^4 \\ v_2^1 & v_2^2 & v_2^3 & v_2^4 \\ v_3^1 & v_3^2 & v_3^3 & v_3^4 \end{vmatrix} , \quad (158)$$

which is obviously skewsymmetric. For three basis vectors, say $\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4$, this trivially gives

$$[\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_4] = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -\mathbf{e}_2 \quad . \quad (159)$$

The above three-bracket may also expressed by¹³

$$[\mathbf{e}_{a_1}, \mathbf{e}_{a_3}, \mathbf{e}_{a_4}] = -\epsilon_{a_1 a_2 a_3 a_4} \mathbf{e}_{a_4} \quad \text{or} \quad [\mathbf{e}_1, \mathbf{e}_{a-1}, \hat{\mathbf{e}}_a, \mathbf{e}_{a+1}, \mathbf{e}_4] = (-1)^{a+1} \mathbf{e}_a, \quad a = 1, 2, 3, 4, \quad (160)$$

where the hatted $\hat{\mathbf{e}}_a$ is absent. To check that the above three-bracket (and similarly for higher order ones) satisfies the FI one may use the Schouten identities technique as in Ex. 56 below. Since all three-brackets (158) follow from (160) by linearity, the above shows that the euclidean four-vector space becomes the four-algebra A_4 .

The A_4 case generalizes easily to the n -Lie algebra A_{n+1} defined on a $(n+1)$ -dimensional euclidean space of ordered basis $\{\mathbf{e}_a\}, a = 1 \dots, n+1$. The ‘vector product’ of n vectors $v_1, v_2, \dots, v_n, v_s = v_s^a \mathbf{e}_a$, is defined by the determinant

$$[v_1, v_2, \dots, v_n] = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n & \mathbf{e}_{n+1} \\ v_1^1 & v_1^2 & \cdots & v_1^n & v_1^{n+1} \\ v_2^1 & v_2^2 & \cdots & v_2^n & v_2^{n+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ v_n^1 & v_n^2 & \cdots & v_n^n & v_n^{n+1} \end{vmatrix}, \quad (161)$$

which defines the n -bracket. In terms of the elements of the basis $\{\mathbf{e}_a\}$ of the vector space, the algebra is defined by

$$[\mathbf{e}_1, \dots, \mathbf{e}_{a-1}, \hat{\mathbf{e}}_a, \mathbf{e}_{a+1}, \dots, \mathbf{e}_{n+1}] = (-1)^{a+1} \mathbf{e}_a \quad (a = 1, \dots, n+1) \quad , \quad (162)$$

The above expression has a different sign factor than eq. (2) in [16] and, further, it does not depend on n due to the different determinant arrangement. Equivalently,

$$[v_1, v_2, \dots, v_n] = \mathbf{e}_b \epsilon^b{}_{a_1 \dots a_n} v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} \quad . \quad (163)$$

The euclidean A_{n+1} algebra above is simple [16]. In fact, all finite simple n -Lie algebras are known and, in contrast with the plethora of simple Lie algebras provided by the Cartan classification, the $n > 2$ FAs are easily characterized. It was shown by Filippov [16] that every $(n+1)$ -dimensional n -Lie algebra is simple and isomorphic to one of the form in eq. (164) below. Further, Ling showed [25] that every finite-dimensional simple n -Lie algebra is of dimension $(n+1)$, thus completing the classification of simple FAs:

Theorem 53. (*Classification of simple FAs*)

A simple real Filippov n -algebra ($n \geq 3$) is isomorphic to one of the $(n+1)$ -dimensional Filippov n -algebras defined by

$$[\mathbf{e}_1 \dots \hat{\mathbf{e}}_a \dots \mathbf{e}_{n+1}] = (-1)^{a+1} \epsilon_a \mathbf{e}_a \quad \text{or} \quad [\mathbf{e}_{a_1} \dots \mathbf{e}_{a_n}] = (-1)^n \epsilon_{a_{n+1}} \epsilon_{a_1 \dots a_{n+1}} \mathbf{e}_{a_{n+1}} \quad , \quad (164)$$

where the ϵ_a are signs (if the FA is not real, the signs may be absorbed by redefining its basis).

If all $\epsilon_a = 1$, the above algebras are the euclidean A_{n+1} FAs; if there are minus signs, the $(n+1)$ -Lie algebras are Lorentzian. For instance, $n = 3$ and $\epsilon_a = 1$ defines the algebra A_4 ; if there is one minus sign, eq. (164) characterizes $A_{1,3}$ and, in general and for arbitrary n , it defines the simple pseudoeuclidean Filippov algebras $A_{p,q}$ with $p+q = n+1$. Note that we might equally well have used in eq. (164) the $\epsilon_{a_1 \dots a_n}^a$ without the ϵ_a in the *r.h.s.* by taking

¹³We will use here $\{\mathbf{e}_a\}$ rather than $\{X_a\}$ for the basis of the A FAs as in [16].

$\epsilon_{a_1 \dots a_n}^a = \eta^{ab} \epsilon_{a_1 \dots a_n b}$ where $\epsilon_{1 \dots n(n+1)} = +1$ and η is a $(n+1) \times (n+1)$ diagonal metric with $+1$ and -1 in the places indicated by the ϵ_a 's. We shall keep nevertheless the customary ϵ_a factors as in [16].

It is easy to check, for instance, that if the basis $\{\mathbf{e}_a\}$ ($a=1,2,3,4$) defines the real euclidean three-algebra A_4 , the (complex) redefinitions $\mathbf{e}'_1 = -i\mathbf{e}_1$, $\mathbf{e}'_2 = i\mathbf{e}_2$, $\mathbf{e}'_3 = -\mathbf{e}_3$, $\mathbf{e}'_4 = \mathbf{e}_4$ define a basis $\{\mathbf{e}'_a\}$ for the real $A_{1,3}$ Lorentzian three-algebra. Thus, in general (and since the real orthogonal and Lorentzian FAs have the same complex form) the *complex* simple n -Lie algebras are given by the n -brackets (164) above without any ϵ_a signs.

Ling's theorem above shows that the class of finite, simple n -Lie algebras is very restricted since there is essentially one simple finite-dimensional FA for $n > 2$. It has recently been shown [178] that the class of simple linearly compact n -Lie algebras contains four types for $n > 2$, the A_{n+1} FAs (called O^n in [178]) plus three infinite-dimensional n -Lie algebras. Further, for $n > 2$ there are no simple linearly compact n -Lie superalgebras which are not n -Lie algebras.

For the classification of the $(n+1)$ - and $(n+2)$ -dimensional n -Lie algebras see [179].

Remark 54.

It is worth mentioning at this stage an important difference between the simple FAs and the GLAs in Th. 28, which are given by Lie algebra cohomology cocycles and have an underlying *Lie* group manifold. Since the only three-dimensional simple compact Lie algebra is $su(2)$, and has ϵ_{ijk} as its structure constants, the above theorem states that all the euclidean simple FAs are just direct generalizations to dimension $n+1$, $n \geq 3$, of the $n=2$ $su(2)$ Lie algebra, and that their structure constants are given by the fully antisymmetric tensor $\epsilon_{a_1 \dots a_{n+1}}$. By adding the corresponding minus signs, all the $n \geq 3$ Lorentzian FAs algebras are, similarly, direct generalizations of $so(1,2)$.

This simple observation hints at the rigidity of the $n \geq 3$ simple FAs and is behind the proofs [180] of Thms. 74 and 80.

The simple FAs admit a realization in terms of the matrices of a Clifford algebra by

Lemma 55. (*GLA multibracket realization of the simple FAs*)

All simple real n -Lie algebras can be realized in terms of even multibrackets (eq. (87)) involving the Dirac matrices of the Clifford algebra of an even D -dimensional vector space.

Proof.

a) n odd:

Here $D = n + 1$. It will be sufficient to consider the n odd euclidean simple FA

$$[\mathbf{e}_{a_1} \dots \mathbf{e}_{a_n}] = -\epsilon_{a_1 \dots a_{n+1}} \mathbf{e}_{a_{n+1}} \quad (165)$$

for the $n=3$, $D=4$ case. Let $\{\gamma_a\}$, $a=1, \dots, 4$ be the gamma matrices $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ of the four-dimensional euclidean space. The γ_5 matrix is given by $\gamma_5 \equiv \gamma_1 \gamma_2 \gamma_3 \gamma_4$ and squares to one, $\gamma_5^2 = 1$. Then, the three-bracket (see [27])

$$[\gamma_a, \gamma_b, \gamma_c] := \frac{1}{4!} [\gamma_a, \gamma_b, \gamma_c, \gamma_5] \quad , \quad (166)$$

defined in terms of the GLA multibracket of eq. (87), provides a gamma matrices realization of the euclidean FA A_4 .

To see that the above defines the three-bracket of A_4 , it is sufficient to notice that the full antisymmetrization of the product $\gamma_5 \gamma_a \gamma_b \gamma_c$ gives

$$[\gamma_5, \gamma_{a_1}, \gamma_{a_2}, \gamma_{a_3}] = 4\gamma_5 [\gamma_{a_1}, \gamma_{a_2}, \gamma_{a_3}] = 4\gamma_5 3! \gamma_5 \epsilon_{a_1 a_2 a_3 a_4} \gamma_{a_4} = 4! \epsilon_{a_1 a_2 a_3 a_4} \gamma_{a_4} \quad ,$$

where the second bracket is also given by the full antisymmetrization of its three entries. Therefore, the three-bracket defined by

$$[\gamma_{a_1}, \gamma_{a_2}, \gamma_{a_3}] := [\gamma_{a_1}, \gamma_{a_2}, \gamma_{a_3}, \gamma_5]' = -\epsilon_{a_1 a_2 a_3 a_4} \gamma_{a_4} \quad , \quad (167)$$

where the prime indicates that the skewsymmetrization of the four-dimensional multibracket is taken there with unit weight, realizes¹⁴ the simple euclidean algebra A_4 with basis $e_a = \gamma_a$, eq. (165). Eq. (166), that realizes the 3-bracket of a FA in terms of a multibracket of order four, uses the type of bracket that appears in the Basu-Harvey equation [181] (see also [26, 182] in the context of the BL model), to be dicussed in Sec 14.3.

Moving to arbitrary odd n simply requires an even $D = (n + 1)$ -dimensional space and a trivial generalization of eq. (167) involving a D -dimensional multibracket $[\gamma_{a_1}, \dots, \gamma_{a_n}, \gamma_{D+1}]$, where γ_{D+1} is the ‘gamma five’ matrix of the D even Clifford algebra. The even Lorentzian simple algebras are obtained similarly, by replacing δ_{ab} by the Minkowski metric $\eta_{\mu\nu}$, $\epsilon_{a_1 a_2 a_3 a_4}$ by $\epsilon_{\mu_1 \mu_2 \mu_3 \mu_4}$, etc.

It is clear why the above construction does not work for even n : the ‘ γ_5 ’ of an odd-dimensional space is a scalar matrix and *e.g.* $[\gamma_5, \gamma_a, \gamma_b, \gamma_c] = 0$. It is sufficient in this case, however, to consider even multibrackets with all the n entries free.

b) n even:

For n even, the realization of A_{n+1} is given in terms by the matrices of the $D = n$ Clifford algebra defined by $\{\gamma^a, \gamma^b\} = (-1)^{n/2} 2 \delta^{ab}$, $a = 1, \dots, n$, plus the matrix $\gamma^{n+1} = \gamma^1 \dots \gamma^n$. Taking the $n + 1$ matrices $\gamma^A = (\gamma^a, \gamma^{n+1})$ as the basis of the $(n + 1)$ -dimensional euclidean FA vector space, it follows that

$$[\gamma^{A_1}, \dots, \gamma^{A_n}]' = \epsilon^{A_1 \dots A_{n+1}} \gamma^{A_{n+1}} \quad (A = 1, \dots, n + 1) \quad , \quad (168)$$

where the multibracket is again that of eq. (87) but with weight one antisymmetrization as before. Note that by taking the bracket (168) with its last entry fixed to be *e.g.* $\gamma^{A_n} = \gamma^{n+1}$ (see Prop. 43), the same even order n multibracket determines the odd \tilde{n} -Lie bracket, $\tilde{n} = n - 1$, of the odd FAs $A_{\tilde{n}+1}$ of the previous case, constructed on even dimensional $D = \tilde{n} + 1$ vector spaces. \square

7.6. General metric n -Lie algebras.

Filippov algebras may be endowed with a scalar product. In the physical literature, metric 3-Lie algebras have been discussed in the context of the BLG model of Sec. 14 (see *e.g.* [182, 56, 31, 32, 183]). In fact, the Lie algebras metricity condition (12) may be naturally extended to the FAs. Since it is the fundamental objects \mathcal{X} that induce derivations, a scalar product on \mathfrak{G} $\langle Y, Z \rangle = g_{ab} Y^a Z^b$, where Y^a, Z^b are the coordinates of $Y, Z \in \mathfrak{G}$, will be invariant when

$$\begin{aligned} \mathcal{X} \cdot \langle Y, Z \rangle &= \langle \mathcal{X} \cdot Y, Z \rangle + \langle Y, \mathcal{X} \cdot Z \rangle \\ &= \langle [X_1, \dots, X_{n-1}, Y], Z \rangle + \langle Y, [X_1, \dots, X_{n-1}, Z] \rangle = 0 \quad . \end{aligned} \quad (169)$$

This may also be expressed as the condition

$$(-1)^n \langle [Y, X_1, \dots, X_{n-1}], Z \rangle = \langle Y, [X_1, \dots, X_{n-1}, Z] \rangle \quad \text{or} \quad (-1)^n \langle Y \cdot \mathcal{X}, Z \rangle = \langle Y, \mathcal{X} \cdot Z \rangle \quad ,$$

which is the analogue of eq. (12) for the Lie algebra case. In coordinates, condition (169) reads

$$f_{a_1 \dots a_{n-1} b}{}^l g_{lc} + f_{a_1 \dots a_{n-1} c}{}^l g_{bl} = 0 \quad , \quad (170)$$

which reflects that the metric g_{ab} is InDer \mathfrak{G} -invariant.

¹⁴For $n = 3$, a gamma matrices realization of A_4 may be given (see *e.g.* [31]) in terms of ordinary two-brackets by means of the double commutator $[[\gamma_a, \gamma_b] \gamma_5, \gamma_c]$, which is (a, b, c) skewsymmetric; it is equivalent to the one above since $3! [[\gamma_a, \gamma_b] \gamma_5, \gamma_c] = [\gamma_5, \gamma_a, \gamma_b, \gamma_c]$.

An n -Lie algebra endowed with a metric g as above is called a *metric Filippov algebra*. If \mathfrak{G} is a metric n -Lie algebra, those obtained by Prop. 43 are metric $(n-1)$ -Lie algebras since eq. (169) remains satisfied for fixed $X_1 = A$.

7.6.1. The structure constants of a metric FA as invariant antisymmetric polynomials.

Using the non-degenerate metric g to lower indices, it follows that the structure constants of the FA with all the indices down are fully skewsymmetric, since they are already in the first $(n-1)$ indices and the above expression gives

$$\begin{aligned} f_{a_1 \dots a_{n-1} bc} + f_{a_1 \dots a_{n-1} cb} &= 0 \quad , \\ f_{a_1 \dots a_n a_{n+1}} &= \langle [X_{a_1}, \dots, X_{a_n}], X_{a_{n+1}} \rangle \quad . \end{aligned}$$

The above may be considered as the coordinates of the $(n+1)$ -form \mathfrak{f} on \mathfrak{G} defined by

$$\mathfrak{f}(X_{a_1}, \dots, X_{a_n}, X_{a_{n+1}}) = \langle [X_{a_1}, \dots, X_{a_n}], X_{a_{n+1}} \rangle \quad .$$

Using now the metric g_{ab} to move indices, we see that the FI, eq. (150), may also be written in the form

$$\sum_{i=1}^{n+1} f_{a_1 \dots a_{n-1} b_i}{}^l f_{b_1 \dots b_{i-1} l b_{i+1} \dots b_{n+1}} = 0 \quad \text{or} \quad f_{a_1 \dots a_{n-1} [b_1}{}^l f_{b_2 \dots b_{n+1}] l} = 0 \quad (171)$$

on account of the skewsymmetry. This shows that the fully antisymmetric structure constants determine an antisymmetric covariant tensor \mathfrak{f} on \mathfrak{G} of rank $(n+1)$ (*cf.* eq. (58)) which is Lie \mathfrak{G} -invariant; its invariance may be expressed as $L_{\mathcal{X}} \cdot \mathfrak{f} = 0$.

7.7. Nambu algebras.

The Nambu algebra ($n=3$) and, in general, the Nambu-Poisson algebras for arbitrary n , are infinite-dimensional FAs that follow closely the pattern of the simple FAs where the n -bracket defined by the determinant ‘vector product’ of n vectors of a $(n+1)$ -dimensional space is replaced by the Jacobian determinant of n functions on an n -dimensional manifold¹⁵. This is why it is possible to give a Nambu bracket version of the original BLG model, as will be shown in Sec. 15.

Example 56. (n -algebras on functions on \mathbb{R}^n and canonical Nambu bracket)

Let f_1, f_2, \dots, f_n be functions on \mathbb{R}^n with coordinates $\{x^i\}, i = 1, \dots, n$. The n -bracket $\{f_1, \dots, f_n\}$ or Nambu-Poisson bracket is defined by the Jacobian determinant

$$\{f_1, f_2, \dots, f_n\} := \epsilon_{1 \dots n}^{i_1 \dots i_n} \partial_{i_1} f^1 \dots \partial_{i_n} f^n = \left| \frac{\partial(f_1, f_2, \dots, f_n)}{\partial(x^1, x^2, \dots, x^n)} \right| \quad , \quad f_i = f_i(x^1, x^2, \dots, x^n) \quad . \quad (172)$$

Introducing the multivector $\Lambda_{(n)} = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n}$ it follows that $\{f_1, \dots, f_n\} = \Lambda_{(n)}(df_1, \dots, df_n)$; $\Lambda_{(n)}$ is the standard Nambu tensor (see further Sec. 13.4). For $n=3$, eq. (172) is the expression of the Nambu bracket $\{f_1, f_2, f_3\}$ studied in [38] although, of course, Nambu also mentioned the general n case. The fact that the Jacobian defines an n -Lie algebra structure was already noticed by Filippov in his original paper [16] (see also [42] and [186] for further analysis).

We can check explicitly at this stage that the Jacobian bracket above satisfies the FI. Consider the simplest $n=3$ case. Then, any antisymmetrization of more than three indices gives zero, a trick that leads to the so-called ‘Schouten identities’. Therefore, as far as the skewsymmetry among the k and the j_1, j_2, j_3 indices is concerned,

$$0 \equiv \epsilon_{123}^{i_1 i_2 [k} \epsilon_{123}^{j_1 j_2 j_3]} \sim \left(\epsilon_{123}^{i_1 i_2 k} \epsilon_{123}^{j_1 j_2 j_3} - \epsilon_{123}^{i_1 i_2 j_1} \epsilon_{123}^{k j_2 j_3} - \epsilon_{123}^{i_1 i_2 j_2} \epsilon_{123}^{j_1 k j_3} - \epsilon_{123}^{i_1 i_2 j_3} \epsilon_{123}^{j_1 j_2 k} \right) \quad .$$

¹⁵The question of whether another interesting determinant, the Wronskian, generates an n -Lie algebra is discussed in [184]; see further [185].

This expression leads to the FI because

$$\begin{aligned} & \epsilon_{123}^{i_1 i_2 k} \epsilon_{123}^{j_1 j_2 j_3} \partial_{i_1} f_1 \partial_{i_2} f_2 \partial_k (\partial_{j_1} g_1 \partial_{j_2} g_2 \partial_{j_3} g_3) = \\ & \left(\epsilon_{123}^{i_1 i_2 j_1} \epsilon_{123}^{k j_2 j_3} + \epsilon_{123}^{i_1 i_2 j_2} \epsilon_{123}^{j_1 k j_3} + \epsilon_{123}^{i_1 i_2 j_3} \epsilon_{123}^{j_1 j_2 k} \right) \partial_{i_1} f_1 \partial_{i_2} f_2 \partial_k (\partial_{j_1} g_1 \partial_{j_2} g_2 \partial_{j_3} g_3) \end{aligned}$$

implies

$$\{f_1, f_2, \{g_1, g_2, g_3\}\} = \{\{f_1, f_2, g_1\}, g_2, g_3\} + \{g_1, \{f_1, f_2, g_2\}, g_3\} + \{g_1, g_2, \{f_1, f_2, g_3\}\}. \quad (173)$$

Actually, there are three more terms in the *r.h.s.* of the above equation but these cancel since, using the again a Schouten identity, they add up to $-\epsilon_{123}^{i_1 i_2 k} \epsilon_{123}^{j_1 j_2 j_3} \partial_k (\partial_{i_1} f_1 \partial_{i_2} f_2) \partial_{j_1} g_1 \partial_{j_2} g_2 \partial_{j_3} g_3$ which obviously vanishes.

The FI (173) is an essential ingredient of the Nambu-Poisson mechanics [38,41] to be discussed in Sec 13.4. The time evolution of a dynamical quantity in a ($n = 3$) Nambu mechanical system is governed [38] by two ‘hamiltonian functions’ H_1, H_2 and given by $\dot{f} = \{f, H_1, H_2\}$ and, in particular, that of $\{f, g, h\}$ is given by $\{\{f, g, h\}, H_1, H_2\}$. Thus, the FI guarantees consistency since

$$\begin{aligned} \frac{d}{dt} \{f, g, h\} &= \{\dot{f}, g, h\} + \{f, \dot{g}, h\} + \{f, g, \dot{h}\} \quad \Leftrightarrow \\ \{f, g, h, \{H_1, H_2\}\} &= \{\{f, H_1, H_2\}, g, h\} + \{f, \{g, H_1, H_2\}, h\} + \{f, g, \{h, H_1, H_2\}\}. \end{aligned} \quad (174)$$

The above identity was introduced as a consistency condition for Nambu mechanics in [39,40]. It establishes that the time evolution (given by the Hamiltonian vector field X_{H_1, H_2}) is a derivation of the Nambu bracket.

Example 57. (*Nambu-Poisson bracket on the ring $\mathcal{F}(M)$ of functions on a compact manifold*)

Let M be a compact, oriented manifold without boundary, with volume form μ . The previous Nambu n -bracket may be defined (see *e.g.* [149, 49]) for $g_1, \dots, g_n \in \mathcal{F}(M)$ by adopting

$$\{g_1, \dots, g_n\} \mu = dg_1 \wedge \dots \wedge dg_n \quad (175)$$

(obviously, the Nambu bracket defined by the Jacobian determinant in Ex. 56 corresponds to taking $\mu = dx^1 \wedge \dots \wedge dx^n$). Let now $X_{g_1, \dots, g_{n-1}}$ be the vector field defined by $X_{g_1, \dots, g_{n-1}} \cdot g_n = \{g_1, \dots, g_{n-1}, g_n\} = dg_n(X_{g_1, \dots, g_{n-1}})$, Eq. (175) clearly implies

$$i_{X_{g_1, \dots, g_{n-1}}} \mu = (-1)^{n+1} dg_1 \wedge \dots \wedge dg_{n-1} \quad ,$$

since there is only a non-zero contribution from its *r.h.s.* producing the bracket $\{g_1, \dots, g_{n-1}, g_n\}$, which then may be factored out from both sides. This also shows that the volume element is invariant under the action of the *hamiltonian vector field* $X_{g_1, \dots, g_{n-1}}$ since $L_{X_{g_1, \dots, g_{n-1}}} \mu \equiv (i_{X_{g_1, \dots, g_{n-1}}} d + d i_{X_{g_1, \dots, g_{n-1}}}) \mu = 0$. The application of the Lie derivative to the two sides of eq. (175) now gives

$$\begin{aligned} L_{X_{f_1, \dots, f_{n-1}}} (\{g_1, \dots, g_n\} \mu) &= \{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} \mu = \\ \sum_{i=1}^n dg_1 \wedge \dots \wedge d(L_{X_{f_1, \dots, f_{n-1}}} g_i) \wedge \dots \wedge dg_n &= \sum_{i=1}^n \{g_1, \dots, \{f_1, \dots, f_{n-1}, g_i\}, \dots, g_n\} \mu \quad , \end{aligned}$$

where in the last line $[d, L_X] = 0$ has been used on the *r.h.s.* of (175). Then, the FI for the n -bracket $\{g_1, \dots, g_n\}$ defined by eq. (175) follows.

The above defines the Nambu n -Lie algebra \mathfrak{N} , an example of an infinite-dimensional FA. An $n = 3$ \mathfrak{N} will appear in Sec. 15 in the context of the BLG-NB model.

7.7.1. *The Nambu bracket of functions on a compact manifold as a metric n -Lie algebra.*

Consider smooth functions $h, g, f \in \mathcal{F}(M)$ on a compact, oriented n -dimensional manifold M without boundary as in Ex. 57. The volume n -form μ allows us to define a scalar product in $\mathcal{F}(M)$ by

$$\langle h, g \rangle := \int_M \mu h g \quad . \quad (176)$$

Then, it follows that the Nambu algebra is a metric n -FA. Indeed, the equivalent to eq. (169) here reads $X_{f_1, \dots, f_{n-1}} \langle h, g \rangle = 0$. Thus,

$$\langle \{f_1, \dots, f_{n-1}, h\}, g \rangle + \langle h, \{f_1, \dots, f_{n-1}, g\} \rangle = 0 \quad ,$$

which is satisfied since, by eq. (175),

$$\begin{aligned} \int_M \mu \{f_1, \dots, f_{n-1}, h\} g &= \int_M df_1 \wedge \dots \wedge df_{n-1} \wedge dh g \\ &= - \int_M h df_1 \wedge \dots \wedge df_{n-1} \wedge dg = - \int_M \mu h \{f_1, \dots, f_{n-1}, g\} , \end{aligned}$$

which follows by integrating by parts and by using Stokes theorem for the boundaryless M .

Lemma 58. *(The simple algebras as finite-dimensional subalgebras of Nambu FAs)*

The simple FAs (eq. (164)) are finite subalgebras of the n -Nambu algebras \mathfrak{N} of functions on suitable n -dimensional manifolds M .

Proof. Let e.g. M_n be the unit sphere $S^n \subset \mathbb{R}^{n+1}$ defined by the euclidean metric on \mathbb{R}^{n+1} as $y^a y_a = 1$. Its points are characterized by $n+1$ Cartesian coordinate functions subject to the constraint $y^a y_a = 1$, $y^a = (\mathbf{y}, y^{n+1} = \sqrt{1 - \mathbf{y}^2})$. Then, taking $\frac{d\mathbf{y}^n}{\sqrt{1 - \mathbf{y}^2}}$ as the invariant measure on S^n it follows that

$$\{y^{a_1}, \dots, y^{a_n}\} = \epsilon^{a_1 \dots a_{n+1}} y^{a_{n+1}} ,$$

which defines the A_{n+1} euclidean algebra (up to an irrelevant global sign with respect to (164)) with basis elements $(y^1 \dots y^{n+1})$. Thus, the A_{n+1} FA is a subalgebra of the infinite-dimensional Nambu algebra of functions on the compact sphere S^n .

The other simple algebras are obtained by replacing S^n by the non-compact hyperboloids Ω^n defined by suitable pseudo-euclidean metrics on \mathbb{R}^{n+1} , μ by the invariant measure on Ω^n , say $\mu = \frac{d^n \mathbf{y}}{\sqrt{1 + \mathbf{y}^2}}$, to obtain the n -bracket of the Lorentzian simple $A_{1,n}$ $(n+1)$ -Lie algebras. \square

The above lemma extends easily to the case of $n+1$ functions $\phi^a(y)$ on a manifold M , subject to the constraint

$$\phi_1^2 + \dots + \phi_{n+1}^2 = 1 . \quad (177)$$

There is, however, an interesting topological subtlety that we analyze below¹⁶. Consider $M = S^n$; then, the functions $\phi^a(y)$ define a map $\phi: S^n \rightarrow S^n$, where the first sphere is parametrized by the $n+1$ coordinates y^a , $y^a y_a = 1$, and the second sphere by ϕ^a . Clearly, the possible maps ϕ fall into disjoint homotopy classes characterized by a winding number and thus the degree of the map ϕ should be reflected in the definition of the Nambu bracket.

Let μ be the volume form on the first sphere S^n , for which we take the expression

$$\mu(y) = \sum_{i=1}^{n+1} (-1)^{n+i-1} y^i dy^1 \wedge \dots \wedge \widehat{dy^i} \wedge \dots \wedge dy^{n+1} \quad , \quad y^a y_a = 1 \quad ; \quad (178)$$

¹⁶We thank Paul Townsend for a helpful comment on this point.

the factors have been chosen to reproduce the measure used above. Similarly, the volume form on the second sphere is given by

$$\mu'(\phi) = \sum_{i=1}^{n+1} (-1)^{n+i-1} \phi_i d\phi_1 \wedge \cdots \wedge \widehat{d\phi_i} \wedge \cdots \wedge d\phi_{n+1} \quad , \quad \phi^a \phi_a = 1 \quad . \quad (179)$$

By eq. (175), the Nambu bracket $\{\phi_{a_1}(y), \dots, \phi_{a_n}(y)\}$ of functions in $\mathcal{F}(M)$ is defined by

$$\{\phi_{a_1}(y), \dots, \phi_{a_n}(y)\} \mu(y) = d\phi_{a_1}(y) \wedge \cdots \wedge d\phi_{a_n}(y) \quad , \quad (180)$$

and we would like to see how it is expressed in terms of $\phi_{a_{n+1}}(y)$. To this aim, it is sufficient to show that the *r.h.s.* of (180) is in fact given by

$$d\phi_{a_1}(y) \wedge \cdots \wedge d\phi_{a_n}(y) = \epsilon_{a_1 \dots a_{n+1}} \phi_{a_{n+1}}(y) \phi^*(\mu'(\phi)) \quad , \quad (181)$$

where $\phi^*(\mu')$ is the pull-back of the volume form $\mu'(\phi)$ to the first sphere, hence given by

$$\begin{aligned} (\phi^*(\mu'))(y) &= \sum_{i=1}^{n+1} (-1)^{n+i-1} \phi_i(y) d\phi_1(y) \wedge \cdots \wedge \widehat{d\phi_i(y)} \wedge \cdots \wedge d\phi_{n+1}(y) \\ &= \frac{(-1)^n}{n!} \epsilon^{b_1 \dots b_{n+1}} \phi_{b_1}(y) d\phi_{b_2}(y) \wedge \cdots \wedge d\phi_{b_{n+1}}(y) \quad , \end{aligned} \quad (182)$$

with $\phi^*(\phi^a \phi_a) = \phi^a(y) \phi_a(y) = 1$. Then, and omitting the y dependence, the *r.h.s.* of (181) is equal to

$$\begin{aligned} &\epsilon_{a_1 \dots a_{n+1}} \phi_{a_{n+1}} \frac{(-1)^n}{n!} \epsilon^{b_1 \dots b_{n+1}} \phi_{b_1} d\phi_{b_2} \wedge \cdots \wedge d\phi_{b_{n+1}} \\ &= \frac{(-1)^n}{n!} \phi_{a_{n+1}} \phi_{b_1} \epsilon_{a_1 \dots a_{n+1}}^{b_1 \dots b_{n+1}} d\phi_{b_2} \wedge \cdots \wedge d\phi_{b_{n+1}} \\ &= \frac{(-1)^n}{n!} \phi_{a_{n+1}} \phi_{b_1} \sum_{i=1}^{n+1} (-1)^{n-i+1} \delta_{a_{n+1}}^{b_i} \epsilon_{a_1 \dots \widehat{a_i} \dots a_n}^{b_1 \dots \widehat{b_i} \dots b_{n+1}} d\phi_{b_2} \wedge \cdots \wedge d\phi_{b_{n+1}} \\ &= \frac{1}{n!} \phi_{a_{n+1}} \phi_{b_1} \delta_{a_{n+1}}^{b_1} \epsilon_{a_1 \dots a_n}^{b_2 \dots b_{n+1}} d\phi_{b_2} \wedge \cdots \wedge d\phi_{b_{n+1}} \\ &= d\phi_{a_1}(y) \wedge \cdots \wedge d\phi_{a_n}(y) \quad , \end{aligned} \quad (183)$$

as we wanted to show. Above we have used eq. (89) and, in the third and fourth lines of the expression, that $\phi^{b_i} d\phi_{b_i} = 0$ (in the terms with $i = 2, \dots, n+1$) and $\phi^{b_1} \phi_{b_1} = 1$, respectively.

Now, as is well known, the degree $\deg \phi$ of the mapping ϕ is an integer that may be expressed by the (Kronecker) integral

$$\deg \phi = \frac{1}{\text{vol}(S^n)} \int_{S^n} \phi^*(\mu') \quad . \quad (184)$$

Thus, since $\int \mu(y) = \text{vol}(S^n)$, the above is tantamount to $(\phi^*(\mu'))(y) = (\deg \phi) \mu(y)$. Therefore, inserting eq. (181) into eq. (180) we finally obtain

$$\{\phi_{a_1}(y), \dots, \phi_{a_n}(y)\} = (\deg \phi) \epsilon_{a_1 \dots a_{n+1}} \phi_{a_{n+1}}(y) \quad . \quad (185)$$

Thus, the Nambu-brackets of the S^n -constrained functions $\phi^a(y)$ are classified by the Brouwer degree of the map ϕ . For instance, $\deg \phi = 1$ for the identity map, which gives the standard form of the A_{n+1} simple algebras as realized by S^n -constrained functions on S^n .

7.8. Fundamental objects: definition and properties.

The relevance of the fundamental objects \mathcal{X} for an n -Lie algebra \mathfrak{G} stems from the fact that, as already seen, they define inner derivations $ad_{\mathcal{X}} \in \text{InDer } \mathfrak{G}$. The \mathcal{X} 's will also play a crucial rôle in FA cohomology. Let us recall their definition to analyze their properties.

Definition 59. (*Fundamental objects \mathcal{X} for a FA*)

A fundamental object \mathcal{X} of a FA is determined by $(n-1)$ elements X_1, X_2, \dots, X_{n-1} of \mathfrak{G} on which it is skewsymmetric; thus¹⁷, $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$. The fundamental objects act on the elements of the FA by (left) multiplication:

$$\mathcal{X} \in \wedge^{n-1}\mathfrak{G} \quad , \quad ad_{X_1 \dots X_{n-1}} \equiv ad_{\mathcal{X}} \quad , \quad ad_{\mathcal{X}} \cdot Z \equiv \mathcal{X} \cdot Z := [X_1, X_2, \dots, X_{n-1}, Z]. \quad (186)$$

As a result, $ad : \wedge^{n-1}\mathfrak{G} \rightarrow \text{End } \mathfrak{G}$ is characterized by an n -bracket having void its last entry,

$$ad : \mathcal{X} \mapsto ad_{\mathcal{X}} \equiv \mathcal{X} \cdot \equiv [X_1, X_2, \dots, X_{n-1}, \quad] \quad ; \quad (187)$$

a *fundamental object defines an inner derivation* of the FA \mathfrak{G} . For $n=2$, the fundamental objects \mathcal{X} are the elements of the Lie algebra \mathfrak{g} themselves and $ad_{\mathcal{X}}$ reduces to ad_X .

Chosen a basis, we will often use the obvious notation

$$ad_{X_{a_1} \dots X_{a_{n-1}}} \equiv ad_{a_1 \dots a_{n-1}} \quad , \quad ad_{a_1 \dots a_{n-1}} \cdot X_b := [X_{a_1}, \dots, X_{a_{n-1}}, X_b] = f_{a_1 \dots a_{n-1} b}{}^l X_l. \quad (188)$$

Clearly the $(\dim \mathfrak{G} \times \dim \mathfrak{G})$ -dimensional matrix $ad_{a_1 \dots a_{n-1}} \in \text{End } \mathfrak{G}$ is given by

$$(ad_{a_1 \dots a_{n-1}})^l{}_b = f_{a_1 \dots a_{n-1} b}{}^l \quad , \quad (189)$$

to be compared with the \mathfrak{g} case at the end of Sec. 2.1.

Definition 60. (*Composition of fundamental objects*) [149]

Given two fundamental objects $\mathcal{X}, \mathcal{Y} \in \wedge^{n-1}\mathfrak{G}$, the (non-associative) composition $\mathcal{X} \cdot \mathcal{Y} \in \wedge^{n-1}\mathfrak{G}$ of the two is the bi- and i -linear map $\wedge^{n-1}\mathfrak{G} \otimes \wedge^{n-1}\mathfrak{G} \rightarrow \wedge^{n-1}\mathfrak{G}$ given by the sum

$$\begin{aligned} \mathcal{X} \cdot \mathcal{Y} &:= \sum_{i=1}^{n-1} (Y_1, \dots, Y_{i-1}, \mathcal{X} \cdot Y_i, Y_{i+1}, \dots, Y_{n-1}) \\ &= \sum_{i=1}^{n-1} (Y_1, \dots, Y_{i-1}, [X_1, X_2, \dots, X_{n-1}, Y_i], Y_{i+1}, \dots, Y_{n-1}) \quad , \end{aligned} \quad (190)$$

which is the natural extension of the adjoint derivative $ad_{\mathcal{X}}$ action on \mathfrak{G} to $\wedge^{n-1}\mathfrak{G}$.

Thus, the dot composition $\mathcal{X} \cdot \mathcal{Y}$ defines the inner derivation given by $(\mathcal{X} \cdot \mathcal{Y}) \cdot Z = [\mathcal{X} \cdot \mathcal{Y}, Z] := \sum_{i=1}^{n-1} [Y_1, \dots, Y_{i-1}, \mathcal{X} \cdot Y_i, \dots, Y_{n-1}, Z]$; the composition of a fundamental object with itself always determines the trivial derivation.

The following Lemma follows from the FI:

Lemma 61. (*Properties of the composition of fundamental objects*)

The dot product of fundamental objects \mathcal{X} of an n -Lie algebra \mathfrak{G} satisfies the relation

$$\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) = (\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} \quad \forall \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \wedge^{n-1}\mathfrak{G}. \quad (191)$$

¹⁷The notation $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ merely reflects that the fundamental object $\mathcal{X} = (X_1, \dots, X_{n-1}) \in \mathfrak{G} \times \dots \times \mathfrak{G}$ is antisymmetric in its arguments; it does not imply that \mathcal{X} is a $(n-1)$ -multivector obtained by the associative wedge product of vectors.

As a result, the images $ad_{\mathcal{X}}$ of the fundamental objects by the adjoint map $ad : \wedge^{n-1}\mathfrak{G} \rightarrow InDer\mathfrak{G}$ determine inner derivations that satisfy

$$\mathcal{X} \cdot (\mathcal{Y} \cdot Z) - \mathcal{Y} \cdot (\mathcal{X} \cdot Z) = (\mathcal{X} \cdot \mathcal{Y}) \cdot Z \quad (192)$$

$$\text{or} \quad ad_{\mathcal{X}}ad_{\mathcal{Y}}Z - ad_{\mathcal{Y}}ad_{\mathcal{X}}Z = ad_{\mathcal{X} \cdot \mathcal{Y}}Z \quad \forall \mathcal{X}, \mathcal{Y} \in \wedge^{n-1}\mathfrak{G}, \forall Z \in \mathfrak{G} \quad ,$$

which is equivalent to the FI for $[X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, Z]]$. It then follows that the inner derivations $ad_{\mathcal{X}}$ of a FA \mathfrak{G} generate an ordinary Lie algebra, $Lie\mathfrak{G} \equiv InDer\mathfrak{G} \equiv ad\mathfrak{G}$.

Proof. Let us compute first $\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z})$.

$$\begin{aligned} \mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) &= \sum_{i=1}^{n-1} \mathcal{X} \cdot (Z_1, \dots, Z_{i-1}, [Y_1, \dots, Y_{n-1}, Z_i], Z_{i+1}, \dots, Z_{n-1}) \\ &= \sum_{i=1}^{n-1} \sum_{j \neq i, j=1}^{n-1} (Z_1, \dots, Z_{j-1}, [X_1, \dots, X_{n-1}, Z_j], Z_{j+1}, \dots, Z_{i-1}, [Y_1, \dots, Y_{n-1}, Z_i], Z_{i+1}, \dots, Z_{n-1}) \\ &\quad + \sum_{i=1}^{n-1} (Z_1, \dots, Z_{i-1}, [X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, Z_i]], Z_{i+1}, \dots, Z_{n-1}) \quad . \end{aligned}$$

The first term in the *r.h.s* is symmetric in \mathcal{X}, \mathcal{Y} ; hence,

$$\begin{aligned} \mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) &= \\ \sum_{j=1}^{n-1} (Z_1, \dots, Z_{j-1}, \{ [X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, Z_j]] - [Y_1, \dots, Y_{n-1}, [X_1, \dots, X_{n-1}, Z_j]] \}, Z_{j+1}, \dots, Z_{n-1}) & \quad (193) \end{aligned}$$

On the other hand, using definition (190), we find

$$(\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} (Z_1, \dots, Z_{j-1}, [Y_1, \dots, [X_1, \dots, X_{n-1}, Y_i], \dots, Y_{n-1}, Z_j], Z_{j+1}, \dots, Z_{n-1}) \quad (194)$$

Now, using the FI for $[X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, Z_j]]$, we see that the above expression reproduces (193).

The above proof obviously carries forward to the simplest case where the fundamental object \mathcal{Z} in (191) is replaced by a FA element Z as in (192). It is sufficient to note that, by the FI,

$$\begin{aligned} ad_{\mathcal{X} \cdot \mathcal{Y}}Z &= \sum_{i=1}^{n-1} [Y_1, \dots, Y_{i-1}, [X_1, \dots, X_{n-1}, Y_i], Y_{i+1}, \dots, Y_{n-1}, Z] \\ &= [X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, Z]] - [Y_1, \dots, Y_{n-1}, [X_1, \dots, X_{n-1}, Z]] \\ &= ad_{\mathcal{X}}ad_{\mathcal{Y}}Z - ad_{\mathcal{Y}}ad_{\mathcal{X}}Z \quad \forall Z \in \mathfrak{G} \quad . \end{aligned} \quad (195)$$

□

This expression also shows, by exchanging the X s and the Y s, that $ad_{\mathcal{X} \cdot \mathcal{Y}}Z = -ad_{\mathcal{Y} \cdot \mathcal{X}}Z$ on any $Z \in \mathfrak{G}$, and hence that

$$ad_{\mathcal{X} \cdot \mathcal{Y}} = -ad_{\mathcal{Y} \cdot \mathcal{X}} \quad \text{or, equivalently,} \quad (\mathcal{X} \cdot \mathcal{Y}) \cdot = -(\mathcal{Y} \cdot \mathcal{X}) \cdot \quad . \quad (196)$$

Note the dots after the brackets: $\mathcal{X} \cdot \mathcal{Y} \neq -\mathcal{Y} \cdot \mathcal{X}$ but $(\mathcal{Y} \cdot \mathcal{X}) \cdot Z = -(\mathcal{Y} \cdot \mathcal{X}) \cdot Z$.

As exhibited by eq. (191) or (192), the composition law $\mathcal{X} \cdot \mathcal{Y}$ is not associative; in fact, eq. (191) measures the lack of associativity, $\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) - (\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} = \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z})$ as in the standard Lie algebra case. Indeed, for $n=2$ (192) reproduces the JI written in the

form $[X, [Y, Z]] - [[X, Y], Z] = [Y, [X, Z]]$, which exhibits the obvious lack of associativity of the Lie bracket. For $n > 2$, eqs. (191) and (192) (and (230) below) are a consequence of the characteristic identity that defines the n -Lie algebra \mathfrak{G} , the FI.

7.9. Kasymov's criterion for semisimplicity of a FA.

Kasymov's analogue of the Cartan-Killing metric for the case of a FA \mathfrak{G} is the $2(n-1)$ -linear generalization

$$k(\mathcal{X}, \mathcal{Y}) = k(X_1, \dots, X_{n-1}, Y_1, \dots, Y_{n-1}) := \text{Tr}(ad_{\mathcal{X}} ad_{\mathcal{Y}}) \quad , \quad \mathcal{X}, \mathcal{Y} \in \wedge^{n-1} \mathfrak{G} . \quad (197)$$

Then, a FA is semisimple if it satisfies the following generalization [24] of the Cartan criterion:

Theorem 62. (Semisimplicity of a FA)

An n -Lie algebra is semisimple iff the Kasymov form above is non-degenerate in the sense that

$$k(Z, \mathfrak{G}, \overset{n-2}{\mathfrak{G}}, \mathfrak{G}, \overset{n-1}{\mathfrak{G}}, \mathfrak{G}) = 0 \Rightarrow Z = 0 , \quad (198)$$

where the $2n-3$ arguments besides Z are arbitrary elements of \mathfrak{G} .

Remark 63. (Kasymov form and simple FA algebras)

One might have thought of another generalization of Cartan's criterion by looking at the form k in eq. (197) as a bilinear form on $\wedge^{n-1} \mathfrak{G}$,

$$k : \wedge^{n-1} \mathfrak{G} \times \wedge^{n-1} \mathfrak{G} \longrightarrow \mathbb{R} ,$$

to analyze the consequences of non-degeneracy in the usual sense, $k(\mathcal{X}, \mathcal{Y}) = \text{Tr}(ad_{\mathcal{X}} ad_{\mathcal{Y}}) = 0 \quad \forall \mathcal{Y} \in \wedge^{n-1} \mathfrak{G} \Rightarrow \mathcal{X} = 0$, perhaps thinking of extending to FAs the semisimplicity proof that holds for Lie algebras with a non-degenerate Cartan-Killing form. But we see immediately that this does not work for $n \geq 3$. Any semisimple n -Lie algebra is the direct sum of its simple ideals [25], eq. (155). As a consequence, an n -bracket $[\dots, X, \dots, Y, \dots]$ is zero whenever X and Y belong to different simple ideals. Then, if X_1, \dots, X_{n-2} and Y are in different ideals, it follows that $ad_{\mathcal{X}}$ for $\mathcal{X} = (X_1, \dots, X_{n-2}, Y)$ is the zero derivation, so that $k(\mathcal{X}, \mathcal{Y}) = 0$ for any \mathcal{Y} without \mathcal{X} itself being zero (*cf.* Th. 62). This cannot happen for an $n = 2$ Filippov (Lie) algebra \mathfrak{g} , since its fundamental objects are determined by a single element of \mathfrak{g} .

Nevertheless, for *simple* n -Lie algebras k is nondegenerate as a bilinear metric on $\wedge^{n-1} \mathfrak{G}$. To show this, we may use that a real simple n -Lie algebra is [25, 16] one of the FAs in eq. (164). Since k on $\wedge^{n-1} \mathfrak{G}$ is determined by its values on a basis, we take $\mathcal{X} = (\mathbf{e}_{a_1}, \dots, \mathbf{e}_{a_{n-1}})$, $\mathcal{Y} = (\mathbf{e}_{b_1}, \dots, \mathbf{e}_{b_{n-1}})$. Using eq. (186), the action of $ad_{\mathcal{X}} ad_{\mathcal{Y}}$ on the vector \mathbf{e}_c is found to be

$$ad_{\mathcal{X}} ad_{\mathcal{Y}} \mathbf{e}_c = \sum_{d,g=1}^{n+1} \varepsilon_d \varepsilon_g \epsilon_{b_1 \dots b_{n-1} c}^d \epsilon_{a_1 \dots a_{n-1} d}^g \mathbf{e}_g , \quad (199)$$

so that the trace of the matrix $ad_{\mathcal{X}} ad_{\mathcal{Y}}$ is given by

$$k(\mathcal{X}, \mathcal{Y}) = \text{Tr}(ad_{\mathcal{X}} ad_{\mathcal{Y}}) = \sum_{d,g=1}^{n+1} \varepsilon_d \varepsilon_g \epsilon_{b_1 \dots b_{n-1} g}^d \epsilon_{a_1 \dots a_{n-1} d}^g \equiv k_{(a_1 \dots a_{n-1})(b_1 \dots b_{n-1})} . \quad (200)$$

The numbers appearing in the *r.h.s.* of (200), seen as a $\binom{n+1}{n-1} \times \binom{n+1}{n-1}$ matrix of indices $(a_1 \dots a_{n-1})(b_1 \dots b_{n-1})$ characterized by the fundamental objects above, determine a diagonal one with non-zero diagonal elements. Indeed, given an index determined by a certain set $(a_1 \dots a_{n-1})$, the antisymmetric tensor $\epsilon_{a_1 \dots a_{n-1} d}^g$ fixes the remaining d and g (and $\varepsilon_d \varepsilon_g$) so that the other matrix index (b_1, \dots, b_{n-1}) has to be given by a reordering of $(a_1 \dots a_{n-1})$. For this reason the only non-zero elements of the $k_{(a_1 \dots a_{n-1})(b_1 \dots b_{n-1})}$ matrix are all its diagonal ones,

and hence it is non-degenerate. For $n = 2$ one just finds, of course, that the Kasymov matrix is proportional to $-\mathbb{I}_3$ and that $su(2)$ is simple.

8. THE LIE ALGEBRAS ASSOCIATED TO n -LIE ALGEBRAS

8.1. Preliminaries: the Lie algebra $\text{Lie } \mathfrak{G}$ associated with a 3-Lie algebra \mathfrak{G} .

Let \mathfrak{G} be a 3-Lie algebra. Its three-bracket satisfies the FI,

$$[X_1, X_2, [Y_1, Y_2, Z]] = [[X_1, X_2, Y_1], Y_2, Z] + [Y_1, [X_1, X_2, Y_2], Z] + [Y_1, Y_2, [X_1, X_2, Z]] . \quad (201)$$

Moving the last term to the *l.h.s.* and writing it in terms of the adjoint map, $ad_{X_1, X_2} : Z \mapsto [X_1, X_2, Z]$, the above equation reads

$$ad_{X_1, X_2}(ad_{Y_1, Y_2} Z) - ad_{Y_1, Y_2}(ad_{X_1, X_2} Z) = ad_{([X_1, X_2, Y_1], Y_2) + (Y_1, [X_1, X_2, Y_2])} Z \quad (202)$$

or, equivalently (see Sec. 7.8),

$$ad_{\mathcal{X}}(ad_{\mathcal{Y}} Z) - ad_{\mathcal{Y}}(ad_{\mathcal{X}} Z) = ad_{\mathcal{X} \cdot \mathcal{Y}} Z \quad \forall \mathcal{X}, \mathcal{Y} \in \wedge^2 \mathfrak{G}, Z \in \mathfrak{G} \quad (203)$$

which, as already discussed (see eq. (196)) is $\mathcal{X} \leftrightarrow \mathcal{Y}$ skewsymmetric¹⁸ albeit $\mathcal{X} \cdot \mathcal{Y} \neq -\mathcal{Y} \cdot \mathcal{X}$ in the *r.h.s* for $n > 2$.

Clearly, the commutator of two inner derivations is another one. This may be more explicitly seen by rewriting the above expressions as

$$[[X_1, X_2, \quad], [Y_1, Y_2, \quad]] = [[X_1, X_2, Y_1], Y_2, \quad] + [Y_1, [X_1, X_2, Y_2], \quad] . \quad (204)$$

Further, *the inner derivations of a 3-algebra \mathfrak{G} determine an ordinary Lie algebra* since the Lie bracket of two derivations $[[X_1, X_2, \quad], [Y_1, Y_2, \quad]]$ satisfies the JI

$$\sum_{XYZ \text{ cycl.}} [[X_1, X_2, \quad], [[Y_1, Y_2, \quad], [Z_1, Z_2, \quad]]] = 0 \quad , \quad (205)$$

as it is evident from Prop. 43 above and will be seen for general n below. Since $ad : \wedge^2 \mathfrak{G} \mapsto \text{End } \mathfrak{G}$ may have a non-trivial kernel, the map $\mathcal{X}_{a_1 a_2} \in \wedge^2 \mathfrak{G} \rightarrow ad_{\mathcal{X}_{a_1 a_2}}$ is not injective.

The Lie algebra associated with a 3-Lie algebra will become essential to define the symmetries of the BLG model [182, 27, 28, 31] in Sec. 14.

8.1.1. Coordinate expressions for $n=3$.

Let $n = 3$. The coordinates of the $\dim \mathfrak{G} \times \dim \mathfrak{G}$ matrices $[X_{a_1}, X_{a_2}, \quad] \equiv \mathcal{X}_{a_1 a_2} \cdot \equiv (\mathcal{X}_{a_1 a_2}) \equiv ad_{a_1 a_2} \in \text{End } \mathfrak{G}$ are given by

$$ad_{a_1 a_2}{}^l{}_k \equiv (\mathcal{X}_{a_1 a_2}){}^l{}_k = f_{a_1 a_2 k}{}^l \quad , \quad \mathcal{X}_{a_1 a_2} \cdot X_k = [X_{a_1}, X_{a_2}, X_k] = f_{a_1 a_2 k}{}^l X_l \quad . \quad (206)$$

Then, the coordinates expression for eq. (204) reads

$$(\mathcal{X}_{a_1 a_2}){}^s{}_l (\mathcal{X}_{b_1 b_2}){}^l{}_k - (\mathcal{X}_{b_1 b_2}){}^s{}_l (\mathcal{X}_{a_1 a_2}){}^l{}_k \equiv f_{a_1 a_2 l}{}^s f_{b_1 b_2 k}{}^l - f_{b_1 b_2 l}{}^s f_{a_1 a_2 k}{}^l = f_{a_1 a_2 b_1}{}^l f_{l b_2 k}{}^s + f_{a_1 a_2 b_2}{}^l f_{b_1 l k}{}^s \quad , \quad (207)$$

and the last equality (with $k = b_3$) reproduces the FI (151) as it should.

The above equation may be written in the form

$$[(\mathcal{X}_{a_1 a_2}), (\mathcal{X}_{b_1 b_2})]{}^s{}_k = -f_{a_1 a_2 [b_1}{}^l f_{b_2] l k}{}^s$$

which means that we may express the above Lie commutators as [28]

$$[(\mathcal{X}_{a_1 a_2}), (\mathcal{X}_{b_1 b_2})]{}^s{}_k = \frac{1}{2} C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} (\mathcal{X}_{c_1 c_2}){}^s{}_k \quad (208)$$

¹⁸For $n=2$, where $[X, Y] = XY - YX$, this equation is the familiar $[[X, \quad], [Y, \quad]] = [[X, Y], \quad]$, eq. (28), which is manifestly $X \leftrightarrow Y$ skewsymmetric in both sides.

taking, for instance,

$$C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} = f_{a_1 a_2 [b_1}{}^{[c_1} \delta_{b_2]}^{c_2]} \quad , \quad (209)$$

However, this does not mean that the above C 's are the structure constants of Lie \mathfrak{G} on two counts. First, although the *r.h.s.* of expressions eq. (207)-(208) are $(a_1 a_2) \leftrightarrow (b_1 b_2)$ skewsymmetric as mandated by their *l.h.s.*, this does not necessarily imply that the constants in eq. (209) retain this property since the sum over $(c_1 c_2)$ has been removed. One may, of course, write antisymmetric C 's in eq. (208) by taking

$$C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} = \frac{1}{2} \left(f_{a_1 a_2 [b_1}{}^{[c_1} \delta_{b_2]}^{c_2]} - (a \leftrightarrow b) \right) \quad (210)$$

but, secondly, this is not sufficient for them to be the structure constants of Lie \mathfrak{G} since, in general, the (c_1, c_2) -labeled matrices $(\mathcal{X}_{c_1 c_2})$ are not a basis¹⁹ of Lie \mathfrak{G} .

When the 3-Lie algebra is *simple*, however, the constants (209) are already skewsymmetric in the lower indices,

$$f_{a_1 a_2 [b_1}{}^{[c_1} \delta_{b_2]}^{c_2]} = -f_{b_1 b_2 [a_1}{}^{[c_1} \delta_{a_2]}^{c_2]}$$

(as shown for arbitrary n in Sec. 8.2.1 below) and, further, define the structure constants of Lie \mathfrak{G} (see Th. (66)).

Example 64. (*The Lie algebra associated with the euclidean FA A_4*)

The metric FA A_4 ($f_{a_1 a_2 a_3}{}^l = -\epsilon_{a_1 a_2 a_3}{}^l$) is simple, and the C 's in (209) define the structure constants of Lie A_4 . To identify this algebra easily we may use eq. (208) to derive the commutation relations for the dual $\tilde{M}^{a_1 a_2} = \frac{1}{2} \epsilon^{a_1 a_2 b_1 b_2} (\mathcal{X}_{b_1 b_2})$ generators acting on the A_4 vector space. Using eq. (89), the commutation relations become

$$[\tilde{M}^{a_1 a_2}, \tilde{M}^{b_1 b_2}] = -\delta^{a_1 b_2} \tilde{M}^{a_2 b_1} - \delta^{a_2 b_1} \tilde{M}^{a_1 b_2} + \delta^{a_1 b_1} \tilde{M}^{a_2 b_2} + \delta^{a_2 b_2} \tilde{M}^{a_1 b_1} \quad , \quad (211)$$

which are immediately recognized as those of the (semisimple) $so(4) = so(3) \oplus so(3)$ algebra.

This was already evident from the second equation in eq. (206) for A_4 which, taking the dual and with all indices down, gives the familiar action of $so(4)$ on a vector $\mathbf{e}_k \in \mathbb{R}^4$,

$$\tilde{M}_{a_1 a_2} \cdot \mathbf{e}_k = -(\delta_{a_1 k} \mathbf{e}_{a_2} - \delta_{k a_2} \mathbf{e}_{a_1}) \quad . \quad (212)$$

Of course, this does not mean that the 3-bracket $[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{a_3}]$ for A_4 may be given by the *r.h.s.* of the above equation, which is antisymmetric in its first two (a_1, a_2) indices only. Three-brackets that are not necessarily antisymmetric may define 3-Leibniz algebras (Sec. 9). We shall see on Sec. 10.1 that the *r.h.s.* of eq. (212) does indeed define a particular example of 3-Leibniz algebra and, more specifically, the *metric Lie-triple system* of Ex. 70 .

8.1.2. Lie \mathfrak{G} -invariant tensors associated with a simple $n = 3$ metric FA.

It was seen in eq. (171) that the structure constants of a metric FA determine an invariant, fully antisymmetric tensor on \mathfrak{G} of rank $n + 1$. For $n = 3$, they also provide a *symmetric* invariant tensor for Lie \mathfrak{G} with coordinates

$$k_{(a_1 a_2) (b_1 b_2)}^{(2)} = k^{(2)}(\mathcal{X}_{a_1 a_2}, \mathcal{X}_{b_1 b_2}) \equiv k^{(2)}((\mathbf{e}_{a_1}, \mathbf{e}_{a_2}), (\mathbf{e}_{b_1}, \mathbf{e}_{b_2})) := \langle [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2} \rangle = f_{a_1 a_2 b_1 b_2} \quad , \quad (213)$$

¹⁹Of course, the $(a_1 a_2) \leftrightarrow (b_1 b_2)$ antisymmetry of the C 's and the JI hold on the matrices $(\mathcal{X}_{e_1 e_2})$,

$$\begin{aligned} & (C_{a_1 a_2 b_1 b_2}{}^{e_1 e_2} + C_{b_1 b_2 a_1 a_2}{}^{e_1 e_2}) (\mathcal{X}_{e_1 e_2})^s{}_l = 0 \quad , \\ & \sum_{\text{cycl.}(a_1 a_2), (b_1 b_2), (c_1 c_2)} (C_{a_1 a_2 b_1 b_2}{}^{d_1 d_2} C_{c_1 c_2 d_1 d_2}{}^{e_1 e_2}) (\mathcal{X}_{e_1 e_2})^s{}_l = 0 \quad , \end{aligned}$$

since this is what follows from (208) and the JI in End \mathfrak{G} , $\sum_{\text{cycl.}} [[(\mathcal{X}_{a_1 a_2}), (\mathcal{X}_{b_1 b_2})], (\mathcal{X}_{c_1 c_2})] = 0$. However, the $(\mathcal{X}_{e_1 e_2})$ cannot be removed in the above equations.

which is obviously symmetric under the $(a_1 a_2) \leftrightarrow (b_1 b_2)$ exchange. Similarly, the familiar *Killing form* k of Lie \mathfrak{G} is given by eq. (197)

$$k_{a_1 a_2 b_1 b_2}^{(1)} = k^{(1)}(\mathcal{X}_{a_1 a_2}, \mathcal{Y}_{b_1 b_2}) := \text{Tr}(ad_{\mathcal{X}_{a_1 a_2}} ad_{\mathcal{Y}_{b_1 b_2}}) = f_{a_1 a_2 s}{}^l f_{b_1 b_2 l}{}^s \quad , \quad (214)$$

and may be seen to be proportional to

$$\kappa_{(a_1 a_2)(b_1 b_2)}^{(1)} = C_{a_1 a_2 c_1 c_2}{}^{d_1 d_2} C_{b_1 b_2 d_1 d_2}{}^{c_1 c_2} \quad .$$

It is not difficult to check the Lie \mathfrak{G} -invariance of these tensors in the formalism of fundamental objects. This reads

$$\mathcal{Z} \cdot k(\mathcal{X}, \mathcal{Y}) = k(\mathcal{Z} \cdot \mathcal{X}, \mathcal{Y}) + k(\mathcal{X}, \mathcal{Z} \cdot \mathcal{Y}) = 0 \quad . \quad (215)$$

Indeed, with $\mathcal{Z} = \mathcal{Z}_{c_1 c_2} \equiv (\mathbf{e}_{c_1}, \mathbf{e}_{c_2})$, $\mathcal{X} = \mathcal{X}_{a_1 a_2}$, $\mathcal{Y} = \mathcal{Y}_{b_1 b_2}$, the *r.h.s.* of the above expression for $k^{(2)}$ gives, using (190),

$$k^{(2)} \left((f_{c_1 c_2 a_1}{}^d \mathbf{e}_d, \mathbf{e}_{a_2}) + (\mathbf{e}_{a_1}, f_{c_1 c_2 a_2}{}^d \mathbf{e}_d), (\mathbf{e}_{b_1}, \mathbf{e}_{b_2}) \right) \\ + k^{(2)} \left((\mathbf{e}_{a_1}, \mathbf{e}_{a_2}), (f_{c_1 c_2 b_1}{}^d \mathbf{e}_d, \mathbf{e}_{b_2}) + (\mathbf{e}_{b_1}, f_{c_1 c_2 b_2}{}^d \mathbf{e}_d) \right) = 0 \quad ,$$

which indeed is zero since it yields eq. (171) for $n = 3$,

$$f_{c_1 c_2 a_1}{}^d f_{d a_2 b_1 b_2} + f_{c_1 c_2 a_2}{}^d f_{a_1 d b_1 b_2} + f_{c_1 c_2 b_1}{}^d f_{a_1 a_2 d b_2} + f_{c_1 c_2 b_2}{}^d f_{a_1 a_2 b_1 d} = 0 \quad . \quad (216)$$

Similarly, $k^{(1)}$ also satisfies eq. (215) since, using the *ad* representation property (eq. 203 or Th. 65 below)

$$k^{(1)}(\mathcal{Z} \cdot \mathcal{X}, \mathcal{Y}) + k^{(1)}(\mathcal{X}, \mathcal{Z} \cdot \mathcal{Y}) = \text{Tr}(ad_{\mathcal{Z} \cdot \mathcal{X}} ad_{\mathcal{Y}}) + \text{Tr}(ad_{\mathcal{X}} ad_{\mathcal{Z} \cdot \mathcal{Y}}) \\ = \text{Tr}((ad_{\mathcal{Z}} ad_{\mathcal{X}} - ad_{\mathcal{X}} ad_{\mathcal{Z}}) ad_{\mathcal{Y}}) + \text{Tr}(ad_{\mathcal{X}} (ad_{\mathcal{Z}} ad_{\mathcal{Y}} - ad_{\mathcal{Y}} ad_{\mathcal{Z}})) = 0 \quad . \quad (217)$$

For the euclidean A_4 FA, $f_{a_1 a_2 a_3 a_4} = \epsilon_{a_1 a_2 a_3 a_4}$ and the metric $k^{(1)}$ in (214) becomes the *so*(4) Cartan-Killing metric of coordinates

$$k_{a_1 a_2 b_1 b_2}^{(1)} = -(\delta_{a_1 b_1} \delta_{a_2 b_2} - \delta_{b_1 a_2} \delta_{a_1 b_2}) \quad , \quad (218)$$

which is negative definite. The metric $k^{(2)}$ has signature (3,3), and will play an important rôle in the Chern-Simons term of the BLG model in Sec. 14.4.

8.2. The Lie algebra associated to an n -Lie algebra.

The case of the three-algebra in Sec. 8.1 is readily extended to an n -Lie algebra using the properties of the fundamental objects and Lemma 61. The result may be re-stated as the following

Theorem 65. (*Lie algebra associated with a n -Lie algebra*)

The inner derivations $ad_{\mathcal{X}}$ define [187, 21] an ordinary Lie algebra for the bracket

$$ad_{\mathcal{X}} ad_{\mathcal{Y}} - ad_{\mathcal{Y}} ad_{\mathcal{X}} = [ad_{\mathcal{X}}, ad_{\mathcal{Y}}] = ad_{\mathcal{X} \cdot \mathcal{Y}} \quad \text{or} \quad [[\mathcal{X}, -], [\mathcal{Y}, -]] = [\mathcal{X} \cdot \mathcal{Y}, -] \quad . \quad (219)$$

The subalgebra of $\text{End } \mathfrak{G}$ defined by the Lie bracket (219) is the Lie algebra $\text{Lie } \mathfrak{G} = \text{InDer } \mathfrak{G}$ associated with the FA \mathfrak{G} .

Proof. The JI,

$$[ad_{\mathcal{X}}, [ad_{\mathcal{Y}}, ad_{\mathcal{Z}}]] + [ad_{\mathcal{Y}}, [ad_{\mathcal{X}}, ad_{\mathcal{Z}}]] + [ad_{\mathcal{X}}, [ad_{\mathcal{Y}}, ad_{\mathcal{Z}}]] = 0 \quad , \quad (220)$$

is of course satisfied for any elements in $\text{End } \mathfrak{G}$; what we have to check is the consistency with the Lie bracket defined above. This is so because the *l.h.s.* of the above equation is

$$= [ad_{\mathcal{X}}, ad_{\mathcal{Y} \cdot \mathcal{Z}}] + [ad_{\mathcal{Y}}, ad_{\mathcal{X} \cdot \mathcal{Z}}] + [ad_{\mathcal{X}}, ad_{\mathcal{X} \cdot \mathcal{Y}}] \\ = ad_{\mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z})} + ad_{\mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z})} + ad_{\mathcal{X} \cdot (\mathcal{X} \cdot \mathcal{Y})} \quad , \quad (221)$$

which is zero by eq. (191). \square

When \mathfrak{G} is simple ad is injective. If \mathfrak{G} is only semisimple, all fundamental objects with two entries in two different simple components of \mathfrak{G} determine the zero derivation and ad is no longer injective (see Rem. 63).

8.2.1. The Lie algebras associated to the simple FAs.

In general, chosen a basis of \mathfrak{G} the commutation relations for different elements $ad_{\mathcal{X}}$ may be written as

$$[ad_{\mathcal{X}_{a_1 \dots a_{n-1}}}, ad_{\mathcal{X}_{b_1 \dots b_{n-1}}}] = \frac{1}{(n-1)!} C_{a_1 \dots a_{n-1} b_1 \dots b_{n-1}}^{c_1 \dots c_{n-1}} ad_{\mathcal{X}_{c_1 \dots c_{n-1}}} \quad (222)$$

with *e.g.* antisymmetric C 's given by (*cf.* eq. (210))

$$C_{a_1 \dots a_{n-1} b_1 \dots b_{n-1}}^{c_1 \dots c_{n-1}} = \frac{1}{(n-2)! 2} \left(f_{a_1 \dots a_{n-1} [b_1}^{[c_1} \delta_{b_2}^{c_2} \dots \delta_{b_{n-1}}^{c_{n-1}]} - (a \leftrightarrow b) \right) \quad (223)$$

When \mathfrak{G} is simple (Th. 53), the first half of the C 's above is already antisymmetric and provides the structure constants of Lie \mathfrak{G} . To check their antisymmetry explicitly, we write

$$\begin{aligned} C_{a_1 \dots a_{n-1} b_1 \dots b_{n-1}}^{c_1 \dots c_{n-1}} &\propto \epsilon_{a_1 \dots a_{n-1} [b_1}^d \delta_d^{[c_1} \delta_{b_2}^{c_2} \dots \delta_{b_{n-1}}^{c_{n-1}]} \\ &\propto \epsilon_{a_1 \dots a_{n-1} [b_1}^d \epsilon_{b_2 \dots b_{n-1}] d e_1 e_2} \epsilon^{c_1 \dots c_{n-1} e_1 e_2} \end{aligned} \quad (224)$$

(note that in the first line above the lower index d is unaffected by the antisymmetrization imposed by the square bracket, which acts on the $(n-1)$ indices c and b only). We now use a Schouten-type identity obtained by antisymmetrizing the $(n+2)$ indices $(b_1 \dots, b_{n-1}, d, e_1, e_2)$ in the last term above to obtain

$$\epsilon_{a_1 \dots a_{n-1} [b_1}^d \epsilon_{b_2 \dots b_{n-1}] d e_1 e_2} \epsilon^{c_1 \dots c_{n-1} e_1 e_2} - (n-2)! 2 \epsilon_{a_1 \dots a_{n-1} e_1}^d \epsilon_{b_2 \dots b_{n-1} d b_1 e_2} \epsilon^{c_1 \dots c_{n-1} e_1 e_2} = 0 \quad (225)$$

This means that the structure constants are proportional to the last term. Therefore,

$$C_{a_1 \dots a_{n-1} b_1 \dots b_{n-1}}^{c_1 \dots c_{n-1}} \propto \epsilon^{c_1 \dots c_{n-1} e_1 e_2} \epsilon_{e_1 a_1 \dots a_{n-1}}^d \epsilon_{e_2 b_1 \dots b_{n-1} d} \quad (226)$$

which is clearly antisymmetric under the interchange of the a and b indices.

For $n > 3$ one proceeds as for the A_4 case in Ex. 64. For instance, for the euclidean A_5 , $f_{a_1 a_2 a_3 a_4}^{a_5} = \epsilon_{a_1 a_2 a_3 a_4}^{a_5}$ and $(M_{a_1 a_2 a_3})^s_k = f_{a_1 a_2 a_3 k}^s$. Then one finds that the dual generators $\tilde{M}^{a_1 a_2} = \frac{1}{3!} \epsilon^{a_1 a_2 b_1 b_2 b_3} M_{b_1 b_2 b_3}$ ($M_{b_1 b_2 b_3} = \frac{1}{2} \epsilon_{b_1 b_2 b_3 a_1 a_2} \tilde{M}^{a_1 a_2}$) reproduce exactly the commutation relations for the $\binom{5}{2} = 10$ skewsymmetric matrices that are the generators of the orthogonal algebra, eq. (211). Thus, the Lie algebra associated with A_5 is $so(5)$.

This was to be expected: since the A_{n+1} algebras are simple, the algebra Lie A_{n+1} of inner derivations coincides with the Lie algebra of the group of automorphisms (Th. 46), and therefore Lie A_{n+1} is the algebra of the $SO(n+1)$ group. Indeed, in the general case the matrices $f_{a_1 \dots a_n}^{a_{n+1}}$, $a_i = 1 \dots, n+1$, determine the $\binom{n+1}{n-1} = \binom{n+1}{2}$ generators of $so(n+1)$ algebra; they are obviously rotations, since the $\binom{n+1}{2}$ $(n+1)$ -dimensional matrices are antisymmetric, $f_{a_1 \dots a_{n-1} bc} = -f_{a_1 \dots a_{n-1} cb}$. The more familiar $so(n+1)$ commutators follow immediately from eqs. (222) and (226) by moving to the dual generators $\tilde{M}^{a_1 a_2}$.

As a result, the following theorem follows:

Theorem 66. (*Lie algebras associated with the simple euclidean A_{n+1} algebras*)

The Lie algebra associated with the euclidean A_{n+1} is the algebra $so(n+1)$ of its inner derivations.

Since the A_{n+1} algebras are simple, all their derivations are inner (Th. 49) and determine semisimple Lie algebras (actually, simple for $n > 3$). Thus, it is not surprising that there is an analogue of the Whitehead's lemma for FAs (Thms. 74 and 80 below) which, accordingly, holds true [180] for all n -Lie algebras, $n \geq 2$.

8.3. Representations of Filippov algebras in the sense of Kasymov.

To motivate the definition below, let us note first that the expression

$$[ad_{X_1 \dots X_{n-1}}, ad_{Y_1 \dots Y_{n-1}}] = \sum_{i=1}^{n-1} ad_{Y_1 \dots ad_{X_1, \dots, X_{n-1}} Y_i \dots Y_{n-1}} \quad (227)$$

reproduces eq. (192) acting on $Z \in \mathfrak{G}$, and that

$$ad_{X_1 \dots X_{n-2} [Y_1, \dots, Y_n]} = \sum_{i=1}^n (-1)^{n-i} (ad_{Y_1 \dots \widehat{Y}_i \dots Y_n}) (ad_{X_1 \dots X_{n-2} Y_i}) \quad (228)$$

reproduces eq. (139) acting on X_{n-1} . The above properties -the FI- satisfied by the $ad_{\mathcal{X}}$ may be extended by replacing the representation space \mathfrak{G} by a vector space V and ad by a general ρ subject to analogous conditions to guarantee the preservation of the FA structure as dictated by the FI. This leads to

Definition 67. (*Representations of Filippov algebras* [23, 170]; see also [25])

Let V be a vector space. A representation ρ of a FA on V is a multilinear map $\rho : \wedge^{n-1} \mathfrak{G} \rightarrow \text{End} V$, $\rho : \mathcal{X} \rightarrow \rho(X_1, \dots, X_{n-1})$ such that

$$\begin{aligned} [\rho(\mathcal{X}), \rho(\mathcal{Y})] &= \rho(\mathcal{X} \cdot \mathcal{Y}) \quad , \\ \rho(X_1, \dots, X_{n-2}, [Y_1, \dots, Y_n]) &= \sum_{i=1}^n (-1)^{n-i} \rho(Y_1, \dots, \widehat{Y}_i, \dots, Y_n) \rho(X_1, \dots, X_{n-2}, Y_i) \quad . \end{aligned} \quad (229)$$

The representation space V is said to be a (left) \mathfrak{G} -module. Note that the arguments of ρ are fundamental objects and not elements in \mathfrak{G} so that, strictly speaking, ρ is not representing \mathfrak{G} itself.

The subspace $\ker \rho = \{Z \in \mathfrak{G} | \rho(Z, \mathfrak{G}, \dots, \mathfrak{G}) = 0\} \subset \mathfrak{G}$ is the kernel of the representation ρ ; it follows that it is an ideal of \mathfrak{G} . When $\ker \rho = 0$ (resp. \mathfrak{G}) the representation is *faithful* (resp. *trivial*). In the intermediate cases, ρ is a faithful representation of the quotient FA $\mathfrak{G}/\ker \rho$ [23]. When ρ is the adjoint map and the kernel of ad is defined *as above* for $\rho = ad$, it coincides with the centre $Z(\mathfrak{G})$ of the FA since, if $ad(Z, \mathfrak{G}, \dots, \mathfrak{G})$ is the null element in $\text{End } \mathfrak{G}$, it follows that $[Z, \mathfrak{G}, \dots, \mathfrak{G}] = 0$, which determines the centre $Z(\mathfrak{G})$ (Def. 45).

The above defining properties are also readily obtained (see [25]) by imposing an n -Lie algebra structure on the vector space $\mathfrak{G} \oplus V$ with the condition that V be an abelian ideal, $[\mathfrak{G}, \dots, \mathfrak{G}, V] \subset V$, $[\mathfrak{G}, \dots, \mathfrak{G}, V, V] = 0$. Indeed, the FI applied to $[X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, v]]$ with the notation $[X_1, \dots, X_{n-1}, v] = \rho(X_1, \dots, X_{n-1}) \cdot v$ gives

$$[X_1, \dots, X_{n-1}, [Y_1, \dots, Y_{n-1}, v]] - [Y_1, \dots, Y_{n-1}, [X_1, \dots, X_{n-1}, v]] = \rho(\mathcal{X} \cdot \mathcal{Y}) \cdot v \quad ,$$

which gives the first equality in eq. (229), and the second one follows from eq. (139) rewritten in the form

$$[X_1, \dots, X_{n-2}, [Y_1, \dots, Y_n], v] = \sum_{i=1}^n (-1)^{n-i} [Y_1, \dots, \widehat{Y}_i, \dots, Y_n, [X_1, \dots, X_{n-2}, Y_i, v]]$$

and factoring out v . In fact, all manipulations are as before where $[X_1, \dots, X_{n-1}, v] = \mathcal{X} \cdot v$ indicates $\rho(X_1, \dots, X_{n-1}) \cdot v = \rho(\mathcal{X}) \cdot v$ or $\mathcal{X} \cdot v$ for short. For instance, on $v \in V$ the first expression in eq. (229) gives

$$\mathcal{X} \cdot (\mathcal{Y} \cdot v) - \mathcal{Y} \cdot (\mathcal{X} \cdot v) = (\mathcal{X} \cdot \mathcal{Y}) \cdot v \quad \forall v \in V \quad , \quad (230)$$

where the dot indicates the ρ -action. For $V = \mathfrak{G}$, $v = Z$ this is eq. (192) and corresponds to $\rho = ad$; its matrix representation is given in eq (189). Note that, although V is called a \mathfrak{G} -module, ρ above is *not* a map of \mathfrak{G} in $\text{End } V$; V might have been called as well a ' $\rho(\wedge^{n-1}\mathfrak{G})$ -module' under the action ρ of the fundamental objects \mathcal{X} on V , consistent with the FA structure of \mathfrak{G} through the expressions above. It is only for $n = 2$ that the map $\rho : \mathfrak{g} \rightarrow \text{End } V$ is a homomorphism of (Lie) algebras.

The *coadjoint representation* is obtained by taking $V = \mathfrak{G}^*$, $coad : \wedge^{n-1}\mathfrak{G} \rightarrow \text{End } \mathfrak{G}^*$. ad and $coad$ are related in the usual manner,

$$coad_{\mathcal{X}} : \nu \in \mathfrak{G}^* \mapsto coad_{\mathcal{X}} \cdot \nu \in \mathfrak{G}^* \quad , \quad (coad_{\mathcal{X}} \cdot \nu)(Y) = -\nu(ad_{\mathcal{X}} \cdot Y) \quad \forall \nu \in \mathfrak{G}^* \quad , \quad \mathcal{X} \in \wedge^{n-1}\mathfrak{G} \quad , \quad Y \in \mathfrak{G} \quad .$$

Summarizing, the discussion in this section has exhibited two important aspects of Filippov algebras:

1) Due to Th. 65, any n -Lie algebra \mathfrak{G} has an *associated ordinary Lie algebra* $\text{Lie } \mathfrak{G} = \text{InDer } \mathfrak{G}$ defined through the fundamental objects \mathcal{X} and the commutator of eq. (219) of adjoint endomorphisms $ad_{\mathcal{X}}$ of \mathfrak{G} . This follows from the obvious fact that the composition of endomorphisms in $\text{End } V$ has a Lie rather than a FA structure. As a 'representation' (in the sense of the previous definition) of the FA on itself, $\text{Lie } \mathfrak{G}$ is consistent with the FI for \mathfrak{G} (see eqs. (219), (192)). When the carrier vector space is an arbitrary one V , the representation ρ on V induced by the fundamental objects is that of Def. 67.

2) The fundamental rôle (hence their name) of the objects $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$, characterized by $(n-1)$ elements of the n -Lie algebra \mathfrak{G} . Their relevance will be apparent again when discussing the FA cohomology in Sec 11. For $n = 2$ the *fundamental objects* $\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ and the *elements* $X \in \mathfrak{G}$ of a FA are one and the same object, but for $n > 2$ they emerge as separate entities.

9. n -LEIBNIZ ALGEBRAS

There are occasions in which the skewsymmetry of the FA bracket (eq. (134)) is not demanded, but the FI still holds. This is the case of the n -Leibniz algebras \mathfrak{L} [21, 22] (see also [188]), which generalize the Leibniz algebras \mathcal{L} of Sec. 4 to the n -ary case.

Definition 68. (*n -Leibniz algebras and their fundamental objects*)

An n -Leibniz algebra \mathfrak{L} is a vector space endowed with an n -linear application $[\quad , \quad \overset{n}{\cdot} \quad , \quad] : \mathfrak{L} \times \overset{n}{\cdot} \times \mathfrak{L} \rightarrow \mathfrak{L}$, the Leibniz n -bracket, such that the derivation property (136) is satisfied. For $n=2$, \mathfrak{L} reduces to the ordinary Loday/Leibniz algebra \mathcal{L} [18, 19] of Sec. 4.

An n -Leibniz algebra that satisfies

$$[X_1, \dots, X_i, \dots, X_j, \dots, X_n] = 0 \quad \forall X_i = X_j \quad 1 \leq i, j \leq n \quad ,$$

has an anticommutative n -bracket and is also a FA algebra.

It also proves convenient to introduce *fundamental objects* \mathcal{X} for n -Leibniz algebras. This is simply done by relaxing the skewsymmetry condition ($\mathcal{X} \in \wedge^{n-1}\mathfrak{G}$ for a FA) since the Leibniz n -bracket is not antisymmetric. Accordingly, the *fundamental objects of an n -Leibniz algebra* \mathfrak{L}

are now defined as elements $(X_{a_1}, \dots, X_{a_{n-1}}) \equiv \mathcal{X}_{a_1 \dots a_{n-1}}$, where no anticommutativity is now implied. Thus $\mathcal{X} \in \otimes^{n-1} \mathfrak{L}$ for an n -Leibniz algebra, but again $\mathcal{X} \cdot \equiv [X_1, X_2, \dots, X_{n-1}, \] \in \text{End } \mathfrak{L}$ defines an inner derivation of the Leibniz n -bracket as a result of the FI and $ad_{\mathcal{X}} Z \equiv \mathcal{X} \cdot Z := [X_1, \dots, X_{n-1}, Z] \in \mathfrak{L}$. We shall use the same notation $\mathcal{X} = (X_1, \dots, X_{n-1})$ for the fundamental objects of both Filippov and Leibniz n -algebras when there is no risk of confusion, without making explicit that the fundamental objects of a FA \mathfrak{G} are skewsymmetric in their arguments and not necessarily so for a LA \mathfrak{L} : $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$, $\otimes^{n-1} \mathfrak{L}$, respectively, for a \mathcal{X} of \mathfrak{G} , \mathfrak{L} .

The representations of Leibniz algebras were reviewed in Sec. 4.1; see also [19], where the construction of the universal enveloping algebra of a \mathcal{L} as well as the proof of a Poincaré-Birkhoff-Witt theorem was given. Those of n -Leibniz algebras \mathfrak{L} were considered in [189], where a PBW-type theorem for the universal enveloping algebras of finite-dimensional n -Leibniz algebras was established.

9.1. Leibniz algebra \mathcal{L} associated with an n -Leibniz algebra \mathfrak{L} .

An n -Leibniz algebra \mathfrak{L} is the non-antisymmetric analogue of an n -Lie algebra \mathfrak{G} : the Leibniz n -bracket in \mathfrak{L} satisfies the FI but it is not necessarily skewsymmetric (when it is, the n -Leibniz algebra becomes an n -Lie algebra). The composition of two fundamental objects of \mathfrak{L} is defined again by eq. (190) in which the bracket that appears there is now the n -bracket in \mathfrak{L} . Although in Sec. 7.8 we had the FAs in mind, to obtain properties (191) and (230) from the composition of the fundamental objects in eq. (190), only a vector space closed under an n -bracket $[X_1, \dots, X_n]$ satisfying the FI was required; neither the antisymmetry of the fundamental objects \mathcal{X} nor that of the n -bracket were assumed. Consequently, as far as the proof of eqs. (191) and (230) is concerned, nothing changes if the fundamental objects \mathcal{X} are the possibly non-skewsymmetric ones of an n -Leibniz algebra \mathfrak{L} .

Let \mathfrak{L} be an n -Leibniz algebra. Then, eq. (191) also holds for \mathfrak{L} , and using the notation $\mathcal{X} \cdot \mathcal{Y} \equiv [\mathcal{X}, \mathcal{Y}]$ it takes the form

$$[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]] = [[\mathcal{X}, \mathcal{Y}], \mathcal{Z}] + [\mathcal{Y}, [\mathcal{X}, \mathcal{Z}]] \quad , \quad \mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \otimes^{n-1} \mathfrak{L} \quad , \quad (231)$$

where $[\mathcal{X}, \mathcal{Y}] \neq -[\mathcal{Y}, \mathcal{X}]$ is a non-antisymmetric two-bracket. Eq. (231) constitutes a particular example of the derivation property $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ that defines an ordinary Leibniz algebra structure [18, 19] (Sec. 4) for a non-antisymmetric bilinear bracket $[\ , \]$ in which the two entries in $[\ , \]$ are fundamental objects $\mathcal{X} \in \otimes^{n-1} \mathfrak{L}$. Hence, given an n -Leibniz algebra \mathfrak{L} , the linear space of the fundamental objects endowed with the dot operation (190) which defines a non-antisymmetric two-bracket, $[\mathcal{X}, \mathcal{Y}] \equiv \mathcal{X} \cdot \mathcal{Y}$, becomes an $n = 2$ Leibniz algebra [21, 187]. This is the *ordinary Leibniz algebra \mathcal{L} associated with an n -Leibniz algebra \mathfrak{L}* .

When the fundamental objects are those of a FA, $\mathcal{X}, \mathcal{Y} \in \wedge^{n-1} \mathfrak{G}$, and the n -bracket involved in the definition of $\mathcal{X} \cdot \mathcal{Y}$ is therefore that of an n -Lie algebra, the resulting \mathcal{L} is the *Leibniz algebra associated with the Filippov algebra \mathfrak{G}* . For $n=2$, the Leibniz algebra associated to the FA is in fact an ordinary Lie algebra since the FA bracket is skewsymmetric and the FA itself is an ordinary Lie algebra.

Additionally, the inner endomorphisms $ad_{\mathcal{X}} : Z = [X_1, \dots, X_{n-1}, Z]$ of an n -Leibniz algebra \mathfrak{L} generate a *Lie algebra* under the commutator, since the validity of eq. (192) depends only on the FI. Thus, in spite of relaxing the full anticommutativity of the n -bracket, Lie \mathfrak{L} is obtained as usual. This is relevant to define the gauge transformations in BLG-type models (Sec. 14.1) that use 3-algebras algebras with brackets that are not fully antisymmetric (hence, 3-Leibniz

algebras) as in [30, 190].

10. LIE-TRIPLE SYSTEMS

Since triple systems \mathfrak{T} constitute a particular example of 3-Leibniz algebras, we recall their definition here. Lie (and Jordan) triple systems have been the subject of extensive study in mathematics (see *e.g.* [191–193, 148]; see also [194]) and in physics, as *e.g.* in connection with parastatistics or the Yang-Baxter equation [195–197]. Further triple (and supertriple) systems generalizations may be found in [198–200] and references therein. For the algebra of inner derivations of \mathfrak{T} see, in particular, [193, 148].

Definition 69. (*Lie-triple systems*)

A Lie-triple system is a vector space \mathfrak{T} plus a trilinear map $\text{map } \mathfrak{T} \times \mathfrak{T} \times \mathfrak{T} \rightarrow \mathfrak{T}$ satisfying the properties

- a) $[X, Y, X] = -[Y, X, Z]$
- b) $[X_1, X_2, [Y_1, Y_2, Y_3]] = [[X_1, X_2, Y_1], Y_2, Y_3] + [Y_1, [X_1, X_2, Y_2], Y_3] + [Y_1, Y_2, [X_1, X_2, Y_3]]$
- c) $[X, Y, Z] + [Y, Z, X] + [Z, X, Y] = 0 \quad \forall X_i, Y_i, Z \in \mathfrak{T} \quad .$

Thus, Lie triple systems are 3-Leibniz algebras with brackets skewsymmetric in their first two arguments which, in addition, satisfy the cyclic property (c). When the space \mathfrak{T} is Z_2 -graded, a suitable addition of signs in the above expression accounting for the Z_2 -grading leads to the definition of super-triple systems [200, 97, 197]. There are also *anti*-Lie triple systems, defined by putting a plus sign in condition a) above. An elementary example of Lie-triple system is that of a Lie algebra \mathfrak{g} ; defining the three-bracket [191] by $[X, Y, Z] := [[X, Y], Z]$, all properties above are fulfilled (the last one being simply the JI for \mathfrak{g}). In fact, any associative algebra \mathfrak{A} is a Lie-triple system for the bracket $[x, y, z] = [[xy]z]$ where $[xy] = xy - yx$, $x, y, z \in \mathfrak{A}$.

10.1. Lie-triple systems and 3-Leibniz algebras.

To see how certain 3-Leibniz algebras and Lie-triple systems are related, let us consider here briefly the approach of [183], which is based in a construction of metric 3-Leibniz algebras due to Faulkner [201]; we refer to [183] for details and to [200, 197]. To make things simpler, consider the euclidean A_4 . Clearly, $ad : \wedge^2 \mathfrak{G} \rightarrow so(4)$ is one-to-one, and the $so(4)$ ad algebra is metric with respect to the scalar product

$$(ad_{a_1 a_2}, ad_{b_1 b_2}) = \langle [e_{a_1}, e_{a_2}, e_{b_1}], e_{b_2} \rangle = (ad_{b_1 b_2}, ad_{a_1 a_2}) \quad , \quad a = 1, 2, 3, 4$$

i.e., with respect the metric $k^{(2)}$ in (213) for $\epsilon_{a_1 a_2 b_1 b_2}$. We already know that this scalar product is not degenerate, which also follows trivially from the above expression, since $(ad_{a_1 a_2}, ad_{b_1 b_2}) = 0 \quad \forall b_1 b_2$ implies that $[e_{a_1}, e_{a_2}, \quad]$ has to be the trivial endomorphism. Since $(ad_{a_1 a_2}, ad_{a_1 a_2}) = 0$ obviously, the algebra has a basis of null vectors, and thus the inner endomorphisms algebra $\text{InDer } A_4 = so(4)$ is even-dimensional (which of course we knew) and the scalar product (\quad, \quad) above has signature (3,3). Further, since $so(4)$ preserves the scalar product $\langle \quad, \quad \rangle$ in A_4 ,

$$\langle [e_{a_1}, e_{a_2}, e_{b_1}], e_{b_2} \rangle = - \langle e_{b_1}, [e_{a_1}, e_{a_2}, e_{b_2}] \rangle \quad .$$

In general, it follows [183] that a real metric 3-Lie algebra \mathfrak{G} gives rise to a bilinear map $\wedge^2 \mathfrak{G} \rightarrow \text{InDer } \mathfrak{G} \subset so(\dim \mathfrak{G})$ and that the Lie algebra of the ad derivations is metric. Thus, a real metric 3-Lie algebra determines a metric vector space (\mathfrak{G} itself) and an inner derivations algebra endowed with a non-degenerate, symmetric scalar product.

Reciprocally, given a real metric algebra $\mathfrak{g} \subset so(\dim V)$ where V is a metric vector space carrying a faithful representation L of \mathfrak{g} , it is possible to construct a metric 3-Leibniz algebra.

Given an orthonormal basis $\{\mathbf{e}_a\}$ of V , these 3-Leibniz structures are characterized by a three bracket defined by

$$[\mathbf{e}_{a_1}, \mathbf{e}_{a_1}, \mathbf{e}_{a_1}] := L_{a_1 a_2} \cdot \mathbf{e}_{a_3} . \quad (232)$$

which then satisfies the following properties:

a) *scalar product preservation*

$$\langle [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2} \rangle + \langle \mathbf{e}_{b_1}, [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_2}] \rangle = 0 , \quad (233)$$

b) *symmetry condition*

$$\langle [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2} \rangle = \langle [\mathbf{e}_{b_1}, \mathbf{e}_{b_2}, \mathbf{e}_{a_1}], \mathbf{e}_{a_2} \rangle \quad (234)$$

It follows from eqs. (233), (234) that the 3-bracket is antisymmetric in the first two entries,

$$[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{a_3}] = -[\mathbf{e}_{a_2}, \mathbf{e}_{a_1}, \mathbf{e}_{a_3}]$$

and, using the composition properties of $L_{a_1 a_2}$, that it satisfies the FI.

Real 3-Leibniz algebras satisfying eqs. (233), (234) and the FI were introduced in [31, 190] in the context of the BLG model (see Sec. 14.2.2) and called *generalized 3-Lie algebras*. Thus, these algebras are in one-to-one correspondence with the pairs (\mathfrak{g}, V) where \mathfrak{g} is a metric subalgebra of $so(\dim V)$ and V a faithful \mathfrak{g} -module [183] (we refer also to [183] for the case of the hermitian (complex) BL algebras in [30], and to [202, 203] in connection with Jordan-triple systems). The above symmetry properties imply that the structure constants of the generalized 3-Lie algebras satisfy the relations [31]

$$f_{a_1 a_2 b_1 b_2} = -f_{a_2 a_1 b_1 b_2} = f_{b_1 b_2 a_1 a_2} = -f_{a_1 a_2 b_2 b_1} .$$

Since the above 3-bracket is a map $[\ , \] : \wedge^2 V \otimes V \rightarrow V$, it has the general symmetry of $\mathbb{H} \otimes \mathbb{H} = \mathbb{H} \oplus \mathbb{H}$ (the second term is not irreducible since it contains a trace). It then follows [183] that the above ternary algebras have two special subcases associated with the above decomposition: the first part, for which the 3-bracket is totally antisymmetric, corresponds to an ordinary FA ($V = \mathfrak{G}$); the second one determines a bracket with a mixed symmetry. In this last case, it follows that the 3-bracket of the 3-Leibniz algebra also has the additional property c) in Def. 10 above and hence defines a metric three-Lie system ($V = \mathfrak{T}$). In general, however, the 3-Leibniz algebra associated with the pair (\mathfrak{g}, V) is neither a 3-Lie algebra nor a Lie-triple system.

Example 70. (*The Lie-triple system associated with $(so(4), \mathbb{R}^4)$*)

Consider [200, 183] $so(4)$ acting faithfully on the euclidean \mathbb{R}^4 with basis $\{\mathbf{e}_a\}$ as in eq. (212). By eq. (232) above, $M_{a_1 a_2} \cdot \mathbf{e}_{a_4}$ defines the 3-bracket

$$[\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{a_3}] = -(\delta_{a_1 a_3} \mathbf{e}_{a_2} - \delta_{a_3 a_2} \mathbf{e}_{a_1}) \quad (235)$$

This bracket corresponds now to the $so(4)$ scalar product given by

$$(M_{a_1 a_2}, M_{b_1 b_2}) = \langle [\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \mathbf{e}_{b_1}], \mathbf{e}_{b_2} \rangle = -(\delta_{a_1 b_1} \delta_{a_2 b_2} - \delta_{b_1 a_2} \delta_{a_1 b_2}) , \quad (236)$$

which is the $so(4)$ Cartan-Killing metric in eq. (218). The bracket (235) satisfies the cyclic property c) in Def. 69 and thus defines a Lie-triple system which, in fact, appeared very long ago (see [191]).

Lie-triple systems are important in the theory of symmetric spaces (see *e.g.* [204]), the reason being that $\mathfrak{g} \oplus V$ may be given a metric Lie algebra structure (see further [148] for triple systems as tangent spaces to totally geodesic spaces; recall that, in contrast, general Filippov and Leibniz algebras do not have a ‘linear approximation’ interpretation). The canonical procedure to construct Lie (and Lie super-) algebras from Lie- (and anti-Lie-) triple systems is detailed in [200], where the Lie algebra $so(N + 1)$ (from $\mathfrak{g} = so(N)$, $\dim V = N$) and the $osp(1, N)$

superalgebra (from $\mathfrak{g} = sp(N)$) are obtained. For the $so(4)$, $\dim A_4=4$ example above, the *embedding Lie algebra* is thus $so(5)$ and the Lie triple system corresponds to the symmetric space $SO(5)/SO(4) \sim S^4$; we refer to [200, 183] for further details and references.

11. COHOMOLOGY AND HOMOLOGY FOR FILIPPOV ALGEBRAS

We discuss now the cohomology and homology for FA's. We shall see that there is more than one cohomology complex relevant in applications. In analogy with Lie algebras, where the central extensions and the infinitesimal deformations are characterized, respectively, by the Lie algebra cohomology groups $H_0^2(\mathfrak{g}, \mathbb{R})$ and $H_{ad}^2(\mathfrak{g}, \mathfrak{g})$ for the trivial and ad representations, the cohomology groups relevant in the central extensions and infinitesimal deformations of FA will be given by $H_0^1(\mathfrak{G}, \mathbb{R})$ and $H_{ad}^1(\mathfrak{G}, \mathfrak{G})$, albeit defined on cohomology complexes that are not the 'natural' analogues of the Lie algebra ones, as will be shown below.

11.1. Introduction: 3-Lie algebras and central extensions.

To motivate the general cohomology problem and the definitions that will follow in the general case, let us consider first the simple problem of extending centrally a 3-Lie algebra \mathfrak{G} in close analogy with the Lie algebra case discussed in Sec. 3.2. The fundamental objects are determined by two elements of \mathfrak{G} , $\mathcal{X} = (X_1, X_2)$. The existence of such a central extension, denoted $\tilde{\mathfrak{G}}$, implies that the original three-bracket in \mathfrak{G} may be modified by the presence of a term in the additional central generator Ξ , so that the commutation relations of $\tilde{\mathfrak{G}}$ read

$$[\tilde{X}_a, \tilde{X}_b, \tilde{X}_c] = f_{abc}{}^d \tilde{X}_d + \alpha(X_a, X_b, X_c) \Xi \quad , \quad [\tilde{X}_a, \tilde{X}_b, \Xi] = 0 \quad \forall \tilde{X}_a \in \tilde{\mathfrak{G}} \quad , \quad (237)$$

where α is a (necessarily antisymmetric) trilinear map $\alpha : (X_a, X_b, X_c) \mapsto \alpha(X_a, X_b, X_c) \in \mathbb{R}$. The fact that (237) is a 3-Lie algebra means that $\alpha(X, Y, Z)$ must be such that the FI holds for the extended FA $\tilde{\mathfrak{G}}$. This clearly constrains the possible $\alpha(X, Y, Z)$ in (237) to those that are consistent with the structure of the extended 3-Lie algebra.

This may be formalized in terms of 3-Lie algebra cohomology in the following way. First, let us call *one-cochains* α to the \mathbb{R} -valued antisymmetric trilinear maps that appear in (237),

$$\alpha^1(Y_1, Y_2, Z) \equiv \alpha^1(\mathcal{X}, Z) \quad , \quad (238)$$

because they depend on *one* fundamental object (plus an element of \mathfrak{G}). Thus, we may think of α^1 as an element in $\wedge^2 \mathfrak{G}^* \wedge \mathfrak{G}^*$ (this is of course $\wedge^3 \mathfrak{G}^*$, but it proves convenient here to use this split notation). We may now introduce a coboundary operator δ that takes p -cochains to $(p+1)$ -cochains by increasing the number of fundamental objects in the argument of the cochain by one. In the present case, this means that the action on α gives a two-cochain, a five-linear element of $\wedge^2 \mathfrak{G}^* \otimes \wedge^2 \mathfrak{G}^* \wedge \mathfrak{G}^*$. Let us define specifically the action of δ on the one-cochain α^1 by

$$\begin{aligned} (\delta \alpha^1)(\mathcal{X}, \mathcal{Y}, Z) &:= -\alpha^1(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha^1(\mathcal{Y}, \mathcal{X} \cdot Z) + \alpha^1(\mathcal{X}, \mathcal{Y} \cdot Z) \\ &= -\alpha^1([X_1, X_2, Y_1], Y_2, Z) - \alpha^1(Y_1, [X_1, X_2, Y_2], Z) + \\ &\quad - \alpha^1(Y_1, Y_2, [X_1, X_2, Z]) + \alpha^1(X_1, X_2, [Y_1, Y_2, Z]) \quad . \end{aligned} \quad (239)$$

A one-cochain on the three-algebra \mathfrak{G} will be a one-cocycle if the above expression is zero, in which case

$$\alpha^1(X_1, X_2, [Y_1, Y_2, Z]) = \alpha^1([X_1, X_2, Y_1], Y_2, Z) + \alpha^1(Y_1, [X_1, X_2, Y_2], Z) + \alpha^1(Y_1, Y_2, [X_1, X_2, Z]) \quad . \quad (240)$$

The structure of this expression (or of eq. (239)) is easily justified: it follows by imposing the FI on (237). Thus, the one-cochain in (238) that defines a central extension must be a one-cocycle,

since this guarantees that the extended algebra (237) satisfies the FI. Hence, α defines a 3-Lie algebra central extension if $\delta\alpha = 0$.

As for ordinary Lie algebras, the above central extension may be trivial. This means that it is possible to make a redefinition of the basis of $\tilde{\mathfrak{G}}$ that removes the central element Ξ from the *r.h.s.* of (237) so that Ξ appears in $\tilde{\mathfrak{G}}$ as a fully independent addition, $\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathbb{R}$. Such a redefinition is possible when the one-cocycle α is trivial (it is a one-coboundary), which means that it is generated by a zero-cochain *i.e.*, by a linear map on \mathfrak{G} , $\beta : X \mapsto \beta(X) \in \mathbb{R}$. Then, $\alpha^1(X, Y, Z) = (\delta\beta)(X, Y, Z) = -\beta([X, Y, Z])$. The last equality in this expression follows from the general definition of coboundary operator for \mathfrak{G} algebras to be given in the next section, but at this stage it may be justified by the fact that, using (239), any one-cochain thus defined is a (trivial) one-cocycle by virtue of the FI (as it should since δ must be nilpotent) and, further, because a central extension is actually a trivial one when the one-cocycle is a one-coboundary as defined above. In fact, if $\alpha = \delta\beta$,

$$\alpha(\mathcal{X}, Z) = (\delta\beta)(\mathcal{X}, Z) = -\beta(\mathcal{X} \cdot Z) = -\beta([X, Y, Z]), \quad (241)$$

and it is sufficient to define the new basis generators \tilde{X}'_d of $\tilde{\mathfrak{G}}$ by $\tilde{X}'_d = \tilde{X}_d - \beta(X_d)\Xi$ to obtain

$$[\tilde{X}'_a, \tilde{X}'_b, \tilde{X}'_c] = f_{abc}{}^d \tilde{X}_d - \beta([X_a, X_b, X_c]) = f_{abc}{}^d (\tilde{X}_d - \beta(X_d)) = f_{abc}{}^d \tilde{X}'_d \quad , \quad [\tilde{X}'_a, \tilde{X}'_b, \Xi] = 0 \quad , \quad (242)$$

which exhibits explicitly the triviality of the extension, $\tilde{\mathfrak{G}} = \mathfrak{G} \oplus \mathbb{R}$.

Note that, in naming the order of the three-algebra cohomology cocycles, we are counting the number of fundamental objects that they contain. Thus, a 3-Lie algebra p -cocycle has $p(3-1)+1$ elements of \mathfrak{G} as arguments²⁰ and a general n -Lie algebra p -cocycle contains $p(n-1)+1$ arguments, since the fundamental objects have $(n-1)$ elements of \mathfrak{G} as (antisymmetric) arguments.

Example 71. (*The $n = 3$ Nambu-Heisenberg-Weyl algebra*) [38]

Consider an abelian $n = 3$ FA of dimension $3N$, with basis determined by three subsets of N generators each, (X_a, Y_a, Z_a) , $a = 1, \dots, N$. Since it is abelian,

$$[X_a, Y_b, Z_c] = 0 \quad \text{and, of course,} \quad [X, X, Y] = 0 \quad \text{etc.} \quad .$$

Let α^1 be the one-cochain defined by $\alpha^1(X_a, Y_b, Z_c) = \kappa$ for $a = b = c$ and $\alpha^1(X_a, Y_b, Z_c) = 0$ otherwise; α^1 also gives zero when two or more entries come from the same basis subset, $\alpha^1(X_a, X_b, Z_c) = 0$ etc., and is, of course, antisymmetric in its three entries. The values of the constant $\kappa \in \mathbb{R}$ characterize distinct cohomology classes, but, since any non-zero value will lead to extensions that are isomorphic as 3-Lie algebras, we may take $\kappa = 1$ (or \hbar). Eq. (240) is obviously satisfied, and thus α^1 is the one-cocycle that defines the Nambu-Heisenberg-Weyl 3-Lie algebra

$$[X_a, Y_b, Z_c] = \mathbb{I} \quad \text{for } a = b = c \quad \text{and } = 0 \quad \text{in all other cases} \quad (243)$$

as *e.g.*, $[X_a, X_b, Z_c] = 0$.

It is worth noting that this central extension of the above abelian $3N$ -dimensional 3-Lie algebra is realized by the bracket $\sum_{a=1}^N \frac{\partial(f^1, f^2, f^3)}{\partial(x^a, y^a, z^a)}$ [38] of a triplet of coordinates (x^a, y^b, z^c) , since $\{x^a, y^b, z^c\} = 1$ for $a = b = c$ and $\{x^a, y^b, z^c\} = 0$ otherwise, etc. This constitutes the $n = 3$ counterpart of the PB realization of the classical abelian algebra of dynamical variables $[q^a, p_b] = 0$, which produces the Heisenberg-Weyl Lie algebra $\{q^a, p_b\} = \delta_b^a$ ($[\hat{q}^a, \hat{p}_b] = i\hbar\delta_b^a\mathbb{I}$

²⁰Thus, an ordinary Lie algebra \mathfrak{g} cohomology (Def. 2) *two*-cocycle corresponds to a *one*-cocycle from the present n -Lie algebra \mathfrak{G} point of view since for $n = 2$ the fundamental objects contain just an element of the algebra. For a Lie algebra \mathfrak{g} , $\mathcal{X} = X$ and $\Omega(\mathcal{X}, Y) = \Omega(X, Y)$.

upon Dirac quantization). It is well known that Poisson brackets realize ‘projective’ (central) extensions of Lie algebras (see [205]). In this simple $n = 3$ example, the Nambu bracket realization of the above abelian 3-Lie algebra leads to the Nambu-Heisenberg-Weyl 3-algebra of eq. (243) although, of course, an interpretation of the Nambu bracket algebra in terms of transformations would associate these to pairs $\{x_a, y_b, \}$ etc. corresponding to antisymmetric fundamental objects and not to single algebra elements. This, we note in passing, is also reflected in the way one-cocycles depend on the elements of the ($n = 3$) FA as shown below and explains why we wrote $\alpha^1 \in \wedge^2 \mathfrak{G}^* \wedge \mathfrak{G}^*$ rather than $\alpha^1 \in \wedge^3 \mathfrak{G}^*$.

Let us now consider the general case.

11.2. FA cohomology complex adapted to the central extension problem.

This is the cohomology complex for the trivial action that follows from the previous discussion. We shall take here \mathfrak{G} -valued cochains for later convenience; since the action is trivial, this will not modify the form of the action of the coboundary operator on the corresponding p -cochains.

Definition 72. (FA cohomology p -cochains)

A p -cochain is a linear map $\alpha \in C^p(\mathfrak{G}, \mathfrak{G}) = \text{Hom}(\wedge^{(n-1)} \mathfrak{G} \otimes \overset{p}{\cdot} \otimes \wedge^{(n-1)} \mathfrak{G} \wedge \mathfrak{G}, \mathfrak{G})$

$$\alpha : (\mathcal{X}_1, \dots, \mathcal{X}_p, Z) \mapsto \alpha(\mathcal{X}_1, \dots, \mathcal{X}_p, Z) \equiv \alpha(X_1^1, \dots, X_1^{n-1}, \dots, X_p^1, \dots, X_p^{n-1}, Z) . \quad (244)$$

Thus, α takes p fundamental objects \mathcal{X} and an element of \mathfrak{G} as arguments; in all, $(p(n-1) + 1)$ elements of the n -algebra \mathfrak{G} .

Definition 73. (Coboundary operator δ for the trivial action)

The coboundary operator for the trivial action of \mathfrak{G} is the map $\delta : C^p(\mathfrak{G}, \mathfrak{G}) \rightarrow C^{p+1}(\mathfrak{G}, \mathfrak{G})$ defined (see [60]) by its action on a p -cochain $\alpha \in C^p$ by

$$\begin{aligned} (\delta\alpha)(\mathcal{X}_1, \dots, \mathcal{X}_{p+1}, Z) = & \\ & \sum_{1 \leq i < j}^{p+1} (-1)^i \alpha(\mathcal{X}_1, \dots, \hat{\mathcal{X}}_i, \dots, \mathcal{X}_i \cdot \mathcal{X}_j, \dots, \mathcal{X}_{p+1}, Z) \\ & + \sum_{i=1}^{p+1} (-1)^i \alpha(\mathcal{X}_1, \dots, \hat{\mathcal{X}}_i, \dots, \mathcal{X}_{p+1}, \mathcal{X}_i \cdot Z) , \end{aligned} \quad (245)$$

cf. eq. (27).

To check that $\delta^2 \equiv 0$, it is sufficient to recall that eq. (191) and $\mathcal{X} \cdot (\mathcal{Y} \cdot Z) - \mathcal{Y} \cdot (\mathcal{X} \cdot Z) = (\mathcal{X} \cdot \mathcal{Y}) \cdot Z$ hold by virtue of the FI. Let us illustrate the nilpotency of δ with a couple of examples.

Let \mathfrak{G} be a 3-Lie algebra. A zero-cochain is given by a map $\alpha^0 : Z \mapsto \alpha^0(Z) \in \mathfrak{G}$, and the action of δ on α^0 defines the one-cochain $\delta\alpha^0$ given by

$$(\delta\alpha^0)(\mathcal{X}, Z) = -\alpha^0(\mathcal{X} \cdot Z) = -\alpha^0([X_1, X_2, Z]) , \quad (246)$$

which is fully antisymmetric in its three arguments X_1, X_2, Z as any *one*-cochain should be. Similarly, a one-cochain α^1 is a \mathfrak{G} -valued map $\alpha^1 \in \text{Hom}(\wedge^2 \mathfrak{G} \wedge \mathfrak{G}, \mathfrak{G})$, $\alpha^1 : (\mathcal{X}, Z) \mapsto \alpha^1(\mathcal{X}, Z) \in \mathfrak{G}$. The action of the coboundary operator on α^1 is the two-cochain $\delta\alpha^1$ defined like in eq. (239). The one-cochain α^1 is a one-cocycle if the two-cochain $\delta\alpha^1 = 0$, in which case its coordinates satisfy

$$(\delta\alpha)_{a_1 a_2 b_1 b_2 k}{}^s = f_{b_1 b_2 k}{}^l \alpha_{a_1 a_2 l}{}^s - f_{a_1 a_2 b_1}{}^l \alpha_{l b_2 k}{}^s - f_{a_1 a_2 b_2}{}^l \alpha_{b_1 l k}{}^s - f_{a_1 a_2 k}{}^l \alpha_{b_1 b_2 l}{}^s = 0 . \quad (247)$$

This is necessarily the case for any coboundary $\alpha^1 = \delta\alpha^0$ since $(\delta\alpha^0)(X_1, X_2, Z) = \alpha^0([X_1, X_2, Z])$ and then eq. (239) is zero $\forall \alpha^0$ by linearity and the FI.

We may compare this with the Lie $n = 2$ case, for which $\mathcal{X}, \mathcal{Y} = X, Y$ and eq. (239) produces the three-coboundary given by the fourth line in eq. (26) ($\rho = 0$). A Lie algebra one-cochain Ω^1 for the trivial action generates the two-coboundary $(\delta\Omega^1)(X_1, X_2) = -\Omega^1([X_1, X_2])$, and then $(\delta^2\Omega^1)(X_1, X_2, X_3) = 0$ implies (or is guaranteed by) the JI. The two-coboundary generated by the \mathfrak{g} -valued cochain (trivial action) defined by $\Omega^1(X_i) = -X_i$, has for coordinates the \mathfrak{g} structure constants, $(\delta\Omega)_{ij}{}^k = C_{ij}{}^k$. Similarly, for $n = 3$, a zero cochain α^0 generates the one-coboundary $\delta\alpha^0$ in eq. (246) and $\delta^2\alpha^1 = 0 \forall \alpha^0$ is guaranteed by the FI as shown above. If we now take the \mathfrak{G} -valued one-cochain of coordinates $\alpha_{a_1 a_2 l}^s = f_{a_1 a_2 l}^s$, we see that α^1 is a one-cocycle for the trivial action since $(\delta\alpha^1)_{a_1 a_2 b_1 b_2 k}^s = 0$, as given by eq. (247), is the FI. Actually, α^1 is the one-coboundary $\alpha^1 = \delta\alpha^0$ generated by the zero cochain defined by $\alpha^0(X_a) = -X_a$ by eq. (246). The corresponding Lie algebra expressions, written in terms of the canonical form θ on a Lie group G , and within the present labelling of the cochain orders, correspond to $\alpha^0 = -\theta$, the one-coboundary to $\delta\alpha^0 = -d\theta = \theta \wedge \theta$ (eq. (20)) and $d^2\theta = 0$ follows by the JI.

As a second example, consider a FA cohomology two-cochain, $\alpha^2(\mathcal{X}_1, \mathcal{X}_2, Z)$. Writing all terms, $\delta\alpha^2$ is the three-cochain given by

$$\begin{aligned} (\delta\alpha^2)(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, Z) &= -\alpha^2(\mathcal{X}_1 \cdot \mathcal{X}_2, \mathcal{X}_3, Z) - \alpha^2(\mathcal{X}_2, \mathcal{X}_1 \cdot \mathcal{X}_3, Z) \\ &\quad + \alpha^2(\mathcal{X}_1, \mathcal{X}_2 \cdot \mathcal{X}_3, Z) - \alpha^2(\mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_1 \cdot Z) \\ &\quad + \alpha^2(\mathcal{X}_1, \mathcal{X}_3, \mathcal{X}_2 \cdot Z) - \alpha^2(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \cdot Z) \quad . \end{aligned}$$

We may now check that $\delta^2 = 0$ by assuming that α^2 is actually a two-coboundary, $\alpha^2 = \delta\alpha^1$. Then, the above expression becomes

$$\begin{aligned} &= \alpha^1((\mathcal{X}_1 \cdot \mathcal{X}_2) \cdot \mathcal{X}_3, Z) + \alpha^1(\mathcal{X}_3, (\mathcal{X}_1 \cdot \mathcal{X}_2) \cdot Z) - \alpha^1(\mathcal{X}_1 \cdot \mathcal{X}_2, \mathcal{X}_3 \cdot Z) \\ &+ \alpha^1(\mathcal{X}_2 \cdot (\mathcal{X}_1 \cdot \mathcal{X}_3), Z) + \alpha^1(\mathcal{X}_1 \cdot \mathcal{X}_3, \mathcal{X}_2 \cdot Z) - \alpha^1(\mathcal{X}_2, (\mathcal{X}_1 \cdot \mathcal{X}_3) \cdot Z) \\ &- \alpha^1(\mathcal{X}_1 \cdot (\mathcal{X}_2 \cdot \mathcal{X}_3), Z) - \alpha^1(\mathcal{X}_2 \cdot \mathcal{X}_3, \mathcal{X}_1 \cdot Z) + \alpha^1(\mathcal{X}_1, (\mathcal{X}_2 \cdot \mathcal{X}_3) \cdot Z) \\ &+ \alpha^1(\mathcal{X}_2 \cdot \mathcal{X}_3, \mathcal{X}_1 \cdot Z) + \alpha^1(\mathcal{X}_3, \mathcal{X}_2 \cdot (\mathcal{X}_1 \cdot Z)) - \alpha^1(\mathcal{X}_2, \mathcal{X}_3 \cdot (\mathcal{X}_1 \cdot Z)) \\ &- \alpha^1(\mathcal{X}_1 \cdot \mathcal{X}_3, \mathcal{X}_2 \cdot Z) - \alpha^1(\mathcal{X}_3, \mathcal{X}_1 \cdot (\mathcal{X}_2 \cdot Z)) + \alpha^1(\mathcal{X}_1, \mathcal{X}_3 \cdot (\mathcal{X}_2 \cdot Z)) \\ &+ \alpha^1(\mathcal{X}_1 \cdot \mathcal{X}_2, \mathcal{X}_3 \cdot Z) + \alpha^1(\mathcal{X}_2, \mathcal{X}_1 \cdot (\mathcal{X}_3 \cdot Z)) - \alpha^1(\mathcal{X}_1, \mathcal{X}_2 \cdot (\mathcal{X}_3 \cdot Z)) = 0 \quad , \end{aligned}$$

since all terms cancel out. Indeed, three pairs of them cancel each other directly, and the remaining terms can be collected in four groups of three sharing a common argument, \mathcal{X} or Z . Then, each group is seen to add up to zero on account of eqs. (191), (192), both a result of the FI. Higher orders proceed similarly.

Thus, the cohomology complex $(C^\bullet(\mathfrak{G}), \delta)$ for the trivial action is the relevant one for the central extension problem of FAs. The following analogue of the Whitehead lemma for Lie algebras, that we state here without proof, holds [180]:

Theorem 74. *(Semisimple FAs and central extensions)*

Let \mathfrak{G} be a semisimple FA. Then all its central extensions are trivial.

Remark 75.

The above coboundary operator does not allow for a formulation of the n -Lie algebra cohomology *à la* Chevalley-Eilenberg for $n > 2$ (Sec. 3.5), since the existence of a group-like manifold associated with FAs, allowing for the ‘localization’ process [104] of *e.g.* the linear maps α^0 so that they become \mathfrak{G} -valued covariant vector *fields* on such a manifold, is an open question. In

fact, as we shall see below, the operator that allows us to write MC-like equations for a FA is not an exterior differential, as it was the case for the generalized MC equations of the GLAs \mathcal{G} . Further, for $n \geq 3$ it is Lie \mathfrak{G} -invariance (and not ‘ \mathfrak{G} -invariance’) what is defined, a fact related to the previous problem. Finally, there is no matrix realization for the elements of \mathfrak{G} in general allowing us to construct invariant polynomials along the pattern that gives them in the Lie algebra case; in fact, as soon as one speaks of invariance, it is Lie \mathfrak{G} rather than \mathfrak{G} that appears, since the notion of invariance is associated to a *Lie* group of transformations.

11.3. MC-like equations for FAs.

The discussion in Rem. 4 prompts us to see whether it is possible to define MC-like equations for $n > 2$ FAs as we did for the GLAs in eqs. (122), (123). The key difference is that, in contrast with the Lie algebra cohomology, the p -cochains for the FA cohomology are no longer antisymmetric in all their \mathfrak{G} arguments for $p \geq 2$ and, further, there is no underlying group manifold. Consider linear \mathfrak{G} -valued maps ω on \mathfrak{G} , $\omega : \mathfrak{G} \rightarrow \mathfrak{G}$, $\omega = \omega^a \circ X_a$, as zero-cochains for the cohomology complex of Def. 73 (in the $n = 2$ Lie case, ω would be the canonical one-form θ on the group manifold). Then, the action of the coboundary operator in eq. (245) on $\omega \in C^0(\mathfrak{G}, \mathfrak{G})$ gives $\delta\omega(\mathcal{X}, Z) = -\omega([X_{a_1}, \dots, X_{a_{n-1}}, Z])$.

Let us now define the n -bracket of \mathfrak{G} -valued maps in terms of the FA by

$$[\omega, \dots, \omega] := \omega^{a_1} \wedge \dots \wedge \omega^{a_n} \circ [X_{a_1}, \dots, X_{a_n}] \quad i.e. \quad [\omega, \dots, \omega]^c = f_{a_1 \dots a_n}^c \omega^{a_1} \wedge \dots \wedge \omega^{a_n}, \quad (248)$$

since $[\omega, \dots, \omega] = [\omega, \dots, \omega]^c X_c$. Then, in analogy with the MC eqs. (20), we may write in terms of the coboundary operator δ

$$\delta\omega = -\frac{1}{n!}[\omega, \dots, \omega] \quad \text{or} \quad (\delta\omega)^c = -\frac{1}{n!}f_{a_1 \dots a_n}^c \omega^{a_1} \wedge \omega^{a_2} \dots \wedge \omega^{a_n}. \quad (249)$$

Note that in the FA case we cannot write $[\omega, \dots, \omega] \propto \omega \wedge \dots \wedge \omega$ since the \wedge product of the ω 's would generate a multibracket $[X_{a_1}, \dots, X_{a_n}]$, given by the full antisymmetrization of the products X_{a_1}, \dots, X_{a_n} , rather than a FA n -bracket. Eqs. (249) constitute the *MC equations for a FA \mathfrak{G}* as defined by the FA cohomology coboundary operator δ . For $n = 2$, of course, $\delta \rightarrow d$, $\omega \rightarrow \theta$, $d\theta = -\frac{1}{2}[\theta, \theta] = -\theta \wedge \theta \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ and the MC-like equations for \mathfrak{G} above become the MC ones of a Lie algebra \mathfrak{g} .

$\delta\omega$ is clearly a \mathfrak{G} -valued one-coboundary. The expression stating that $\delta^2\omega = 0$, as it follows from Def. 73 or eq. (239), may be formally rewritten in the form

$$\begin{aligned} & [\omega, \overset{n-1}{\cdot}, \omega, [\omega, \overset{n}{\cdot}, \omega]] - [\omega, [\omega, \overset{n}{\cdot}, \omega], \omega, \overset{n-2}{\cdot}, \omega] - \\ & [\omega, \omega, [\omega, \overset{n}{\cdot}, \omega], \omega, \overset{n-3}{\cdot}, \omega] - \dots - [\omega, \overset{n-1}{\cdot}, \omega, [\omega, \overset{n}{\cdot}, \omega]] = 0, \end{aligned} \quad (250)$$

where *e.g.*, the second double bracket means

$$[\omega, [\omega, \overset{n}{\cdot}, \omega], \omega, \overset{n-2}{\cdot}, \omega] \propto \omega^{b_1} \omega^{a_1} \dots \omega^{a_{n-1}} \omega^{b_2} \dots \omega^{b_n} [X_{b_1}, [X_{a_1}, \dots, X_{a_{n-1}}, X_{b_2}], X_{b_3}, \dots, X_{b_n}].$$

Obviously, the wedge product for the ω 's *cannot* be used above since there is antisymmetry only for the groups of ω 's involving the appropriate indices (*cf.* eq. (248)). The symmetry properties of the \mathfrak{G} arguments of any two-cochain follow from its definition, $\alpha^2 \in \text{Hom}(\wedge^{n-1} \mathfrak{G} \otimes \wedge^{n-1} \mathfrak{G} \wedge \mathfrak{G}, \mathfrak{G})$; the FI, unlike the GJI, does not involve a full antisymmetrization.

If we now define the action of $[\omega, \overset{n-1}{\cdot}, \omega, \cdot]$ by

$$[\omega, \overset{n-1}{\cdot}, \omega, \cdot] : \omega \mapsto [\omega, \dots, \omega]. \quad (251)$$

we see that eq. (250) states that this operator acts on the n -bracket of \mathfrak{G} -valued elements of \mathfrak{G}^* , $[\omega, \overset{n}{\cdot}, \omega]$, as a *derivation*, and that this property follows directly from the FI/nilpotency of δ .

For $n = 3$, the MC-like equation (249) for FAs above has the structure of the Basu-Harvey [181] equation. This has appeared [26, 206, 182, 56, 207] in the description of M-branes and in the BLG model and has its $n = 2$ MC precedent in the Nahm equation [208].

11.4. Dual FA homology complex for the trivial action.

The homology operator for n -algebras was introduced by Takhtajan [209, 21] in the context of the Nambu-Lie algebras [41] to be described in Sec. 13.4. We shall consider here (see [60]) the homology dual to the cohomology complex of Def. 73 adapted to the central extension problem. Let the p -chains C_p be defined as $(\mathcal{X}_1, \dots, \mathcal{X}_p, Z) \in \wedge^{n-1} \mathfrak{G} \otimes \cdot^p \otimes \wedge^{n-1} \mathfrak{G} \wedge \mathfrak{G}$.

Definition 76. (*n-algebra homology operator*)

On a C_1 chain, the homology operator $\partial : C_1 \rightarrow C_0 \equiv \mathfrak{G}$ is defined by $\partial : (\mathcal{X}, Z) \mapsto \partial(\mathcal{X}, Z) \equiv \partial(X_1, \dots, X_{n-1}, Z) := [X_1, \dots, X_{n-1}, Z]$. In general, $\partial : C_p \rightarrow C_{p-1}$ is given by

$$\begin{aligned} \partial(\mathcal{X}_1, \dots, \mathcal{X}_p, Z) &= \sum_{1 \leq i < j}^p (-1)^i (\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_i \cdot \mathcal{X}_j, \dots, \mathcal{X}_p, Z) \\ &+ \sum_{i=1}^p (-1)^i (\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_p, \mathcal{X}_i \cdot Z) \quad . \end{aligned} \quad (252)$$

It is not difficult to check that ∂ is the dual of δ . Using the definition of δ in (245), we find for a $(p-1)$ -cochain α

$$\begin{aligned} \alpha(\partial(\mathcal{X}_1, \dots, \mathcal{X}_p, Z)) &= \sum_{1 \leq i < j}^p (-1)^i \alpha(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_{j-1}, \mathcal{X}_i \cdot \mathcal{X}_j, \dots, \mathcal{X}_p, Z) \\ &+ \sum_{i=1}^p (-1)^i \alpha(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_p, \mathcal{X}_i \cdot Z) \quad , \end{aligned} \quad (253)$$

which shows that, on $(\mathcal{X}_1, \dots, \mathcal{X}_p, Z)$,

$$\alpha \partial = \delta \alpha \quad . \quad (254)$$

11.5. Other FA cohomology complexes.

We may also consider cochains of the type $\alpha(\mathcal{X}_1, \dots, \mathcal{X}_p)$, depending on p fundamental objects or on $p(n-1)$ elements of \mathfrak{G} *i.e.*, taking arguments in $(\wedge^{n-1} \mathfrak{G}) \otimes \cdot^p \otimes (\wedge^{n-1} \mathfrak{G})$. It is also possible to have cochains taking the same arguments but that are valued on a FA left module V in the sense of eq. (230), elements of $\text{Hom}(\otimes^p (\wedge^{n-1} \mathfrak{G}), V)$. In this case, there is a FA cohomology complex that closely mimetizes the Lie algebra cohomology one when the fundamental object arguments are replaced by Lie algebra arguments. The coboundary operator for this \mathfrak{G} -module cohomology complex is defined [149] by

$$\begin{aligned} \delta \alpha(\mathcal{X}_1, \dots, \mathcal{X}_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \mathcal{X}_i \cdot \alpha(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_{p+1}) \\ &+ \sum_{\substack{i,j \\ 1 \leq i < j}}^{p+1} (-1)^i \alpha(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_i, \dots, \mathcal{X}_i \cdot \mathcal{X}_j, \dots, \mathcal{X}_{p+1}) \quad , \end{aligned} \quad (255)$$

where the dot in the first line above indicates the action of the fundamental object on the \mathfrak{G} -module where the cochains takes values (eqs. (192), (230)).

The proof of the nilpotency of δ above follows the same pattern as that of the Lie algebra cohomology operator s (eqs. (25),(27)). The key issue is that all terms in $\delta(\delta \alpha^p)$ may be

collected in groups of two types, with three terms each group. The groups of the first type share a p -cochain with equal arguments, on which two fundamental objects act in the form $\mathcal{X} \cdot (\mathcal{Y} \cdot \alpha^p) - \mathcal{Y} \cdot (\mathcal{X} \cdot \alpha^p) - (\mathcal{X} \cdot \mathcal{Y}) \cdot \alpha^p = 0$, and thus add up to zero because α^p takes values on a \mathfrak{G} -module, eq. (230). Due to p -linearity, the groups of the second type add up to α^p taking values on equal fundamental objects but for one, which turns out to be the addition of $(\mathcal{X} \cdot \mathcal{Y}) \cdot \mathcal{Z} + \mathcal{Y} \cdot (\mathcal{X} \cdot \mathcal{Z}) - \mathcal{X} \cdot (\mathcal{Y} \cdot \mathcal{Z}) = 0$, again zero by (191).

When the action is non-trivial, other FA cohomology complexes are possible, as the one that is relevant for deformations of FA algebras to described in Sec. 11.6 below. There, both the left and right actions enter, as was seen in Sec. 4.2 and will be found again in Sec. 11.7. In fact, the cohomology of Leibniz algebras underlies that of the FAs, as will be discussed in Sec. 12.

11.6. Deformations of Filippov algebras.

We now look at the cohomology complex that governs the deformations of FAs, in close parallel with the Lie algebra deformations considered in Sec. 3.3.

11.6.1. Infinitesimal deformations of FAs.

Let us begin with the simple example of a 3-Lie algebra; the results, when written in terms of the fundamental objects, will apply immediately to any n -Lie algebra. Proceeding as in the case of the Lie algebra in Sec. 3.3, an infinitesimal deformation requires the existence of a deformed three-bracket $[X_1, X_2, Z]_t$,

$$[X_1, X_2, Z]_t = [X_1, X_2, Z] + t\alpha(X_1, X_2, Z) + O(t^2) \quad , \quad (256)$$

where $\alpha(X_1, X_2, Z)$ is a three-linear skewsymmetric \mathfrak{G} -valued map $\alpha : \wedge^2 \mathfrak{G} \wedge \mathfrak{G} \rightarrow \mathfrak{G}$, $\alpha : (\mathcal{X}, Z) \mapsto \alpha(\mathcal{X}, Z)$, depending on one fundamental object plus an element of \mathfrak{G} . The bracket $[X_1, X_2, Z]_t$ satisfies the deformed FI, which in terms of the fundamental objects reads

$$[\mathcal{X}, (\mathcal{Y} \cdot Z)]_t = [(\mathcal{X} \cdot \mathcal{Y})_t, Z]_t + [\mathcal{Y}, (\mathcal{X} \cdot Z)]_t \quad , \quad (257)$$

where the subindex t indicates that the n -brackets $(\mathcal{X} \cdot Z)_t$, $(\mathcal{Y} \cdot Z)_t$ and those that appear in the definition of $\mathcal{X} \cdot \mathcal{Y}$ (eq. (190)) are deformed as in eq. (256). At first order, this gives the condition

$$\begin{aligned} ad_{\mathcal{X}}\alpha(\mathcal{Y}, Z) - ad_{\mathcal{Y}}\alpha(\mathcal{X}, Z) - [\alpha(\mathcal{X}, Y_1), Y_2, Z] - [Y_1, \alpha(\mathcal{X}, Y_2), Z] \\ - \alpha(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha(\mathcal{Y}, \mathcal{X} \cdot Z) + \alpha(\mathcal{X}, \mathcal{Y} \cdot Z) = 0 \quad . \end{aligned} \quad (258)$$

Looking at the square brackets above we may introduce $\alpha(\mathcal{X}, \quad) \cdot \mathcal{Y}$ as the fundamental object defined by

$$\begin{aligned} \alpha(\mathcal{X}, \quad) \cdot \mathcal{Y} &:= (\alpha(\mathcal{X}, \quad) \cdot Y_1, Y_2) + (Y_1, \alpha(\mathcal{X}, \quad) \cdot Y_2) \\ &= (\alpha(\mathcal{X}, Y_1), Y_2) + (Y_1, \alpha(\mathcal{X}, Y_2)) \quad [\alpha(\mathcal{X}, \quad) \cdot Y_i := \alpha(\mathcal{X}, Y_i)] \quad . \end{aligned} \quad (259)$$

Then, the condition that $\alpha(\mathcal{X}, Z)$ in (256) defines a FA bracket may be written as (*cf.* eq. (239))

$$\begin{aligned} (\delta\alpha)(\mathcal{X}, \mathcal{Y}, Z) &= ad_{\mathcal{X}}\alpha(\mathcal{Y}, Z) - ad_{\mathcal{Y}}\alpha(\mathcal{X}, Z) - (\alpha(\mathcal{X}, \quad) \cdot \mathcal{Y}) \cdot Z \\ &\quad - \alpha(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha(\mathcal{Y}, \mathcal{X} \cdot Z) + \alpha(\mathcal{X}, \mathcal{Y} \cdot Z) = 0 \quad , \end{aligned} \quad (260)$$

which may be simplified by writing $ad_{\mathcal{X}}\alpha(\mathcal{Y}, Z) = \mathcal{X} \cdot \alpha(\mathcal{Y}, Z)$, etc. (for the trivial action, $\delta\alpha$ reproduces eq. (239)). The identification of this expression with $(\delta\alpha)(\mathcal{X}, \mathcal{Y}, Z)$, as written in

its *l.h.s.*, indicates that $\alpha(\mathcal{X}, Z)$ has to be a one-cocycle $\alpha \equiv \alpha^1$ for the \mathfrak{G} -valued cohomology that will be defined below²¹.

Using now the ordinary Lie algebra experience, we may define a zero-cocycle as a \mathfrak{G} -valued map $\alpha^0 : \mathfrak{G} \rightarrow \mathfrak{G}$, $\alpha^0 : Z \rightarrow \alpha^0(Z)$, which does not contain any fundamental object as an argument. It will generate a one-coboundary by

$$(\delta\alpha^0)(\mathcal{X}, Z) = \mathcal{X} \cdot \alpha^0(Z) - \alpha^0(\mathcal{X} \cdot Z) + (\alpha^0(\quad) \cdot \mathcal{X}) \cdot Z \quad , \quad (261)$$

where the last term is $(\alpha^0(\quad) \cdot \mathcal{X}) \cdot Z = [\alpha^0(X_1), X_2, Z] + [X_1, \alpha^0(X_1), Z]$. Indeed, a calculation using eqs. (260) and (191) shows that $(\delta(\delta\alpha^0))(\mathcal{X}, \mathcal{Y}, Z) \equiv 0$. To check that (261) is a sensible definition, we now look at the expression of the one-cocycle of an infinitesimal deformation when the deformation is actually trivial. Let $X'_i = X_i - t\alpha^0(X_i)$ the redefinition of the basis of the apparently deformed FA which removes the $t\alpha(\mathcal{X}, Z)$ term in (256). Then (we may assume directly that we are dealing with an n -Lie algebra) the new bracket for the primed generators reads, to order t ,

$$\begin{aligned} (\mathcal{X}' \cdot Z')_t &\equiv [X'_1, \dots, X'_{n-1}, Z']_t = [X_1 - t\alpha^0(X_1), \dots, X_{n-1} - t\alpha^0(X_{n-1}), Z - t\alpha^0(Z)]_t \\ &= [X_1, \dots, X_{n-1}, Z]_t - t \left(\sum_{i=1}^{n-1} [X_1, \dots, X_{i-1}, \alpha^0(X_i), X_{i+1}, \dots, X_{n-1}, Z] + [X_1, \dots, X_{n-1}, \alpha^0(Z)] \right) \\ &= [X_1, \dots, X_{n-1}, Z] - t\alpha^0(\mathcal{X} \cdot Z) + t(\alpha^1(\mathcal{X}, Z) + \alpha^0(\mathcal{X} \cdot Z) - (\alpha^0(\quad) \cdot \mathcal{X}) \cdot Z - \mathcal{X} \cdot \alpha^0(Z)) \\ &= [X_1, \dots, X_{n-1}, Z]' + t(\alpha^1(\mathcal{X}, Z) - (\delta\alpha^0)(\mathcal{X}, Z)) \quad , \end{aligned} \quad (262)$$

where in the third line we have added and subtracted $t\alpha^0(\mathcal{X} \cdot Z) = t\alpha^0([X_1, \dots, X_{n-1}, Z])$ and used definition (261) in the fourth, the form of which now appears justified. We see that the t term above disappears, and with it the infinitesimal deformation, if α^1 is the one-coboundary $\alpha^1 = \delta\alpha^0$. Thus, the infinitesimal deformations of n -Lie algebras are governed by the first²² cohomology group H_{ad}^1 for (260) where *ad* refers generically to the action of the fundamental objects on the cochains (the general cocycle condition will be given in Sec. 11.7 below); a FA algebra is *stable*, or *rigid*, if $H_{ad}^1 = 0$. As defined here, the p -cochains have p fundamental objects plus an element of \mathfrak{G} as their arguments; in all, $p(n-1) + 1$ elements of \mathfrak{G} .

Comparing eqs. (261), (260) with the equivalent formulae for previous cohomology complexes, we see that the coboundary operator producing these expressions (and therefore adapted to the deformation problem), leads to a cohomology complex different from *e.g.*, that defined by eq. (255). This is because the cochain spaces C^p for (255) are $\text{Hom}(\otimes^p(\wedge^{n-1}\mathfrak{G}), \mathfrak{G})$ and thus automatically unsuitable for deformations since (as it was also the case for the central extensions cohomology) the cochains defined here naturally include a solitary element of \mathfrak{G} as an argument. As a result, the situation for the deformation of FAs is different from that for the Lie algebras, since the relevant cohomology leads to (260) rather than to the corresponding

²¹We note that for the $n = 2$ case, where the fundamental objects are simply the elements of the FA algebra, $(\alpha(X, \quad) \cdot Y) \cdot Z = \alpha(X, Y) \cdot Z$ and $X \cdot Y = [X, Y]$. Then, eq. (260) may be written as

$$\begin{aligned} (\delta\alpha)(X, Y, Z) &= X \cdot \alpha(Y, Z) - Y \cdot \alpha(X, Z) - \alpha(X, Y) \cdot Z \\ &\quad - \alpha([X, Y], Z) - \alpha(Y, [X, Z]) + \alpha(X, [Y, Z]) = 0 \quad , \end{aligned}$$

which reproduces eq. (80), which is the two-cocycle condition for the Leibniz algebra cohomology. Further, since an $n = 2$ FA is actually an ordinary Lie algebra, the third term $-\alpha(X, Y) \cdot Z$ may be rewritten as $Z \cdot \alpha(X, Y)$ and the above expression corresponds to the second equation in (26).

²²As mentioned, the order p is defined by the number of fundamental objects contained in the p -cochain. Within this terminology, the infinitesimal deformations of a Lie algebra \mathfrak{g} are governed by the \mathfrak{g} -valued one-cocycles for the adjoint action, H_{ad}^1 , where the action of \mathcal{X} is defined as above and in Def. 78 below.

one-cocycle expression from (255), which is the cohomology complex that follows the pattern of the ordinary Lie algebra cohomology coboundary operator s in eq. (25). The key difference is the definition of the action of the fundamental object in the third terms in the right hand sides of eqs. (260) and (261). In all, this reflects the different dependence of the cochains on the fundamental objects and the single element of \mathfrak{G} on the one hand and, on the other, of the actions induced by these on the cochains.

11.6.2. Higher order deformations of FAs.

Let us now look at the problem of extending an infinitesimal deformation to second order. Consider

$$[X_1, \dots, X_{n-1}, Z]_t \equiv (\mathcal{X} \cdot Z)_t = (\mathcal{X} \cdot Z) + t\alpha_1(\mathcal{X}, Z) + t^2\alpha_2(\mathcal{X}, Z) + O(t^3) \quad , \quad (263)$$

where the subindices 1, 2 refer to the deformation order; note that $\alpha_1, \alpha_2 \in \text{Hom}(\wedge^{n-1}\mathfrak{G} \wedge \mathfrak{G}, \mathfrak{G})$ *i.e.*, both are \mathfrak{G} -valued one-cochains and, further, α_1 is already assumed to be a one-cocycle. We now ask ourselves whether the infinitesimal deformation α_1 can be expanded to a second order one determined by a certain α_2 . Again, the condition is that the FI (eqs. (139), (257) for the deformed FA), that we shall now write in the form

$$(\mathcal{X} \cdot (\mathcal{Y} \cdot Z))_t - ((\mathcal{X} \cdot \mathcal{Y})_t \cdot Z)_t - (\mathcal{Y} \cdot (\mathcal{X} \cdot Z))_t = 0 \quad , \quad (264)$$

be satisfied up to order $O(t^3)$. Since α_1 is an infinitesimal deformation, the terms of order t are zero since $\delta\alpha_1 = 0$. The terms of order t^2 give the condition

$$\begin{aligned} & \alpha_1(\mathcal{X}, \alpha_1(\mathcal{Y}, Z)) - \alpha_1(\alpha_1(\mathcal{X}, \mathcal{Y}), Z) - \alpha_1(\mathcal{Y}, \alpha_1(\mathcal{X}, Z)) \\ & \quad + \mathcal{X} \cdot \alpha_2(\mathcal{Y}, Z) - \alpha_2(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha_2(\mathcal{Y}, \mathcal{X} \cdot Z) \\ & \quad - \mathcal{Y} \cdot \alpha_2(\mathcal{X}, Z) + \alpha_2(\mathcal{X}, \mathcal{Y} \cdot Z) - (\alpha_2(\mathcal{X}, \mathcal{Y})) \cdot Z \\ & \equiv \gamma(\mathcal{X}, \mathcal{Y}, Z) + (\delta\alpha_2)(\mathcal{X}, \mathcal{Y}, Z) = 0 \end{aligned} \quad (265)$$

using the one cocycle condition (260). The above expression implies $\delta\gamma = 0$ and thus $\gamma(\mathcal{X}, \mathcal{Y}, Z)$ has to be a two-cocycle. The two-cochain γ in (265) is

$$\gamma(\mathcal{X}, \mathcal{Y}, Z) := \alpha_1(\mathcal{X}, \alpha_1(\mathcal{Y}, Z)) - \alpha_1(\alpha_1(\mathcal{X}, \mathcal{Y}), Z) - \alpha_1(\mathcal{Y}, \alpha_1(\mathcal{X}, Z)) \quad . \quad (266)$$

The action of the coboundary operator δ on γ is the case $p = 2$ in eq. (269) below. A non completely trivial calculation shows that

$$\delta\gamma(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, Z) = \sum_{a=1}^{n-1} \delta\alpha_1((X_{(2)1}, \dots, \alpha_1(\mathcal{X}_1, X_{(2)a}), \dots, X_{(2)n-1}), \mathcal{X}_3, Z) = 0 \quad ,$$

so that γ is indeed a two-cocycle since $\delta\alpha_1 = 0$.

Thus, if γ is a two-coboundary generated by a one-cochain α' , $\gamma = \delta\alpha'$, it is sufficient to take the second order one-cochain as $\alpha_2 = -\alpha'$ to have the above condition fulfilled as in the Lie algebra case (Sec. 3.3.2). Proceeding in this way, we may conclude that there is no obstruction if $H_{ad}^2 = 0$, where the second cohomology group is defined with respect to the cohomology complex defined in the next section.

11.7. Cohomology complex for deformations of n -Lie algebras.

The above discussion leads us naturally to introducing the ingredients of the deformation theory of FAs.

Definition 77. (*\mathfrak{G} -valued p -cochains*)

A p -cochain for the deformation cohomology is a map $\alpha^p : \wedge^{(n-1)} \mathfrak{G} \otimes \cdots \otimes \wedge^{(n-1)} \mathfrak{G} \wedge \mathfrak{G} \rightarrow \mathfrak{G}$ so that

$$\alpha : (\mathcal{X}_1, \dots, \mathcal{X}_p, Z) \mapsto \alpha(\mathcal{X}_1, \dots, \mathcal{X}_p, Z) \equiv \alpha(X_1^1, \dots, X_1^{n-1}, \dots, X_p^1, \dots, X_p^{n-1}, Z) \in \mathfrak{G} . \quad (267)$$

We shall refer generically to the space of the above p -cochains as C_{ad}^p , since the (left) action of the fundamental objects on the \mathfrak{G} -valued α^p is given by

$$\mathcal{X} \cdot \alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, Z) = [X_1, \dots, X_{n-1}, \alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, Z)] .$$

To define the action of the coboundary operator, we need first the equivalent of (259) for an n -Lie algebra. This is given by the sum of fundamental objects

$$\alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, \quad) \cdot \mathcal{Y} = \sum_{i=1}^{n-1} (Y_1, \dots, \alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, Y_i), \dots, Y_{n-1}) , \quad (268)$$

since, by definition, $\alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, Y_i) \in \mathfrak{G}$.

The previous discussion has allowed us to define the action of the coboundary operator on zero-cochains α^0 and one-cochains α^1 by eqs. (261), (260) respectively. The natural generalization of the above expressions to the p -cochain case leads us to the deformation cohomology complex below.

Definition 78. (*Coboundary operator for the deformation cohomology*)

The coboundary operator $\delta : C_{ad}^p \rightarrow C_{ad}^{p+1}$ is given by

$$\begin{aligned} (\delta\alpha^p)(\mathcal{X}_1, \dots, \mathcal{X}_p, \mathcal{X}_{p+1}, Z) = & \\ & \sum_{1 \leq j < k}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_j, \dots, \mathcal{X}_{k-1}, \mathcal{X}_j \cdot \mathcal{X}_k, \mathcal{X}_{k+1}, \dots, \mathcal{X}_{p+1}, Z) \\ & + \sum_{j=1}^{p+1} (-1)^j \alpha^p(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_j, \dots, \mathcal{X}_{p+1}, \mathcal{X}_j \cdot Z) \\ & + \sum_{j=1}^{p+1} (-1)^{j+1} \mathcal{X}_j \cdot \alpha^p(\mathcal{X}_1, \dots, \widehat{\mathcal{X}}_j, \dots, \mathcal{X}_{p+1}, Z) \\ & + (-1)^p (\alpha^p(\mathcal{X}_1, \dots, \mathcal{X}_p, \quad) \cdot \mathcal{X}_{p+1}) \cdot Z , \end{aligned} \quad (269)$$

where the last term is defined by eq. (268). For $p = 0, 1$ this reproduces eqs. (261) and the first equality in (260), with which one may check that $(\delta^2\alpha^0)(\mathcal{X}, \mathcal{Y}, Z) = 0$, already an altogether non-trivial calculation. Explicitly, on a one-cochain α^1 (see eq. (260)),

$$\begin{aligned} (\delta\alpha^1)(\mathcal{X}, \mathcal{Y}, Z) = & \mathcal{X} \cdot \alpha^1(\mathcal{Y}, Z) - \mathcal{Y} \cdot \alpha^1(\mathcal{X}, Z) - (\alpha^1(\mathcal{X}, \quad) \cdot \mathcal{Y}) \cdot Z \\ & - \alpha^1(\mathcal{X} \cdot \mathcal{Y}, Z) - \alpha^1(\mathcal{Y}, \mathcal{X} \cdot Z) + \alpha^1(\mathcal{X}, \mathcal{Y} \cdot Z) , \end{aligned}$$

an expression (set equal to zero) repeatedly used in the previous calculation for the one-cocycle defining the first order deformation of a FA.

It may be seen that the cohomology complex defined above is essentially equivalent to the one introduced by Gautheron in an important paper [149]. There, he was the first to consider the deformation cohomology for Nambu algebras, a particular case of FAs (Ex. 56 and Sec. 13.4);

see also [21, 188]. By defining \mathfrak{G} -valued cocycles Z_{ad}^p and coboundaries B_{ad}^p in the usual manner, the previous results may be formulated as the following

Theorem 79. (*Deformations of FA algebras*)

Let \mathfrak{G} be a FA. The first cohomology group H_{ad}^1 for the above coboundary operator governs the infinitesimal deformations of \mathfrak{G} . If $H_{ad}^1 = 0$, the FA is rigid. If it admits an infinitesimal deformation, the obstructions to move to higher order result from $H_{ad}^2 \neq 0$.

In analogy with the Lie algebra case, the following theorem [180] holds

Theorem 80. (*Semisimple FAs and deformations*)

Semisimple n -Lie algebras are rigid.

This theorem and Th. 74 constitute the extension of the standard Whitehead's lemma for Lie algebras (see [96]) to n -Lie algebras. Since Lie algebras are $n = 2$ FAs, this result [180] states that Whitehead's lemma holds true for any semisimple Filippov algebra, $n \geq 2$. The simple FAs inherit, so to speak, the rigidity of their Lie ancestors $so(3)$ and $so(1, 2)$.

12. n -LEIBNIZ ALGEBRA COHOMOLOGY. LA vs. FA COHOMOLOGY

As discussed in Sec. 9.1, the composition $\mathcal{X} \cdot \mathcal{Y}$ of fundamental objects, rewritten as $[\mathcal{X}, \mathcal{Y}]$, has the properties of a Leibniz algebra commutator. Further, since the skewsymmetry of the FA n -bracket is not needed for the nilpotency of δ in, say, Def. 73, δ also defines a coboundary operator for the cohomology of an n -Leibniz algebra \mathfrak{L} , where now the p -cochains may be considered as elements $\alpha \in (\otimes^{(n-1)} \mathfrak{L}^*) \otimes \dots \otimes (\otimes^{(n-1)} \mathfrak{L}^*) \otimes \mathfrak{L}^* = \otimes^{p(n-1)+1} \mathfrak{L}^* = \text{Hom}(\otimes^{p(n-1)+1} \mathfrak{L}, \mathbb{R})$. The key ingredient that guarantees the nilpotency of the coboundary operator is the FI, which implies eqs. (191), (230), etc. The only difference between the n -Lie algebra and n -Leibniz algebra cohomology complexes for the trivial representation, irrelevant for the nilpotency of δ , is that for an n -Lie algebra $\mathcal{X} \in \wedge^{n-1} \mathfrak{G}$ and for a Leibniz algebra $\mathcal{X} \in \otimes^{n-1} \mathfrak{L}$. Hence, with the appropriate changes in the definition of the p -cochain spaces C^p , the coboundary operator δ of Def. 73 defines both the corresponding cohomologies for n -Lie \mathfrak{G} and n -Leibniz \mathfrak{L} algebras adapted to the central extension problem, which correspond to the trivial action.

Thus, the FA cohomologies defined by the coboundary operators (245) (and the corresponding homology) constitute simply the application of the n -Leibniz algebra coboundary operators that define the corresponding n -Leibniz algebra cohomology complexes to the n -Lie algebra cohomology. Analogously, an n -Leibniz algebra cohomology p -cochain corresponding to the FA cohomology complex defined by (255) is an element of $\text{Hom}(\otimes^{p(n-1)} \mathfrak{L}, V)$, where V is the corresponding \mathfrak{L} -module. In fact, n -Leibniz algebras largely underlie the structural cohomological properties of the FAs.

Similar considerations apply to the n -Leibniz algebra cohomology adapted to the deformation problem already considered for the FAs. To define the appropriate n -Leibniz algebra cohomology complex it is sufficient to take the p -cochains as \mathfrak{L} -valued elements in $(\otimes^{n-1} \mathfrak{L}^*) \cdot \dots \cdot (\otimes^{n-1} \mathfrak{L}^*) \otimes \mathfrak{L}^*$, *i.e.*, as elements of $\text{Hom}(\otimes^{p(n-1)+1} \mathfrak{L}, \mathfrak{L})$. Indeed, for the $n = 2$ case, and reverting to the notation where p indicates the number of algebra elements on which α^p takes arguments,

$\alpha^p \in C^p(\mathcal{L}, \mathcal{L}) = \text{Hom}(\otimes^p \mathcal{L}, \mathcal{L})$, Def. 78 leads to

$$\begin{aligned}
 (s\alpha^p)(X_1, \dots, X_p, X_{p+1}) = & \\
 & \sum_{1 \leq j < k}^{p+1} (-1)^j \alpha^p(X_1, \dots, \widehat{X}_j, \dots, X_{k-1}, [X_j, X_k], X_{k+1}, \dots, X_{p+1}) \\
 & + \sum_{j=1}^p (-1)^{j+1} X_j \cdot \alpha^p(X_1, \dots, \widehat{X}_j, \dots, X_{p+1}) \\
 & + (-1)^{p+1} \alpha^p(X_1, \dots, X_p) \cdot X_{p+1} \quad ,
 \end{aligned} \tag{270}$$

which coincides with the coboundary operator for the Leibniz algebra cohomology complex $(C^\bullet(\mathcal{L}, \mathcal{L}), s)$ of eq. (85) (for Leibniz algebra homology, see [210]).

Alternatively, the FA cohomology complex would follow from that for the n -Leibniz algebras by demanding that the n -Leibniz bracket be skewsymmetric so that it becomes the n -bracket of a FA.

We conclude with a comment on deformations. The proof of the Whitehead Lemma for FAs [180], Ths. 74 and 80, relies on the skewsymmetry of the n -bracket of FAs and it will not hold when the full antisymmetry is relaxed. Thus, one might expect having a richer deformation structure for n -Leibniz algebras and even for Leibniz-type deformations of FAs viewed as Leibniz algebras. This has been observed already for the $n = 2$ case [150] by looking at Leibniz deformations of a Lie algebra and, further, a specific Leibniz deformation of the euclidean 3-Lie algebra has been found [169]. Thus, the next natural step is to look *e.g.* at n -Leibniz deformations of simple n -Lie algebras to see whether this opens more possibilities. It has been shown [151] that for n -Leibniz deformations with brackets that keep the antisymmetry in their first $n - 1$ arguments and thus have antisymmetric fundamental objects, rigidity still holds for any $n > 3$.

13. n -ARY POISSON STRUCTURES

In the previous sections we have mainly reviewed two possible ways of generalizing the Lie algebra structure to algebras endowed with brackets with more than two entries, and studied various aspects of the two generalizations. It is not surprising that, as far as the Poisson bracket (PB) and the standard Poisson structure (PS, see *e.g.* [211] for a classic paper and [212] for a review) share the basic properties of Lie algebras, similar n -ary generalizations of the ordinary PS should also exist. We devote this section to two fully antisymmetric extensions of the standard PS for brackets with more than two entries.

The first generalization of the PS to $n > 2$ [4,5], termed *generalized Poisson structure* (GPS), is naturally defined for brackets with an even number of entries and parallels the properties of higher order generalized Lie algebras, GLA, of Sec. 6. Thus, its characteristic identity is a higher order Poisson bracket version of the GJI satisfied by these higher order Lie algebras. The second one, but earlier in time, is the Nambu-Poisson (N-P) structure [38–41] (sometimes called n -Poisson, see [48]); its characteristic identity is the N-P bracket Filippov identity satisfied by the Filippov algebras. We review below both generalizations; a comparison between both structures, GPS and N-P, may be found in [159] and in the table in Sec. 13.5. Further discussion on GP and N-P structures and related topics may be found *e.g.* in [10, 48, 61, 62, 213, 49, 50] and references therein; references on more physical aspects are given in Sec. 13.5 below. It is worth mentioning that it is possible to drop the full antisymmetry in the definition of the N-P

brackets, in analogy with the situation of n -Leibniz algebras with respect to FAs; this leads to the notion of Nambu-Leibniz brackets and Nambu-Leibniz (or Nambu-Loday) algebras [214], but these will not be discussed here.

Besides having a fully antisymmetric n -ary bracket satisfying the corresponding characteristic identity, both $n > 2$ extensions of the standard Poisson structure have an additional property that has not been required for GLAs and n -Lie algebras: they satisfy Leibniz's rule²³. The skewsymmetry of the n -ary Poisson bracket and the requirement of the Leibniz's rule are tantamount to saying that the two corresponding n -ary PS generalizations may be defined through a skewsymmetric multivector field.

13.1. Standard Poisson structures: a short summary.

Definition 81. (*Poisson structure*) (PS)

Let M be a manifold and $\mathcal{F}(M)$ the ring of smooth functions on M . A Poisson bracket on $\mathcal{F}(M)$ is a bilinear mapping $\{\cdot, \cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$ such that, for any three functions $f, g, h \in \mathcal{F}(M)$, satisfies

a): Skewsymmetry

$$\{f, g\} = -\{g, f\} \quad , \quad (271)$$

b): Leibniz's rule,

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \quad , \quad (272)$$

c): Jacobi identity

$$\frac{1}{2}\text{Alt}\{f, \{g, h\}\} = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \quad . \quad (273)$$

The JI eq. (273) may be read as stating that the full skewsymmetrization of the double PB is zero or as an expression of the derivation property,

$$\{H, \{g, h\}\} = \{\{H, g\}, h\} + \{g, \{H, h\}\} \quad , \quad (274)$$

which is behind the evolution of a mechanical system with hamiltonian H .

Let $\{x^i\}$ be local coordinates on the manifold M and $f, g \in \mathcal{F}(M)$. Conditions a), b) and c) mean that it is possible to characterize a specific Poisson structure by defining the PB in terms of some specific functions $\omega^{ij}(x)$. This is done by defining the PB by

$$\{f(x), g(x)\} = \omega^{ij}\partial_i f\partial_j g \quad , \quad (275)$$

which clearly guarantees that Leibniz's rule b) is satisfied, with $\omega^{ij}(x)$ subject to the conditions

$$\omega^{ij} = -\omega^{ji} \quad , \quad \omega^{jk}\partial_k\omega^{lm} + \omega^{lk}\partial_k\omega^{mj} + \omega^{mk}\partial_k\omega^{jl} = 0 \quad , \quad (276)$$

which take care of a) and c). Since the JI is fully antisymmetric in f, g, h , the *differential condition* in eq. (276) does not contain any second order derivatives.

A PB on $\mathcal{F}(M)$ defines a Poisson structure (PS) on M , usually denoted by the pair (M, ω) . It is possible to define a PS by means of a bivector field or *Poisson bivector*

$$\Lambda = \frac{1}{2}\omega^{jk}\partial_j \wedge \partial_k \quad , \quad (277)$$

²³Since $[XY, Z] = X[Y, Z] + [X, Z]Y$ within the enveloping algebra $\mathcal{U}(\mathfrak{g})$, one may apply similar factorizations for the GLA \mathcal{G} multibracket resolutions in terms of two-brackets. Thus, since the composition of elements in the multibrackets (87) is associative but not commutative, these GLAs do not satisfy a property similar to b) in eq. (283) as it is readily seen from the four-bracket resolution in eq. (97). In contrast, the infinite-dimensional GLAs given by the GPS in Sec. 13.2 incorporate Leibniz's rule in their definition. Similarly, the FAs defined by the Nambu bracket do satisfy Leibniz's rule; other FAs, as Ex. 52, do not. Leibniz's rule is a strong condition: other ternary -say- structures, such as those given in terms of the associator in Sec. 2.2, do not respect it either as already pointed out by Nambu [38].

which takes care of a) and b) above and where $\omega^{ij}(x)$ satisfies eq. (276) so that the PB given by

$$\{f, g\} = \Lambda(df, dg) \quad . \quad (278)$$

does satisfy the JI.

Alternatively, a two-vector Λ defines [215] a PS *i.e.*, it is a Poisson bivector, if it has a vanishing Schouten-Nijenhuis bracket²⁴ [160, 161] with itself,

$$[\Lambda, \Lambda] = 0 \quad , \quad (279)$$

since this condition reproduces (276). Since a Poisson structure on a manifold M is defined by a Poisson bivector Λ , it is denoted (Λ, M) .

Definition 82. (*Linear Lie-Poisson structure*)

When the manifold M is the vector space dual to that of a finite Lie algebra \mathfrak{g} , there always exists a PS. It is obtained by defining the fundamental Poisson bracket $\{x_i, x_j\}$ (where $\{x_i\}$ are coordinates on \mathfrak{g}^*). Since $\mathfrak{g} \sim (\mathfrak{g}^*)^*$, we may think of \mathfrak{g} as a subspace of the ring of smooth functions $\mathcal{F}(\mathfrak{g}^*)$. Then, the Lie algebra commutation relations

$$\{x_i, x_j\} = C_{ij}^k x_k \quad (280)$$

define, by assuming b) above, a mapping $\mathcal{F}(\mathfrak{g}^*) \times \mathcal{F}(\mathfrak{g}^*) \rightarrow \mathcal{F}(\mathfrak{g}^*)$ associated with the two-vector

$$\Lambda = \frac{1}{2} C_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \quad (281)$$

This bivector defines a PS since condition (276) (or (279)) is simply the JI for the structure constants of \mathfrak{g} . The resulting PS is called a *Lie-Poisson* structure.

Definition 83. (*Compatible Poisson structures*)

Two Poisson bivectors Λ, Λ' are called *compatible* if the SNB among themselves is zero,

$$[\Lambda, \Lambda'] = 0 \quad . \quad (282)$$

The compatibility condition is equivalent to requiring that any linear combination $\lambda\Lambda + \mu\Lambda'$ of two Poisson bivectors be a Poisson bivector.

Lemma 84. (*Poisson cohomology*)

A Poisson bivector Λ defines a coboundary operator δ_Λ acting on multivectors A by $\delta_\Lambda : A \mapsto [\Lambda, A]$, where the bracket is the SNB. The resulting cohomology is called *Poisson cohomology* [215] (see also [91]).

Proof. It is sufficient to notice that the operator δ_Λ is nilpotent since, on account of eq. (279) and eq. (395) in the Appendix 2, $\delta_\Lambda^2 A = [\Lambda, [\Lambda, A]] = 0$ on any A (see further [91]). \square

To conclude this section, we would like to recall that the Lie (Sec. 2), Loday/Leibniz (Sec. 4) and Poisson brackets are just important examples of a larger variety of brackets with two entries. For an analysis of various related ‘two’-structures see *e.g.* [94] and references therein.

We now move to discuss two specific generalizations of the PS, the *generalized Poisson structures* [4, 5] and the *Nambu-Poisson structures* [41].

²⁴The definition and properties of the Schouten-Nijenhuis bracket are given in Appendix 2.

13.2. Generalized Poisson structures.

Since eq. (279) for the ordinary PS is simply the JI for a Lie algebra \mathfrak{g} , it is natural to introduce [4, 5] in the even case a generalization of the PS by means of the following

Definition 85. (*Generalized Poisson Structure (GPS)*)

Let n be even. A GPS is defined [4, 5] by an n -linear mapping $\{ \cdot, \dots, \cdot \} : \mathcal{F}(M) \times \dots \times \mathcal{F}(M) \rightarrow \mathcal{F}(M)$, the *generalized Poisson bracket* (GPS) of functions on the manifold M , satisfying the properties

$$\begin{aligned} \text{a) } \{f_1, \dots, f_i, \dots, f_j, \dots, f_n\} &= -\{f_1, \dots, f_j, \dots, f_i, \dots, f_n\} \quad (\text{skewsymmetry}), \\ \text{b) } \{f_1, \dots, f_{n-1}, gh\} &= g\{f_1, \dots, f_{n-1}, h\} + \{f_1, \dots, f_{n-1}, g\}h \quad (\text{Leibniz's rule}). \end{aligned} \quad (283)$$

plus the GJI,

$$\begin{aligned} \text{c) } \text{Alt}\{f_1, \dots, f_{2s-1}, \{f_{2s}, \dots, f_{4s-1}\}\} &\equiv \\ \sum_{\sigma \in S_{4s-1}} (-1)^{\pi(\sigma)} \{f_{\sigma(1)}, \dots, f_{\sigma(2s-1)}, \{f_{\sigma(2s)}, \dots, f_{\sigma(4s-1)}\}\} &= 0 \quad , \end{aligned} \quad (284)$$

which obviously corresponds to eq. (93). In fact, the GPS are an example of infinite-dimensional higher order algebras.

The geometrical nature of the above definition is exhibited by the following

Lemma 86. (*GPS multivectors or GPS tensors*)

An $n = 2s$ even multivector $\Lambda_{(2s)}$,

$$\Lambda_{(2s)} = \frac{1}{(2s)!} \omega_{i_1 \dots i_{2s}} \partial^{i_1} \wedge \dots \wedge \partial^{i_{2s}} \quad , \quad (285)$$

defines a GPS $(\Lambda_{(2s)}, M)$ i.e., is a GPS tensor, iff it has zero SN bracket with itself,

$$[\Lambda_{(2s)}, \Lambda_{(2s)}] = 0 \quad , \quad (286)$$

where $[\cdot, \cdot]$ again denotes the SNB. The generalized Poisson bracket (GPB) is then given by [4, 5]

$$\Lambda_{(2s)}(df_1, \dots, df_{2s}) = \{f_1, \dots, f_{2s}\} = \omega_{i_1, \dots, i_{2s}} \partial^{i_1} f_1 \dots \partial^{i_{2s}} f_{2s} \quad . \quad (287)$$

Proof. First we notice that the Schouten-Nijenhuis bracket (Appendix 2) of an odd n -multivector with itself $[\Lambda_{(n)}, \Lambda_{(n)}]$ vanishes identically and hence the above condition would be empty for n odd.

The fact that $\Lambda_{(2s)}$ is a multivector field automatically guarantees the skewsymmetry of the GPB and the Leibniz rule: the tensorial character of Λ implies that it is linear on functions and then $\Lambda_{(2s)}(d(fg), df_2, \dots, df_{2s}) = g\Lambda_{(2s)}(df, df_2, \dots, df_{2s}) + f\Lambda_{(2s)}(dg, df_2, \dots, df_{2s})$. Thus, to see that $\Lambda_{(2s)}$ defines a GPS is sufficient to check that the GJI is satisfied. Written in terms of the coordinate functions of $\Lambda_{(2s)}$, eq. (286) gives the *differential condition* for the coordinates of $\Lambda_{(2s)}$,

$$\epsilon_{i_1 \dots i_{4s-1}}^{j_1 \dots j_{4s-1}} \omega_{j_1 \dots j_{2s-1}} \sigma \partial^\sigma \omega_{j_{2s} \dots j_{4s-1}} = 0 \quad \text{or} \quad \omega_{\sigma[j_1 \dots j_{2s-1}]} \partial^\sigma \omega_{j_{2s} \dots j_{4s-1}} = 0 \quad , \quad (288)$$

(clearly, for $s=1$ we are left with $\omega_{\sigma[j_1]} \partial^\sigma \omega_{j_2 j_3} = 0$, eq. (276)).

Now, the GJI eq. (284) implies, using eq. (287),

$$\begin{aligned} &\epsilon_{i_1 \dots i_{4s-1}}^{j_1 \dots j_{4s-1}} \{f_{j_1}, \dots, f_{j_{2s-1}}, \omega_{l_{2s} \dots l_{4s-1}} \partial^{l_{2s}} f_{j_{2s}} \dots \partial^{l_{4s-1}} f_{j_{4s-1}}\} \\ &= \epsilon_{i_1 \dots i_{4s-1}}^{j_1 \dots j_{4s-1}} \omega_{l_1 \dots l_{2s-1}} \sigma \partial^{l_1} f_{j_1} \dots \partial^{l_{2s-1}} f_{j_{2s-1}} (\partial^\sigma \omega_{l_{2s} \dots l_{4s-1}} \partial^{l_{2s}} f_{j_{2s}} \dots \partial^{l_{4s-1}} f_{j_{4s-1}} \\ &\quad + 2s \omega_{l_{2s} \dots l_{4s-1}} \partial^\sigma \partial^{l_{2s}} f_{j_{2s}} \partial^{l_{2s+1}} f_{j_{2s+1}} \dots \partial^{l_{4s-1}} f_{j_{4s-1}}) = 0 \quad , \end{aligned} \quad (289)$$

where the last summand groups $2s$ terms that become equal after suitable index relabelling. The second term in the *r.h.s* vanishes because the part multiplying $\partial^\sigma \partial^{l_{2s}} f_{j_{2s}}$ is

antisymmetric with respect to the interchange $\sigma \leftrightarrow l_{2s}$. Thus, the second order derivatives disappear automatically and we are left with (288) because the functions $f_{j_1}, \dots, f_{j_{4s-1}}$ are arbitrary. \square

Lemma 87. (*Generalized Poisson cohomology [5]*)

A a Poisson $2s$ -vector $\Lambda_{(2s)}$ defines a generalized Poisson cohomology, with coboundary operator acting on multivectors $\delta_{\Lambda_{(2s)}} : \wedge^q(M) \rightarrow \wedge^{q+(2s-1)}(M)$ by $\delta_{\Lambda_{(2s)}} : A \mapsto [\Lambda_{(2s)}, A]$.

Proof. It suffices to extend Lemma 84 to the GPS. \square

Remark 88. (*Odd GPS*)

GP structures, like GLAs, are naturally defined for n even. Nevertheless, it is also possible to define a GPS in the *odd* case [159]. For a GPB with an odd number n of arguments, the second summand in the *r.h.s.* of (289) does not vanish, giving rise to another, algebraic condition that has to be satisfied by the coordinates of the GPS multivector. An *algebraic condition* will always be present for the N-P structures [41] below, see Lemma 91).

The above constructions and GPS may also be extended to the \mathbb{Z}_2 -graded (‘supersymmetric’) case [63], including the definition of the *graded GPS tensors*, that may be introduced through a suitable graded SN bracket [63] (see [216] for another construction).

The generalized Poisson cohomology and homology was further considered in [159] (see also [62]).

13.3. Higher order or generalized linear Poisson structures.

It is easy to construct examples [5] of GPS (infinitely many, in fact) in the linear case by extending the construction in Def. 82. They are obtained by applying the earlier construction of GLAs to the GPS. Let \mathcal{G} be a simple Lie algebra of rank l . We know from Table 1 and Theorem 28 that associated with \mathcal{G} there are $(l - 1)$ simple higher order Lie algebras. Their structure constants define a GPB $\{ \cdot, \dots, \cdot \} : \mathcal{G}^* \times \dots \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ by

$$\{x_{i_1}, \dots, x_{i_{2m_l-2}}\} = \Omega_{i_1 \dots i_{2m_l-2}}{}^\sigma x_\sigma \quad , \quad (290)$$

where Ω is the $(2m_l - 1)$ -cocycle for the Lie algebra cohomology of \mathcal{G} associated with an invariant polynomial of rank m_l .

If one now computes the GJI (288) for $\omega_{i_1 \dots i_{2m_l-2}} = \Omega_{i_1 \dots i_{2m_l-2}}{}^\sigma x_\sigma$, or, alternatively, the SNB $[\Lambda, \Lambda]$ for the $(2m - 1)$ -vector

$$\Lambda = \frac{1}{(2m - 2)!} \Omega_{i_1 \dots i_{2m-2}}{}^\sigma x_\sigma \partial^{i_1} \wedge \dots \wedge \partial^{i_{2m-2}} \quad , \quad (291)$$

one finds that $[\Lambda, \Lambda] = 0$ due to the GJI for the higher order structure constants $C = \Omega$ given in (98) and Λ above is called a *linear GPS tensor*. This means that all these simple higher order Lie algebras associated with a simple \mathfrak{g} define linear GP structures (in fact, and more generally, it is also true that the SNB of two $(2m-2)$ - and $(2m'-2)$ -multivectors Λ, Λ' constructed from two $(2m-1)$ -, $(2m'-1)$ -cocycles Ω, Ω' as in eq. (291) is zero, $[\Lambda, \Lambda'] = 0$: [5], Th. 8.3). Conversely, given a linear GPS with fundamental GPB (290), there is a higher order Lie algebra \mathcal{G} associated with it.

It is clear from the discussion in [4, 5, 3], however, that not only the basic, primitive invariants explicitly considered in Sec. 6.1 (and that led to the simple GLAs classification of Th. 28) can be used to generate linear GPS (and GLAs). In fact, it is easy to check (using more general odd antisymmetric invariants, the constructions in [4, 5, 3] and the properties in eqs. (399), (400) of the SN bracket) that these odd polynomials also determine GLAs and linear GPS. This

has also been pointed out independently in [217] using a different approach based on general super-Poisson methods.

We conclude this section by noting that the Jacobi structures considered by Lichnerowicz, which are defined using Jacobi brackets (a generalization of the standard Poisson bracket in which Leibniz's rule is replaced by a weaker condition), may be similarly extended to the n -ary case leading to *generalized Jacobi structures* [218].

Let us turn now to the generalization of the PS that has a FA structure.

13.4. Nambu-Poisson (N-P) structures.

As we saw in Sec. 7.7, Nambu considered [38] already in 1973 the possibility of extending the Poisson brackets of standard Hamiltonian mechanics to brackets of three functions defined by the Nambu Jacobian²⁵. Clearly, the Nambu bracket may be generalized further to a Nambu-Poisson (N-P) one allowing for an arbitrary number of entries.

The first two properties of the N-P bracket, skewsymmetry and Leibniz's rule, are shared with the standard PS and the GPS, and are again automatically guaranteed if the new bracket is defined in local coordinates $\{x_i\}$ on M in terms of an n -vector

$$\Lambda_{(n)} = \frac{1}{n!} \eta_{i_1 \dots i_n}(x) \partial^{i_1} \wedge \dots \wedge \partial^{i_n} \quad , \quad (292)$$

so that, as in (287), the N-P bracket is given by

$$\{f_1, \dots, f_n\} = \Lambda_{(n)}(df_1, \dots, df_n) \quad . \quad (293)$$

The key difference is the condition that generalizes the JI and that completes the definition of the N-P structures; this restricts the allowed $\eta_{i_1 \dots i_n}(x)$ above so that $\Lambda_{(n)}$ is a N-P tensor (Lemma 91 below). The identity behind the $n = 3$ generalized mechanics, which Nambu did not write in his paper [38], is the 'five-point identity' of Sahoo and Valsakumar [39, 40]. They introduced this relation (which is the FI of the Nambu FA \mathfrak{N}) as a necessary consistency requirement for the time evolution of Nambu mechanics. In the general case of N-P brackets with n entries the corresponding Filippov '($2n-1$)-point identity' was written by Takhtajan [41], who studied it in detail and called it the *fundamental identity*. This is simply the FI of the infinite-dimensional n -Lie algebra \mathfrak{N} in Sec. 7.7, already anticipated by Filippov for the n -bracket of functions defined by the Jacobian [16, 42]. Thus, a general N-P structure is given by the following

Definition 89. (*Nambu-Poisson structures*) [41]

A N-P structure is defined by a N-P bracket with n -entries that satisfies the conditions of eq. (283) plus the FI

$$\begin{aligned} \text{c) } \quad \{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} &= \{\{f_1, \dots, f_{n-1}, g_1\}, g_2, \dots, g_n\} \\ &+ \{g_1, \{f_1, \dots, f_{n-1}, g_2\}, g_3, \dots, g_n\} + \dots + \{g_1 \dots, g_{n-1}, \{f_1, \dots, f_{n-1}, g_n\}\} \end{aligned} \quad (294)$$

for the Filippov algebra \mathfrak{N} defined by the N-P n -bracket. To each N-P bracket satisfying the conditions of the definition corresponds a N-P n -tensor such that eq. (293) is satisfied. The manifold M is called a *Nambu-Poisson manifold* $(\Lambda_{(n)}, M)$.

²⁵When the quark model of Gell-Mann and Zweig was proposed (1964), the Fermi statistics for quarks did not mix well with the expected symmetry properties of the ground state wavefunction of the Δ^{++} particle; alternatives were searched for at the time, as Greenberg paraquarks (1964) and Han-Nambu quarks (1965). As is well known, the final answer -QCD- was proposed by Gell-Mann, Fritzsche and Leutwyler in 1971-73; in 1973, Nambu and Han were also analyzing the various existing proposals [219]. It seems that exploring new dynamical alternatives in this context was one of the motivations underlying Nambu mechanics [38].

N-P structures [41] have been discussed in many papers; see [149] and *e.g.* [152, 10, 21, 159, 48, 61, 62, 213, 49, 50, 220, 221] for further information.

The N-P n -bracket, eq. (293), establishes a linear correspondence between $(n - 1)$ -forms and vector fields,

$$df_1 \wedge \cdots \wedge df_{n-1} \mapsto X_{f_1, \dots, f_{n-1}} \quad ,$$

defined by

$$dg(X_{f_1, \dots, f_{n-1}}) = X_{f_1, \dots, f_{n-1}} \cdot g = \Lambda_{(n)}(df_1, \dots, df_{n-1}, dg) = \{f_1, \dots, f_{n-1}, g\} \quad .$$

The hamiltonian vector field $X_{f_1, \dots, f_{n-1}}$ (see Ex. 57) is indeed a vector field as a consequence of the tensorial character of $\Lambda_{(n)}$. Its physical significance is clear: in N-P mechanics, the time evolution of a dynamical magnitude F is determined through $(n - 1)$ ‘Hamiltonians’ H_1, \dots, H_{n-1} by

$$\dot{F} := X_{H_1, \dots, H_{n-1}} \cdot F = \{H_1, \dots, H_{n-1}, F\} \quad \forall F \quad . \quad (295)$$

This expression is the evident dynamical counterpart (eq. (134)) of the $ad_{\mathcal{X}}$ derivation of the n -FAs: due to the FI, $X_{H_1, \dots, H_{n-1}}$ is a derivation of the N-P bracket: the fundamental objects of \mathfrak{N} define inner derivations $ad_{H_1 \dots H_{n-1}}$ of the N-P algebra \mathfrak{N} since the FI implies [41]

$$\frac{d}{dt} \{f_1, \dots, f_n\} = \{\dot{f}_1, \dots, f_n\} + \{f, \dot{f}_2, \dots, f_n\} \cdots + \{f_1, \dots, \dot{f}_n\} \quad . \quad (296)$$

Thus, the FI guarantees that the bracket of any n constants of the motion is itself a constant of the motion.

It follows that a N-P structure is defined by a multivector $\Lambda_{(n)}$ such that the hamiltonian vector fields are a derivation of the N-P n -bracket algebra. It is also easy to see that such a N-P tensor is invariant under the action of a hamiltonian vector field, $L_X \Lambda_{(n)} = 0$. This follows from the FI since

$$\begin{aligned} X_{f_1, \dots, f_{n-1}} \cdot \{g_1, \dots, g_n\} &= L_{X_{f_1, \dots, f_{n-1}}} (\Lambda_{(n)}(dg_1, \dots, dg_n)) \\ &= (L_{X_{f_1, \dots, f_{n-1}}} \Lambda_{(n)})(dg_1, \dots, dg_n) + \sum_{i=1}^n \Lambda_{(n)}(dg_1, \dots, L_{X_{f_1, \dots, f_{n-1}}} dg_i, \dots, dg_n) \\ &= (L_{X_{f_1, \dots, f_{n-1}}} \Lambda_{(n)})(dg_1, \dots, dg_n) + \sum_{i=1}^n \{g_1, \dots, \{f, \dots, f_{n-1}, g_i\}, \dots, g_n\} \quad , \end{aligned}$$

using in the second line that the Lie derivative commutes with contractions and in the third that $L_{X_{f_1, \dots, f_{n-1}}} dg = dL_{X_{f_1, \dots, f_{n-1}}} g = d\{f_1, \dots, f_{n-1}, g\}$. Thus, we have the following

Proposition 90.

The hamiltonian vector fields determine infinitesimal automorphisms of the N-P structure $(\Lambda_{(n)}, M)$.

Lemma 91. *(Conditions that define a N-P tensor) [41]*

Eqs. (292), (293) in eq. (294) determine²⁶ two conditions that the local coordinates $\eta_{i_1 \dots i_n}$ of an n -vector $\Lambda_{(n)}$ have to satisfy to define a N-P structure i.e., to be a N-P tensor. The first one is the differential condition,

$$\eta_{i_1 \dots i_{n-1} \rho} \partial^\rho \eta_{j_1 \dots j_n} - \frac{1}{(n-1)!} \epsilon_{j_1 \dots j_n}^{l_1 \dots l_n} (\partial^\rho \eta_{i_1 \dots i_{n-1} l_1}) \eta_{\rho l_2 \dots l_n} = 0 \quad . \quad (297)$$

The second is the algebraic condition. It reads

$$\Sigma + P(\Sigma) = 0 \quad , \quad (298)$$

²⁶There is a trivial misprint (a term obviously missing) in eq. (5) of [41], accounted for in [45], eq. (10).

where Σ is the $(n+n)$ -tensor

$$\begin{aligned} \Sigma_{i_1 \dots i_n j_1 \dots j_n} = & \eta_{i_1 \dots i_n} \eta_{j_1 \dots j_n} - \eta_{i_1 \dots i_{n-1} j_1} \eta_{i_n j_2 \dots j_n} - \eta_{i_1 \dots i_{n-1} j_2} \eta_{j_1 i_n j_3 \dots j_n} \\ & - \eta_{i_1 \dots i_{n-1} j_3} \eta_{j_1 j_2 i_n j_4 \dots j_n} - \dots - \eta_{i_1 \dots i_{n-1} j_n} \eta_{j_1 j_2 \dots j_{n-1} i_n}, \end{aligned} \quad (299)$$

and P is the permutation operator that interchanges its i_1 and j_1 indices.

Eq. (299) may be rewritten as

$$\Sigma_{i_1 \dots i_n j_1 \dots j_n} = \frac{1}{n!} \epsilon_{i_n j_1 \dots j_n}^{l_1 \dots l_{n+1}} \eta_{i_1 \dots i_{n-1} l_1} \eta_{l_2 \dots l_{n+1}} \quad .$$

Eq. (298) follows²⁷ from requiring the vanishing of the second derivatives in (294), which now do not vanish automatically as for the GPS. Of course, for a standard ($n=2$) PS eq. (297) reproduces eq. (276) and eq. (298) is absent. Thus, the FI is much more constraining than the JI and, as a result, N-P $n \geq 3$ structures are more rigid than the standard $n=2$ Poisson ones (see Lemma 92 below). Conditions (297) and (298) play for the Nambu-Poisson structures the rôle of eq. (288) for the GPS, which follows from the geometrical requirement that the $2s$ -vector that defines a GPS has a vanishing SN bracket with itself [4, 5].

The algebraic condition imposes severe restrictions on the potential N-P tensors for $n > 2$ (for $n=2$, eq. (298) is trivial). It turns out [46, 149, 48] (see also [222, 213, 49]) that (contrarily to initial expectations [41] but confirming a later conjecture [45]), condition (298) implies that the Nambu-Poisson n -vector $\Lambda_{(n)}$ in (292) is *decomposable*. This means that $\Lambda_{(n)}$ can be written as the exterior product of vector fields, a result that Takhtajan has traced to follow from the Weitzenböck condition (see [223], p. 116) in the theory of invariants. Specifically, the N-P tensors satisfy the following

Lemma 92. (*Decomposability of the N-P tensors*)

If $\Lambda_{(n)}$ is a N-P tensor of order $n \geq 3$ on a manifold M , there are local coordinates on $U \subset M$, $\{x^1, \dots, x^n, x^{n+1}, \dots, x^m\}$, such that for $x \in U$,

$$\Lambda_{(n)} = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \cdots \wedge \frac{\partial}{\partial x^n} \quad ,$$

and reciprocally.

This is the form of the canonical the N-P multivector in Euclidean space \mathbb{R}^n that defines the N-P bracket of n $f_i(x_1, \dots, x_n) \in \mathcal{F}(M)$

$$\Lambda_{(n)} = \frac{1}{n!} \epsilon_{i_1 \dots i_n}^{1 \dots n} \partial^{i_1} \wedge \cdots \wedge \partial^{i_n} \quad , \quad \{f_1, \dots, f_n\} = \Lambda_{(n)}(df_1, \dots, df_n) = \left| \frac{\partial(f_1, \dots, f_n)}{\partial(x_{i_1}, \dots, x_{i_n})} \right| \quad , \quad (300)$$

expressed in terms of the Jacobian (Ex. 56). Further, any N-P bracket such as the one given in eq. (175) may be written in the canonical Nambu Jacobian form above using a suitable local coordinate system (see [49]). The net result is that, as mentioned, $n \geq 3$ N-P structures are extremely rigid; the canonicity of the Nambu bracket above parallels the uniqueness of the simple FAs A_{n+1} . This is not surprising on account of the relation among the decomposability of N-P tensors and the Plücker conditions [50], a connection that was used [34] to show the unicity of the $n=3$ A_4 FA in the context of the Bagger-Lambert model of Sec. 14. In fact, one may say that the FAs \mathfrak{N} determined by the Nambu n -brackets above constitute the infinite-dimensional counterparts of the simple n -Lie algebras A_{n+1} , a point that will find a physical

²⁷The presence of the algebraic condition implies that a constant antisymmetric tensor, although it satisfies automatically eq. (297), is not necessarily a N-P tensor since it still has to satisfy eq (298). Also, the direct sum of N-P tensors is not a N-P tensor since it is not decomposable (see Lemma 92 below).

application in Sec. 15. It is thus not surprising that, already in his original paper [16], Filippov considered the Jacobian FAs (see further [42]).

Remark 93. (*Subordinated N-P structures*)

Following Prop. 43 it follows that, given a Nambu n -bracket, one may construct another one of order $(n - 1)$ by fixing one element in the original N-P n -bracket *i.e.*, by defining

$$\{f_1, \dots, f_{n-1}\} = \{f_1, \dots, f_{n-1}, g\} \quad (g \text{ fixed}) \quad ,$$

since the Nambu $(n - 1)$ -bracket in the *l.h.s* will fulfil the FI as in Prop. 43. As a result, a N-P tensor that defines a N-P structure $(\Lambda_{(n)}, M)$ of order n induces a family of $(n - 1)$ N-P tensors $\Lambda_{(n-1)}$ that define $(n - 1)$ -N-P brackets by the relation above so that $(\Lambda_{(n-1)}, M)$ is a N-P structure. Thus, as in the finite-dimensional case, if \mathfrak{N} is an n -FA, the above construction defines a \mathfrak{N}' Nambu subordinated $(n - 1)$ -FA.

Definition 94. (*Linear Nambu-Poisson structures*) [41]

A N-P tensor whose components are linear in x_i (*cf.* Def. 82), $\eta_{i_1 \dots i_n}(x) = \eta_{i_1 \dots i_n}{}^j x_j$, is called a *linear N-P tensor* and defines a *linear N-P structure*. The corresponding bracket is given by

$$\{x_{i_1}, \dots, x_{i_n}\} = \eta_{i_1 \dots i_n}{}^j x_j \quad . \quad (301)$$

The linear N-P structures of eq. (301) play for FAs the rôle of the linear (or Lie-) Poisson ones for Lie algebras. Any linear N-P structure of order n defined by the linear n -N-P tensor $\Lambda_{(n)} = f_{a_1 \dots a_n}{}^b x_b \partial_{a_1} \wedge \dots \wedge \partial_{a_n}$ induces an n -Lie algebra structure on the dual $(\mathbb{R}^m)^*$. The converse, however, may not hold, since a linear n -vector $\Lambda_{(n)} = f_{a_1 \dots a_n}{}^b x_b \frac{\partial}{\partial x_{a_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{a_n}}$, where the $f_{a_1 \dots a_n}{}^b$ are the structure constants of an n -Lie algebra, may give rise to a non-decomposable tensor. In other words, although the differential condition (297) for the n -tensor coordinates $\eta_{a_1 \dots a_n} = f_{a_1 \dots a_n}{}^b x_b$ becomes the FI of the FA (see eq. (154)) and is therefore satisfied, the algebraic condition may not hold, in which case $\Lambda_{(n)}$ does not define a N-P structure and therefore is not a N-P tensor.

The skewsymmetric tensor $\eta_{i_1 \dots i_n}(x) = \epsilon_{i_1 \dots i_n}{}^{i_{n+1}} x_{i_{n+1}}$ is a linear N-P tensor [41, 45]. Accordingly, it defines the *linear Poisson structure*

$$\{x_1, \dots, x_{i_{n+1}}\} = \epsilon_{i_1 \dots i_n}{}^{i_{n+1}} x_{i_{n+1}} \quad ,$$

which (see Sec. 7.5) is associated with A_{n+1} . Clearly, and in contrast with the higher order linear Poisson structures, only for $n = 2$ the above N-P structure corresponds to a linear *Lie-Poisson* structure, since it is only for the standard Poisson case that $\epsilon_{ij}{}^k$ are the *structure constants* of a Lie algebra, $su(2)$ (*cf.* eq. (290)). Since the linear Poisson structures of Def. 82 are called *Lie-Poisson* structures, the linear N-P structures above might be called as well *Filippov-Nambu-Poisson* structures.

Further analysis of linear N-P structures is given in [213, 224, 49] and refs. therein.

It is possible to construct Nambu-Poisson n -tensors on Lie groups G (in fact, left-invariant N-P tensors) by using the LI vector fields that generate a $(n \geq 3)$ -dimensional subalgebra \mathfrak{h} of the Lie algebra \mathfrak{g} of G . This is done by setting $\Lambda_{(n)} = X_1 \wedge \dots \wedge X_n$, where $\{X_i\}$ is a basis of \mathfrak{h} ; $\Lambda_{(n)}$ is then a LI N-P n -tensor [225]. In fact, there is a one-to-one correspondence between the set of the LI N-P n -tensors (up to a constant) on G and the set of n -dimensional subalgebras $\mathfrak{h} \subset \mathfrak{g}$. We shall not discuss this further and refer instead to [225] for details.

There is one question that remains to be answered: the possible connection between the *even* order GP and N-P structures. Writing the FI in the form of eq. (139)

$$\begin{aligned} & \{f_{i_1}, \dots, f_{i_{n-1}}, \{f_{i_n}, \dots, f_{i_{2n-1}}\}\} = \\ & = \{\{f_{i_1}, \dots, f_{i_{n-1}}, f_{i_n}\}, f_{i_{n+1}}, \dots, f_{i_{2n-1}}\} + \dots + \{f_{i_n}, \dots, f_{i_{2n-2}}, \{f_{i_1}, \dots, f_{i_{n-1}}, f_{i_{2n-1}}\}\} = \\ & (-1)^{n-1} \{f_{i_{n+1}}, \dots, f_{i_{2n-1}}, \{f_{i_1}, \dots, f_{i_{n-1}}, f_{i_n}\}\} + (-1)^{n-2} \{f_{i_n}, f_{i_{n+2}}, \dots, f_{i_{2n-1}}, \{f_{i_1}, \dots, f_{i_{n-1}}, f_{i_{n+1}}\}\} \\ & + \dots + (-1)^{n-n} \{f_{i_n}, \dots, f_{i_{2n-2}}, \{f_{i_1}, \dots, f_{i_{n-1}}, f_{i_{2n-1}}\}\} \end{aligned}$$

and contracting the first and last terms of the equality with $\epsilon^{i_1 \dots i_{2n-1}}$ it follows that

$$\epsilon^{i_1 \dots i_{2n-1}} \{f_{i_1}, \dots, f_{i_{n-1}}, \{f_{i_n}, \dots, f_{i_{2n-1}}\}\} = n(-1)^{n-1} \epsilon^{i_1 \dots i_{2n-1}} \{f_{i_1}, \dots, f_{i_{n-1}}, \{f_{i_n}, \dots, f_{i_{2n-1}}\}\} \quad ,$$

from which the GJI in eq. (284) follows (one may also look at the form of the FI and the GJI, for instance in eqs. (152), (99)). Thus, we may state the (expected) following result

Lemma 95.

Every Nambu-Poisson structure of even order is also a generalized Poisson structure, but the converse does not hold.

In fact, this Lemma also follows trivially from the decomposability of the N-P tensors, Lemma. 92, which automatically implies zero SN bracket $[\Lambda, \Lambda] = 0$.

13.5. Brief remarks on the quantization of higher order Poisson structures.

Setting aside the intrinsic mathematical interest of the two n -ary generalizations of the standard Poisson structure above discussed, a first question is to find examples of physical mechanical systems that might be described by them. It is fair to say that there are not too many, particularly for the GPS since the GJI does not reflect a derivation property of the multibracket. The reader interested in finding discussions of mechanical systems described by n -ary Poisson structures may look *e.g.*, at [38, 226, 39, 41, 44, 51, 227–229, 5] and references therein. In field theory, an $n = 3$ infinite-dimensional \mathfrak{N} algebra has recently appeared in the context of M-theory, as it will be described in Sec. 15.

The antisymmetry of the standard Poisson bracket is shared by the higher order N-P structures of Sec. 13.4 and the GPS of Sec. 13.2. As pointed out by Nambu himself [38], the antisymmetry property is necessary to have hamiltonians that are constants of the motion since the time evolution of a dynamical quantity F is governed by $\dot{F} = \{H_1, H_2, F\}$ or, in general [41], by $(n-1)$ hamiltonians, eq. (295). This is also the case for a mechanical system described by a GPS [4,5]; clearly, such a time evolution implies that all the hamiltonian functions are constants of the motion. The derivation property of the N-P bracket encoded in the FI makes the N-P bracket specially suitable for the differential equation describing the evolution of a dynamical quantity, while the lack of this property for the GPS, governed by the GJI, makes less obvious its application to mechanical systems²⁸. Another aspect of the n -ary structures is the Liouville theorem; both N-P mechanics and the linear GPS structures have an n -dimensional analogue (see [159]).

Let us conclude the discussion of n -ary Poisson structures with a few words on quantization, a word often used too loosely, at least from a physical point of view. For the purposes of this review, the case of quantizing standard Poisson structures may be considered ‘solved’ by using *e.g.* the Dirac approach even in the presence of constraints although, as already mentioned,

²⁸Nevertheless, although the GPB of constants of the motion is not a constant of the motion in general, a weaker result exists for any set of functions f_1, \dots, f_q , $q > 2s$, such that the functions in $(H_1, \dots, H_{2s-1}, f_1, \dots, f_{2s-1})$ are in *involution* (see [5], Th. 6.2). Under the restricting conditions of this theorem, one also has that the FI (296) is satisfied; see [5] for further discussion.

more sophisticated approaches to quantization exist. There was also a ‘generalized quantum dynamics’ [230] which, in principle, may lead directly to the equations of motion of the operators without going through the quantization of the classical theory. When moving to higher order, however, the difficulties appear already at a basic level. Indeed, the quantization of the Nambu-Poisson mechanics is fraught with serious difficulties, especially if the word ‘quantization’ is understood physically *i.e.*, as a general procedure that, starting from the classical dynamics of a physical system (as described by N-P mechanics), gives a quantum one that reproduces all the structural properties of the original classical system in the $\hbar \rightarrow 0$ limit. This implies, besides the skewsymmetry of the N-P n -bracket, Leibniz’s rule and the FI.

There is a simple argument against any elementary quantization of N-P mechanics which tries to keep the standard correspondence among dynamical quantities, their associated quantum operators and all the structural relations satisfied by their classical counterparts through the N-P brackets. It is physically natural to assume that the quantum operators \mathcal{O}_i corresponding to the different classical dynamical quantities O_i are associative. Then, it follows that a commutator $[\mathcal{O}_1, \dots, \mathcal{O}_{2s}]$ defined by the antisymmetrized sum of their products, as in eq. (87), will naturally lead to the GJI rather than to the FI. This problem is already underlined by the difficulty in finding matrix realizations of FAs (see Lemma. 55 and Ex. 52 for just two examples). The possibility of considering higher order (*e.g.* cubic) matrices has also been explored [174, 231, 232, 56] but again there are difficulties to mimic all properties of Nambu brackets. Further, for multibrackets of odd order n we saw in eq. (95) that the zero in the *r.h.s.* of the even GJI is replaced by a larger multibracket $[\mathcal{O}_1, \dots, \mathcal{O}_{2s-1}]$. In any case, multibrackets defined by the antisymmetric products of associative operators \mathcal{O} as in eq. (87) will lead to an identity which is *outside* the N-P algebraic scheme. As a result, finding a simple procedure of quantizing Nambu-Poisson mechanics with associative operators that keeps all the three properties of the N-P structure, and especially the FI (139) that regulates the time evolution of the system, is a problem likely without solution, although sophisticated quantization methods have been proposed²⁹. This inherent difficulty for quantizing Nambu-Poisson brackets while preserving their defining properties at the same time *i.e.*, of having a correspondence between the classical and quantized versions of the theory, has been pointed out repeatedly (see, *e.g.* [240, 41, 241, 222, 174, 242, 243]), starting with Nambu himself [38]. In this respect (but setting aside the question of the time evolution), the even GPS satisfy an identity, the GJI, more amenable to a quantum version.

We collect in Table 2 below the main properties of GP and N-P structures (see [60])

The situation may be summarized by saying that the associativity of the quantum operators, which implies the GJI that is more suitable for quantization, is not compatible with the derivation property of the N-P bracket; this last leads to the FI which, in turn, is inconvenient for a quantum version. Such a compatibility only exists for the standard Poisson structures to which both schemes reduce for $n=2$. In his paper, Nambu stated [38] that ‘quantum theory is pretty much unique although its classical analog may not be’. But one might as well take the point of view (see also [43]) that classical mechanics is pretty unique too if the term

²⁹The deformation quantization (star product) approach to Nambu mechanics was investigated in [222]. It was observed there that this approach does not provide a straightforward solution to the quantization problem of Nambu-Poisson structures in general. This led the authors to propose a peculiar modification of the deformation quantization they termed ‘Zariski quantization’ (see [233] in connection with M-theory).

For the deformation quantization approach and $*$ -products, see [234, 235, 64] plus the pioneering work by Berezin [236, 237]; see further [14]. Recent work on quantized Nambu-Poisson structures (in terms of non-commutative geometries) has been done in [238]. For a sourcebook on deformation quantization see [239].

	PS	N-P	GPS (even order)
Characteristic identity (CI):	eq. (273) (JI)	eq. (294) (FI)	eq. (284) (GJI)
Defining conditions:	eq. (276)	Eqs. (297),(298)	eq. (288)
Liouville theorem:	Yes	Yes	Yes
Poisson theorem:	Yes	Yes	No (in general)
CI realization in terms of associative operators:	Yes	No (in general)	Yes

Table 2. Some properties of the Poisson, Nambu-Poisson and generalized Poisson structures.

‘dynamical system’ is restricted to its physical rather than to a mathematical meaning, under which a system of differential equations may be judged sufficient to describe the ‘dynamics’ of a ‘system’. In any case, it seems clear that the quantization of higher order Poisson brackets requires renouncing to some of the standard steps towards quantum mechanics. Of course, all quantization procedures tend to hide the obvious fact that Nature is quantum *ab initio* and that, accordingly, the emphasis should rather be on finding the classical limit of quantum descriptions if these were readily available; the insistence on quantization schemes just reflects the fact that, unfortunately, this is not so.

Having said this, it is worth going back to the n even case, for which the GJI holds for associative operators. It is possible to avoid the loss of correspondence between the classical and the quantum versions of the theory if one considers that the Filippov identity is not the ‘fundamental’ one for the quantization of Nambu-Poisson [41] or Nambu’s generalized hamiltonian systems. The relative virtues of the Nambu bracket given by the Jacobian determinant (300) and that the GJI (93) may be combined if one accepts that the identity that must be satisfied by the quantized bracket is, as argued above, the GJI that follows from the full antisymmetrization. Further, since the above Jacobian determinant is given in terms of the Levi-Civita tensor, it is possible to solve the $n = 2s$ even Nambu bracket in terms of ordinary Poisson brackets much in the same way a multibracket with $2s$ entries is reducible to sums of products of s ordinary two-brackets³⁰. In this way, one may adopt -in fact, following Nambu³¹- that the quantum Nambu bracket is the fully antisymmetric higher order commutator and that it is the GJI of the multibracket, not the FI, the property that is relevant in quantization.

In fact, maximally superintegrable systems are all describable classically by both Hamiltonian, and, equivalently, Nambu mechanics [51, 245, 229]. Thus, their Hamiltonian quantization provides a check for their Nambu quantization. This point of view, which in a way takes the best of the two worlds, has been consistently advocated by Curtright and Zachos [51, 246, 65]. In it, the connection between the quantum bracket (the multibracket, satisfying the GJI and thus suitable for associative quantum operators) and the classical one (the Nambu bracket given by the Nambu Jacobian, eq. (300)) becomes evident. For instance, since in the classical $\hbar \rightarrow 0$

³⁰In general, the expansion of the even $n = 2p$ multibracket clearly mimics the expression of the Pfaffian of an antisymmetric $n \times n$ matrix A , $\text{Pf}(A_{ij}) = \frac{1}{2^p p!} \epsilon_{i_1, \dots, i_{2p}} A_{i_1 i_2} \cdots A_{i_{2p-1} i_{2p}}$, if the non-commutativity of the different two-brackets is ignored, which reduces the number of terms in expansions such as eq. (97) by $\frac{1}{p!}$. PB’s commute, and thus the above PB resolution is in obvious correspondence with the Pfaffian (as also noted by K. Bering, see [244]).

³¹See eqs. (33a) and (33b) in [38] for $n=3$, which are the same as eqs. (91); Nambu already appreciated the especial difficulties of the odd case.

limit $\frac{1}{i\hbar}[\mathcal{O}_1, \mathcal{O}_2] \rightarrow \{O_1, O_2\}$, for $n = 4$ we obtain from eq. (97) that, in that limit,

$$\begin{aligned} \frac{1}{2} \frac{1}{(i\hbar)^{\frac{n}{2}}} [\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4] &\rightarrow \{O_1, O_2\}\{O_3, O_4\} - \{O_1, O_3\}\{O_2, O_4\} + \{O_1, O_4\}\{O_2, O_3\} \\ &= \{O_1, O_2, O_3, O_4\} \end{aligned} \quad (302)$$

by using the decomposition of the $n = 4$ Jacobian into products of ordinary Poisson brackets given by the resolution of the $n=4$ Levi-Civita symbol in terms of products of $n=2$ ones; notice that, by proceeding in this way, one is taking the Nambu Jacobian as the *fundamental* property of Nambu mechanics.

The same correspondence clearly works in the higher order even³² case, for which

$$\frac{1}{(n/2)!} \frac{1}{(i\hbar)^{\frac{n}{2}}} [\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_n] \rightarrow \{O_1, O_2, \dots, O_n\} \quad (303)$$

in the classical limit. The first factorial appears because the reduction of the $n = 2s$ bracket to products of two-brackets contains terms that become the same in the classical limit which replaces commutators by Poisson brackets (*cf.* eqs. (97) and (302)) since the product of functions is commutative. We shall not carry the discussion any further and refer to the papers quoted in this section for details.

14. THE BAGGER-LAMBERT-GUSTAVSSON (BLG) MODEL

We now come to the last part of this review, the appearance of 3-Lie and Nambu FA structures in brane theory. We shall restrict ourselves to the original BLG proposal and to its Nambu bracket extension because of their higher structural simplicity. Other approaches will be mentioned, but perhaps it is fair to say that, at present, there is not a completely satisfactory answer to the questions mentioned below and later in Sec. 14.2.2. In the remaining sections the emphasis will be on the geometry of BLG-like models, rather than on their physical contents, as a way to illustrate the previous n -ary algebraic structures. Some other aspects, as *e.g.* the possible deformations of BLG and related theories (see [247–252] and refs. therein), will not be considered here.

The strong coupling limit of the IIA superstring theory is a $D = 11$ one, M-theory, the low energy limit of which is $D = 11$ supergravity [253, 254]. This admits a fully 32-supersymmetric solution with the geometry of $AdS_4 \times S^7$ and isometry group $OSp(8|4)$. To obtain some insight into the structure of the elusive M-theory it became important, due to the AdS/CFT correspondence [255, 256] (see [257–259] for reviews) to construct the action of the superconformal gauge theories that are AdS/CFT dual to M-theory on the above background. As discussed in [260], these theories were expected to be worldvolume $d = 3$ gauge theories coupled to massless matter with $\mathcal{N} = 8$ linearly realized supersymmetries and $OSp(8|4)$ superconformal symmetry, which is also the symmetry of the M-theory solution. Thus, excluding the possibility of singlets, they were to contain eight $d = 3$ real scalar fields, coming from the eight transverse coordinates of the M2-brane (8=11-[3 M2-worldvolume coord.]) plus 16 real (off-shell) $d = 3$ Goldstone fermions (the other 16 being removed by κ -symmetry) and, since they had a $\mathcal{N} = 8$ supersymmetry, they would present a natural $SO(8)$ R-symmetry. Since this corresponds on-shell to 8 (bosonic) =16/2 (fermionic) degrees of freedom, there is no room left for any more on-shell physical bosonic *d.o.f.* Therefore, it was proposed that the gauge fields should appear

³²It may be possible, in principle, to consider odd cases by embedding the odd $(2s - 1)$ -quantum brackets into even $2n$ quantum brackets [244] to reduce their quantization to the even case.

in the action through a Chern-Simons term and that, accordingly, the theory should be a supersymmetric extension of a gauge theory of Chern-Simons type. It seemed after the analysis in [260], however, that in spite of the theoretical grounds for the existence of such a theory, a superconformal action with the required properties (and specially the $\mathcal{N}=8$ superPoincaré invariance) could not exist. A few years after this apparent ‘no-go’ result, the ground-breaking work of Bagger and Lambert (and Gustavsson) to be described below showed that a $d = 3$ superconformal action with CS term and $\mathcal{N} = 8$ supersymmetry was possible after all, sparking a great interest on the subject.

14.1. Symmetry considerations and ingredients of the BLG model.

The BLG model [26,27,261,182,28] (see also [29] and [262] for recent work involving the background gauge fields of $D = 11$ supergravity) is a three-dimensional maximally supersymmetric superconformal gauge theory aimed, along the lines above, to describing the very low energy effective worldvolume theory of a system of N coincident M2-branes in the $D = 11$ spacetime of M-theory. Although the original goals were not reached in the form they were initially expected, the A_4 BLG model provided the first successful example of an interacting $d = 3$ gauge theory with $\mathcal{N} = 8$ linearly realized supersymmetries (*i.e.*, $d = 3$ maximally supersymmetric) and with superconformal symmetry $OSp(4|8)$. Further, the Noether currents associated with the BLG lagrangian generate [263] the $d = 3$, $\mathcal{N} = 8$ superPoincaré or ‘M2’-algebra with central charges that had already been discussed on general grounds [264],

$$\{Q_\alpha^p, Q_\beta^q\} = -2(\gamma^\mu \gamma^0)_{\alpha\beta} \delta^{pq} P_\mu + \epsilon_{\alpha\beta} Z^{[pq]} + (\gamma^\mu \gamma^0)_{\alpha\beta} Z_\mu^{(pq)} \quad (\mu = 0, 1, 2; \alpha, \beta = 1, 2; p, q, = 1, \dots, 8), \quad (304)$$

where the symmetric central charge is traceless, $\delta_{pq} Z_\mu^{(pq)} = 0$. This algebra has an obvious $SO(8)$ automorphism group under which the eight $d = 3$ two-component Majorana supercharges Q^a form a chiral $Spin(8)$ spinor. The $\binom{17}{2}$ degrees of freedom of the *l.h.s.* of eq. (304) split as $136 = 3 + \binom{8}{2} + 3(\binom{9}{2} - 1) = 3 + 28 + 3 \times 35$. Thus, the worldvolume zero- (one)-form $Z^{[pq]}$ ($Z_\mu^{(pq)}$) transforms under the **28** (**35⁺**) representation of $SO(8)$. In transverse space, these two central charges may also be understood, respectively, as a two- and a self-dual four-form ($\frac{1}{2} \binom{8}{4} = 35$).

The Bagger-Lambert construction was originally based on a three Filippov algebra structure, the properties of which were actually rediscovered from the physical requirements needed to build the model. Bagger and Lambert were led to a *three*-Lie algebra since they wanted to recover the Basu-Harvey equation [181] which may be formulated in terms of a three-bracket (see eq. (317)). Thus, the BLG theory appeared to provide an application of the Filippov algebra structure discussed in previous sections. We give below an outline of the original BL action and its actual relation to the simple euclidean A_4 3-Lie algebra. We shall nevertheless keep occasionally a generic $n = 3$ FA notation when there is no need of identifying \mathfrak{G} with A_4 , for which $f_{abc}{}^d = -\epsilon_{abc}{}^d$.

In order to write certain terms of the worldvolume lagrangian, including the kinetic ones, the $n = 3$ FA was required to be endowed with an invariant metric $\langle \cdot, \cdot \rangle$, so that eq. (169) (or (170)) is satisfied. Further, to avoid states with negative norm in the quantum theory, it was assumed that the metric was positive definite. It turned out that this determined completely the finite 3-Lie algebra [34, 35, 265] to be the simple euclidean A_4 one; of course, it is also possible to have a direct sum of multiple A_4 copies and trivial one-dimensional algebras, as conjectured in [56, 266]. There is, however, a simple argument leading to A_4 : if Lie \mathfrak{G} has to be semisimple, something one would require at least for a gauge group, \mathfrak{G} has to be reductive by Th. 51. Removing then a possible, uninteresting centre, we are left with a semisimple \mathfrak{G} .

Positive definiteness -or the needed compactness of Lie \mathfrak{G} - leads then to a direct sum of A_4 copies and simplicity restricts the 3-Lie algebra to the euclidean A_4 as the only possibility.

To look at the fields of the BLG theory, let us assume that the M2-brane worldvolume coincides with the $D = 11$ hyperplane parametrized by the 0, 1, 2 spacetime coordinates; then, the remaining 3, \dots , 10 coordinates are transverse to the M2-brane. This splitting is preserved by a $SO(1, 2) \times SO(8)$ subgroup of $SO(1, 10)$. The fields describing a single M2-brane depend on the $d = 3$ Minkowski worldvolume membrane coordinates x^μ , $\mu = 0, 1, 2$, and are given by two sets of ‘matter’ fields, bosonic and fermionic, plus additional gauge fields. The bosonic fields describe the transverse fluctuations of the membrane and are given by eight worldvolume scalar transverse coordinate fields $X^I(x)$ labelled by $I = 3, \dots, 10$. The 16 fermionic fields are eight two-component worldvolume spinors that may be described in terms of a $D = 11$ 32-component Majorana spinor $\Psi(x)$ subject to the condition $\Gamma^{012}\Psi = -\Psi$, where the Γ ’s are the gamma matrices of the eleven-dimensional spacetime (Γ^μ, Γ^I). Since the $D = 11$ ‘ γ^5 ’ is the unity (there is no chirality in odd dimensions), spinors that are Γ^{012} chiral have also a definite $Spin(8)$ chirality since this is determined by $\Gamma^{34\dots 9(10)}$.

The M2-brane breaks half of the supersymmetries, and the preserved ones are taken to be chiral, $\Gamma^{012}\epsilon = \epsilon$. The antichiral Ψ fields are the goldstinos corresponding to the broken supersymmetries and the X^I are the Goldstone scalar fields that correspond to the eight broken translations. The Ψ fields have sixteen independent real components; from the point of view of the $Spin(1, 2) \times Spin(8)$ subgroup of $Spin(1, 10)$, they are bidimensional $SO(1, 2)$ spinors; the $\mathcal{N} = 8$ supersymmetries refer to this $d = 3$ worldvolume description (both the $SO(1, 2)$ and the chiral $SO(8)$ spinorial indices of Ψ are omitted). Thus, the eight scalars $X^I(x)$ and the eight fermionic spinors $\Psi(x)$ transform, respectively, under the eight-dimensional vector and chiral spinor representations of the R-symmetry group $SO(8)$ which preserves the M2 superalgebra (304).

In order to describe a stack of M2-branes, Bagger and Lambert introduced bosonic and fermionic fields taking values in a FA \mathfrak{G} (which, as mentioned, turned out to be A_4 in their first proposal). Thus, the matter fields of the BLG-type models $X^I(x) = X^{Ia}(x)e_a$, $\Psi(x) = \Psi^a(x)e_a$ ($a = 1, \dots, 4 = \dim \mathfrak{G}$), where $\{e_a\}$ is a basis of \mathfrak{G} , carry a $\dim \mathfrak{G}$ -dimensional representation of Lie \mathfrak{G} . As for the gauge fields A_μ^{ab} , they are Lie \mathfrak{G} -valued worldvolume vector fields with $A_\mu^{ab}(x) = -A_\mu^{ba}(x)$. The two indices ab refer (see Sec. 8.1.1) to those that determine the elements of Lie \mathfrak{G} through the fundamental objects of \mathfrak{G} . This means that, assuming simplicity ($\mathfrak{G} = A_4$), we have a one-to-one correspondence between fundamental objects $\mathcal{X} \in \wedge^2 \mathfrak{G}$ and elements $T_{ab} \in \text{Lie } \mathfrak{G}$, $(T_{ab})_c^d = f_{abc}^d$ so that $A = A^{ab}T_{ab}$, $A_c^d = A^{ab}f_{abc}^d$. Thus, in spite of being given through the structure constants f_{abc}^d of a *three*-Lie algebra, the vector fields A_μ (which may be seen as $\wedge^2 \mathfrak{G}$ -valued) are of course ordinary Lie algebra, Lie \mathfrak{G} -valued gauge fields. Thus, the gauge fields are in the adjoint representation of the gauge group as usual (but not the matter ones which, unlike in [260], take values in \mathfrak{G} itself). The fact that Lie \mathfrak{G} determines the gauge group explains the rôle and the physical importance of the Lie algebra associated with the FA \mathfrak{G} .

14.2. The BLG lagrangian.

The BLG model is given by the worldvolume lagrangian density [27,182] (see also [261,36,29])

$$\begin{aligned} \mathcal{L}_{BL} = & -\frac{1}{2} \langle \mathcal{D}_\mu X^I, \mathcal{D}^\mu X^I \rangle + \frac{i}{2} \langle \bar{\Psi}, \Gamma^\mu \mathcal{D}_\mu \Psi \rangle \\ & -g \frac{i}{4} \langle [\bar{\Psi}, X^I, X^J], \Gamma_{IJ} \Psi \rangle - \frac{g^2}{2 \cdot 3!} \langle [X^I, X^J, X^K], [X^I, X^J, X^K] \rangle \end{aligned}$$

$$+\frac{1}{g}\mathcal{L}_{CS} \quad . \quad (305)$$

The corresponding action may be split, following the above three lines, as

$$I_{BLG} = \int d^3x \mathcal{L}_{kin} + \int d^3x \mathcal{L}_{int} + \frac{1}{g} \int d^3x \mathcal{L}_{CS} . \quad (306)$$

The covariant derivative \mathcal{D}_μ above is defined for a generic \mathfrak{G} -valued matter field $V = V^a e_a$ by

$$(\mathcal{D}_\mu V)^a = \partial_\mu V^a - f_{cb}{}^a A_\mu^{cd} V^b , \quad (307)$$

and \mathcal{L}_{CS} has the form

$$\mathcal{L}_{CS} = \frac{1}{2} \epsilon^{\mu\nu\rho} \left(f_{abcd} A_\mu^{ab} \partial_\nu A_\rho^{cd} + \frac{2}{3} f_{cda}{}^g f_{efgb} A_\mu^{ab} A_\nu^{cd} A_\rho^{ef} \right) . \quad (308)$$

The Chern-Simons term \mathcal{L}_{CS} was called ‘twisted’ because it did not seem to have the standard CS expression (but see Sec. 14.4 below).

The BLG action is scale-invariant provided that the gauge fields have length dimension $A = L^{-1}$ and the constant g is dimensionless. Then, the kinetic terms for the worldvolume matter fields are also scale-invariant with $[X] = L^{-\frac{1}{2}}$ and $[\Psi] = L^{-1}$, the expected dimensions for a $d = 3$ theory with no dimensionful constants. The dimension of A , consistent with its rôle as part of a covariant derivative, would be unnatural for the kinetic term of a $d = 3$ field theory, but there is no such a term for the gauge field in eq. (305). It may be seen that these dimensions also fix the form of the possible interaction terms in the lagrangian, which cannot depend on any dimensional coupling constant. In spite of the appearance of a Chern-Simons term in the lagrangian and that standard CS terms are parity odd, the theory is parity invariant due to the composite nature of the ‘twisted’ \mathcal{L}_{CS} term as we shall see later.

14.2.1. Gauge and supersymmetry transformations.

The BLG action is invariant under both gauge symmetry and supersymmetry; it has $OSp(4|8)$ superconformal symmetry [267]. The non-propagating gauge fields A^{ab} are needed for the closure of the supersymmetry algebra transformations [27] which are given below but that will not be discussed here. By standard arguments [268, 269], the invariance of the quantum theory under ‘large’ gauge transformations implies the quantization of the coefficient of the CS term [182]. For the original A_4 BLG model, the geometry of the \mathcal{L}_{CS} term (sec. 14.4) leads to the quantization condition $\frac{1}{g} = \frac{k\hbar}{2\pi}$ where k is an integer. As a result of this quantization, the theory does not contain any continuous parameter, and thus it must be conformally invariant [182] to all orders of perturbation theory since there are no coupling constants to run. Further, it is possible to redefine the three-bracket and the gauge fields so that g disappears from eq. (305). In fact, it had been known for some time that three-dimensional Chern-Simons gauge theories were themselves conformally invariant [270, 271], both the pure gauge theories and those coupled to massless matter fields. The problem, thus, was how to incorporate the extended supersymmetries needed to give a dual description of M2-branes, and this is what the gauge BLG model succeeded in doing.

The gauge transformations of the different fields are given by (they will be rewritten in a more geometrically transparent way in Sec. 14.4)

$$\begin{aligned} \delta X^{Ia} &= \lambda^{cd} f_{cdb}{}^a X^{Ib} \\ \delta \Psi^a &= \lambda^{cd} f_{cdb}{}^a \Psi^b \\ \delta(f_{cdb}{}^a A_\mu^{cd}) &= \partial_\mu(f_{cdb}{}^a \lambda^{cd}) + 2f_{cdb}{}^a f_{efg}{}^c \lambda^{dg} A_\mu^{ef} \end{aligned} \quad (309)$$

These transformations actually correspond to the Lie \mathfrak{G} (Sec. 14.4) gauge group algebra. For $\mathfrak{G} = A_4$, $\text{Lie } A_4 = su(2) \oplus su(2)$ (Ex. 64); the gauge group of the $\mathcal{N} = 8$ BLG theory with positive definite metric is thus $SU(2) \times SU(2)$. Further, for A_4 , the $f_{cdb}{}^a$ can be removed in the third line of eq. (309) above and eq. (310) below.

The supersymmetry transformations are given by

$$\begin{aligned} \delta_\epsilon X^I &= i\bar{\epsilon}\Gamma^I\Psi \\ \delta_\epsilon\Psi &= \mathcal{D}_\mu X^I\Gamma^\mu\Gamma^I\epsilon - \frac{g}{3!}[X^I, X^J, X^K]\Gamma^{IJK}\epsilon \\ \delta_\epsilon(f_{cdb}{}^a A_\mu^{cd}) &= igf_{cdb}{}^a\bar{\epsilon}\Gamma_\mu\Gamma^I X^{Ic}\Psi^d, \end{aligned} \tag{310}$$

where the supersymmetry parameter ϵ has the standard dimensions $[\epsilon] = L^{\frac{1}{2}}$.

14.2.2. Physical considerations and ternary algebras.

It was soon realized that the non-trivial gauge symmetry of the model, $\text{Lie } A_4 = SU(2) \times SU(2)$ (short, *e.g.*, of a $SU(N)$ one), could not give rise to the moduli space of a stack of M2-branes (see [30, 29]) as initially hoped for. In fact, it was argued in [272, 273] that the A_4 BLG model at level $k = 1$ describes two M2-branes propagating in a \mathbb{R}^8/Z_2 orbifold background (for general k , on a ‘M-fold’). Thus, the original BL model does not describe a number $N > 2$ of M2-branes, a fact that might have been expected [274] from the smallness of its gauge group.

There is, nevertheless, the possibility of relaxing the assumption of positive definiteness of the metric. There are models with Lorentzian signature on the 3-Lie algebra [32, 275, 276], although these immediately raise the question of the unitarity of the quantized theory (see also [277] for this point and [278] for n -Lie algebras with Lorentzian metric). The Lorentzian FAs are obtained from a semisimple Lie algebra \mathfrak{g} from which the (fully antisymmetric) structure constants of the FA are constructed. As for the Lorentzian theory in [32], it can be recast as an ordinary gauge theory, but the Lie algebra associated with the FA is no longer semisimple, with its Levi factor given by the original semisimple \mathfrak{g} ; as a result, the theory has features reminiscent of those of WZW models [279] and gauge theories [280] based on non-semisimple Lie groups. It is possible to remove the ghosts of the 3-Lie Lorentzian theories, but the modification breaks the conformal invariance spontaneously and reduces them to maximally supersymmetric $d = 3$ YM theories [281, 282].

One may also relax the full antisymmetry of the three-bracket³³ as is the case of the hermitian algebras [30] to be mentioned below and that of the real ‘generalized three-algebras’ in [31] (see further [190, 250]); the gauge symmetries are generated, of course, by ordinary Lie groups. These algebras lead to superconformal field theories that accommodate a higher number of M2-branes in exchange for a reduced amount of supersymmetry with respect to the A_4 BLG model. Nevertheless, these and other algebras (as in [286]) will not be considered here, although we refer to [183] for a discussion of the Lie-algebraic structure of the hermitian [30] and real ‘generalized three-algebras’ [31]; see also Sec. 10 for the real case. To conclude the above discussion we will mention that, very recently, the algebras appearing in BLG-type models have been discussed in the context of Jordan-triple systems (*Lie*-triple systems were considered in Sec. 10); we just refer here to [202, 203] for further information.

Setting aside the above Filippov and related three-algebras approach, it has been shown [37] that, giving up the full $\mathcal{N} = 8$ manifest supersymmetry, it is possible to find $d = 3$ superconformal Chern-Simons theories with $U(N) \times U(N)$ (and $SU(N) \times SU(N)$) gauge groups, thus providing a more general description of the N M2-brane system (see [287] and [288]

³³Other aspects of three-Lie algebras have been discussed in [265, 283, 284, 275, 285]; see also [174] for early work.

for further discussion of the ABJM theory and [289] for a different, alternative approach to BLG theory). The family of ABJM models [37] have matter fields in the bi-fundamental representation of the gauge group and a double set of gauge fields in the adjoint; they present explicit $\mathcal{N} = 6$ supersymmetry and, furthermore, the 3-Lie algebra structure does not play any rôle at all in their formulation. The $SU(2) \times SU(2)$ gauge symmetry and the original BLG model appears as a particular $N = 2$ example of the sequence of theories in [37], for which the original $\mathcal{N} = 6$ supersymmetry of twelve real supercharges is enhanced to a $\mathcal{N} = 8$ one with sixteen (see also [267, 290]) because the fundamental representation of $SU(2)$ is equivalent to its conjugate. It is stated in [37] that at levels $k = 1$ and $k = 2$ the ABJM theories describe, respectively, the low energy limit of N M2-branes in a flat and on a \mathbb{R}^8/Z_2 space and, generically, on \mathbb{R}^8/Z_k at level k (see further [291] for the equivalence of the $U(2) \times U(2)$ ABJM theory at $k = 1, 2$ with the $\mathcal{N} = 8$ BLG one and [292] for the most general three-dimensional $\mathcal{N} = 5$ superconformal CS theories based on three algebras).

It was then shown by Bagger and Lambert [30] that it was possible to recover $\mathcal{N} = 6$ models of the type considered in [37] by making the three-algebra complex, relaxing the full skewsymmetry of the three-bracket by making it skewsymmetric and linear in the first two entries and antilinear in the last and, in so doing, by moving effectively from a 3-Lie algebra to a kind of complex (right) 3-Leibniz one (Sec. 9). In this approach, the standard Filippov identity changes to a ‘hermitean FI’ and accordingly its expression in terms of the structure constants due to the appearance of complex conjugation. The resulting generalized BL model [30] presents $\mathcal{N} = 6$ supersymmetry, $SU(4)$ R-symmetry and $U(1)$ global symmetry. The extended worldvolume superalgebra of the $\mathcal{N} = 6$ BL model has been computed in [293], where for a particular choice of the three-bracket the ABJM model superalgebra with central charges has been derived.

In order to describe a large number N or condensate of M2-branes, it is also possible to move to the infinite-dimensional three-algebra defined by the Nambu bracket. This can be done in a natural way that implies, in particular, making the range of the FA indices of the BLG model infinite as we will show in Sec. 15.

14.3. The BLG model and the Basu-Harvey (B-H) equation.

The B-H equation [181] arises as a BPS condition for the BLG theory; reproducing it was, in fact, one of its motivations [26, 182]. Indeed, the model given by the lagrangian (305) has a BPS solution that corresponds to the $D = 11$ M2-brane-M5-brane system as seen from M2, and the condition that such a bosonic configuration satisfies is the B-H equation [181], a counterpart of the Nahm equation [208] that appears in the D1-D3 system in $D = 10$ (for generalizations of The B-H equation see [294, 295] in the context of general calibrations, ref. [207] for arbitrary p -algebras and [296] for the B-H equations as ‘homotopy M-C equations’). This bosonic solution may be found in the usual way by requiring that it preserves some supersymmetries. These configurations saturate a *Bogomol’nyi*-type bound, which is a lower bound on the energy given in terms of some charges that, in general, appear in the supersymmetry algebra as central (when the Lorentz part is ignored). The invariance of the BLG action under the supersymmetry transformations (310) leads [263] to charges that generate the $d = 3$ $\mathcal{N} = 8$ supersymmetry algebra (304). The presence of central charges is due to the fact that the Lagrangian density is only quasi-invariant (invariant up to a worldvolume total derivative) under supersymmetry transformations, and for that reason they are topological (see [297]) *i.e.*, relevant in topologically non-trivial situations. Some of the topological central charges of the enlarged $d = 3$, $\mathcal{N} = 8$ superalgebra [264] appear in the Bogomol’nyi bound that corresponds to the M2-M5-brane brane system.

Let us see how the B-H equation appears when the preservation of a fraction of the supersymmetries is required; actually, it will be seen that the solutions of the B-H equation are $\frac{1}{2}$ -BPS solitons. The bosonic configurations searched for are of the type

$$\begin{aligned} M2 : & 0 \ 1 \ 2 \\ M5 : & 0 \ 1 \quad 3 \ 4 \ 5 \ 6 \quad , \end{aligned} \quad (311)$$

which means that the membranes are extended along the 1,2 spatial directions and that the spatial directions of the M5 brane worldvolume coincide with the (13456) hyperplane. Thus, x^2 is the coordinate of the M2-brane that is transverse to the M5-brane and x^1 parametrizes the M2-M5 intersection. In terms of the fields appearing in the BLG lagrangian (305), the configuration that describes the system (311) has to have X^3, X^4, X^5 and X^6 as the only non-constant coordinate fields X^I because these are the fivebrane coordinates transverse to the M2 worldvolume as seen in (311) so that $I = 3, \dots, 10 \rightarrow J = 3, 4, 5, 6$. Also, these fields must depend only on x^2 , which parametrizes the ‘distance’ of the M2 points to the M5 brane.

The condition that a fraction of the supersymmetries is preserved for a bosonic configuration, $\Psi = 0$, reduces to $\delta_\epsilon \Psi = 0$ in the second equation in (310), which gives

$$\partial_2 X^J \Gamma^2 \Gamma^J \epsilon - \frac{g}{6} [X^J, X^{\mathcal{J}}, X^{\mathcal{K}}] \Gamma^{\mathcal{J}\mathcal{K}} \epsilon = 0 \quad , \quad \mathcal{J}, \mathcal{J}, \mathcal{K}, \mathcal{L} = 3, 4, 5, 6 \quad , \quad (312)$$

where the gauge field is also taken to be zero ($\mathcal{D}_2 \rightarrow \partial_2$), an *ansatz* the consistency of which will be shown below. The B-H equation arises as a condition that the bosonic fields have to satisfy if the above equation has to have a non-trivial $\epsilon \neq 0$ solution. To see this, we notice that $\Gamma^{\mathcal{J}\mathcal{K}}$ can be written as

$$\Gamma^{\mathcal{J}\mathcal{K}} = \Gamma^{3456} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}} \Gamma^{\mathcal{L}} \quad , \quad (313)$$

where $\epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}}$ is completely antisymmetric and defined by $\epsilon^{3456} = 1$. Inserting eq. (313) in eq. (312) we obtain

$$\partial_2 X^J \Gamma^J \epsilon = \frac{g}{3!} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}} [X^J, X^{\mathcal{J}}, X^{\mathcal{K}}] \Gamma^{23456} \Gamma^{\mathcal{L}} \epsilon \equiv B^{\mathcal{L}} \Gamma^{23456} \Gamma^{\mathcal{L}} \epsilon \quad , \quad (314)$$

where the \mathfrak{G} -valued $B^{\mathcal{L}}$ above is introduced to simplify the expressions below. Since $\{\Gamma^J, \Gamma^{\mathcal{J}}\} = 2\delta^{\mathcal{J}J}$, it is seen that Γ^{23456} above commutes with the Γ^J ($J = 3, 4, 5, 6$) and squares to the unit matrix. Now, we compute

$$\begin{aligned} \langle \partial_2 X^J, \partial_2 X^{\mathcal{J}} \rangle \epsilon &= \langle \partial_2 X^J, \partial_2 X^{\mathcal{J}} \rangle \Gamma^J \Gamma^{\mathcal{J}} \epsilon = \langle \partial_2 X^J \Gamma^J, \partial_2 X^{\mathcal{J}} \Gamma^{\mathcal{J}} \rangle \epsilon \\ &= \langle \partial_2 X^J \Gamma^J, \Gamma^{23456} B^{\mathcal{J}} \Gamma^{\mathcal{J}} \rangle \epsilon = \langle \partial_2 X^J, B^{\mathcal{J}} \rangle \Gamma^{23456} \Gamma^{\mathcal{J}} \Gamma^J \epsilon \\ &= -\langle \partial_2 X^J, B^{\mathcal{J}} \rangle \Gamma^{23456} \Gamma^{\mathcal{J}} \Gamma^J \epsilon + 2 \langle \partial_2 X^J, B^{\mathcal{J}} \rangle \Gamma^{23456} \epsilon \\ &= -\langle B^{\mathcal{J}}, B^{\mathcal{J}} \rangle \epsilon + 2 \langle \partial_2 X^J, B^{\mathcal{J}} \rangle \Gamma^{23456} \epsilon \quad , \end{aligned} \quad (315)$$

where eq. (314) has been used in the second and forth lines. The above equality is equivalent to $\Gamma^{23456} \epsilon = \left(\frac{\langle \partial_2 X^J, \partial_2 X^{\mathcal{J}} \rangle + \langle B^{\mathcal{J}}, B^{\mathcal{J}} \rangle}{2 \langle \partial_2 X^J, B^{\mathcal{J}} \rangle} \right) \epsilon$ and, since $(\Gamma^{23456})^2 = 1$, implies that $\Gamma^{23456} \epsilon = \pm \epsilon$ and

$$\langle \partial_2 X^J \mp B^{\mathcal{J}}, \partial_2 X^{\mathcal{J}} \mp B^{\mathcal{J}} \rangle = 0 \quad . \quad (316)$$

Since the metric is positive definite, this means that the configurations that preserve a fraction of the supersymmetries (actually, half of the sixteen) have to satisfy

$$\partial_2 X^J - \frac{g}{3!} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}J} [X^{\mathcal{J}}, X^{\mathcal{K}}, X^{\mathcal{L}}] = 0 \quad , \quad (317)$$

selecting the upper sign.

We have still to check that the bosonic configurations that solve (317) are actual solutions of the bosonic field equations derived from the lagrangian (305). These are given by

$$\begin{aligned} \mathcal{D}^\mu \mathcal{D}_\mu X^I + \frac{g^2}{2} [X^J, X^K, [X^I, X^J, X^K]] &= 0, \\ f_{cdb}{}^a F_{\mu\nu}^{cd} &= -g f_{cdb}{}^a \epsilon_{\mu\nu\rho} X^{Jc} \mathcal{D}^\rho X^{Jd}, \end{aligned} \quad (318)$$

where F^{cd} are the curvatures corresponding to the gauge fields in (309), the form of which will be given in Sec. 14.4 below. For the configurations that we are considering, the above field equations become

$$\begin{aligned} \partial^2 \partial_2 X^J + \frac{g^2}{2} [X^J, X^K, [X^J, X^J, X^K]] &= 0, \\ f_{cdb}{}^a F_{01}^{cd} &= 0 = -g f_{cdb}{}^a X^{\mathcal{J}c} \partial^2 X^{\mathcal{J}d}. \end{aligned} \quad (319)$$

Inserting (317) into the first equation of (319) one obtains an identity. We check now the consistency of setting $A = 0$ in the covariant derivatives in eq. (312), (319). Since this implies $F = 0$, it requires that the *r.h.s.* of the last equation above be also zero. Using (317), the *r.h.s.* becomes

$$\begin{aligned} &\frac{g}{3!} f_{cdb}{}^a X^{\mathcal{J}c} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}\mathcal{J}} X^{\mathcal{J}e} X^{\mathcal{K}f} X^{\mathcal{L}g} f_{efg}{}^d \\ &= \frac{g}{3 \cdot 4!} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}\mathcal{J}} X^{\mathcal{J}c} X^{\mathcal{J}e} X^{\mathcal{K}f} X^{\mathcal{L}g} f_{[efg}{}^d f_{c]db}{}^a = 0, \end{aligned} \quad (320)$$

which is indeed zero by the FI (eq. (152)) (and, in this particular point, any \mathfrak{G} would do). Therefore, one may use ordinary rather than covariant derivatives in eq. (317).

Equation (317) is the B-H equation [181] as it was written in [26]. The original B-H equation has the form

$$\partial_2 X^{\mathcal{J}} - \frac{g}{3!} \epsilon^{\mathcal{J}\mathcal{K}\mathcal{L}\mathcal{J}} [G, X^{\mathcal{J}}, X^{\mathcal{K}}, X^{\mathcal{L}}] = 0, \quad (321)$$

where here $G, X^{\mathcal{J}}, X^{\mathcal{K}}, X^{\mathcal{L}}, G^2 = 1$, are matrices and the *four* entries bracket is the *multibracket* of a GLA *i.e.*, it is given by the complete antisymmetrization of the products of its entries, eq. (87). The matrix G (see [181] for its definition and its connection with the construction of the fuzzy 3-spheres) has the property $\{G, X^{\mathcal{J}}\} = 0$ and the $X^{\mathcal{J}}$ may be realized as $X^{\mathcal{J}} = X^{\mathcal{J}a} \Gamma_a$, as in Th. 55, where the two forms of the B-H equation, (317) and (321), are related.

Equation (317) is satisfied by the ‘fuzzy three-funnel’ solution, which is given by a fuzzy 3-sphere [181, 298] with a radius that increases as the worldvolume coordinate x^2 approaches the value ($x^2 = 0$, say) where the M5-brane is located with respect to M2, so that it matches with the 5-brane worldvolume field theory solution that describes the M2-M5 brane system from the M5-brane point of view and that is known to be [299] a self-dual string soliton. We refer to the literature (see [181] and references therein) for details. As stated earlier, the above supersymmetric fuzzy funnel solution saturates a Bogomol’nyi bound on the energy that involves the one-form ‘central’ charge of the supersymmetry algebra [264, 263].

To conclude this section, we note that eq. (317) has the structure of the MC-like equations (Sec. 11.3) that hold for an arbitrary n -Lie algebra; thus, the B-H equation, which describes M2-branes ending on M5 branes, are intimately tied to the FA structure. This is also the case for their $n = 2$ precedent, the Nahm equation [208], which is a MC-type equation for the D1-D3 system. The solution of the Nahm equation is a ‘fuzzy two-funnel’, a fuzzy two-sphere with a radius depending on the transverse distance from the D1 brane that gets larger as it approaches the D3 brane. In the light of other possibilities (see [300]), one might think of extending the above $n = 2, 3$ pattern to higher n generalizations [207] which in the present

context would be tied to a higher MC-type equation for the FA (see eq. (249)), but this will not be discussed here any further (see also [296]). Finally, we mention that both the Nahm and the B-H equations have been interpreted in terms of non-commutative geometry, where the realizations of the worldvolume coordinate functions of the D3-brane and the M5-brane satisfy, respectively, non-commuting relations $[X^i, X^j] = i\theta^{ij}$, $[X^i, X^j, X^k] = i\theta^{ijk}$, where the *r.h.s.* of the brackets that describe the quantum geometry over the D3- and the M5-branes are constant, completely antisymmetric matrices; see [301].

14.4. The geometry of the A_4 BLG model CS term.

Let $\mathfrak{G} = A_4$ explicitly. Its three-bracket is given by $[e_a, e_b, e_c] = -\epsilon_{abc}{}^d e_d$, (eq. (160)) where indices are raised and lowered with the euclidean metric. We also know that the Lie $\mathfrak{G} = \text{Lie} A_4$ bracket has the generic form of eq. (208) with structure constants given by eq. (209) since A_4 is simple. Further, there is a one-to-one correspondence between the fundamental objects \mathcal{X}_{ab} of A_4 and the A_4 endomorphisms $\mathcal{X}_{ab} = ad_{\mathcal{X}_{ab}}$ that determine $\text{Lie} A_4 = so(4)$. Thus, for notational simplicity, we may use here \mathcal{X}_{ab} to denote the elements of $so(4)$ as well (Ex. 64), the commutation relations of which are expressed in the familiar form (211) using the dual basis.

To introduce $SO(4)$ gauge fields, we start from the $so(4)$ Maurer-Cartan equations written for one-forms ω^{ab} dual to the $so(4)$ basis elements $\mathcal{X}_{ab} = -\mathcal{X}_{ba}$,

$$\omega^{a_1 a_2}(\mathcal{X}_{b_1 b_2}) = \delta_{b_1}^{[a_1} \delta_{b_2}^{a_2]} \quad \text{or} \quad \omega^A(\mathcal{X}_B) = \delta_B^A. \quad (322)$$

The capital letters A, B label the pairs $(a_1 a_2)$, $(b_1 b_2)$ with $a_1 < a_2$, $b_1 < b_2$. Then, the MC equations (17) for Lie A_4 read

$$\begin{aligned} d\omega^{c_1 c_2} &= d\omega^C = -\frac{1}{2} C_{AB}{}^C \omega^A \wedge \omega^B \\ &= -\frac{1}{2} \frac{1}{4} C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} \omega^{a_1 a_2} \wedge \omega^{b_1 b_2} = \frac{1}{4} \epsilon_{a_1 a_2 b_1}{}^{[c_1} \delta_{b_2}^{c_2]} \omega^{a_1 a_2} \wedge \omega^{b_1 b_2}. \end{aligned}$$

The gauge fields $A^{c_1 c_2}$ are the $so(4)$ -valued connection one-forms obtained by ‘softening’ the MC forms $\omega^{c_1 c_2}$ (all forms are assumed to be defined on the appropriate manifolds, the coordinates of which are omitted). The Cartan structural equations then provide the curvatures or field strengths F ,

$$\begin{aligned} F^{c_1 c_2} &= F^C = (dA + A \wedge A)^C = dA^C + \frac{1}{2} C_{AB}{}^C A^A \wedge A^B \\ &= dA^{c_1 c_2} + \frac{1}{2} \frac{1}{4} C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} A^{a_1 a_2} \wedge A^{b_1 b_2} \\ &= dA^{c_1 c_2} - \frac{1}{4} \epsilon_{a_1 a_2 b_1}{}^{[c_1} \delta_{b_2}^{c_2]} A^{a_1 a_2} \wedge A^{b_1 b_2}. \end{aligned} \quad (323)$$

These curvatures are covariant under the Lie A_4 gauge transformations of A , which are given by

$$\begin{aligned} \delta_\lambda A &= \mathcal{D}\lambda = d\lambda + [A, \lambda], \\ \delta_\lambda A^{c_1 c_2} &= d\lambda^{c_1 c_2} + \frac{1}{4} C_{a_1 a_2 b_1 b_2}{}^{c_1 c_2} A^{a_1 a_2} \lambda^{b_1 b_2}. \end{aligned} \quad (324)$$

The covariant derivatives of zero-forms such as the matter fields, objects of the form $V = V^a e_a$ where the basis $\{e_a\}$ carries a representation of the gauge group, are defined easily. In the BLG model the V ’s denote bosonic or fermionic worldvolume fields, $V = X^I, \Psi$. Under Lie A_4 , V transforms by

$$\delta V = \delta V^d e_d = \frac{1}{2} (-\lambda^{ab}) [e_a, e_b, e_c] V^c$$

$$\begin{aligned}
&= \frac{1}{2}(-\lambda^{ab})\mathcal{X}_{ab} \cdot \mathbf{e}_c V^c \equiv (-\lambda) \cdot V \\
&= \frac{1}{2}\lambda^{ab}\epsilon_{abc}{}^d V^c \mathbf{e}_d \quad \Rightarrow \quad \delta V^d = \frac{1}{2}\lambda^{ab}\epsilon_{abc}{}^d V^c, \tag{325}
\end{aligned}$$

where the dot is the usual adjoint action (see eq. (186)), and the λ^{ab} parameter is -2 times the one that appears in the BLG literature because there the $1/2$ factor is not added. Then, the covariant derivative of the matter fields is defined by

$$\mathcal{D}V = dV + A \cdot V, \tag{326}$$

where $A = A^A \mathcal{X}_A = \frac{1}{2}A^{ab} \mathcal{X}_{ab}$. Note that the gauge fields used here are minus twice the ones used in the original BL and subsequent papers, because their covariant derivative is defined with a minus sign and there is no compensating $1/2$ factor in the BL definition of the components of the gauge field.

The above gauge fields (connection forms) and field strengths (curvatures) may now be used to construct a Chern-Simons (CS) three-form by the standard Chern-Weil theorem. First, an invariant four-form $P(F)$ is introduced with the help of an invariant symmetric tensor, $k_{AB} = k_{a_1 a_2 b_1 b_2} = k_{BA}$,

$$P(F) = k_{AB} F^A \wedge F^B = \frac{1}{4} k_{a_1 a_2 b_1 b_2} F^{a_1 a_2} \wedge F^{b_1 b_2} \equiv H, \tag{327}$$

denoted H for short (not need of worrying here about factors). The Lie \mathfrak{G} -invariance of the polynomial guarantees that the four-form H is closed and, since gauge free differential algebras are contractible, that $H = d\Omega$. The *Chern-Simons three-form* Ω for the symmetric invariant polynomial k_{AB} is given by

$$\Omega = k_{AB} \left(A^A \wedge dA^B + \frac{2}{3} A^A \wedge (A \wedge A)^B \right),$$

so that

$$\begin{aligned}
\Omega &= k_{AB} \left(A^A \wedge dA^B + \frac{1}{3} C_{CD}{}^B A^A \wedge A^C \wedge A^D \right) \\
&= \frac{1}{4} k_{a_1 a_2 b_1 b_2} \left(A^{a_1 a_2} \wedge dA^{b_1 b_2} + \frac{1}{3} C_{c_1 c_2 d_1 d_2}{}^{b_1 b_2} A^{a_1 a_2} \wedge A^{c_1 c_2} \wedge A^{d_1 d_2} \right). \tag{328}
\end{aligned}$$

We now look for rank two $SO(4)$ -invariant polynomials; they were given in Sec. 8.1.2. The first and obvious one $k_{AB}^{(1)}$ is the Killing metric (eq. (214)):

$$k_{a_1 a_2 b_1 b_2}^{(1)} = \delta_{AB} = \delta_{b_1 [a_1} \delta_{a_2] b_2}. \tag{329}$$

Additionally, there is also the possibility of taking (eq. (213))

$$k_{a_1 a_2 b_1 b_2}^{(2)} = \epsilon_{a_1 a_2 b_1 b_2} = k_{b_1 b_2 a_1 a_2}^{(2)}. \tag{330}$$

This independent metric exists because $so(4)$ is not simple (see Table in Sec. 3.9). The above $k^{(1)}$ and $k^{(2)}$ are obviously invariant; to see it explicitly, it suffices to check (eq. (50)) with the C 's for A_4 that

$$C_{c_1 c_2 d_1 d_2 a_1 a_2} = k_{a_1 a_2 b_1 b_2} C_{c_1 c_2 d_1 d_2}{}^{b_1 b_2}$$

is antisymmetric under the interchange of $(a_1 a_2)$ and $(d_1 d_2)$ (or of $(a_1 a_2)$ and $(c_1 c_2)$).

The CS term that appears in the action of the BLG model, eq. (308), is the one obtained by using $k^{(2)}$ in eq. (330) [28] *i.e.*, the polynomial that does not admit a $so(n)$ generalization³⁴. Indeed, using the results of Sec. 8.1.1, we obtain

$$\Omega^{(2)} = \frac{1}{4}\epsilon_{a_1 a_2 b_1 b_2} \left(A^{a_1 a_2} \wedge dA^{b_1 b_2} - \frac{1}{3}\epsilon_{c_1 c_2 d_1}{}^{b_1} A^{a_1 a_2} \wedge A^{c_1 c_2} \wedge A^{d_1 b_2} \right), \quad (331)$$

which coincides, up to a global factor, with the CS term (308) of the BLG model [27, 30], once the second term of (331) is multiplied by the factor -2 that appears when moving to the gauge fields commonly used in the BLG literature.

Let us now see explicitly that the CS terms for $k^{(1)}$ and $k^{(2)}$ are the sum and the difference, respectively, of the CS terms for the two $so(3)$ components of $so(4) = so(3) \oplus so(3)$. Define

$$\omega_{\pm}^1 = \omega^{14} \pm \omega^{23}, \quad \omega_{\pm}^2 = \omega^{24} \pm \omega^{31}, \quad \omega_{\pm}^3 = \omega^{34} \pm \omega^{12} \quad \left(\omega_{\pm}^i = \omega^{i4} \pm \frac{1}{2}\epsilon_{iab4}\omega^{ab} \right), \quad (332)$$

$i = 1, 2, 3$. This new basis splits explicitly $so(4)$ into its two (plus and minus) $so(3)$ components since, using (323), we find,

$$d\omega_{\pm}^1 = -\omega_{\pm}^2 \wedge \omega_{\pm}^3, \quad d\omega_{\pm}^2 = -\omega_{\pm}^3 \wedge \omega_{\pm}^1, \quad d\omega_{\pm}^3 = -\omega_{\pm}^1 \wedge \omega_{\pm}^2, \quad (333)$$

which are the MC equations of two $so(3)$ copies in the standard basis. In this basis the two $so(3)$ gauge fields A_{\pm}^i and corresponding curvatures F_{\pm}^i are given by

$$A_{\pm}^i = A^{i4} \pm \frac{1}{2}\epsilon_{iab4}A^{ab}, \quad F_{\pm}^i = F^{i4} \pm \frac{1}{2}\epsilon_{iab4}F^{ab}. \quad (334)$$

To show that $\Omega^{(1,2)}$, the two CS forms obtained by using the invariant polynomials $k^{(1,2)}$ respectively, can be written in terms of two $so(3)$ CS three-forms for the A_{\pm}^i and A_{\mp}^i gauge fields, we write the invariant four-forms $H^{(1)}$ and $H^{(2)}$ in terms of

$$H_{\pm} = \sum_{i=1}^3 F_{\pm}^i \wedge F_{\pm}^i = d\Omega_{\pm}. \quad (335)$$

Consider first $H^{(1)}$. Using (327) and (329) we obtain

$$\begin{aligned} H^{(1)} &= \frac{1}{4}\delta_{b_1[a_1}\delta_{a_2]b_2}F^{a_1 a_2} \wedge F^{b_1 b_2} = \frac{1}{2}F^{a_1 a_2} \wedge F_{a_1 a_2} \\ &= F^{12} \wedge F_{12} + F^{13} \wedge F_{13} + F^{14} \wedge F_{14} + F^{23} \wedge F_{23} + F^{24} \wedge F_{24} + F^{34} \wedge F_{34}. \end{aligned} \quad (336)$$

Similarly, $H^{(2)}$ is given by

$$H^{(2)} = \frac{1}{4}\epsilon_{a_1 a_2 a_3 a_4}F^{a_1 a_2} \wedge F^{a_3 a_4} = 2(F^{12} \wedge F^{34} + F^{13} \wedge F^{42} + F^{14} \wedge F^{23}). \quad (337)$$

Now, computing the sum and the difference of H_+ and H_- in eq (335),

$$\begin{aligned} H_+ \pm H_- &= (F^{14} + F^{23}) \wedge (F^{14} + F^{23}) + (F^{24} + F^{31}) \wedge (F^{24} + F^{31}) \\ &\quad + (F^{34} + F^{12}) \wedge (F^{34} + F^{12}) \\ &\quad \pm (F^{14} - F^{23}) \wedge (F^{14} - F^{23}) \pm (F^{24} - F^{31}) \wedge (F^{24} - F^{31}) \\ &\quad \pm (F^{34} - F^{12}) \wedge (F^{34} - F^{12}), \end{aligned} \quad (338)$$

³⁴The A_4 -based BLG model which is considered here selects the Lie algebra $so(4)$. It is easy to see why the polynomial (330) does not generalize for arbitrary n . The existence of a metric determined by the fully skewsymmetric tensor of a $(n+1)$ -dimensional space would require $2(n-1) = n+1$, hence $n=3$. This is the obstruction which prevents moving from the original BLG $SO(4)$ model to a $SO(n+1)$ one.

it is seen that for the plus sign the crossed terms cancel and the squares add, giving

$$\frac{1}{2}(H_+ + H_-) = H^{(1)}. \quad (339)$$

For the minus sign it is instead the crossed terms that survive so that

$$\frac{1}{2}(H_+ - H_-) = H^{(2)}. \quad (340)$$

Therefore, the CS term for the Killing form $k^{(1)}$ is the sum of two ordinary $so(3)$ CS terms, whereas the ‘twisted’ CS term (331) for $k^{(2)}$ is given by their difference [36, 29]. The relative minus sign in $\Omega_+ - \Omega_-$ solves the problem of parity invariance [302], since the odd parity of the standard CS term is compensated by requiring that parity interchanges the two different $SU(2)$ gauge fields so that $H_+ \leftrightarrow H_-$, something that could not be done if the CS term were the single one associated with a simple group.

15. A BLG MODEL BASED ON THE NAMBU BRACKET INFINITE-DIMENSIONAL FA \mathfrak{N}

So far we have described the BLG model having in mind that the vector space of the FA \mathfrak{G} is finite-dimensional and, actually, the four-dimensional A_4 . As we have seen, the A_4 -based BLG first model is equivalent to one based on a semisimple Lie group, $SU(2) \times SU(2)$, and describes only two M2-branes. Although, as mentioned in Sec. 14.2.2, there have been other proposals to overcome this limitation, the problem of finding an action describing an arbitrary number N of coincident M2-branes cannot be considered fully closed (for instance, the ABJM proposal [37] - which does not require an n -Lie algebra- has only $\mathcal{N} = 6$ manifest supersymmetry, see Sec. 14.1). The positive definiteness of the metric of the 3-Lie algebra led to the A_4 , $SU(2) \times SU(2)$ BLG first model of the previous section but, still keeping the positivity, there is one more option: one may move to infinite-dimensional FAs.

Let us then consider the model that results by replacing the previous A_4 FA by the infinite-dimensional $n = 3$ FA based on the Nambu bracket [38] of functions (Ex. 57). This leads to a BLG-Nambu bracket (BLG-NB) model describing the low energy limit of a ‘condensate’ of $N \rightarrow \infty$ M2-branes. To derive the BLG-NB action [303, 58] we shall take the original BLG one in eq. (305) and replace the A_4 FA by the infinite-dimensional Nambu algebra \mathfrak{N} of functions on a (compact) three-dimensional manifold M_3 that will be identified with S^3 (the isometry group of which is $SO(4)$) for the BLG-NB model, although we will often maintain a generic M_3 notation. The transition from A_4 to \mathfrak{N} will be achieved by replacing the A_4 three-brackets by Nambu brackets, and the sums over the a indices by sums over infinite, discrete indices \mathfrak{a} , following as much as possible the finite-dimensional BLG A_4 case. The use of the Nambu bracket in the context of the BLG model was already mentioned in [182], initiated in [57, 303] and studied in general in [58] (see also [304]). The novelty introduced by the Nambu FA \mathfrak{N} is that, in this case, $\text{Lie } \mathfrak{N}$ turns out to be the infinite-dimensional Lie algebra of the volume-preserving diffeomorphisms group of M_3 , $SDiff(M_3)$ (S for ‘special’ or volume preserving).

These (rigid) diffeomorphisms, when they are made local by making them to depend on the $d = 3$ spacetime variables, become characterized by functions ξ of six coordinates (x^μ, y^i) , where x^μ are the spacetime variables, $\mu = 1, 2, 3$, and $y^i, i = 1, 2, 3$, are the coordinates of $y \in M_3$. The fact that the matter fields take values in the infinite vector space where the adjoint derivatives act, will imply that they have the form $\phi(x, y)$, $\phi = (X, \Psi)$. It turns out that that the BLG-NB Lagrangian provides a *gauge* theory of volume preserving diffeomorphisms (of M_3) which has $\mathcal{N} = 8$ supersymmetry and superconformal invariance. It has been conjectured [58] that the $N \rightarrow \infty$ limit of the ABJM model [37] might lead to the $\mathcal{N} = 8$ supersymmetric BLG-NB one,

in which case the ABJM model would be a discretization of the Nambu bracket approach to be considered below.

The study of branes as gauge theories of volume preserving diffeomorphisms goes back to the work of Hoppe [305, 54, 55] who found that the full diffeomorphisms group for membranes with the topology of a sphere may be thought of as the $N \rightarrow \infty$ limit of $SU(N)$. The case of the supermembranes was taken up in [52], where it was shown that the light cone gauge-fixed action could be considered equivalent to that of a super-Yang-Mills theory of ‘symplectic’ or area preserving diffeomorphisms of the membrane surface. This is so because the $N \rightarrow \infty$ limit [54] of $SU(N)$ Yang-Mills theories may be considered as leading to $SDiff(M_2)$ gauge theories in which the Lie algebra commutator becomes the Poisson bracket of functions on M_2 [306]. This was extended to general $p > 2$ super- p -branes in [53], where it was shown that these may be considered as ‘exotic’ gauge theories (but see Sec. 15.2.3), the gauge group being $SDiff(M_p)$. It is then natural to look for volume preserving diffeomorphisms gauge actions using general Nambu-Poisson brackets (Sec. 13.4): after all, the Poisson bracket is simply the $n = 2$ Nambu-Poisson one.

But, before discussing the BLG-NB model, let us go back to the Nambu algebra as an infinite-dimensional FA.

15.1. The Nambu algebra \mathfrak{N} as an infinite-dimensional 3-Lie algebra.

Let M_3 be a three-dimensional compact oriented manifold without boundary, $M_3 = S^3$. Let $y^i = (y^1, y^2, y^3)$ be local coordinates on M_3 , and let $\mu(y) = e(y)dy^1 \wedge dy^2 \wedge dy^3$ be the volume form on M_3 . A volume preserving diffeomorphism is defined by a vector field $\xi(y) = \xi^i(y)\partial_i$ on M_3 such that $L_\xi \mu = 0$ where L_ξ is the Lie derivative; this implies that the components ξ^i of $\xi \in sdiff(M_3)$ satisfy the condition $\partial_i(e\xi^i) = 0$. By eq. (175), the Nambu bracket of three functions $\phi_1(y), \phi_2(y), \phi_3(y) \in \mathcal{F}(M_3)$ is locally given by

$$\{\phi_1(y), \phi_2(y), \phi_3(y)\} = e^{-1}(y)\epsilon^{ijk}\partial_i\phi_1(y)\partial_j\phi_2(y)\partial_k\phi_3(y) , \quad (341)$$

where ϵ^{ijk} is defined from $\epsilon^{123} = 1$; the scalar density e is not necessarily derived from a metric (*i.e.*, that $e = \sqrt{g}$), since no metric on M_3 is assumed. However, the Nambu algebra itself is taken to be metric in the sense of Sec. 7.7.1. For the antisymmetric symbol with indices down we take ϵ_{ijk} given by $\epsilon_{123} = e(y)$ as in [58]; then, $\epsilon^{i_1i_2i_3}\epsilon_{j_1j_2j_3} = e\epsilon_{j_1j_2j_3}^{i_1i_2i_3}$.

Since M_3 is compact we shall assume that there is a complete set of functions $\mathbf{e}_\alpha(y)$, where α denotes a set of discrete indices, that is orthonormal with respect to the metric (176) such that, for a scalar function $\phi \in \mathcal{F}(M_3)$,

$$\phi(y) = \sum_{\alpha} \phi^{\alpha} \mathbf{e}_{\alpha}(y) , \quad (342)$$

which is completed with

$$\langle \mathbf{e}_a(y), \mathbf{e}_b(y) \rangle = \delta_{ab} , \quad \phi^{\alpha} = \langle \phi(y), \mathbf{e}_a(y) \rangle , \quad \langle \phi_1(y), \phi_2(y) \rangle = \int_{M_3} \mu(y)\phi_1(y)\phi_2(y) , \quad (343)$$

as in (176), together with the ‘resolution of the identity’ on $\mathcal{F}(M_3)$

$$\sum_{\alpha} \mathbf{e}_{\alpha}(y)\mathbf{e}_{\alpha}(y') = \delta^3(y, y') , \quad \int_{M_3} \mu(y)\phi(y)\delta^3(y, y') = \phi(y') . \quad (344)$$

We do not need specifying $\delta^3(y, y')$; it is sufficient to know that it exists for a compact M_3 as S^3 (for instance, the spherical harmonics satisfy $\sum_{l=1}^{\infty} \sum_{m=-l}^l Y_{lm}(\theta, \phi)Y_{lm}(\theta', \phi')^* = \delta^2(\theta, \theta'; \phi, \phi') = \delta(\cos \theta - \cos \theta')\delta(\phi - \phi')$). For S^3 , which is the $SU(2)$ group manifold, eq. (342) just expresses the Peter-Weyl theorem for the harmonic analysis on compact groups with

$y \in S^3 \sim SU(2)$. For higher dimensional spheres S^p , $p > 3$ (no longer group manifolds), such an expression is related to general hyperspherical harmonics expansions. The case of S^3 has been discussed in [307]; that *e.g.* of the spheres S^{3A-4} , where A is the mass number, appeared very long ago in the n -body problem in nuclear physics (see *e.g.* [308]; see also [309] in the context of fuzzy spheres).

The linearity of the bracket implies, after replacing the vectors ϕ by their expansions in eq. (342), that the above three-bracket is also given by

$$\{\phi_1(y), \phi_2(y), \phi_3(y)\} = \sum_{abc} \phi_1^a(y) \phi_2^b(y) \phi_3^c(y) \{\mathbf{e}_a(y), \mathbf{e}_b(y), \mathbf{e}_c(y)\} . \quad (345)$$

The structure constants of this infinite-dimensional FA, referred to the basis $\{\mathbf{e}_a\}$ are, by definition, the coefficients $f_{abc}{}^d$ that appear in the expression

$$\{\mathbf{e}_a(y), \mathbf{e}_b(y), \mathbf{e}_c(y)\} = \sum_d f_{abc}{}^d \mathbf{e}_d(y) . \quad (346)$$

Therefore, using eqs. (342), (346) and (341) we find that the structure constants of the Nambu algebra \mathfrak{N} in the $\{\mathbf{e}_a\}$ basis are given by

$$f_{abcd} = \langle \{\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c\}, \mathbf{e}_d \rangle = \int_{M_3} \mu(y) e^{-1}(y) \epsilon^{ijk} \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) \partial_k \mathbf{e}_c(y) \mathbf{e}_d(y) \quad (347)$$

which exhibits an obvious $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ antisymmetry which becomes a full one in $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ on account of partial integration. Since the basis is orthonormal, there is no distinction between the up and down indices in f .

15.2. The BLG-NB model.

That some forms of p -brane actions can be formulated by using Nambu-Poisson brackets has been known for some time [55, 310]. This is so because the determinant of the induced metric g that appears in the original actions can be rewritten in terms of Nambu $(p+1)$ -brackets. Here we shall restrict ourselves to the Nambu bracket realization of the BLG theory, which may be viewed as the low energy limit of a condensate of nearly coincident M2-branes. To construct its lagrangian in a direct way, we have to extend the A_4 BLG one to the Nambu algebra \mathfrak{N} case (see below eq. (300)) which, in essence, means that we have to look at the consequences of replacing the previous finite range index $a = 1, \dots, \dim \mathfrak{G}$ by the discrete infinite \mathbf{a} . One might think of constructing the lagrangian below using a $(n > 3)$ -dimensional manifold M_3 and the corresponding n -Nambu-Poisson bracket, but there are difficulties for $n \geq 4$ [58]. From the present point of view, $n = 3$ is also natural by an extension of the dimensional arguments of the finite 3-Lie algebra case. As a result, the BLG-NB theory should be an infinite-dimensional Nambu bracket algebra version of the BLG model, in which the CS term would be constructed from the Nambu algebra analogue of the invariant polynomial $k^{(2)}$ in eq. (330).

To show this, let us begin with the matter fields appearing in the BLG-NB model. These depend on the three-dimensional worldvolume Minkowski coordinates $x^\mu = (x^0, x^1, x^2)$ as before but now include the \mathfrak{N} -algebra basis index \mathbf{a} . Therefore, as vectors in \mathfrak{N} depending on the worldvolume coordinates, these fields have the coordinate expansions

$$X^I(x, y) = \sum_{\mathbf{a}} X^{I\mathbf{a}}(x) \mathbf{e}_{\mathbf{a}}(y) , \quad \Psi^I(x, y) = \sum_{\mathbf{a}} \Psi^{I\mathbf{a}}(x) \mathbf{e}_{\mathbf{a}}(y) , \quad (348)$$

in which the sum over the index a for A_4 has been replaced by a sum over the set of indices \mathbf{a} for \mathfrak{N} . The remaining fields of the BLG-NB model are the vector spacetime fields. Their components will be given by $A_\mu(x)^{ab} = -A_\mu(x)^{ba}$, in parallel with $A_\mu^{ab}(x) = -A_\mu^{ab}(x)$ for the

finite case. This is because the A_μ fields depend on the indices that characterize the fundamental objects in $\mathfrak{N} \wedge \mathfrak{N}$, hence their double, antisymmetric \mathfrak{ab} indices.

15.2.1. The BLG-NB lagrangian.

To construct the BLG-NB lagrangian [58, 303] an invariant metric on \mathfrak{N} is needed. It was seen in Sec. 7.7.1 that the Nambu algebra may be made metric. Using the scalar product (176) the BLG lagrangian in eq. (305), when the Filippov bracket is taken to be the Nambu one, leads to

$$\begin{aligned} \mathcal{L}_{BL-NB}(x) &= \int_{M_3} d^3y e(y) \left(-\frac{1}{2} \mathcal{D}_\mu X^I(x, y) \mathcal{D}^\mu X^I(x, y) + \frac{i}{2} \bar{\Psi}(x, y) \Gamma^\mu \mathcal{D}_\mu \Psi(x, y) \right. \\ &\quad - g \frac{i}{4} \{ \bar{\Psi}(x, y), X^I(x, y), X^J(x, y) \} \Gamma_{IJ} \Psi(x, y) \\ &\quad \left. - \frac{g^2}{2 \cdot 3!} \{ X^I(x, y), X^J(x, y), X^K(x, y) \} \{ X^I(x, y), X^J(x, y), X^K(x, y) \} \right) \\ &\quad + \frac{1}{g} \mathcal{L}_{CS} \quad ; \end{aligned} \quad (349)$$

the explicit form of \mathcal{L}_{CS} will be derived below.

The covariant derivative of the matter $\phi(x, y)$ fields in eq. (349) is given by (*cf.* eq. (307))

$$\begin{aligned} \mathcal{D}_\mu \phi(x, y) &= \sum_{\mathfrak{d}} \mathcal{D}_\mu \phi^{\mathfrak{d}}(x) \mathbf{e}_{\mathfrak{d}}(y) \\ &= \sum_{\mathfrak{d}} \left(\partial_\mu \phi^{\mathfrak{d}}(x) - \sum_{\mathfrak{abc}} f_{\mathfrak{abc}}{}^{\mathfrak{d}} A_\mu^{\mathfrak{ab}}(x) \phi^{\mathfrak{c}}(x) \right) \mathbf{e}_{\mathfrak{d}}(y) = \partial_\mu \phi(x, y) - \\ &\quad - \sum_{\mathfrak{abc}\mathfrak{d}} \epsilon^{ijk} \int_{M_3} \mu(y') e^{-1}(y') \partial_{y^i} \mathbf{e}_a(y') \partial_{y^j} \mathbf{e}_b(y') \partial_{y^k} \mathbf{e}_c(y') \mathbf{e}_{\mathfrak{d}}(y') A_\mu^{\mathfrak{ab}}(x) \phi^{\mathfrak{c}}(x) \mathbf{e}_{\mathfrak{d}}(y) \\ &= \partial_\mu \phi(x, y) - e^{-1}(y) \epsilon^{ijk} \sum_{\mathfrak{ab}} \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) A_\mu^{\mathfrak{ab}}(x) \partial_k \phi(x, y) \\ &\equiv \partial_\mu \phi(x, y) + s_\mu^k(x, y) \partial_k \phi(x, y) \quad , \end{aligned} \quad (350)$$

where we have used (344) and expression (347) for the structure constants of the Nambu algebra to compute the term multiplying $\partial_k \phi$ above. This is given by

$$s_\mu^k(x, y) = -e^{-1}(y) \epsilon^{ijk} \sum_{\mathfrak{ab}} \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) A_\mu^{\mathfrak{ab}}(x) \quad . \quad (351)$$

The above expression may be equivalently written as

$$s_\mu^k(x, y) = e^{-1}(y) \epsilon^{ijk} \partial_i A_{j\mu} \quad \text{with} \quad A_{j\mu}(x, y) := - \sum_{\mathfrak{ab}} \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) A_\mu^{\mathfrak{ab}}(x) \quad , \quad (352)$$

where A_j is globally defined since $M_3 = S^3$. Also,

$$\sum_{\mathfrak{ab}} \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) A_\mu^{\mathfrak{ab}}(x) = -\frac{1}{2} \epsilon_{ijk} s_\mu^k(x, y) \quad , \quad \partial_{[i} A_{j]\mu}(x, y) = \epsilon_{ijk} s_\mu^k(x, y) \quad (353)$$

on account of the antisymmetry in i, j of the *l.h.s.*

Summarizing, the covariant derivative of the matter fields is given by

$$\mathcal{D}_\mu \phi = (\partial_\mu + s_\mu^i \partial_i) \phi \quad \text{i.e.,} \quad \mathcal{D} := d + s = d + s^i \partial_i \quad , \quad (354)$$

where d acts on spacetime forms; the form of \mathcal{D} is thus a consequence of the form of the \mathfrak{N} structure constants in eq. (350). Thus, the spacetime one-form $s = s^i \partial_i$, $s^i(x, y) = s^i_\mu(x, y) dx^\mu$, plays the rôle of a gauge connection. Since its explicit expression follows from that of $A_{i\mu}$ in eq. (352), the spacetime one-form $s^i(x, y)$ defines $A_j(x, y)$ up to a total ∂_i derivative: $A_j(x, y)$ and $A'_j(x, y) = A_j(x, y) + \partial_j \alpha(x, y)$ for some one-form $\alpha(x, y)$ lead to the same s . As the true gauge field is $s^i_\mu(x, y)$, $A_{i\mu}$ plays the rôle of a *pre-potential* [58].

Eq. (352) implies that the components es_μ^i of the spacetime one-form es^i satisfy the condition

$$\partial_i(es_\mu^i) = 0 \quad . \quad (355)$$

Hence the gauge field defines vector fields $s_\mu^i \partial_i$ on M_3 that are (local) volume preserving diffeomorphisms and $s = s_\mu dx^\mu$ is a $sdiff(M_3)$ -valued connection one-form. Since two pre-potentials $A'(x)$ and $A(x)$ lead to the same $s^i_\mu(x, y)$ if they are ‘pre-gauge’ related, the freedom in choosing $A_i(x, y)$ corresponds to taking elements $A^{ab}(x)$ in $\wedge^2 \mathfrak{N}$ that differ in one that belongs to the kernel of the ad mapping, since this difference will produce a zero s .

Let us now write the CS part of the BLG-NB action. This is obtained by moving from the $\mathcal{L}_{CS}(x)$ in (308) to the Nambu case. Using the structure constants of \mathfrak{N} , the resulting CS lagrangian is the spacetime three-form $L_{CS}(x) = \mathcal{L}_{CS}(x) d^3x$ given by

$$\begin{aligned} L_{CS}(x) &= \frac{1}{2} \sum_{abc\bar{d}} \left(f_{abc}{}^{\bar{d}} A^{ab}(x) \wedge dA^{\bar{c}d}(x) + \frac{2}{3} \sum_{efg} f_{\bar{c}da}{}^g f_{efgb} A^{ab}(x) \wedge A^{\bar{c}d}(x) \wedge A^{ef}(x) \right) \\ &= -\frac{1}{2} \int_{M_3} \mu(y) \left(s^k(x, y) \wedge dA_k(x, y) - \frac{1}{3} \epsilon_{ijk} s^i(x, y) \wedge s^j(x, y) \wedge s^k(x, y) \right), \end{aligned} \quad (356)$$

where eq. (353) has been used and $A^{ab}(x) = A^{ab}_\mu(x) dx^\mu$, and s are spacetime one-forms; recall that $\mu(y) = e(y) d^3y$ and that the structure constants $f_{abc\bar{d}}$ (eq. (347)) are fully antisymmetric.

Note that L_{CS} is not entirely written in terms of the gauge field $s^i(x, y)$: it also requires the pre-potential $A_i(x, y)$ defined by the last equation in (352). For this reason, the above L_{CS} was called ‘CS-like’ in [58]; its structure will be exhibited in Sec. 15.2.3. L_{CS} is of course well defined because (pre-gauge) related pre-potentials A'_k, A_k lead to the same L_{CS} due to condition (355).

15.2.2. Gauge and supersymmetry transformations of the fields of the BLG-NB model.

The gauge and supersymmetry transformations of the fields in the BLG-NB action can be readily obtained by looking at eqs. (309), (310). First, we notice that the derivations determined by Nambu brackets with one void entry as expressed in terms of the fundamental objects of \mathfrak{N} come from coefficients $\lambda^{ab} = -\lambda^{ba}$, where the \mathbf{a}, \mathbf{b} antisymmetry reflects that of the fundamental objects of the Nambu algebra (as the cd skewsymmetry in eq. (309)). These (rigid) infinitesimal Lie algebra transformations are made local by making λ^{ab} to depend on the $d = 3$ Minkowski coordinates, $\lambda^{ab} \rightarrow \lambda^{ab}(x)$. Using the expression of the \mathfrak{N} structure constants, it is found that the gauge transformations are actually determined by local functions $\xi(x, y)$ on the M_3 manifold. In fact, we demonstrate below that the variations of the fields under the gauge transformations of parameter $\lambda^{ab}(x)$ are local $SDiff(M_3)$ transformations

$$\begin{aligned} \delta X^I(x, y) &= -\xi^k(x, y) \partial_k X^I(x, y) \\ \delta \Psi(x, y) &= -\xi^k(x, y) \partial_k \Psi(x, y) \\ \delta s^i(x, y) &= d\xi^i(x, y) - \xi^j(x, y) \partial_j s^i(x, y) + \partial_j \xi^i(x, y) s^j(x, y), \end{aligned} \quad (357)$$

determined by $\xi^k(x, y)$.

To see how $\xi(x, y)$ appears, let us compute $\delta\phi$ to show how the original (\mathbf{a}, \mathbf{b}) dependence of λ leads to a y dependence of ξ . This will also identify $\xi(y)$ as a volume preserving diffeomorphism. The variation is given by the adjoint derivative $\sum_{\mathbf{ab}} \lambda^{\mathbf{ab}}(x) \{ \mathbf{e}_\mathbf{a}(y), \mathbf{e}_\mathbf{b}(y), \}$; with $\phi = X, \Psi$ we obtain from eqs. (342) and (347)

$$\begin{aligned} \delta\phi(x, y) &= \sum_{\mathbf{ab}} \lambda^{\mathbf{ab}}(x) \{ \mathbf{e}_\mathbf{a}(y), \mathbf{e}_\mathbf{b}(y), \phi(x, y) \} = \sum_{\mathbf{abc}\mathfrak{d}} \lambda^{\mathbf{ab}}(x) f_{\mathbf{abc}\mathfrak{d}} \mathbf{e}_\mathfrak{d}(y) \phi^\mathfrak{c}(x) \\ &= \sum_{\mathbf{abc}\mathfrak{d}} \lambda^{\mathbf{ab}}(x) \int_{M_3} \mu(y') e^{-1}(y') \epsilon^{ijk} \partial_{y'^i} \mathbf{e}_\mathbf{a}(y') \partial_{y'^j} \mathbf{e}_\mathbf{b}(y') \partial_{y'^k} \mathbf{e}_\mathfrak{c}(y') \mathbf{e}_\mathfrak{d}(y') \mathbf{e}_\mathfrak{d}(y) \phi^\mathfrak{c}(x) \\ &= e^{-1}(y) \epsilon^{ijk} \sum_{\mathbf{ab}} \lambda^{\mathbf{ab}}(x) \partial_i \mathbf{e}_\mathbf{a}(y) \partial_j \mathbf{e}_\mathbf{b}(y) \partial_k \phi(x, y) . \end{aligned} \quad (358)$$

Therefore,

$$\delta\phi(x, y) = -\xi^k(x, y) \partial_k \phi(x, y) , \quad (359)$$

with

$$\xi^k(x, y) = -e^{-1}(y) \epsilon^{ijk} \sum_{\mathbf{ab}} \partial_i \mathbf{e}_\mathbf{a}(y) \partial_j \mathbf{e}_\mathbf{b}(y) \lambda^{\mathbf{ab}}(x) \Rightarrow \partial_k (e(y) \xi^k(x, y)) = 0 \quad (360)$$

(cf. eq. (351)) so that $\xi \in \text{sdiff}(M_3)$ and is local since $\lambda^{\mathbf{ab}} = \lambda^{\mathbf{ab}}(x)$.

The last expression in eq. (357) for the gauge fields follows similarly. The extension of the corresponding formula in eq. (309) to the Nambu algebra case reads

$$\delta \left(\sum_{\mathfrak{cd}} f_{\mathfrak{cd}\mathfrak{b}}{}^\mathfrak{a} A_\mu^{\mathfrak{cd}}(x) \right) = \partial_\mu \left(\sum_{\mathfrak{cd}} f_{\mathfrak{cd}\mathfrak{b}}{}^\mathfrak{a} \lambda^{\mathfrak{cd}}(x) \right) + 2 \sum_{\mathfrak{cdefg}} f_{\mathfrak{cd}\mathfrak{b}}{}^\mathfrak{a} f_{\mathfrak{efg}}{}^\mathfrak{c} \lambda^{\mathfrak{deg}}(x) A_\mu^{\mathfrak{ef}}(x) . \quad (361)$$

Let us now compute the expression between brackets in the *l.h.s.* of (361). It is given by

$$\begin{aligned} &\sum_{\mathfrak{cd}} \int_{M_3} \mu(y) e^{-1}(y) \epsilon^{ijk} \partial_i \mathbf{e}_\mathfrak{c}(y) \partial_j \mathbf{e}_\mathfrak{d}(y) \partial_k \mathbf{e}_\mathbf{b}(y) \mathbf{e}_\mathbf{a}(y) A_\mu^{\mathfrak{cd}}(x) \\ &= - \int_{M_3} \mu(y) s_\mu^k(x, y) \partial_k \mathbf{e}_\mathbf{b}(y) \mathbf{e}_\mathbf{a}(y) = - \langle s_\mu^k(x) \partial_k \mathbf{e}_\mathbf{b} , \mathbf{e}_\mathbf{a} \rangle , \end{aligned} \quad (362)$$

where we have used (351). Similarly, the expression between brackets appearing in the first term of the *r.h.s.* of (361) is given by

$$- \langle \xi^k(x) \partial_k \mathbf{e}_\mathbf{b} , \mathbf{e}_\mathbf{a} \rangle . \quad (363)$$

We now compute the second term of the *r.h.s.* of (361). After integrating by parts with respect to y^i in the integral corresponding to $f_{\mathfrak{cd}\mathfrak{b}}{}^\mathfrak{a}$ and using (344), it gives

$$- 2 \epsilon^{ijk} \epsilon^{rst} \int_{M_3} \mu(y) e^{-1}(y) \sum_{\mathfrak{dgef}} \partial_j \mathbf{e}_\mathfrak{d}(y) \partial_k \mathbf{e}_\mathbf{b}(y) \partial_i \mathbf{e}_\mathbf{a}(y) \partial_r \mathbf{e}_\mathfrak{c}(y) \partial_s \mathbf{e}_\mathfrak{f}(y) \partial_t \mathbf{e}_\mathfrak{g}(y) \lambda^{\mathfrak{deg}}(x) A_\mu^{\mathfrak{ef}}(x) . \quad (364)$$

Now we use (351) and

$$\sum_{\mathbf{ab}} \partial_i \mathbf{e}_\mathbf{a}(y) \partial_j \mathbf{e}_\mathbf{b}(y) \lambda^{\mathbf{ab}}(x) = -\frac{1}{2} \epsilon_{ijk} \xi^k(x, y) , \quad (365)$$

which follows from eq. (360) since $\lambda^{\mathbf{ab}} = -\lambda^{\mathbf{ba}}$ (see the analogue for $A_\mu^{\mathbf{ab}}(x)$ and $s_\mu(x, y)$ in eq. (353)). Then, (364) is equal to

$$\int_{M_3} \mu(y) e^{-1}(y) \epsilon^{ijk} \epsilon_{jti} \partial_i (e(y) \xi^l(x, y) s^t(x, y)) \partial_k \mathbf{e}_\mathbf{b}(y) \mathbf{e}_\mathbf{a}(y)$$

$$\begin{aligned}
&= - \int_{M_3} \mu(y) (\partial_i \xi^k(x, y) s_\mu^i(x, y) - \xi^i(x, y) \partial_i s_\mu^k(x, y)) \partial_k \mathbf{e}_b(y) \mathbf{e}_a(y) \\
&= - \langle (\partial_i \xi^k(x) s_\mu^i(x) - \xi^i(x) \partial_i s_\mu^k(x)) \partial_k \mathbf{e}_b, \mathbf{e}_a \rangle, \tag{366}
\end{aligned}$$

using in the last equality that $\partial_i(e(y)s_\mu^i(x, y)) = 0$ and $\partial_i(e(y)\xi^i(x, y)) = 0$. Substituting (362), (363) and (366) in (361), we finally arrive at

$$\langle (\delta s_\mu^k(x) - (\partial_\mu \xi^k(x) + \partial_i \xi^k(x) s_\mu^i(x) - \xi^i(x) \partial_i s_\mu^k(x))) \partial_k \mathbf{e}_b, \mathbf{e}_a \rangle = 0 \tag{367}$$

which, since \mathbf{a} and \mathbf{b} are arbitrary, reproduces the gauge transformations δs^i of the gauge fields in eq. (357).

Thus, the infinitesimal gauge transformations for the matter $\phi = (X, \Psi)$ and gauge s^i fields in the BLG-NB model have all the expected standard form

$$\delta \phi = -L_\xi \phi \quad , \quad \delta s^i = d\xi + [s, \xi] \quad \text{with} \quad \xi = \xi^i \partial_i \quad , \quad \partial_i(e\xi^i) = 0 \quad \Rightarrow \quad \xi \in \mathit{sdiff}(M_3) \tag{368}$$

and $\delta(\mathcal{D}\phi) = -L_\xi(\mathcal{D}\phi)$ guarantees the invariance of the action under local diffeomorphisms. We have thus recovered the known fact [54, 55] that the adjoint action or derivation defined by the Nambu three-bracket generates local, M_3 -volume preserving diffeomorphisms. As a result, the BLG-NB action defines a $\mathit{SDiff}(S^3)$ gauge theory. The field strength is given by

$$\begin{aligned}
F &= \mathcal{D}s := ds + \frac{1}{2}[s, s] \quad , \quad F = F^i \partial_i \quad , \\
F^i(x, y) &= ds^i(x, y) + s^j(x, y) \wedge \partial_j s^i(x, y) \quad \text{with} \quad \partial_i(e(y)F^i(x, y)) = 0 \quad , \tag{369}
\end{aligned}$$

where d acts on the spacetime functions. Under gauge transformations, $\delta F = [F, \xi]$.

Let us look again at the $\delta\phi$ in eq. (358) to analyze further the gauge transformations. These are adjoint transformations -derivations- given in terms of the fundamental objects $\mathcal{X}_{\mathbf{e}_a \mathbf{e}_b} = (\mathbf{e}_a(y), \mathbf{e}_b(y))$ (as in (187)) with local parameters $\lambda^{ab}(x)$, and determine elements of the gauged Lie \mathfrak{N} by eq. (360). Given an arbitrary $\xi^k(x, y)$ satisfying $\partial_k(e\xi^k) = 0$ we may find a $\lambda^{ab}(x)$ that generates it. The fact that it is the functions $\xi^k(x, y)$ rather than the $\lambda^{ab}(x)$ that determine Lie \mathfrak{N} illustrates again that, for general FAs, different fundamental objects may induce the same element in $\text{InDer}\mathfrak{G} = \text{Lie}\mathfrak{G}$ (see [21] in the present Nambu algebra context). The elements of Lie \mathfrak{N} are obtained by taking the quotient by the kernel of the ad map, here characterized by (see (358))

$$\begin{aligned}
\sum_{ab} \lambda^{ab}(x) (\mathbf{e}_a, \mathbf{e}_b) \in \ker ad &\Leftrightarrow \sum_{ab} \lambda^{ab}(x) \{ \mathbf{e}_a(y), \mathbf{e}_b(y), \mathbf{e}_c(y) \} = 0 \quad \forall \mathbf{e}_c(y) \\
\Leftrightarrow \xi^k &= -e^{-1}(y) \epsilon^{ijk} \sum_{ab} \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) \lambda^{ab}(x) = 0 \quad . \tag{370}
\end{aligned}$$

Thus, all elements of $\wedge^2 \mathfrak{N}$ determined by λ^{ab} s that satisfy the above condition produce the trivial diffeomorphism of M_3 or zero element of $\text{Lie}\mathfrak{N} = \mathit{sdiff}(M_3)$.

We conclude this section with the supersymmetry transformations of the BLG-NB fields. These may be similarly found, and are given by

$$\begin{aligned}
\delta_\epsilon X^I(x, y) &= i\bar{\epsilon} \Gamma^I \Psi(x, y) \\
\delta_\epsilon \Psi(x, y) &= \mathcal{D}_\mu X^I(x, y) \Gamma^\mu \Gamma^I \epsilon - \frac{g}{3!} \{ X^I(x, y), X^J(x, y), X^K(x, y) \} \Gamma^{IJK} \epsilon \\
\delta_\epsilon s_\mu^k(x, y) &= -ig e^{-1}(y) \epsilon^{ijk} \bar{\epsilon} \Gamma_\mu \Gamma^I \partial_i X^I(x, y) \partial_j \Psi(x, y) \quad , \tag{371}
\end{aligned}$$

where again the last equality follows from a calculation similar to that leading to (367).

15.2.3. *Structure of the BLG-NB Chern-Simons term.*

As mentioned, the L_{CS} piece includes both the gauge field s^i and pre-potential A_k one-forms [58]. This is also reflected in the four-form which is obtained by taking the exterior differential of L_{CS} in (356), which is given by

$$dL_{CS} = -\frac{1}{2} \int_{M_3} \mu(y) F^i(x, y) \wedge G_i(x, y) , \quad (372)$$

where G_i and F^i are given by

$$G_i(x, y) = dA_i(x, y) - \frac{1}{2} \epsilon_{ijk} s^j(x, y) \wedge s^k(x, y) , \quad F^i(x, y) = e^{-1}(y) \epsilon^{ijk} \partial_j G_k(x, y) ; \quad (373)$$

G_i is a ‘pre-field strength’ two-form, $G_i = dA_i + \frac{1}{2} s^j \wedge \partial_j A_i$. In spite of the mixed appearance of the gauge field and its pre-potential in L_{CS} as well as that of the field and pre-field strengths in dL_{CS} , we show below that the L_{CS} term of the BLG-NB model may be constructed using only the curvature F and a suitable symmetric bilinear (metric) on the relevant Lie algebra, $\text{Lie } \mathfrak{N} = \text{sdiff}(M_3)$, much as the standard CS odd forms are obtained from the finite Lie algebra invariant symmetric polynomials and the curvature of the connection.

We saw in Sec. 14.4 that the CS term of the euclidean A_4 BLG lagrangian was in fact an ordinary CS three-form obtained from the polynomial in eq. (330) given by the structure constants with all indices down, eq. (213). In the present infinite-dimensional situation, the analogue of $k^{(2)}$ in (213) is given by the structure constants of the Nambu FA \mathfrak{N} , namely

$$\begin{aligned} k^{(2)}((\mathbf{e}_a, \mathbf{e}_b), (\mathbf{e}_c, \mathbf{e}_d)) &= f_{abcd} = f_{cdab} = \\ &= \epsilon^{ijk} \int_{M_3} \mu(y) e^{-1}(y) \partial_i \mathbf{e}_a(y) \partial_j \mathbf{e}_b(y) \partial_k \mathbf{e}_c(y) \mathbf{e}_d(y) . \end{aligned} \quad (374)$$

This metric is however degenerate and its kernel is given by the fundamental objects of \mathfrak{N} determined by λ ’s that satisfy

$$k^{(2)}\left(\sum_{ab} \lambda^{ab}(x) (\mathbf{e}_a, \mathbf{e}_b), (\mathbf{e}_c, \mathbf{e}_d)\right) = 0 \quad \forall \mathbf{e}_c, \mathbf{e}_d . \quad (375)$$

It may be checked that the above condition is equivalent to eq. (370), which determines the kernel of the ad map as defined there. Thus, taking the quotient of the space $\mathfrak{N} \wedge \mathfrak{N}$ by $\ker ad$, we conclude that $k^{(2)}$ is well defined on $\text{sdiff}(M_3)$. Let us then introduce, using our $k^{(2)}$ polynomial, the four-form $P(F)$

$$P(F) = k^{(2)}(F, F) , \quad (376)$$

where we denote by F the $\text{sdiff}(M_3)$ -valued curvature corresponding to $F^i(x, y)$ given by (369). As argued, the choice of the $\wedge^2 \mathfrak{N}$ representative for F does not have any effect on $P(F)$ and we may use for $F(x, y)$ any $\wedge^2 \mathfrak{N}$ -valued

$$F(x, y) = \sum_{ab} F^{ab}(x) (\mathbf{e}_a(y), \mathbf{e}_b(y)) \quad (377)$$

provided that $F^{ab}(x)$ gives rise to F^i ,

$$F^i(x, y) = -e^{-1} \epsilon^{ijk} \sum_{ab} \partial_j \mathbf{e}_a(y) \partial_k \mathbf{e}_b(y) F^{ab}(x) , \quad (378)$$

since then $F(x, y) \phi(x, y) = F^{ab}(x) \sum_c \{\mathbf{e}_a(y), \mathbf{e}_b(y), \mathbf{e}_c(y)\} \phi^c(x) = F^i(x, y) \partial_i \phi(x, y)$ using eq. (347). In this way, we move from any $\wedge^2 \mathfrak{N}$ -valued representative $F(x, y)$ to $F(x, y) = F^i(x, y) \partial_i$, which

is $sdiff(M_3)$ -valued since $\partial_j(eF^j) = 0$. Inserting (377) into (376) we compute $P(F)$ to be

$$\begin{aligned} P(F) &= \sum_{abcd} f_{abcd} F^{ab}(x) \wedge F^{cd}(x) \\ &= \epsilon^{ijk} \int_{M_3} \mu(y) e^{-1}(y) \sum_{abcd} \partial_i e_a(y) \partial_j e_b(y) \partial_k e_c(y) e_d(y) F^{ab}(x) \wedge F^{cd}(x) \\ &= - \int_{M_3} \mu(y) F^k(x, y) \wedge G_k(x, y), \end{aligned} \tag{379}$$

with (see eqs. (373), (378)) $G_k(x, y) = - \sum_{ab} e_a(y) \partial_k e_b(y) F^{ab}(x)$. It then follows that dL_{CS} in (372) may be written as $dL_{CS} = \frac{1}{2}P(F)$ and hence has the standard Chern-Weil expression for the infinite $sdiff(M_3)$ algebra. Thus, the BLG-NP theory is in this sense an ordinary (rather than ‘exotic’, *cf.* [58]) $sdiff(M_3)$ gauge theory: as in the A_4 case, the three-form lagrangian $L_{CS}(x)$ on $d = 3$ Minkowski space given by eq. (356) is obtained from an invariant polynomial $P(F)$ on the curvature.

As the BLG CS term, the CS term of the BLG-NP model is parity even [58], since the parity change in the three-dimensional spacetime can be compensated by a ‘parity flip’ in the ‘internal’ M_3 three-space. The BLG-NB model, eq. (349), also leads to a BPS solution corresponding to field configurations $X^I(x^2, y)$ that determine a three-sphere³⁵ the radius of which goes to infinity as x^2 goes to zero. This is consistent with the idea that the fuzzy sphere solution of the B-H equation should become [182] a smooth one in the $N \rightarrow \infty$ limit, in agreement with the fact that the BLG-NP model, which is a $d = 3$ superconformal $\mathcal{N} = 8$ supersymmetric theory, describes a condensate of M2-branes [58].

A superfield formulation of the BLG-NP model based on a set of eight scalar superfields constrained by a superembedding-like equation has been given in [311]. We shall not discuss this nor comment on the possible connection between the BLG-NB model and the large N limit of the ABJM proposal [37], and refer to [58] instead. We conclude by mentioning that the BLG-NB model has also been conjectured to describe a single M5-brane in the strong (constant) three-form field of $D = 11$ supergravity, an interpretation proposed in [57, 303] and further developed in [312] and [313].

16. n -ARY STRUCTURES: A BRIEF PHYSICAL OUTLOOK

We have described in this review the general structure of GLAs and FAs (and other related structures) as well as some possible applications in physics, with special emphasis in the Filippov algebras. In general, the n -bracket of a FA is not defined through a certain combination of associative products of its entries and, in fact, this is the reason that makes it difficult to give *e.g.*, matrix realizations of FAs. The simple $n > 2$ FAs are rather few, far less than those of the $n = 2$ Cartan classification: in fact, just one type of FA for each $n > 2$ if we ignore the signature of the $(n + 1)$ -dimensional metric real vector space on which the simple n -Lie algebras may be constructed (or we consider complex simple FAs).

A question that immediately arises is whether there is a Filippov ‘group-like’ structure associated with the n -Lie algebra one (besides the obvious Lie group associated with $\text{Lie } \mathfrak{G}$) *i.e.*, whether there is an integrated version of FAs of which these would be the linear approximation.

³⁵The authors of [58] assumed implicitly that the maps X of $M(S^3)$ on the unit 3-sphere had degree one; one might think of the physical consequences of having other degrees, see eq. (185).

The answer to this question is unknown; in fact, as far as we are aware, the problem of finding a Filippov ‘group-like’ manifold associated to general FAs has not even been discussed. The case of the ($n > 2$)-Leibniz generalizations should be even more difficult to tackle, since already for $n = 2$ it gives rise to the *coquecigrue* problem of Loday’s algebras mentioned in the main text. It is completely unclear whether such a notion exists in general.

For GLAs, the characteristic identity of the GLA bracket, the GJI, is a necessary result of the associativity of the composition of the elements in the (even) multibracket. We have seen that there is an infinite number of examples which may be constructed from the non-trivial cocycles for the cohomology of the simple compact Lie algebras. Although these simple GLAs take advantage of the existence of an underlying Lie group manifold, there is still room for other examples. But, in general, one might argue that a consequence of the present analysis is the ‘rigidity’ of the ordinary Lie algebra structure with respect to any possible n -ary generalizations. These entail losing different, but essential, parts of the properties associated with Lie algebras (as reflected by the Y-type bifurcation that leads either to the GJI or to the FI when $n > 2$) and which the physical world seems to like.

It is well appreciated that mathematics is full of developments prompted by or related to advances in physics. It may be that the full physical usefulness of these higher order n -ary algebras lays ahead. Nevertheless, we have already seen that, as far as specific applications to generalized mechanical systems are concerned (through their associated n -ary General Poisson and Nambu-Poisson structures), it is fair to say that there are not so many. At the same time, the quantization of n -ary Poisson structures is fraught with the difficulties discussed in this review. This is not an isolated case; there have been other mathematically interesting attempts to quantization, as *e.g.* geometric quantization, which, albeit geometrically attractive, have not met much success from the physical/practical point of view. As for the applications of n -Lie algebras to problems in M-theory, those associated with the BLG model have motivated the present renewed interest in Filippov algebras. Some specific types of 3-Leibniz algebras might be relevant here since, as discussed, the full anticommutativity of the finite ($n=3$)-Lie algebras is too restrictive. Although, as already mentioned, there exist alternatives to using three-algebras, it seems that these may nevertheless provide a natural way to encode various desired symmetry properties of the theory, at the root of which is the delicate interplay between the three-algebras and their associated Lie algebras. There is also, of course, the infinite-dimensional Nambu FA BLG-NP version of the BLG model described in the previous section, which retains full anticommutativity for its (Nambu) 3-bracket.

Having said that, it is worth recalling that the relation between mathematics and physics is as deep as full of surprises, as exhibited *e.g.* by the unexpected and recent application of mathematical aspects of M-theory holography to condensed matter physics (see [314, 315] for reviews). Thus, and in spite of the fact that the initial hopes have not been fulfilled, it might still turn out that n -ary structures in general and Filippov and Filippov-like algebras in particular have come to M-theory physics to stay, although it is quite unclear at present that this will be so.

But, in any case, n -ary algebraic structures are of course interesting by themselves.

17. APPENDIX 1: TWO FORMS OF THE FILIPPOV IDENTITY

We present here the proof of the equivalence of the expressions (141), (142) of the FI for general n , using the anticommuting ghosts b^a and c^a of eq. (142).

Let us show first that the form (141) of the FI implies (142). Eq. (141) may be rewritten as

$$[X_{b_1}, \dots, X_{b_{n-2}}, X_{b_{n-1}}, [X_{a_1}, \dots, X_{a_n}]]$$

$$= \sum_{l=1}^n [X_{b_1}, \dots, X_{b_{n-2}}, X_{a_l}, [X_{a_1}, \dots, X_{a_{l-1}}, X_{b_{n-1}}, X_{a_{l+1}}, \dots, X_{a_n}]] . \quad (380)$$

Now we contract this equation with $c^{b_1} \dots c^{b_k} b^{b_{k+1}} \dots b^{b_{n-2}} b^{b_{n-1}} b^{a_1} \dots b^{a_k} c^{a_{k+1}} \dots c^{a_n}$, for $k = 0, \dots, n-2$. This results in the following $n-1$ equations:

$$\begin{aligned} [C, \dots, C, B, \overset{n-2-k}{\cdot}, B, B, [B, \overset{k}{\cdot}, B, C, \overset{n-k}{\cdot}, C]] = \\ -k[C, \overset{k}{\cdot}, C, B, \overset{n-2-k}{\cdot}, B, B, [B, \overset{k}{\cdot}, B, C, \overset{n-k}{\cdot}, C]] \\ -(n-k)[C, \overset{k}{\cdot}, C, B, \overset{n-2-k}{\cdot}, B, C[B, \overset{k}{\cdot}, B, B, C, \overset{n-k-1}{\cdot}, C]] , \end{aligned} \quad (381)$$

or

$$(k+1)R_k = -(n-k)R_{k+1} \quad (k = 0, \dots, n-2) , \quad (382)$$

where we have introduced

$$R_k \equiv [C, \overset{k}{\cdot}, C, B, \overset{n-1-k}{\cdot}, B, [B, \overset{k}{\cdot}, B, C, \overset{n-k}{\cdot}, C]] , \quad (k = 0, \dots, n-1) . \quad (383)$$

The recurrence (382) has the solution

$$R_k = (-1)^k \frac{k!(n-k)!}{n!} R_0 \quad (k = 1, \dots, n-1) \quad (384)$$

which, taking $k = n-1$, implies $R_0 = n(-1)^{n-1}R_{n-1}$. But this is the FI written in the form (142) because R_0 is the bracket at its *l.h.s.* and $(-1)^{n-1}R_{n-1}$ is the one at the *r.h.s.* (for instance, for a 3-Lie algebra, it gives $[B, B, [C, C, C]] = 3[C, C, [B, B, C]]$, eq. (142)).

Conversely, let us assume that the FI in the form eq. (142) is satisfied. We use it on the set of $n-1$ double brackets ($k = 0, \dots, n-2$)

$$P_k \equiv [B, \overset{k}{\cdot}, B, C, \overset{n-1-k}{\cdot}, C, [B, \overset{n-2-k}{\cdot}, B, C, \overset{k+2}{\cdot}, C]] , \quad (385)$$

and obtain that each of the $n-1$ P_k may be expressed as

$$\begin{aligned} P_k = (n-2-k)[[B, \overset{k}{\cdot}, B, C, \overset{n-1-k}{\cdot}, C, B], B, \overset{n-3-k}{\cdot}, B, C, \overset{k+2}{\cdot}, C] \\ + (-1)^{(n-1)(n-2-k)}(k+2)[B, \overset{n-2-k}{\cdot}, B, [B, \overset{k}{\cdot}, B, C, \overset{n-1-k}{\cdot}, C], C, \overset{k+1}{\cdot}, C] . \end{aligned} \quad (386)$$

Reordering the entries in the *r.h.s.* so that parts in it are identified with P_s , this gives

$$P_k = (-1)^{n-1}(n-2-k)P_{n-3-k} + (-1)^{n-1}(k+2)P_{n-2-k} , \quad k = 0, \dots, n-2 . \quad (387)$$

This system of equations actually implies that all P_k vanish and, in particular, that $P_{n-2} = 0$, which is equivalent to (141). To see it, first notice that the above equation gives, for $k = n-2$,

$$P_{n-2} = (-1)^{n-1}nP_0 . \quad (388)$$

Secondly, inserting the expressions of P_{n-3-k} and P_{n-2-k} in the *r.h.s.* of (387) for $k = 0, \dots, n-3$ leads to the recurrence relation

$$P_k = (n-2-k)[(k+1)P_k + (n-k-1)P_{k+1}] + (k+2)[kP_{k-1} + (n-k)P_k] . \quad (389)$$

It can be checked that its solution is

$$P_k = (-1)^k \frac{(k+2)!(n-2-k)!}{2(n-2)!} P_0 , \quad k = 1, \dots, n-2 . \quad (390)$$

For $k = 2$ this gives $P_{n-2} = (-1)^{n-2} \binom{n-1}{2} P_0$. This relation, together with (388), implies $P_k = 0$ for all k as stated which, for $k = n-2$, reproduces the FI in the form of eq. (141) as $P_{n-2} = 0$. For instance, with $n = 3$, $P_1 = 0$ in eq. (385) gives $[B, C, [C, C, C]] = 0$.

18. APPENDIX 2: THE SCHOUTEN–NIJENHUIS BRACKET

Let $\wedge(M) = \bigoplus_{j=0}^n \wedge^j(M)$ ($\wedge^0 = \mathcal{F}(M)$, $n = \dim M$), be the contravariant exterior algebra of skewsymmetric contravariant (*i.e.*, tangent) tensor fields (*multivectors* or *j-vectors*) over a manifold M . Then the Schouten-Nijenhuis bracket (SNB) of $A \in \wedge^p(M)$ and $B \in \wedge^q(M)$ is the unique (up to a constant) extension of the Lie bracket of two vector fields on M to a \mathbb{R} -bilinear mapping $\wedge^p(M) \times \wedge^q(M) \rightarrow \wedge^{p+q-1}(M)$ in such a way that $\wedge(M)$ becomes a graded superalgebra. We start by defining the SNB for multivectors given by wedge products of vector fields.

Definition 96.

Let $X_1, \dots, X_p, Y_1, \dots, Y_q$ be vector fields over M . Then

$$\begin{aligned} & [X_1 \wedge \dots \wedge X_p, Y_1 \wedge \dots \wedge Y_q] = \\ & = \sum (-1)^{t+s} X_1 \wedge \dots \hat{X}_s \dots \wedge X_p \wedge [X_s, Y_t] \wedge Y_1 \wedge \dots \hat{Y}_t \dots \wedge Y_q \quad , \end{aligned} \tag{391}$$

where $[\ , \]$ is the SNB and \hat{X} indicates the omission of X^{36} . It is easy to check that (391) is equivalent to original definition [160, 161].

Theorem 97.

Let $M = G$ be the group manifold of a Lie group, and let the above vector fields X, Y be LI (*resp.* RI) vector fields on G . Then, the SNB of LI (*resp.* RI) skew multivector fields is also LI (*resp.* RI).

Proof. It suffices to recall that if X is LI and Z is the generator of the left translations, $L_Z X = [Z, X] = 0$ by the first equation in (16). □

Definition 98. (*Schouten-Nijenhuis bracket*)

Let $A \in \wedge^p(M)$ and $B \in \wedge^q(M)$, $p, q \leq n$, be the p - and q -vectors given in a local chart by

$$A(x) = \frac{1}{p!} A^{i_1 \dots i_p}(x) \partial_{i_1} \wedge \dots \wedge \partial_{i_p} \quad , \quad B(x) = \frac{1}{q!} B^{j_1 \dots j_q}(x) \partial_{j_1} \wedge \dots \wedge \partial_{j_q} \quad . \tag{392}$$

The SNB of A and B is the skewsymmetric contravariant tensor field $[A, B] \in \wedge^{p+q-1}(M)$

$$\begin{aligned} & = \frac{1}{(p+q-1)!} [A, B]^{k_1 \dots k_{p+q-1}} \partial_{k_1} \wedge \dots \wedge \partial_{k_{p+q-1}} \quad , \\ [A, B]^{k_1 \dots k_{p+q-1}} & = \frac{1}{(p-1)!q!} \epsilon_{i_1 \dots i_{p-1} j_1 \dots j_q}^{k_1 \dots k_{p+q-1}} A^{\nu i_1 \dots i_{p-1}} \partial_\nu B^{j_1 \dots j_q} \\ & \quad + \frac{(-1)^p}{p!(q-1)!} \epsilon_{i_1 \dots i_p j_1 \dots j_{q-1}}^{k_1 \dots k_{p+q-1}} B^{\nu j_1 \dots j_{q-1}} \partial_\nu A^{i_1 \dots i_p} \quad , \end{aligned} \tag{393}$$

where ϵ is the usual Kronecker symbol in eq. (39).

The SNB is graded-commutative,

$$[A, B] = (-1)^{pq} [B, A] \quad . \tag{394}$$

³⁶It is not difficult to see intuitively the origin of this generalization of the Lie algebra of vector fields to multivectors. If $B = Y_1 \wedge Y_2 \cdots \wedge Y_q$ is a q -vector field, it is natural to define $[X_1 \wedge X_2 \cdots \wedge X_p, B] = \sum_{i=1}^p (-1)^{i+1} X_1 \wedge \dots \wedge \hat{X}_i \cdots \wedge X_p \wedge [X_i, B]$ and then take $[X, Y_1 \wedge Y_2, \dots \wedge Y_q] = L_X(Y_1 \wedge Y_2, \dots \wedge Y_q) = \sum_{j=1}^q Y_1 \wedge Y_2 \wedge \dots \wedge Y_{j-1} \wedge [X, Y_j] \wedge Y_{j+1} \wedge \dots \wedge Y_q$, which leads to (391).

As a result, the SNB is identically zero if $A = B$ are of odd order (or even *degree*; $\text{degree}(A) \equiv \text{order}(A) - 1$). Since $[A, B]$ is a $(p + q - 1)$ -vector, the SNB is also zero if $p + q - 1 > \dim M$ and, of course, when A, B are constant multivectors, $A^{i_1 \dots i_p} \neq A^{i_1 \dots i_p}(x)$, $B^{j_1 \dots j_q} \neq B^{j_1 \dots j_q}(x)$.

The SNB satisfies the graded Jacobi identity,

$$(-1)^{pr} [[A, B], C] + (-1)^{qp} [[B, C], A] + (-1)^{rq} [[C, A], B] = 0 \quad , \quad (395)$$

where (p, q, r) denote the order of (A, B, C) respectively (thus, if Λ is of even order and $[\Lambda, \Lambda] = 0$ it follows from (395) that $[\Lambda, [\Lambda, C]] = 0$).

Let $A \wedge B \in \wedge^{p+q}(M)$,

$$\begin{aligned} A \wedge B &= \frac{1}{(p+q)!} (A \wedge B)^{i_1 \dots i_{p+q}} \partial_{i_1} \wedge \dots \wedge \partial_{i_{p+q}} \quad , \\ (A \wedge B)^{i_1 \dots i_{p+q}} &= \frac{1}{p!q!} \epsilon_{j_1 \dots j_{p+q}}^{i_1 \dots i_{p+q}} A^{j_1 \dots j_p} B^{j_{p+1} \dots j_{p+q}} \quad , \end{aligned} \quad (396)$$

and let $\alpha \in \wedge_{p+q-1}(M)$ be an arbitrary $(p+q-1)$ -form, $\alpha = \frac{1}{(p+q-1)!} \alpha_{i_1 \dots i_{p+q-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{p+q-1}}$. Then, the well known formula for one-forms and vector fields, $d\omega(X, Y) = L_X \omega(Y) - L_Y \omega(X) - i_{[X, Y]} \omega$, generalizes to

$$i_{A \wedge B} d\alpha = (-1)^{pq+q} i_A d(i_B \alpha) + (-1)^p i_B d(i_A \alpha) - i_{[A, B]} \alpha \quad , \quad (397)$$

where the contraction $i_A \alpha$ is the $(q-1)$ -form

$$i_A \alpha(\cdot) = \frac{1}{p!} \alpha(A, \cdot) \quad , \quad i_A \alpha = \frac{1}{(q-1)! p!} A^{i_1 \dots i_p} \alpha_{i_1 \dots i_p j_1 \dots j_{q-1}} dx^{j_1} \wedge \dots \wedge dx^{j_{q-1}} \quad , \quad (398)$$

so that, on forms, $i_B i_A = i_{A \wedge B}$. When α is *closed*, eq. (397) provides a definition of the SNB through $i_{[A, B]} \alpha$.

From the definition of the SNB it follows that

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(p-1)q} B \wedge [A, C] \quad , \quad (399)$$

$$[A \wedge B, C] = (-1)^p A \wedge [B, C] + (-1)^{rq} [A, C] \wedge B \quad . \quad (400)$$

In particular, for the case of the SNB among the wedge product of two *vector* fields

$$[A \wedge B, X \wedge Y] = -A \wedge [B, X] \wedge Y + B \wedge [A, X] \wedge Y - B \wedge [A, Y] \wedge X + A \wedge [B, Y] \wedge X \quad , \quad (401)$$

so that

$$[A \wedge B, A \wedge B] = -2A \wedge B \wedge [A, B] \quad . \quad (402)$$

For instance, if Λ is given by $X = X_j \wedge \partial^j$, $X_j = \frac{1}{2} C_{ij}^k \partial_i$ (see (281)), we may apply (401) to find that the condition $[\Lambda, \Lambda] = 0$ leads to the Jacobi identity.

Remark 99. As mentioned, the SNB is the unique extension of the usual Lie bracket of vector fields which makes a Z_2 -graded Lie algebra of the (graded-)commutative algebra of skewsymmetric contravariant tensors: $\text{degree}([A, B]) = \text{degree}(A) + \text{degree}(B)$. In it, the adjoint action is a graded derivation with respect to the wedge product [91] (see eq. (399)). To make this graded structure explicit, it is convenient to define a new SNB, $[\ , \]'$, which differs from the original one $[\ , \]$ by a factor $(-1)^{p+1}$ in the *l.h.s.* of (391), (393):

$$[A, B]' := (-1)^{p+1} [A, B] \quad . \quad (403)$$

This definition modifies (394) to read

$$[A, B]' = -(-1)^{(p-1)(q-1)} [B, A]' \equiv -(-1)^{ab} [B, A]' \quad . \quad (404)$$

where $a = \text{degree}(A) = (p-1)$, etc. Similarly, (395) is replaced by

$$(-1)^{pr+q+1} [[A, B]', C]' + (-1)^{qp+r+1} [[B, C]', A]' + (-1)^{rq+p+1} [[C, A]', B]' = 0 \quad , \quad (405)$$

which in terms of the degrees (a, b, c) of A, B, C adopts the graded JI form

$$(-1)^{ac}[[A, B]', C]' + (-1)^{ba}[[B, C]', A]' + (-1)^{cb}[[C, A]', B]' = 0 \quad . \quad (406)$$

In fact, the multivector algebra with the exterior product and the SNB is a Gerstenhaber algebra³⁷, in which $\deg(A) = p - 1$ if $A \in \wedge^p$. Thus, the multivectors of the form (127) form an abelian subalgebra of this Gerstenhaber algebra, the commutativity (in the sense of the SNB) being a consequence of (99).

The Schouten-Nijenhuis bracket definition of eq. (403) is used in [91, 316–318] and is more adequate to stress the graded structure of the exterior algebra of skew multivector fields; for instance, (404) and (406) have the same form as in supersymmetry (see *e.g.*, [70]). In Sec. 13.2, however, we use Def. 98 for the SNB as in [215, 161, 235, 5] and others.

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³⁷ A Gerstenhaber algebra [93] is a \mathbb{Z} -graded vector space (with homogeneous subspaces \wedge^a) with two bilinear multiplication operators, \cdot and $[\ , \]$, with the following properties ($u \in \wedge^a, v \in \wedge^b, w \in \wedge^c$):

- a) $\deg(u \cdot v) = a + b$,
- b) $\deg[u, v] = a + b - 1$,
- c) $(u \cdot v) \cdot w = u \cdot (v \cdot w)$,
- d) $[u, v] = -(-1)^{(a-1)(b-1)}[v, u]$,
- e) $(-1)^{(a-1)(c-1)}[u, [v, w]] + (-1)^{(c-1)(b-1)}[w, [u, v]] + (-1)^{(b-1)(a-1)}[v, [w, u]] = 0$,
- f) $[u, v \cdot w] = [u, v] \cdot w + (-1)^{(a-1)b}v \cdot [u, w]$.

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