Contributions of Tidal Poisson Terms in the Theory of the Nutation of a Nonrigid Earth

V. Dehant, M. Folgueira, N. Rambaux and S.B. Lambert

Abstract The tidal potential generated by bodies in the solar system contains Poisson terms, i.e., periodic terms with linearly time-dependent amplitudes. The influence of these terms in the Earth's rotation, although expected to be small, could be of interest in the present context of high accuracy modelling. We have studied their contribution in the rotation of a non rigid Earth with elastic mantle and liquid core. Starting from the Liouville equations, we computed analytically the contribution in the wobble and showed that the presently-used transfer function must be supplemented by additional terms to be used in convolution with the amplitude of the Poisson terms of the potential and inversely proportional to $(\sigma - \sigma_n)^2$ where σ is the forcing frequency and σ_n are the eigenfrequencies associated with the retrograde free core nutation and the Chandler wobble. These results have been detailed in a paper that we published in *Astron. Astrophys.* in 2007. In the present paper, we further examine the contribution from the core on the wobble and the nutation. In particular, we examine the contribution on extreme cases such as for wobble frequencies near the Free Core Nutation Frequency (FCN) or for long period nutations. In addition to the analytical computation, we used a time-domain approach through a numerical model to examine the core and mantle motions and discuss the analytical results.

Keywords Precession · Nutation · Poisson terms

V. Dehant
Royal Observatory of Belgium, 1180 Brussels, Belgium,
e-mail: v.dehant@oma.be

M. Folgueira
Instituto de Astronomía y Geodesia (UCM-CSIC), Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Ciudad Universitaria, 28040 Madrid, Spain
Observatoire de Paris, SYRTE/UMR 8630-CNRS, 61, Avenue de l'Observatoire, 75014 Paris, France

N. Rambaux
Royal Observatory of Belgium, 1180 Brussels, Belgium
Department of Mathematics, Facultés Univ. N.D. de la Paix, 5000 Namur, Belgium, e-mail: Nicolas.Rambaux@fundp.ac.be

S.B. Lambert
Royal Observatory of Belgium, 1180 Brussels, Belgium
Observatoire de Paris, SYRTE/UMR 8630-CNRS, 61, Avenue de l'Observatoire, 75014 Paris, France,
e-mail: marta@mat.ucm.es

1 Scientific context

In view of the great improvement in the measurement of the Earth Orientation Parameters, it is necessary to propose further analytical developments of nutation and to consider all the phenomena providing contributions at the microarcsecond level (see e.g., Dehant et al. 2003). In this paper, we show that several purely periodical terms of nutations with periods of 1, 18.6, and 10,467.6 years must be re-estimated at this level of precision, when taking into account the tidal potential generated by bodies in the solar system containing Poisson terms, that is to say, periodic terms with linearly time-dependent amplitudes. As shown in Folgueira et al. (2007), the consideration of a linear time dependence in the amplitude of the forcing potential produces a change in the amplitudes of some of the purely periodic terms. These terms must be added to the adopted model MHB2000 of Mathews et al. (2002, see also McCarthy and Petit 2004). The results have been detailed in the Astron. Astrophys. paper that we published (Folgueira et al., 2007). These results show that the most important contributions are stemming from nutations of high amplitude near the Free Core Nutation Frequency (FCN) or from very long period nutations. In the present paper, we further examine the contribution from the core on these extreme cases by computing analytically these limits. In order to be closer to the numerical integration that we have performed, we have considered here the simplification of a nondeformable Earth.

Our paper is organized as follows. After describing the differential equations of the rotation of an elastic Earth with a liquid core (Sects. 2 and 3), we explain two different approaches, a semi-analytical development (Sect. 4.1) and a numerical integration (Sect. 4.2), to obtain the coefficients of the nutation due to the tidal Poisson terms. This work is also contributing to the objectives of the DESCARTES Sub-project entitled: "Geophysical effects of adopting the new solutions for the Earth's rotation in the framework of the new parameters adopted by the IAU 2000 Resolutions".

2 The Differential Equations

This section attempts to show firstly the differential equations describing the rotation of an elastic Earth with a liquid core of which the solution will be used to determine the influence of the tidal Poisson terms on the Earth's rotation. The formulation and the methodology used for the derivation of the solution are briefly given at the end of this section. For details of the mathematical process for obtaining the solution, we refer the reader to the paper of Folgueira et al. (2007, see also Dehant et al. 1993 and Greff-Lefftz et al. 2002).

The basic equations are the angular momentum balance equations for the whole Earth and the liquid core relating the angular velocity vectors of the whole Earth, $\vec{\omega}$, and of the liquid core, $\vec{\omega}_{\mathrm{f}}$, to the tidal external potential. In an Earth-fixed frame of reference, the equations for the equatorial component $\omega = \omega_1 + i\omega_2$ are (see, e.g., Sasao et al. 1980; Hinderer et al. 1987):

$$A\left[1+\alpha\frac{k}{k_{fluid}}\right]\frac{\mathrm{d}\omega}{\mathrm{d}t}+\left[A_{\mathrm{f}}+A\alpha\frac{\overline{k}_1}{k_{fluid}}\right]\frac{\mathrm{d}\omega_{\mathrm{f}}}{\mathrm{d}t}$$
$$-\,i\Omega A\alpha\left[1-\frac{k}{k_{fluid}}\right]\omega+i\Omega\left[A_{\mathrm{f}}+A\alpha\frac{\overline{k}_1}{k_{fluid}}\right]\omega_{\mathrm{f}}$$
$$=-3i\frac{A\alpha}{a^2}\left\{V_{21}-\frac{k}{k_{fluid}}\left[-\frac{i}{\Omega}\frac{\mathrm{d}V_{21}}{\mathrm{d}t}+V_{21}\right]\right\}, \quad (1)$$

$$\left[1+\frac{q_0}{2}h_{\mathrm{f}}\right]\frac{\mathrm{d}\omega}{\mathrm{d}t}+\left[1+\frac{q_0}{2}\overline{h}_{1\mathrm{f}}\right]\frac{\mathrm{d}\omega_{\mathrm{f}}}{\mathrm{d}t}+i\Omega(1+\alpha_{\mathrm{f}})\omega_{\mathrm{f}}$$
$$=\frac{3}{a^2\Omega}\frac{q_0}{2}h_{\mathrm{f}}\frac{\mathrm{d}V_{21}}{\mathrm{d}t}. \quad (2)$$

where,

V_{21} is the tesseral part of the potential,

A, A_{f}, and A_{m} and C, C_{f}, and C_{m} are the principal moments of inertia of the whole Earth, of the liquid core, and of the mantle, for the equatorial and z-axis (principal) mass repartition,

Ω is the mean angular of the Earth,

$\alpha = \frac{C-A}{A}$ and $\alpha_{\mathrm{f}} = \frac{C_{\mathrm{f}}-A_{\mathrm{f}}}{A_{\mathrm{f}}}$ are the dynamical flattenings of the whole Earth and of the liquid core,

k is the Love number expressing (without dimension) the mass redistribution potential at the surface of the Earth induced by a forcing expressed in terms of a gradient of an external potential and acting on the whole Earth,

$\overline{k}_1$ is the Love number expressing (without dimension) the mass redistribution potential at the surface of the Earth (and evaluated there) induced by a pressure at the core-mantle boundary (CMB),

h_{f} is the Love number expressing (without dimension) the deformation of the CMB induced by a volumic potential evaluated at the surface,

$h_{1\mathrm{f}}$ is the Love number expressing (without dimension) the deformation of the CMB caused by a forcing expressed in terms of a gradiant of an external potential evaluated at the CMB,

$\overline{h}_{1\mathrm{f}}$ is the Love number expressing (without dimension) the deformation of the CMB caused by an inertial pressure on the CMB,

k_{fluid} is the secular Love number also called the fluid Love number $\left(k_{fluid} = 3\frac{\alpha AG}{\Omega^2 a^5}\right)$, and

q_0 is defined as $\frac{\Omega^2 a_{eq}^3}{\mathrm{GM}_E}$ where M_{E} is the Earth's mass. It must be noted that we have used here the notations introduced in Dehant et al. (1993). These notations are perfectly equivalent to those used in MHB2000. Mathews et al. (2002) used the compliances instead of the Love numbers, grouping the Love numbers with $\frac{q_0}{2}$.

We consider that the external potential contains contributions due to the so-called Poisson terms, that is to say, terms of the form $t \times e^{i\sigma t}$. The sets of fundamental arguments employed in tide and nutation theories of the rigid Earth are not linear in time and therefore they do include small nonlinear terms. This causes slow variations of the fundamental frequencies and hence also of the frequencies of the spectral terms. This is particularly true for the arguments arising from planetary perturbations. The Poisson terms stem from the approximation for short time scales of the long term variations arising from planetary perturbations. Usually one indeed prefers to work with constant frequencies; therefore, one splits the argument into a part linear with time and a remaining part involving the second and higher powers of time. This second part being very small, the sine or cosine of the argument can be written as a purely sinusoidal part of which the amplitude has quadratic and higher degree dependences on the time variable. Within a secular theory of the orbital motion of the Earth and of the other planets of the Solar System, linear as well as non-linear changes in the obliquity appears. Terms in the spectral representation which have amplitudes depending polynomially on time are usually referred to as *Poisson terms*. The external tide generating potential can therefore be written:

$$\begin{aligned} V_{21} &= V_{21,0} + V_{21,1}t + V_{21,2}t^2 \\ &\quad + \sum_n (V_{21,n,0} + V_{21,n,1}t)e^{i\sigma_n t}, \end{aligned} \tag{3}$$

where the amplitudes $V_{21,n,0}$ and $V_{21,n,1}$ are both related to the spectral component of frequency σ_n. Moreover, we assume that the solutions of Eqs. (1) and (2) can be written as follows:

$$\begin{aligned} \omega &= \omega_0 + \omega_1 t + \omega_2 t^2 \\ &\quad + \sum_n (\omega_{n,0} + \omega_{n,1}t)e^{i\sigma_n t}, \end{aligned} \tag{4}$$

$$\begin{aligned} \omega_{\mathrm{f}} &= \omega_{\mathrm{f},0} + \omega_{\mathrm{f},1} t + \omega_{\mathrm{f},2} t^2 \\ &\quad + \sum_n (\omega_{\mathrm{f},n,0} + \omega_{\mathrm{f},n,1}t)e^{i\sigma_n t}. \end{aligned} \tag{5}$$

3 Analytical Solutions

The substitution of the above expressions (4) and (5) for ω and ω_{f}, as well as Eq. (3) for the external potential, into Eqs. (1) and (2), provides a set of ten algebraic equations, which allows us to directly obtain the expressions for ω_0, ω_1, ω_2, ω_{n0}, ω_{n1}, $\omega_{\mathrm{f}0}$, $\omega_{\mathrm{f}1}$, $\omega_{\mathrm{f}2}$, $\omega_{\mathrm{f}n0}$ and $\omega_{\mathrm{f}n1}$ in terms of the coefficients of the external potential: $V_{21,0}$, $V_{21,1}$, $V_{21,2}$, $V_{21,n,0}$ and $V_{21,n,1}$. The part related to $V_{21,n,0}$ and $V_{21,n,1}$ reduces to:

$$\begin{aligned} \omega &= \frac{3}{a^2\Omega}\Bigg\{\Bigg[\frac{\alpha A k - \frac{q_0}{2}h_{\mathrm{f}}A_{\mathrm{f}}k_{fluid}}{k_{fluid}A_{\mathrm{m}}} - \frac{\sigma_{\mathrm{CW}}}{\sigma - \sigma_{\mathrm{CW}}} \\ &\quad + \frac{\sigma'_{\mathrm{FCN}}(\alpha - \frac{q_0}{2}h_{\mathrm{f}})A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})}\Bigg] V_{21,n,0} \\ &\quad + \Bigg[-\frac{\sigma_{\mathrm{CW}}}{(\sigma - \sigma_{\mathrm{CW}})^2} \\ &\quad + \frac{\sigma'_{\mathrm{FCN}}(\alpha - \frac{q_0}{2}h_{\mathrm{f}})A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})^2}\Bigg] iV_{21,n,1} \\ &\quad + \Bigg[\frac{\alpha A k - \frac{q_0}{2}h_{\mathrm{f}}A_{\mathrm{f}}k_{fluid}}{k_{fluid}A_{\mathrm{m}}} - \frac{\sigma_{\mathrm{CW}}}{\sigma - \sigma_{\mathrm{CW}}} \\ &\quad + \frac{\sigma'_{\mathrm{FCN}}(\alpha - \frac{q_0}{2}h_{\mathrm{f}})A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})}\Bigg] V_{21,n,1}t\Bigg\} e^{i\sigma_n t}, \\ &= \frac{3}{a^2\Omega}\left\{T_0(\sigma)V_{21,n,0} + T_1(\sigma)iV_{21,n,1}\right. \\ &\quad \left. + T_0(\sigma)V_{21,n,1}t\right\} e^{i\sigma_n t}, \end{aligned} \tag{6}$$

and the reduced solution for the core wobble ω_{f} is:

$$\begin{aligned} \omega_{\mathrm{f}} &= \frac{3}{a^2\Omega^2}\Bigg\{\Bigg[-\frac{(\alpha k - k_{fluid}\frac{q_0}{2}h_{\mathrm{f}})\Omega A}{k_{fluid}A_{\mathrm{m}}} \\ &\quad + \frac{\sigma_{\mathrm{CW}}^2}{\sigma - \sigma_{\mathrm{CW}}} + \frac{(\alpha - \frac{q_0}{2}h_{\mathrm{f}})\Omega^2 A}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})}\Bigg] V_{21,n,0} \\ &\quad + \Bigg[\frac{\sigma_{\mathrm{CW}}^2}{(\sigma - \sigma_{\mathrm{CW}})^2} + \frac{(\alpha - \frac{q_0}{2}h_{\mathrm{f}})\Omega^2 A}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})^2}\Bigg] iV_{21,n,1} \\ &\quad + \Bigg[-\frac{(\alpha k - k_{fluid}\frac{q_0}{2}h_{\mathrm{f}}\Omega A}{k_{fluid}A_{\mathrm{m}}} + \frac{\sigma_{\mathrm{CW}}^2}{\sigma - \sigma_{\mathrm{CW}}} \\ &\quad + \frac{(\alpha - \frac{q_0}{2}h_{\mathrm{f}})\Omega^2 A}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN}})}\Bigg] V_{21,n,1}t\Bigg\} e^{i\sigma_n t} \end{aligned} \tag{7}$$

$$\begin{aligned} &= \frac{3}{a^2\Omega}\left\{T_{f0}(\sigma)V_{21,n,0} + T_{f1}(\sigma)iV_{21,n,1}\right. \\ &\quad \left. + T_{f0}(\sigma)V_{21,n,1}t\right\} e^{i\sigma_n t}. \end{aligned} \tag{8}$$

In the above expressions,

$$\sigma_{\mathrm{FCN}} = -\Omega\left[1 + \frac{A}{A_{\mathrm{m}}}\left(\alpha_{\mathrm{f}} - \frac{q_0}{2}\overline{h}_{1\mathrm{f}}\right)\right] \tag{9}$$

is the frequency of the nearly diurnal free wobble (NDFW) associated with the free core nutation (FCN), and expressed in the terrestrial frame, $\sigma'_{\mathrm{FCN}} = \sigma_{\mathrm{FCN}} + \Omega$ is its counterpart in the celestial reference frame, and

$$\sigma_{\mathrm{CW}} = \frac{A\Omega\alpha}{A_{\mathrm{m}}}\left(1 - \frac{k}{k_{fluid}}\right) \tag{10}$$

is the Chandler wobble (CW) frequency in the terrestrial frame.

If we consider that the Earth is not deformable (*nondef* stands for non-deformable; this is the case considered in the section on Numerical Integration), all Love numbers become zero so that:

$$\begin{aligned}\omega_{nondef} = \frac{3}{a^2\Omega}\Bigg\{&\left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{\sigma - \sigma_{\mathrm{CW_{nondef}}}} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})}\right]V_{21,n,0}\\ &+\left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{(\sigma - \sigma_{\mathrm{CW_{nondef}}})^2} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})^2}\right]iV_{21,n,1}\\ &+\left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{\sigma - \sigma_{\mathrm{CW_{nondef}}}} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})}\right]V_{21,n,1}t\Bigg\}e^{i\sigma_n t}\end{aligned} \tag{11}$$

$$\begin{aligned}= \frac{3}{a^2\Omega}\big\{&T_{nondef0}(\sigma)V_{21,n,0}\\ &+T_{nondef1}(\sigma)iV_{21,n,1}\\ &+T_{nondef0}(\sigma)V_{21,n,1}t\big\}e^{i\sigma_n t}\end{aligned} \tag{12}$$

where

$$T_{nondef0}(\sigma) = -\frac{\sigma_{\mathrm{CW_{nondef}}}}{\sigma - \sigma_{\mathrm{CW_{nondef}}}} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})} \tag{13}$$

$$T_{nondef1}(\sigma) = -\frac{\sigma_{\mathrm{CW_{nondef}}}}{(\sigma - \sigma_{\mathrm{CW_{nondef}}})^2} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})^2} \tag{14}$$

$T_{nondef0}$ and $T_{nondef1}$ are called transfer functions for the non-deformable Earth (*nondef*); with,

$$\sigma_{\mathrm{FCN_{nondef}}} = -\Omega\left[1 + \frac{A}{A_{\mathrm{m}}}\alpha_{\mathrm{f}}\right] \tag{15}$$

$$\sigma_{\mathrm{CW_{nondef}}} = \frac{A\Omega\alpha}{A_{\mathrm{m}}} \tag{16}$$

If we consider that the Earth is completely rigid (*rig*; $\sigma_{CW_{rig}} = \sigma_{Euler}$), the core contribution disappears so that:

$$\begin{aligned}\omega_{rig} = \frac{3}{a^2\Omega}\Bigg\{&\left[-\frac{\sigma_{\mathrm{Euler}}}{\sigma - \sigma_{\mathrm{Euler}}}\right]V_{21,n,0}\\ &+\left[-\frac{\sigma_{\mathrm{Euler}}}{(\sigma - \sigma_{\mathrm{Euler}})^2}\right]iV_{21,n,1}\\ &+\left[-\frac{\sigma_{\mathrm{Euler}}}{\sigma - \sigma_{\mathrm{Euler}}}\right]V_{21,n,1}t\Bigg\}e^{i\sigma_n t},\\ = \frac{3}{a^2\Omega}\big\{&T_{rig0}(\sigma)V_{21,n,0}\\ &+T_{rig1}(\sigma)iV_{21,n,1}\\ &+T_{rig0}(\sigma)V_{21,n,1}t\big\}e^{i\sigma_n t},\end{aligned} \tag{17}$$

with,

$$\sigma_{\mathrm{Euler}} = \Omega\alpha \tag{18}$$

$$\sigma_{\mathrm{CW_{nondef}}} = \frac{A}{A_{\mathrm{m}}}\sigma_{\mathrm{Euler}} \tag{19}$$

where

$$T_{rig0}(\sigma) = -\frac{\sigma_{\mathrm{Euler}}}{\sigma - \sigma_{\mathrm{Euler}}} \tag{20}$$

$$T_{rig1}(\sigma) = -\frac{\sigma_{\mathrm{Euler}}}{(\sigma - \sigma_{\mathrm{Euler}})^2} = \frac{T_{rig0}(\sigma)}{\sigma - \sigma_{\mathrm{Euler}}} \tag{21}$$

T_{rig0} and T_{rig1} are called transfer functions for the rigid Earth (*rig*).

In the next paragraph, we compute the difference between the non-deformable Earth and the rigid Earth response for the wobble ($\omega_{nondef} - \omega_{rig}$). The analytical expression is thus:

$$\begin{aligned}\omega_{nondef} - \omega_{rig} = {}& \frac{3}{a^2\Omega} \\ & \left\{\left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{\sigma - \sigma_{\mathrm{CW_{nondef}}}} + \frac{\sigma_{\mathrm{Euler}}}{\sigma - \sigma_{\mathrm{Euler}}} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})}\right] V_{21,n,0}\right. \\ & + \left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{(\sigma - \sigma_{\mathrm{CW_{nondef}}})^2} + \frac{\sigma_{\mathrm{Euler}}}{(\sigma - \sigma_{\mathrm{Euler}})^2} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})^2}\right] iV_{21,n,1} \\ & \left. + \left[-\frac{\sigma_{\mathrm{CW_{nondef}}}}{\sigma - \sigma_{\mathrm{CW_{nondef}}}} + \frac{\sigma_{\mathrm{Euler}}}{\sigma - \sigma_{\mathrm{Euler}}} + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\alpha A_{\mathrm{f}}}{A_{\mathrm{m}}(\sigma - \sigma_{\mathrm{FCN_{nondef}}})}\right] V_{21,n,1}t\right\} e^{i\sigma_n t}\end{aligned} \tag{22}$$

which can also be written:

$$\begin{aligned}\omega_{nondef} - \omega_{rig} = {}& \frac{3}{a^2\Omega} \\ & \{ (T_{nondef0}(\sigma) - T_{rig0}(\sigma))(V_{21,n,0} + V_{21,n,1}t) \\ & +(T_{nondef1}(\sigma) - T_{rig1}(\sigma))iV_{21,n,1}\} e^{i\sigma_n t}\end{aligned} \tag{23}$$

For the retrograde diurnal wobble corresponding to nutation, Eq. (22) can be written using the fact that $\sigma = -\Omega + \sigma'$:

$$\begin{aligned}\omega_{nondef} - \omega_{rig} = {}& \frac{3}{a^2\Omega}\Bigg\{ \\ & \left[1 + \frac{\sigma'_{\mathrm{FCN_{nondef}}}}{(\sigma' - \sigma'_{\mathrm{FCN_{nondef}}})}\right]\frac{\alpha A_{\mathrm{f}} V_{21,n,0}}{A_{\mathrm{m}}} \\ & + \left[1 + \frac{\sigma'_{\mathrm{FCN_{nondef}}}\Omega}{(\sigma' - \sigma'_{\mathrm{FCN_{nondef}}})^2}\right]\frac{\alpha A_{\mathrm{f}} iV_{21,n,1}}{A_{\mathrm{m}}\Omega} \\ & + \left[1 + \frac{\sigma'_{\mathrm{FCN_{nondef}}}}{(\sigma' - \sigma'_{\mathrm{FCN_{nondef}}})}\right]\frac{\alpha A_{\mathrm{f}} V_{21,n,1}t}{A_{\mathrm{m}}} \\ & \Bigg\} e^{i\sigma_n t}\end{aligned} \tag{24}$$

This equation may be considered in different frequency ranges within the nearly diurnal frequency band:

- First, for σ' close to $\sigma'_{\mathrm{FCN_{nondef}}}$, Eq. (24) can be written:

$$\begin{aligned}\omega_{nondef} - \omega_{rig} = {}& \frac{3\alpha A_{\mathrm{f}}}{a^2\Omega A_{\mathrm{m}}}\frac{\sigma'_{\mathrm{FCN_{nondef}}}}{(\sigma' - \sigma'_{\mathrm{FCN_{nondef}}})} \\ & \left(V_{21,n,0} + \frac{iV_{21,n,1}}{(\sigma' - \sigma'_{\mathrm{FCN_{nondef}}})} + V_{21,n,1}t\right)\end{aligned} \tag{25}$$

  where we see that, for a frequency close to the FCN resonance, the term in $V_{21,n,1}$ may largely be enhanced.
- For a very long nutation period (σ' close to zero), Eq. (24) can be written:

$$\omega_{nondef} - \omega_{rig} = \frac{3\alpha A_{\mathrm{f}} iV_{21,n,1}}{a^2\Omega^2 A_{\mathrm{m}}}\left(\frac{\Omega}{\sigma'_{\mathrm{FCN_{nondef}}}}\right) \tag{26}$$

  which shows that for the precession or the very long nutation period, we may have a contribution from the core due to the Poisson terms in the forcing. It must be noted that Ferrándiz et al. (2004) have also considered the effects of a liquid core on the Poisson terms and have evaluated an additional core contribution to precession related this. They have found an effect at the level of 20 milliarcsecond (mas) per century.
- Starting from Eq. (24), we can also evaluate the effects for the classical large nutations such as at 13.66 days, 0.5 and 1 year, 9.3 and 18.6 years. We have noted that the transfer function for the Poisson part of the potential (the second term of Eq. (24)) is larger than the transfer function for the periodic part of the potential (the first term of Eq. (24)) by a factor of about 200 for both the prograde annual nutation and the retrograde semi-annual nutation, if the frequencies are expressed in cycle per day. The effective contributions to the Earth nutations depend however on the ratio between $V_{21,n,1}$ and $V_{21,n,0}$. For the two highest terms mentioned here and far from long period as well as far from the FNC period, i.e. the semi-annual retrograde nutation and the prograde annual nutation, the final contribution is at the level of 10^{-6}, which is very small.

The only contributions to purely periodic nutation amplitude from the Poisson part of the forcing would then be either for very long periods or for nutation close to the FCN. They come from the existence of a core within the Earth.

4 Contributions to the Nutation: Two Approaches

4.1 Semi-Analytical Development

As the ratio of the non-rigid Earth circular nutation to the rigid Earth circular nutation (the so-called B-ratio of Wahr, 1981) is identical to the ratio of the non-rigid Earth wobble to the rigid Earth wobble, the above expressions obtained for the wobbles can be used for determining the nutation B-ratios. By using a semi-analytical development of the tide generating potential of Roosbeek (1998, see also Roosbeek and Dehant 1998), we then evaluate numerically the amplitude of each rigid Earth nutation. This allows us to compute the non-rigid Earth nutations for the each nutation frequency, either in the prograde part of the spectrum or in the retrograde part of the spectrum. The combination (sum and differences potentially divided by the sine of the mean obliquity) of prograde and retrograde nutations allows to compute the nutation in longitude $\Delta\psi$ and in obliquity $\Delta\varepsilon$. Identifying the purely-periodic terms and the Poisson terms in the tide generating potential it is then possible to compute the contribution to the nutation due to the tidal Poisson terms.

The contributions for the nutation in longitude $\Delta\psi$ and in obliquity $\Delta\varepsilon$ that are above the level of 0.1 μ as (microarcseconds), are written as follows:

$$\begin{aligned}\Delta\psi^{\text{Poisson}} = &-0.4\sin(-l') + 0.5\sin\Omega \\ &+5.9\sin(2l' - 2F + 2D - 2\Omega) \\ &-0.7\sin(-\Omega)\end{aligned}$$

$$\begin{aligned}\Delta\varepsilon^{\text{Poisson}} = &\ 0.2\cos(-l') - 0.2\cos\Omega\Omega \\ &-2.3\cos(2l' - 2F + 2D - 2\Omega) \\ &+0.3\cos(-\Omega)\end{aligned} \qquad (27)$$

where l' is the mean anomaly of the Sun, F is the mean angular distance of the Moon from the ascending node, D is the difference between the mean longitudes of the Sun and the Moon, and Ω is the mean longitude of the ascending node of the lunar orbit. These terms correspond to the retrograde annual nutation, the prograde 18.6 year nutation, the nutation at the very long period of 10,467.6 years appearing in Roosbeek's tidal potential used here, and the retrograde 18.6 year nutations. As explained in Sect. 3, they correspond to very long period nutation or to a nutation close to the FCN.

Figures 1(a) and 2(a) show the time evolution of each of four terms found to be above 0.1 μas for the nutation in longitude and obliquity. Figures 1(b) and 2(b) represent the global contribution coming from the sum of these four terms.

4.2 Numerical Integration

In parallel to the analytical approach developed in Folgueira et al. (2007) as well as in the previous sections, we have calculated the nutations of the Earth numerically using a numerical integration approach. For obtaining such rotational motions of the Earth, we have used the SONYR model (Bois, Journet, Vokrouhlivkỳ, and Rambaux) allowing to compute rotational and orbital motions of terrestrial planets. SONYR is the acronym of Spin-Orbit *N*-bodY Relativistic model, a dynamical model of the solar System that includes the coupled spin-orbit motions of

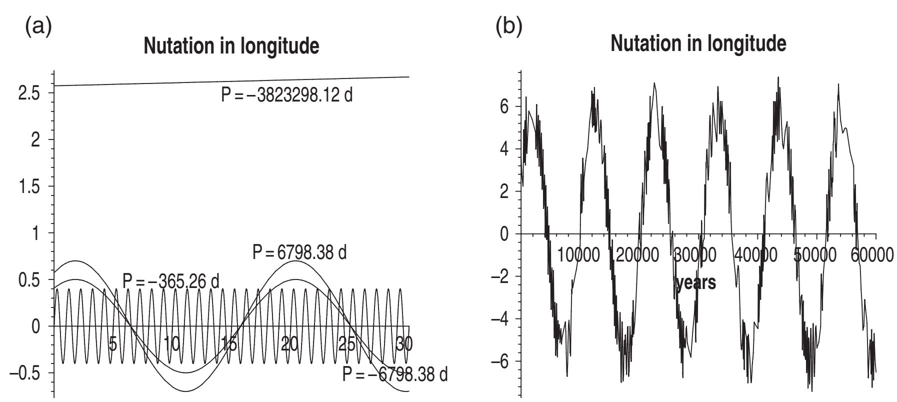

Fig. 1 (**a**) On the left, time evolution of the each contributing term of the purely periodic nutation in longitude coming from the tidal Poisson terms of the forcing. The time is measured in Julian years and the amplitudes in microarcseconds. (**b**) On the right, time evolution of the total contribution to periodic nutation in longitude due to the tidal Poisson terms of the forcing

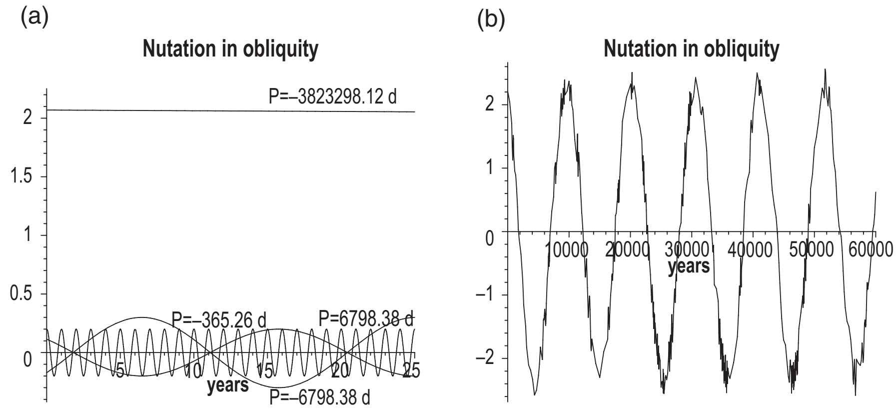

Fig. 2 (**a**) On the left, time evolution of the each contributing term of the purely periodic nutation in obliquity coming from the tidal Poisson terms of the forcing. The time is measured in Julian years and the amplitudes in microarcseconds. (**b**) On the right, time evolution of the total contribution to periodic nutation in obliquity due to the tidal Poisson terms of the forcing

the terrestrial planets and of the Moon. It is described in a series of papers (see a review in Bois (2000) and reference therein; Bois and Vokrouhlickỳ (1995); Rambaux et al. (2007)).

In the frame of the present paper and in order to compare with the analytical results developed in the previous sections (see Eq. (22) for instance), we have in particular calculated the difference between the nutation of a rigid Earth and the nutation of a two-layered Earth model. As seen from Eq. (22), there are three contributions from the existence of a core, among which one contribution related to $V_{21,n,0}$ and two new contributions related to $V_{21,n,1}$, the Poisson part of the forcing potential. As the numerical approach is based on a forced model of the celestial bodies motions and not on potential-form forcing, it is impossible to separate the three contributions. It is however interesting to compare the results at the global level as, in principal, if the time-dependent amplitudes of the nutations can be separated from their constant amplitudes, it is possible to isolate the term in $V_{21,n,1}t$ from the two other terms in $V_{21,n,0}$ and $V_{21,n,1}$. However, the terms coming from the coupling between the core and the mantle (i.e. the non-rigid part of the transfer function in the frequency domain) and from the periodic forcing (the terms in $V_{21,n,0}$ in Eq. (22)) are mixed with the contributions of the purely-periodic terms coming from the Poisson terms in the rotation of the mantle and the core (the terms in $V_{21,n,1}$ in Eq. (22)).

To that aim, we have taken the following approach. First, we neglect in a first approximation the purely relativistic terms in the post-newtonian development. We then compute the dynamical behavior of the two rotational motions of the mantle and the core, for an Earth consisting of a completely homogeneous body and for an Earth consisting of two layers. In order to be able to consider core-mantle coupling, we have included the Poincaré model for the core in the SONYR model (see, e.g., Poincaré 1910; Moritz 1980). The coupling mechanism between the mantle and core in this model is called inertial coupling, and is due to the pressure of the fluid on the core-mantle boundary. It is possible at this level to compare the results from the two Earth models. It must be noted that in our numerical approach we have ignored the deformation contributions and did only consider the core existence for the geophysical part of the equations.

Table 1 provides the ratios between the non-rigid contributions and the rigid contributions for the main prograde and retrograde nutations of the Earth. In this case the terms coming from the transfer function resulting from the existence of a core and the terms coming from the Poisson part of the forcing are mixed. As our analytical developments have shown in Sects. 2 and 3, the main contribution is not coming from the Poisson terms in the forcing, but rather from the core FCN resonance. We find that the contribution of the prograde 18.6 years nutation is smaller than the unity whereas the ratio for all other nutations are greater than 1 as in the case of Wahr's B-ratio (Wahr 1981).

Table 1 Ratio non-deformable nutation amplitude over the rigid nutation amplitude

Nutation period	Prograde nondef/rig	Retrograde nondef/rig
18.6 years	0.9914806597	1.007487569
Semi-annual	1.220587363	1.088899310
Annual	1.851327564	1.068475799
Monthly	1.130579228	1.122725790

For the next step, we want to determine the amplitudes of the purely Poisson terms (the terms in $V_{21,n,1}t$) in the Earth orientation and the amplitudes of the purely periodic terms (the two other terms in $V_{21,n,0}$ and $V_{21,n,1}$). This determination involves the Fourier analysis of the Earth orientation obtained in a *N*-body integration in terms of constant and time-dependent amplitudes for each nutation. This formal decomposition of long-term motions in Fourier series and Poisson terms is however very tricky as the length of the time series used for this decomposition is of importance in this approach. For most of the terms, we can only estimate the sum of the two contributions, and we can only estimate the ratios and the difference between the total non-rigid contribution and the rigid contribution as done above for the construction of the table. For the terms of which the period is smaller than the time span of the time series, it is in principal possible to separate the Poisson terms from the periodic contribution to nutations by fitting an amplitude and a term proportional to the time for each frequency. However, as seen from the previous sections, the time dependent parts are very small. Additionally, many terms interfere with one another and the identification of purely-periodic and Poisson amplitudes becomes impossible.

One possibility to obtain a clear separation, and thus to obtain the impact on the purely-periodic nutations of the Poisson terms in the forcing, would be to compute two orbital motions of the Earth reconstructed using a tidal potential for the Earth with and without Poisson terms and to compare the results from the numerical integration approach for both cases. This will allow obtaining their respective contributions to the Earth rotation.

Additionally, for the nutation close to the FCN, it is impossible to perfectly clear the he nutation time series from the free core nutation excitation if the initial conditions are not properly chosen or if no dissipation is taken into account. A method for getting independence from the initial conditions has been developed for Mercury by Bois and Rambaux (2007) and could possibly be applied for the Earth.

Acknowledgments We are really thankful to Nicole Capitaine and Chengli Huang for their fruitful reviews of our paper.

The research was carried out in the Section 'Time, Earth Rotation, and Pace Geodesy' of the Royal Observatory of Belgium and received financial support from the Descartes Prize Allowance (MF, NR), for which we express our sincere appreciation.

References

Bois, E., 2000. Connaissance de la libration lunaire à l'ère de la télémétrie laser-Lune, *C. R. Acad. Sci. Paris, t.* 1, Série IV, 809–823.

Bois, E., and D. Vokrouhlickỳ, 1995. Relativistic spin effects in the Earth-Moon system, *A & A* **300**, 559.

Bois, E., and N. Rambaux, 2007. On the oscillations in Mercury's obliquity, *Icarus*, in press.

Dehant, V., J. Hinderer, H. Legros, and M. Lefftz, 1993. Analytical approach to the computation of the Earth, the outer core and the inner core rotational motions, *Phys. Earth Planet. Inter*. **76**, 259–282.

Dehant, V., M. Feissel-Vernier, O. de Viron, C. Ma, M. Yseboodt, and C. Bizouard, 2003. Remaining error sources in the nutation at the submilliarcsecond level, *J. Geophys. Res*. **108(B5)**, 2275, DOI: 10.1029/2002JB001763.

Ferrándiz, J.M., J.F. Navarro, A. Escapa, and J. Getino, 2004. Precession of the nonrigid Earth: effect of the fluid outer core, *Astron. J*. **128**, 1407–1411.

Folgueira, M., V. Dehant, S.B. Lambert, and N. Rambaux, 2007. Impact of tidal Poisson terms to non-rigid Earth rotation, *A & A* **469(3)**, 1197–1202, DOI: 10.1051/0004-6361:20066822.

Greff-Lefftz, M., H. Legros, and V. Dehant, 2002. Effect of inner core viscosity on gravity changes and spatial nutations induced by luni-solar tides. *Phys. Earth Planet. Inter*. **129(1–2)**, 31–41.

Hinderer, J., H. Legros, and M. Amalvict, 1987. Tidal motions within the earth's fluid core: resonance process and possible variations, *Phys. Earth Planet. Inter*. **49(3–4)**, 213–221.

Mathews, P. M., T. A. Herring, and B. A. Buffett, 2002. Modeling of nutation and precession: new nutation series for nonrigid Earth and insights into the Earth's interior, *J. Geophys. Res*. **107(B4)**, DOI: 10.1029/2001JB000390.

McCarthy, D.D., and G. Petit (Eds.), 2004. Conventions 2003. *IERS Technical Note* **32**, Publ. Frankfurt am Main: Verlag des Bundesamts für Kartographie und Geodäsie.

Moritz, H. and I.I. Mueller, 1987. *Earth Rotation: Theory and Observation*. The Ungar Publishing Company, New York.

Poincaré H., 1910. Sur la précession des corps déformables. *Bulletin Astronomique*, Serie I, **27**, 321–356.

Rambaux, N., T. Van Hoolst, V. Dehant, and E. Bois, 2007. Inertial core-mantle coupling and libration of Mercury, *A & A* **468(2)**, 711–719.

Roosbeek, F., 1998. Analytical developments of rigid Mars nutation and tide generating potential series, *Celest. Mech. Dynamical Astron*. **75**, 287–300.

Roosbeek, F., and V. Dehant, 1998. RDAN97: An analytical development of rigid Earth nutations series using the torque approach, *Celest. Mech. Dynamical Astron*. **70**, 215–253.

Sasao, T., S. Okubo, and M. Saito, 1980. A simple theory on dynamical effects of stratified fluid core upon nutational motion of the Earth, *Proc. IAU Symposium* **78**, 'Nutation and the Earth's Rotation', Dordrecht, Holland, Boston, D. Reidel Pub. Co., 165–183.

Wahr, J.M., 1981. The forced nutations of an elliptical, rotating, elastic and oceanless earth, *Geophys. J. R. Astron. Soc*. **64**, 705–727.