Anti-Pasch optimal coverings with triples

Fatih Demirkale*
Department of Mathematics
Yıldız Technical University
Esenler, İstanbul 34220
TURKEY
(fatihd@yildiz.edu.tr)

Diane Donovan
ARC Centre of Excellence for Plant Success in
Nature and Agriculture
School of Mathematics and Physics
University of Queensland, St. Lucia 4072
AUSTRALIA
(dmd@maths.uq.edu.au)

Mike Grannell†
School of Mathematics and Statistics
The Open University, Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM
(m.j.grannell@open.ac.uk)

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Abstract

It is shown that for $v \neq 7, 8, 11, 12$ or 13, there exists an optimal covering with triples on v points that contains no Pasch configurations.

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†Corresponding author

1 Introduction

The background to this paper is the anti-Pasch problem for Steiner triple systems. A Steiner triple system of order v, STS(v), is a pair $(V, \mathcal{B})$ where V is a set of v elements (called *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (called *blocks* or *triples*) with the property that each 2-element subset of V is contained in exactly one block. An STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6) [10], and such values are called *admissible*. A *Pasch configuration*, also known as a *quadrilateral*, is a set of 3-element sets on six points having the form

$$\{\{a, b, c\}, \{a, y, z\}, \{x, b, z\}, \{x, y, c\}\}.$$

The anti-Pasch conjecture, originally made by Paul Erdős [6] in a more general form, was that for all sufficiently large admissible v there exists an STS(v) that contains no Pasch configurations among its blocks. The conjecture was finally established in a series of papers [1, 8, 9, 11] culminating in [7]. So it is now known that there exists an STS(v) that contains no Pasch configurations provided v is admissible and $v \neq 7, 13$. Such systems are called *anti-Pasch* and are denoted as APSTS(v).

When v is not admissible, there is no STS(v). However, there will still be a maximum or optimal packing with triples on v points, and there will also be a minimum or optimal covering with triples on v points. In a previous paper we determined the anti-Pasch result for optimal packings [4]. In the current paper we determine the anti-Pasch result for optimal coverings. These packings and coverings are as close as it is possible to get to an STS(v) when v is not admissible.

Our proofs require the checking of a lot of details, so before embarking on them, we give an overview. In broad terms, the proofs are based on establishing the existence of anti-Pasch Steiner triple systems and anti-Pasch optimal packings that satisfy certain additional properties, essentially excluding "forbidden" blocks. These additional properties enable us to adjoin extra blocks that do not introduce Pasch configurations, and thereby obtain anti-Pasch coverings. Most of the complexity in the proofs lies in establishing the existence of these specific anti-Pasch designs.

For $v \equiv 1$ or 3 (mod 6) (with $v \neq 7$ or 13) there exists an APSTS(v), and anti-Pasch optimal coverings for the cases $v \equiv 2$ or 4 (mod 6) are easily obtained from these as explained in Section 3. The most difficult case appears to be $v \equiv 5$ (mod 6). It is easy to obtain an optimal covering by triples in this case, by adding two additional triples to an optimal packing, but it appears to be difficult to ensure that the resulting design avoids Pasch configurations. However, once this case has been resolved we can obtain, relatively easily, the result for $v \equiv 0$ (mod 6), and we explain this in Section 4.

There are two constructions presented in [4] and we show in Section 5 how these may be modified to form the basis of an inductive argument to establish the existence of anti-Pasch optimal coverings by triples on v points when $v \equiv 5 \pmod 6$. The modifications require us to prove the existence of an APSTS(v) having an additional property (called G4a) for each admissible v apart from $v = 7$ or 13. In Section 5 we proceed on the assumption that such STS(v)s exist and defer the proof of this to Section 6 where we make extensive use of some of the constructions presented in [7] and [11].

2 Definitions and terminology

A *partial triple system* of order v, PTS(v), is a pair $(V, \mathcal{B})$ and is defined similarly to an STS(v), except that each 2-element subset of V is required to be contained in *at most* one block. A PTS(v) $= (V, \mathcal{B})$ for which there is no PTS(v) $= (V, \mathcal{B}')$ with $|\mathcal{B}'| > |\mathcal{B}|$ and $\mathcal{B} \subseteq \mathcal{B}'$ is called a *maximal* partial triple system, MPTS(v). An MPTS(v) with the largest possible set of blocks is called a *maximum* maximal partial triple system, MMPTS(v). The name is generally shortened to "maximum partial triple system". Such systems are also known as *optimal* or *maximal* packings with triples, and they give rise to optimal constant weight error-correcting codes (see [2, Section VI.40]).

We often have to write pairs and triples. When no confusion is likely, we may omit brackets $\{\}$ and commas. For example, we may write $\{a, b, c\}$ as abc. Thus a Pasch configuration has the form $\{abc, ayz, xbz, xyc\}$. We use the letters AP (Anti-Pasch) to denote a design without Pasch configurations as in APSTS(v), APPTS(v), APMMPTS(v), etc.

The *leave* of an MMPTS(v) is the set of pairs that are not covered by the blocks. We will be particularly concerned with MMPTS(v) in the case $v \equiv 5 \pmod 6$, and then the leave comprises four pairs, having the form $\{ab, bc, cd, da\}$. This can be represented as a graph by taking each pair as an edge, and so the leave for $v \equiv 5 \pmod 6$ can be expressed as a 4-cycle (a, b, c, d). It may or may not happen that the blocks containing the pairs ac and bd have a common third point e. When this does happen, the MMPTS(v) is said to be of *quintuple type* and, by removing the two blocks ace and bde, a PTS(v) can be formed with a hole $\{a, b, c, d, e\}$.

A *covering by triples* on v points, CT(v) is again a pair $(V, \mathcal{B})$ and is defined similarly to an STS(v), except that now each 2-element subset of V is required to be contained in *at least* one block. A CT(v) $= (V, \mathcal{B})$ for which there is no CT(v) $= (V, \mathcal{B}')$ with $|\mathcal{B}'| < |\mathcal{B}|$ and $\mathcal{B}' \subseteq \mathcal{B}$ is called a *minimal* covering by triples, MCT(v). An MCT(v) with the smallest possible set of blocks is called a *minimum* minimal covering by triples, MMCT(v). The name is generally shortened to "minimum covering by triples". Such systems are also known as *optimal* coverings by triples.

The number of triples in an MMCT(v) is given by

$$|\mathcal{B}| = \left\lceil \frac{v}{3} \left\lceil \frac{v-1}{2} \right\rceil \right\rceil .$$

The *excess* is the multiset of pairs $\{x, y\}$ with multiplicity given by $|B \in \mathcal{B} : \{x, y\} \subseteq B| - 1$. This can be represented as a multigraph by taking each pair of the excess as an edge (with appropriate multiplicity). Table 1 gives the structure of MMCT(v) for different values of v modulo 6, see [2, Section VI.11.5].

v	Number of triples	Excess graph
$1, 3 \pmod 6$	$\dfrac{v^2 - v}{6}$	Empty
$0 \pmod 6$	$\dfrac{v^2}{6}$	$\frac{v}{2} K_2$
$2, 4 \pmod 6$	$\dfrac{v^2 + 2}{6}$	$K_{1,3} \cup \frac{v-4}{2} K_2$
$5 \pmod 6$	$\dfrac{v^2 - v + 4}{6}$	One edge of multiplicity 2

Table 1: The structure of optimal coverings by triples.

In Section 6 we make extensive use of m-bipartite systems first introduced in [7]. These are defined in the following manner. An STS($u, -m$) is a triple $(U, M, \mathcal{B})$, where U is a set of points having cardinality u, $M \subseteq U$ has cardinality m, and $\mathcal{B}$ is a collection of triples of points with the property that every pair of points $\{\alpha, \beta\}$, with $\alpha \in U$, $\beta \in U \setminus M$ appears in precisely one triple from $\mathcal{B}$, and no pairs $\{\alpha, \beta\}$ with $\alpha, \beta \in M$ appear in any triple from $\mathcal{B}$. The set M is called the *hole*. For u and m both admissible (the only cases we will consider here), an STS($u, -m$) is an STS(u) with an STS(m) subsystem removed. An APSTS($u, -m$) is an STS($u, -m$) containing no Pasch configurations. An STS($u, -m$) is said to be *m-bipartite* if the points of $U \setminus M$ can be partitioned into two sets X and Y, each of cardinality n (so that $u = m + 2n$), in such a way that the design has no triples of the forms $\{\mu, x_1, x_2\}$ or $\{\mu, y_1, y_2\}$ where $\mu \in M$, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. An m-bipartite APSTS($u, -m$) will be denoted by BAPSTS($u, -m$). It is easily seen that for admissible $u = m + 2n$, an m-bipartite STS($u, -m$) can only exist if $m \leq n$.

Also in Section 6, use is made of group divisible designs (GDDs) and transversal designs (TDs). For definitions and associated terminology relating to these, the reader is directed to [2].

3 The cases $v \equiv 2$ or $4 \pmod 6$

For $v \equiv 2$ or $4 \pmod 6$ and $v \neq 8, 14$, let $\mathcal{S} = (V, \mathcal{B})$ be an APSTS$(v-1)$ with $V = \{0, 1, \ldots, v-2\}$ and where $\mathcal{B}$ contains the blocks $\{0, 2i-1, 2i\}$ for $i = 1, 2, \ldots, (v-2)/2$. Enlarge V by adding a new point ∞ to form $V^* = V \cup \{\infty\}$ and enlarge $\mathcal{B}$ by adding blocks to form $\mathcal{B}^*$. First add the blocks $\{\infty, 2i-1, 2i\}$ for $i = 2, 3, \ldots, (v-2)/2$. Addition of these blocks cannot generate a Pasch configuration since any Pasch configuration containing two of the blocks with ∞ would imply the existence of a Pasch configuration containing the two corresponding blocks containing the point 0. Then add the blocks $\infty 01, \infty 02$ to complete the formation of $\mathcal{B}^*$. The repeated pairs are now $0\infty, 01, 02, 34, 56, \ldots, (v-3)(v-2)$. So $\mathcal{S}^* = (V^*, \mathcal{B}^*)$ is an MMCT(v) with excess graph $K_{1,3} \cup (\frac{v-4}{2})K_2$.

To prove this MMCT(v) is anti-Pasch it suffices to consider a potential Pasch configuration P containing either $\infty 01$ or $\infty 02$ (but not both). If P contains $\infty 01$ then, without loss of generality, the other block of P containing ∞ is $\infty 34$. So the blocks are either

1. $\infty 01, \infty 34, 03Z, 14Z$, or
2. $\infty 01, \infty 34, 04Z, 13Z$.

In the former case the third block gives $Z = 4$, and in the latter case the third block gives $Z = 3$. So in either case we have a contradiction. In a similar way the possibility of a Pasch configuration containing the block $\infty 02$ can be eliminated. Hence we obtain an APMMCT(v) for all $v \equiv 2$ or 4 (mod 6) with the possible exceptions of $v = 8$ and $v = 14$. By computation it is easy to show that there is no APMMCT(8), but an APMMCT(14) does exist and is given by the following 33 blocks:

$$\begin{array}{c} 012,\ 014,\ 026,\ 038,\ 03X,\ 059,\ 07Y,\ 0ZW,\ 136,\ 157,\ 189, \\ 1XW,\ 1YZ,\ 235,\ 247,\ 28W,\ 29Z,\ 2XY,\ 34Y,\ 379,\ 3ZW,\ 456, \\ 45W,\ 48Z,\ 49X,\ 58Y,\ 5XZ,\ 67Z,\ 67W,\ 689,\ 6XY,\ 78X,\ 9YW. \end{array}$$

As a consequence we may state the following result.

Theorem 3.1 *There exists an APMMCT(v) for all $v \equiv 2$ or 4 (mod 6) with the single exception of $v = 8$.*

4 The case $v \equiv 0 \pmod 6$

We will show in Theorem 5.4 of Section 5 that for $s \geq 2$, there exists an APPTS$(6s+5)$ on the point set $V = \{0, 1, \ldots, 6s+4\}$, with leave K_5 on the set $F = \{0, 1, 2, 3, 4\}$, containing blocks $\{0, 2i-1, 2i\}$ for $i = 3, 4, \ldots, 3s+2$, and having the additional property that adding the five blocks $012, 013, 014, 023$ and 234 does not generate a Pasch configuration.

So take such an APPTS$(6s+5)$. Enlarge V by adding a new point ∞ to form $V^* = V \cup \{\infty\}$ and enlarge $\mathcal{B}$ by adding blocks to form $\mathcal{B}^*$. First add the blocks $012, 013$ and 234. The additional property of the APPTS$(6s+5)$ ensures that these do not create a Pasch configuration. Next add the blocks $\{\infty, 2i-1, 2i\}$ for $i = 3, 4, \ldots, 3s+2$. Addition of these blocks cannot generate a Pasch configuration since any Pasch configuration containing two of the blocks with ∞ would imply the existence of a Pasch configuration containing the two corresponding blocks containing the point 0. Next add the blocks $\infty 14$ and $\infty 23$. Again, by referring to the triples 014 and 023, it can be seen that these additional blocks cannot generate a Pasch configuration. Finally complete $\mathcal{B}^*$ by adding the block $\infty 04$. This cannot lie in a Pasch configuration with $\infty 14$. If it lay in a Pasch configuration with $\infty 23$ then the other two blocks would be either $02Z, 34Z$ or $03Z, 24Z$. In the former case the first block gives $Z = 1$ and the second block gives $Z = 2$, while in the latter case the first block gives $Z = 1$ and the second block gives $Z = 3$, a contradiction in both cases. The only remaining possibility is that $\infty 04$ lies in a Pasch configuration with a block $\{\infty, 2i-1, 2i\}$ for some $i = 3, 4, \ldots, 3s+2$, and without loss of generality we can assume that $i = 3$ so that the second block is $\infty 56$. But then the remaining two blocks would be $05Z, 46Z$ or $06Z, 45Z$. The former case implies $Z = 6$ and the latter case implies $Z = 5$, so in either case we have a contradiction.

So the resulting design is anti-Pasch with point set V^*. In addition to triples of the PTS it has blocks $\{\infty, 2i-1, 2i\}$ for $i = 3, 4, \ldots, 3s+2$, and also 012, 013, 234, $\infty 14$, $\infty 23$ and $\infty 04$. These cover all remaining pairs not covered by triples of the PTS, and the repeated pairs are $\infty 4$, 01, 23, and $\{2i-1, 2i\}$ for $i = 3, 4, \ldots, 3s+2$. So this design is an APMMCT(v) for $v = 6s+6$ with $s \geq 2$.

It remains to consider the cases $s = 0$ and $s = 1$. An APMMCT(6) does exist and is given by the following six blocks:

$$012,\ 013,\ 045,\ 234,\ 235,\ 145.$$

Exhaustive searches by two independently written computer programs confirm that there is no APMMCT(12). Consequently we may state the following result.

Theorem 4.1 *There exists an APMMCT(v) for all $v \equiv 0 \pmod 6$ with the single exception of $v = 12$.*

5 The case $v \equiv 5 \pmod 6$

Here we use two constructions taken from [4]. In each case we make some specific choices and additional assumptions.

Construction 1 starts with an APSTS($n+2$) having an additional property and produces an APMMPTS($3n+2$) for $n \equiv 1$ or 5 (mod 6) that itself has additional properties. Some of these properties make it possible to adjoin extra blocks to this design to yield an APMMCT(v) with $v \equiv 5$ or 17 (mod 18). Construction 2 starts with an APMMPTS($n+2$) having additional properties and produces an APMMPTS($3n+2$) for $n \equiv 3$ (mod 6) that itself has additional properties. Some of these properties make it possible to adjoin extra blocks to this design to yield an APMMCT(v) for $v \equiv 11$ (mod 18). In order to cover all cases $v \equiv 5, 11$ and 17 (mod 18) it is necessary to combine Construction 2 recursively with Construction 1 in order to ensure the existence of "starting" systems APMMPTS($n+2$) having the required additional properties. Both constructions omit a small number of small values of v that can be resolved by computer searches; these small systems are given in [5]. We start by explaining the terminology employed in these constructions.

Both constructions depend on the cycle structure of STS(v) and MMPTS(v) designs. For such a design $(V, \mathcal{B})$, define the double neighbourhood of $x, y \in V$ (with $x \neq y$) as

$$N(x,y) = \{\{z,w\} : xzw \in \mathcal{B} \text{ or } yzw \in \mathcal{B}, \text{ and } \{z,w\} \cap \{x,y\} = \emptyset\}.$$

A double neighbourhood $N(x,y)$ can be represented as a graph $G(x,y)$ by taking the pairs of $N(x,y)$ as edges. In the case of an STS(v) the graph $G(x,y)$ is 2-regular and so it is the union of simple cycles, each of even length at least four. We refer to these as the cycles on the pair $\{x,y\}$, or as the $\{x,y\}$-cycles. In the case of an MMPTS(v) with $v \equiv 5 \pmod 6$, if the pair $\{x,y\}$ lies in the leave, so that the leave has the form (x,y,z,w), then the points z and w have degree one in $G(x,y)$, and therefore the graph contains a path with end points z and w, which we refer to as the path on the pair $\{x,y\}$, or as the $\{x,y\}$-path. If this path has length $v-3$ (i.e. it has $v-2$ vertices) then there will be no cycles on $\{x,y\}$, but if its length is less than $v-3$, there will also be cycles on $\{x,y\}$. If an STS(v) or an MMPTS(v) contains a Pasch configuration, say $\{xpq, xrs, ypr, yqs\}$, then the graph $G(x,y)$ contains the 4-cycle (p,q,s,r). Conversely, if $G(x,y)$ contains a 4-cycle, then the corresponding four blocks form a Pasch configuration. Consequently, an APSTS(v) or an APMMPTS(v) cannot give rise to a cycle of length four on any pair of points $\{x,y\}$.

For a positive integer n denote the set $\{0,1,\ldots,n-1\}$ by N. If $a, b \in N$, define the *difference* $d = |a-b| \pmod n$ to be the minimum of $(a-b) \pmod n$ and $(b-a) \pmod n$, so that $d \in \{0,1,\ldots,\lfloor \frac{n}{2} \rfloor\}$. Very occasionally it is convenient to refer to a difference $d > n/2$ (but with $d < n$); such a difference is to be interpreted as $n - d$. Now suppose that $\mathcal{S} = (V, \mathcal{B})$ is an STS$(n+2)$ or an MMPTS$(n+2)$ on the point set $V = \{A, B\} \cup N$. If $\{A, a, b\} \in \mathcal{B}$ with $a, b \in N$ then we say that A has an associated difference $d = |a - b| \pmod{n}$ in $\mathcal{S}$ and that d is a difference associated with A. The set of all differences associated with A in $\mathcal{S}$ is denoted by D^A. Note that a block $\{A, B, x\}$ does not generate a difference. The set of all differences associated with B in $\mathcal{S}$ is defined in a similar fashion and is denoted by D^B.

Constructions 1 and 2 given below start by combining three STS$(n+2)$s or three MMPTS$(n+2)$s, which we now describe.

For a positive integer n denote the set $\{0_i, 1_i, \ldots, (n-1)_i\}$ by N_i for $i = 0, 1, 2$. Now suppose that for $i = 0, 1, 2$, $\mathcal{S}_i = (V_i, \mathcal{B}_i)$ is an STS$(n+2)$ or an MMPTS$(n+2)$, where $V_i = \{A, B\} \cup N_i$. Then the sets of associated differences D_i^A and D_i^B are formed as described above as subsets of N (not N_i), so that (for example) $d \in D_i^A$ if and only if there exists a block $\{A, a_i, b_i\} \in \mathcal{B}_i$ such that $|a - b| \equiv d \pmod{n}$. If $D_i^A \cap D_j^A = \emptyset$ and $D_i^B \cap D_j^B = \emptyset$ for $i, j = 0, 1, 2$, with $i \neq j$, then we say that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have *different differences* with respect to $\{A, B\}$. Both Constructions 1 and 2 require systems with different differences and in each case these can be formed from a single system with a generic labelling.

A *generic* labelling of an APSTS$(n+2)$ is a labelling of its points by A, B and the elements of N with the following properties.

(i) One block is labelled $AB0$,

(ii) every block labelled Axy with $x, y \in N$ has $|x - y| = 1$ (absolute arithmetic, not just modulo n),

(iii) every block labelled Bxy with $x, y \in N$ has $|x - y| = 3$ or 5 (absolute arithmetic, not just modulo n),

(iv) each $\{A, B\}$-cycle is labelled with a subset of consecutive integers from N.

It was shown in [4] that every APSTS$(n+2)$ has a generic labelling. This was established by choosing any two points to be labelled as A and B, and then labelling the points of each $\{A, B\}$-cycle using an appropriate subset of N. We may conveniently refer to an individual $\{A, B\}$-cycle $(x_1, y_1, x_2, y_2, ..., x_m, y_m)$ as being generically labelled if the blocks are Ax_iy_i and By_ix_{i+1}, the set $\{x_i, y_i : 1 \leq i \leq m\}$ is a subset of consecutive integers from N, and $|x_i - y_i| = 1$, $|y_i - x_{i+1}| = 3$ or 5, where x_{m+1} is taken to be x_1. Note that in a generically labelled APSTS$(n+2)$, apart from the block $AB0$, every block Axy or Bxy has x and y of different parities.

A *generic* labelling of an APMMPTS($n+2$) with $n \equiv 3 \pmod 6$ is a labelling of its points by A, B and the elements of N with the following properties.

(i) The leave is labelled $(A, 0, 1, B)$,

(ii) every block labelled Axy with $x, y \in N$ has $|x-y| = 1$ (absolute arithmetic, not just modulo n),

(iii) every block labelled Bxy with $x, y \in N$ has $|x-y| = 2$, 3 or 5 (absolute arithmetic, not just modulo n),

(iv) the $\{A, B\}$-path and each $\{A, B\}$-cycle (if any) is labelled with a subset of consecutive integers from N.

Again, it was shown in [4] that every APMMPTS($n+2$) with $n \equiv 3$ (mod 6) has a generic labelling.

In both cases, as a convention, when an $\{A, B\}$-cycle or path is listed as $(a, b, c, \ldots)$, we assume that A comes first, i.e. there is a block Aab, a block Bbc, etc. We can now describe the first construction, taken from [4].

Construction 1 Suppose that for $i = 0, 1, 2$, $\mathcal{S}_i = (V_i, \mathcal{B}_i)$ is an APSTS($n+2$) (so $n \equiv 1$ or 5 (mod 6)) on the point set $V_i = \{A, B\} \cup N_i$, with $AB0_i \in \mathcal{B}_i$. Suppose also that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have different differences with respect to $\{A, B\}$. Then an APMMPTS($3n+2$), say $\mathcal{S}^*$, can be formed on the point set $V^* = \{A, B\} \cup N_0 \cup N_1 \cup N_2$ with block set $\mathcal{B}^*$ containing the following triples:

- *Horizontal blocks*: All triples from $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, except for the three triples $AB0_i$, $i = 0, 1, 2$.
- *Vertical blocks*: All triples $x_0 y_1 z_2$ where $x_0 \in N_0$, $y_1 \in N_1$, $z_2 \in N_2$ and $x + y + z \equiv 0 \pmod n$, except for the triple $0_0 0_1 0_2$.
- *Mixed blocks*: The two triples $A0_0 0_1$ and $B0_0 0_2$.

Construction 1 was used in [4] to prove that for $n \equiv 1$ or 5 (mod 6) and $n \geq 5$, there exists an APMMPTS($3n+2$) of quintuple type.

In the current paper we will assume that $n \geq 17$ and that the three systems $\mathcal{S}_i$ are formed from a single APSTS($n+2$), say $\mathcal{S} = (V, \mathcal{B})$, where $V = \{A, B\} \cup N$, $AB0 \in \mathcal{B}$. In order that the resulting APMMPTS($3n+2$)

can be extended to an APMMCT($3n+2$), and that it can be used as an ingredient in Construction 2, we make one additional assumption about $\mathcal{S}$, namely that it has a certain property G4a that we define below. **Section 6 establishes that an APSTS(v) with property G4a exists for all admissible $v \neq 7, 13$.**

For $i = 0, 1, 2$, the system $\mathcal{S}_i$ is formed from $\mathcal{S}$ by applying the mapping $\psi_i : V \mapsto V_i$ defined by

$$\begin{aligned}
&\psi_0(A) = A, \quad \psi_0(B) = B, \quad \psi_0(x) = x_0,\\
&\psi_1(A) = B, \quad \psi_1(B) = A, \quad \psi_1(x) = x_1,\\
&\psi_2(A) = A, \quad \psi_2(B) = B, \quad \psi_2(x) = (2x)_2,
\end{aligned}$$

for $x \in N$ and arithmetic modulo n. This ensures that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have different differences with respect to $\{A, B\}$. In fact, $D_0^A = \{1\}$, $D_0^B \subseteq \{3, 5\}$, $D_1^B = \{1\}$, $D_1^A \subseteq \{3, 5\}$, $D_2^A = \{2\}$ and $D_2^B \subseteq \{6, 10\}$. (In the case $n = 17$, the difference 10 is equivalent to 7, and in the case $n = 19$ it is equivalent to 9.)

Definition 5.1 *If an APSTS(v) can be generically labelled (we sometimes say* relabelled*) on the point set* $\{A, B, 0, 1, \ldots, v-3\}$ *in such a way that there are no blocks of the form* $0w(w+4)$ *(absolute arithmetic), then we will say that the system has property G4a. If "absolute arithmetic" can be replaced by "arithmetic modulo* $v-2$*", then we will say that the system has property G4m.*

Clearly G4m implies G4a. The distinction between properties G4a and G4m is small: the latter requires G4a plus the non-existence of three possible blocks $01(v-5)$, $02(v-4)$ and $03(v-3)$. Note that any APSTS(v) can be generically relabelled with respect to any pair of points, so these properties assert that the relabelling can be done in such a manner so as to avoid blocks $0w(w+4)$.

Definition 5.2 *In a similar way, if an APMMPTS(v) can be generically labelled (*relabelled*) on the point set* $\{A, B, 0, 1, \ldots, v-3\}$ *with* $\{A, B\}$*-path* $(1, 2, 0)$*, and in such a way that there are no blocks of the form* $2w(w+4)$ *(absolute arithmetic), then we will say that the system has property G4a. If "absolute arithmetic" can be replaced by "arithmetic modulo* $v-2$*", then we will say that the system has property G4m.*

Again, G4m implies G4a. Note that an APMMPTS(v) with property G4a must be of quintuple type since it has an $\{A, B\}$-path of length 2: the leave is the 4-cycle $(A, 0, 1, B)$, and there are blocks $A12$ and $B02$. Note also that in this definition, the forbidden blocks are of the form $2w(w+4)$, unlike in

Definition 5.1 where the forbidden blocks are of the form $0w(w+4)$.

In both cases, APSTS(v) and APMMPTS(v), if the relabelling to establish property G4a (or G4m) is effected by a mapping ϕ, then we will say that ϕ is a G4a (or G4m) mapping of the system.

Theorem 5.1 *For $n \equiv 1$ or $5 \pmod 6$ with $n \geq 17$, the system $\mathcal{S}^*$ of order $3n+2$ produced by Construction 1 (with the specific choices and additional assumption described above) is of quintuple type and it has property G4m. There is a G4m mapping ϕ of $\mathcal{S}^*$ such that if the triples $AB1$ and $A01$ are added to the relabelled system, then the resulting design is an APMMCT($3n+2$). Furthermore, the following triples can all be added as blocks to the relabelled system without generating any Pasch configurations: $AB0, AB1, AB2, A01, A02, B01, B12$, 012.*

Proof It should be clear that $\mathcal{S}^*$ is of quintuple type since it has leave $(A, 0_2, 0_1, B)$ and two mixed blocks $A0_00_1$ and $B0_00_2$. We remark that removal of these two mixed blocks gives rise to an APPTS($3n+2$) with leave K_5 on the set $\{A, B, 0_0, 0_1, 0_2\}$. The $\{A, B\}$-cycles of $\mathcal{S}^*$ are the $\{A, B\}$-cycles of the three systems $\mathcal{S}_i$, which are themselves copies of the $\{A, B\}$-cycles of $\mathcal{S}$. The points of $\mathcal{S}^*$ can be relabelled generically with respect to $\{A, B\}$ using a mapping $\phi : V^* \mapsto W$, where $W = \{A, B, 0, 1, \ldots, 3n-1\}$. The action of ϕ is defined in stages.

First we set $\phi(A) = A$ and $\phi(B) = B$. The $\{A, B\}$-path in $\mathcal{S}^*$ is $(0_1, 0_0, 0_2)$, so we set $\phi(0_0) = 2$, $\phi(0_1) = 1$ and $\phi(0_2) = 0$, and then the relabelled $\{A, B\}$-path is $(1, 2, 0)$. Next put $\phi(x_0) = x+2$ for $x = 1, 2, \ldots, n-1$. Up to this point in the relabelling, and amongst the blocks fully relabelled, there are no blocks $2w(w+4)$ (arithmetic modulo $3n$).

Next we define $\phi(x_1)$ for $x = 1, 2, \ldots, n-1$. These points x_1 are relabelled as $n+2, n+3, \ldots, 2n$. Consider the $\{A, B\}$-cycles of $\mathcal{S}$ with their original generic labelling; if Axy is a block of S then x and y have opposite parities because $|x-y| = 1$ (absolute arithmetic). Consequently the same is true for the original labelling of $\mathcal{S}_1$. We choose our generic relabelling so that $\phi(x_1)$ has opposite parity to x, and we also arrange the cycles so that the point $(n-4)_1$ receives the label $2n$, i.e. $\phi((n-4)_1) = 2n$. This can be done by taking the original labelling of $\mathcal{S}_1$, cycling the cycles, and adding $n+1$ to the labels. The process is best illustrated by an example, so we do this below in Example 5.1. Having done this and since there are no blocks of $\mathcal{S}^*$ of the forms $0_0w_0z_1$ or $0_0w_1z_1$, there are still no blocks of the partially relabelled system that have the form $2w(w+4)$ (arithmetic modulo $3n$).

Finally we define $\phi(x_2)$ for $x \neq 0$. These points x_2 are relabelled as $2n+1, 2n+2, \ldots, 3n-1$. The system $\mathcal{S}_2$ was produced from $\mathcal{S}$ by applying the mapping $\psi_2 : x \to (2x)_2$, so we can return to a generic labelling of the cycles of $\mathcal{S}_2$ by inverting this mapping. Thus we define

$$\phi(x_2) = \begin{cases} 2n + \frac{x}{2} \text{ if } x \text{ is even and } x \neq 0, \\ 2n + \frac{n+x}{2} \text{ if } x \text{ is odd.} \end{cases}$$

This completes the definition of the relabelling mapping ϕ. The blocks of $\mathcal{S}^*$ that contain the point 0_0 and do not contain A or B have the forms $0_0x_0y_0$ or $0_0x_1y_2$. We have already shown that in the relabelled system there are no blocks of the form $2w(w+4)$ originating from the former type. We now show that there are none originating from the latter type. So consider a block $0_0x_1y_2$ of $\mathcal{S}^*$. Then $x + y \equiv 0 \pmod{n}$, and since $1 \leq x, y \leq n-1$, we must have $x+y = n$. We also have $n+2 \leq \phi(x_1) \leq 2n$ and $2n+1 \leq \phi(y_2) \leq 3n-1$ so the possibilities for $|\phi(x_1) - \phi(y_2)| = 4 \pmod{3n}$ are limited to the following four cases.

1. $\phi(x_1) = 2n,\ \phi(y_2) = 2n+4$. The second equality gives $y = 8$ and consequently $x = n-8$. But we have $\phi((n-4)_1) = 2n$, so $\phi(x_1) = \phi((n-8)_1) \neq 2n$, and this case is impossible.

2. $\phi(x_1) = 2n-1,\ \phi(y_2) = 2n+3$. The second equality gives $y = 6$ and consequently $x = n-6$. But then both $\phi(x_1)$ and x are odd, a contradiction.

3. $\phi(x_1) = 2n-2,\ \phi(y_2) = 2n+2$. The second equality gives $y = 4$ and consequently $x = n-4$. But we have $\phi((n-4)_1) = 2n \neq 2n-2$, so this is impossible.

4. $\phi(x_1) = 2n-3,\ \phi(y_2) = 2n+1$. The second equality gives $y = 2$ and consequently $x = n-2$. But then both $\phi(x_1)$ and x are odd, a contradiction.

It follows from this that in the relabelled copy of $\mathcal{S}^*$ there are no blocks $2w(w+4)$ (arithmetic modulo $3n$). The relabelling is generic, so we have $D^A = \{1\}$ and $D^B \subseteq \{2, 3, 5\}$.

Next we check that adding the blocks $AB1$ and $A01$ to the relabelled system gives an MMCT$(3n+2)$. The leave of the relabelled copy of $\mathcal{S}^*$ is $(A, 0, 1, B)$, so adding these two triples covers all the pairs of the leave. Furthermore, there is already a block $A12$ in the system, so the additional two blocks will give an excess of two copies of the pair $A1$. It follows that the resulting design is an MMCT$(3n+2)$. To prove it is anti-Pasch and to prove that we can add all the blocks $AB0, AB1, AB2, A01, A02, B01, B12, 012$ without creating a Pasch configuration, it is simpler to consider the APPTS$(3n+2)$, say $\mathcal{P}^*$, obtained from $\mathcal{S}^*$ by removing the two mixed blocks $A0_00_1$ and $B0_00_2$, and then to consider all ten triples on $F =$

$\{A, B, 0_0, 0_1, 0_2\}$. No two of these triples can lie together in a Pasch configuration Q, since this would imply that the six points of Q lie in the 5-element set F. So we may consider the triples individually. By symmetry, there are just three typical cases that need to be considered, namely the blocks $AB0_0, A0_00_1$, and $0_00_10_2$. Of these, the mixed block $A0_00_1$ is already known not to generate any Pasch configurations. So this leaves just two typical cases.

Suppose first that $AB0_0$ lies in a Pasch configuration Q with three blocks of $\mathcal{P}^*$. Then Q has one of the following forms.

1. Blocks $AB0_0$, Ax_0y_0, Bx_0z_0, $0_0y_0z_0$. But these form a Pasch configuration in $\mathcal{S}_0$, a contradiction.

2. Blocks $AB0_0$, Ax_1y_1, Bx_1z_1, $0_0y_1w_2$. Since $z_1 \neq w_2$, these blocks cannot form a Pasch configuration.

3. Blocks $AB0_0$, Ax_2y_2, Bx_2z_2, $0_0y_2w_1$. Since $z_2 \neq w_1$, these blocks cannot form a Pasch configuration.

Now suppose that $0_00_10_2$ lies in a Pasch configuration Q with three blocks of $\mathcal{P}^*$. Then Q has one of the following forms.

1. Blocks $0_00_10_2$, $0_0x_0y_0$, $0_1x_0z_2$, $0_2y_0w_1$. Since $z_2 \neq w_1$, these blocks cannot form a Pasch configuration.

2. Blocks $0_00_10_2$, $0_0x_1y_2$, $0_1x_1z_1$, $0_2y_2w_2$. Since $z_1 \neq w_2$, these blocks cannot form a Pasch configuration.

3. Blocks $0_00_10_2$, $0_0x_1y_2$, $0_1y_2z_0$, $0_2x_1z_0$. These blocks require $x + y \equiv y+z \equiv x+z \equiv 0 \pmod{n}$, which gives $x = y = z = 0$, a contradiction.

This completes the proof of the theorem. □

Theorem 5.1 establishes the existence of APMMCT(v) designs for $v = 18s + 5$ and $v = 18s + 17$ when $v \geq 53$. The related APMMPTS designs also have property G4m and, with the appropriate generic labelling, certain blocks may be added without generating a Pasch configuration. These additional properties are required in order to deal with the case $v = 18s+11$. We have also constructed such systems for $v = 23, 35$ and 41 by means of computer searches and these are given in [5].

Finally, the blocks $012, 013, 014, 234$ form an APMMCT(5) and, again by a computer search, we have constructed an APMMCT(17) (but without property G4m and the additional blocks property) and this is also given in [5].

Example 5.1
With reference to the preceding theorem, suppose that $\mathcal{S}$ has an $\{A,B\}$-cycle $(1,2,5,6,3,4)$. Then $\mathcal{S}_1$ has an $\{A,B\}$-cycle $(2_1,5_1,6_1,3_1,4_1,1_1)$. This can be generically relabelled by ϕ as $(n+1)+(1,2,5,6,3,4)$ by putting $\phi(2_1)=n+2$, $\phi(5_1)=n+3$, $\phi(6_1)=n+6$, $\phi(3_1)=n+7$, $\phi(4_1)=n+4$ and $\phi(1_1)=n+5$. A similar process will define ϕ for every $\{A,B\}$-cycle of $\mathcal{S}_1$. To deal with the specification that $\phi((n-4)_1)=2n$, consider the $\{A,B\}$-cycle of $\mathcal{S}_1$ that contains the point $(n-4)_1$. Suppose this is $(\ldots,a_1,(n-4)_1,b_1,\ldots)$ of length ℓ. By cycling the cycle it can be re-written as $(b_1,\ldots,a_1,(n-4)_1)$. There will be a generic labelling of a cycle of length ℓ on the integers $2n-\ell+1,2n-\ell+2,\ldots,2n$ that can also be written (by cycling the cycle) as $(\beta,\ldots,\alpha,2n)$. Then we define $\phi(b_1)=\beta,\ \ldots,\ \phi(a_1)=\alpha,\ \phi((n-4)_1)=2n$. Note that in all cases, $\phi(x_1)$ has opposite parity to x.

We now move on to the second construction, also taken from [4].

Construction 2 Suppose that for $i=0,1,2,\ \ \mathcal{S}_i=(V_i,\mathcal{B}_i)$ is an APMMPTS$(n+2)$ with $n\equiv 3 \pmod 6$ on the point set $V_i=\{A,B\}\cup N_i$, such that $\mathcal{S}_0,\mathcal{S}_1$ and $\mathcal{S}_2$ have different differences with respect to $\{A,B\}$, and their leaves are respectively (A,a_0,b_0,B), (A,c_1,d_1,B) and $(A,e_2,\ f_2,B)$. Suppose also that $c-d\equiv f-e \pmod n$. Let δ denote the difference $|c-d|$ modulo n, and let g be such that $g+c+e\equiv g+d+f\equiv 0 \pmod n$. Assume that

(i) $\delta\notin D_0^A\cup D_0^B$, and

(ii) there are no blocks $g_0w_0(w+\delta)_0\in\mathcal{B}_0$ (where $w+\delta$ is taken modulo n).

Then an APMMPTS$(3n+2)$, say $\mathcal{S}^*$, can be formed on the point set $V^*=\{A,B\}\cup N_0\cup N_1\cup N_2$ with block set $\mathcal{B}^*$ containing the following triples:

- *Horizontal blocks*: All triples from $\mathcal{B}_0\cup\mathcal{B}_1\cup\mathcal{B}_2$.
- *Vertical blocks*: All triples $x_0y_1z_2$ where $x_0\in N_0$, $y_1\in N_1$, $z_2\in N_2$ and $x+y+z\equiv 0 \pmod n$, except for the two triples $g_0c_1e_2$ and $g_0d_1f_2$.
- *Mixed blocks*: The four triples Ac_1e_2, Bd_1f_2, $g_0c_1d_1$ and $g_0e_2f_2$.

The leave of $\mathcal{S}^*$ is (A,a_0,b_0,B), the same as that of $\mathcal{S}_0$.

In [4], Construction 2 was combined with Construction 1 to prove that for $n \equiv 3 \pmod 6$ and $n \geq 9$, there exists an APMMPTS($3n+2$) of quintuple type and, consequently, that there exists an APMMPTS(v) for all $v \equiv 5 \pmod 6$ with $v \neq 11$.

In the current paper we will assume that $n \geq 27$ and that the three systems $\mathcal{S}_i$ are formed from a single APMMPTS($n+2$), say $\mathcal{S} = (V, \mathcal{B})$, where $V = \{A, B\} \cup N$, with $\{A, B\}$-path $(1, 2, 0)$, and which is generically labelled with respect to $\{A, B\}$. We make two additional assumptions about $\mathcal{S}$.

1. $\mathcal{S}$ has the property G4m defined above, so there are no blocks $2w(w+4)$ (arithmetic modulo n).

2. The following three triples can be added to $\mathcal{B}$ without generating any Pasch configurations: $AB0, AB1, A01$.

We have shown in Theorem 5.1 that such a system $\mathcal{S}$ exists for $n \equiv 3$ or $15 \pmod{18}$ when $n \geq 51$, and our computational results deal with $n = 33$ and $n = 39$. But whenever such a system $\mathcal{S}$ exists, we can proceed as follows.

For $i = 0, 1, 2$, the system $\mathcal{S}_i$ is formed from $\mathcal{S}$ by applying the mappings $\psi_i : V \mapsto V_i$ defined by

$$\begin{aligned}
\psi_0(A) &= A, \quad \psi_0(B) = B, \quad \psi_0(x) = x_0, \\
\psi_1(A) &= A, \quad \psi_1(B) = B, \quad \psi_1(x) = (4x)_1, \\
\psi_2(A) &= B, \quad \psi_2(B) = A, \quad \psi_2(x) = (4x-6)_2,
\end{aligned}$$

for $x \in N$ and arithmetic modulo n. This ensures that $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2$ have different differences with respect to $\{A, B\}$. In fact, $D_0^A = \{1\}$, $D_0^B \subseteq \{2, 3, 5\}$, $D_1^A = \{4\}$, $D_1^B \subseteq \{8, 12, 20\}$, $D_2^A \subseteq \{8, 12, 20\}$ and $D_2^B = \{4\}$. Since $n \geq 27$, these three systems have different differences with respect to $\{A, B\}$. The three $\{A, B\}$-paths are $(1_0, 2_0, 0_0)$, $(4_1, 8_1, 0_1)$ and $((-6)_2, 2_2, (-2)_2)$. So aligning these choices with the letters used in Construction 2, we have

$$a = 0, \; b = 1, \; c = 0, \; d = 4, \; e = -2, \; f = -6, \; g = 2, \; \delta = 4.$$

The four mixed blocks are now $A0_1(-2)_2$, $B4_1(-6)_2$, $2_00_14_1$ and $2_0(-2)_2(-6)_2$. From the three paths we see there are also blocks $A1_02_0$, $A4_18_1$, $A(-6)_22_2$, $B0_02_0$, $B0_18_1$ and $B(-2)_22_2$.

Theorem 5.2 *For $n \equiv 3 \pmod 6$ with $n \geq 27$, the system $\mathcal{S}^*$ of order $3n+2$ produced by Construction 2 (with the specific choices and additional assumptions described above) is of quintuple type and it has property G4m. There is a G4m mapping ϕ of $\mathcal{S}^*$ such that if the triples $AB1$ and $A01$ are added to the relabelled system, then the resulting design is an APMMCT($3n+2$). Furthermore, the following triples can all be added as blocks to the relabelled system without generating any Pasch configurations: $AB0, AB1, A01$.*

Proof It should be clear that $\mathcal{S}^*$ is of quintuple type since it has leave $(A, 0_0, 1_0, B)$ and two horizontal blocks $A1_02_0$ and $B0_02_0$. We remark that removal of these two horizontal blocks gives rise to an APPTS$(3n+2)$ with leave K_5 on the set $\{A, B, 0_0, 1_0, 2_0\}$. The $\{A, B\}$-cycles of $\mathcal{S}^*$ are the $\{A, B\}$-cycles of the three systems $\mathcal{S}_i$, which are themselves copies of the $\{A, B\}$-cycles of $\mathcal{S}$, together with the 6-cycle

$$C^* = (4_1, 8_1, 0_1, (-2)_2, 2_2, (-6)_2).$$

The points of $\mathcal{S}^*$ can be relabelled generically with respect to $\{A, B\}$ using a mapping $\phi : V^* \mapsto W$, where $W = \{A, B, 0, 1, \ldots, 3n-1\}$. The action of ϕ is defined in stages.

First we set $\phi(A) = A$ and $\phi(B) = B$. Then we set $\phi(x_0) = x$ for $x = 0, 1, \ldots, n-1$. So up to this point in the relabelling and amongst the blocks fully relabelled there are no blocks $2w(w+4)$ (arithmetic modulo $3n$), and the $\{A, B\}$-path is relabelled as $(1, 2, 0)$.

Next we relabel all points x_1 apart from $0_1, 4_1, 8_1$. The relabelling is done generically on the $\{A, B\}$-cycles of $\mathcal{S}_1$ using the integers n, $n+1, \ldots, 2n-4$, with $(-4)_1$ relabelled as $2n-4$, i.e. $\phi((-4)_1) = 2n-4$. This can be achieved by cycling the cycle containing the point $(-4)_1$ in a similar manner to that described in Example 5.1. Then we relabel the cycle C^* in the order shown so that it becomes $(2n+1, 2n+2, 2n-1, 2n, 2n-3, 2n-2)$ so, for example, $\phi(4_1) = 2n+1$. This is a generic labelling of C^*.

Finally we relabel all the remaining points, that is all x_2 apart from $2_2, (-6)_2, (-2)_2$. This relabelling is also done generically on the $\{A, B\}$-cycles of $\mathcal{S}_2$ using the integers $2n+3, 2n+4, \ldots, 3n-1$ with $(-10)_2$ relabelled as $3n-1$, i.e. $\phi((-10)_2) = 3n-1$. This can be achieved by cycling the cycle containing the point $(-10)_2$.

The two mixed blocks $2_00_14_1$ and $2_0(-2)_2(-6)_2$ have been relabelled by ϕ as $\{2, 2n-1, 2n+1\}$ and $\{2, 2n, 2n-2\}$ respectively, neither of which has the form $2w(w+4)$. The blocks containing the pairs 2_08_1 and 2_02_2 are $2_08_1(-10)_2$ and $2_02_2(-4)_1$, which have been relabelled as $\{2, 2n+2, 3n-1\}$ and $\{2, 2n-3, 2n-4\}$. Again, neither of these blocks has the form $2w(w+4)$. Finally, all remaining blocks of $\mathcal{S}^*$ that contain the point 2_0 have the form $2_0x_1y_2$, and $n \leq \phi(x_1) \leq 2n-4$, while $2n+3 \leq \phi(y_2) \leq 3n-1$. So $7 \leq (\phi(y_2) - \phi(x_1)) \leq 2n-1$, and consequently none of these blocks is relabelled in the form $2w(w+4)$. The relabelling is generic, so we have $D^A = \{1\}$ and $D^B \subseteq \{2, 3, 5\}$.

Next we check that adding the blocks $AB1$ and $A01$ to the relabelled system gives an MMCT$(3n+2)$. The leave of the relabelled system is $(A, 0, 1, B)$, so adding these two triples covers all the pairs of the leave. Furthermore, there is already a block $A12$ in the system, so the additional two blocks will give an excess of two copies of the pair $A1$. It follows that the resulting design is an MMCT$(3n+2)$.

The final claim is that the triples $AB0, AB1, A01$ can be added to the relabelled system without creating any Pasch configurations. This is equivalent to claiming that the triples $AB0_0, AB1_0, A0_01_0$ can be added to $\mathcal{S}^*$ without creating any Pasch configurations. No two of these can lie together in a Pasch configuration because they have common pairs, so we may consider these triples individually.

Suppose first that $AB0_0$ lies in a Pasch configuration Q with three blocks of $\mathcal{S}^*$. Then Q has one of the following forms.

1. Blocks $AB0_0$, $A0_1(-2)_2$, $B0_18_1$, $0_0(-2)_22_1$. Since $8_1 \neq 2_1$, these blocks cannot form a Pasch configuration.

2. Blocks $AB0_0$, $A0_1(-2)_2$, $B(-2)_22_2$, $0_00_10_2$. Since $2_2 \neq 0_2$, these blocks cannot form a Pasch configuration.

3. Blocks $AB0_0$, Ax_0y_0, Bx_0z_0, $0_0y_0z_0$. But these form a Pasch configuration in $\mathcal{S}_0$, a contradiction.

4. Blocks $AB0_0$, Ax_1y_1, Bx_1Z, $0_0y_1w_2$. The only possibility for these blocks to form a Pasch configuration is that $Z = w_2$ and consequently from Bx_1w_2 that $x_1 = 4_1$ and $w_2 = (-6)_2$. But then from Ax_1y_1, we find $y_1 = 8_1$, while from $0_0y_1w_2$ we find $y_1 = 6_1$, a contradiction.

5. Blocks $AB0_0$, Ax_2y_2, Bx_2Z, $0_0y_2w_1$. The only possibility for these blocks to form a Pasch configuration is that $Z = w_1$ and consequently from Bx_2w_1 that $x_2 = (-6)_2$ and $w_1 = 4_1$. But then from Ax_2y_2, we find $y_2 = 2_2$, while from $0_0y_2w_1$ we find $y_2 = (-4)_2$, a contradiction.

Now suppose that $AB1_0$ lies in a Pasch configuration Q with three blocks of $\mathcal{S}^*$. Then Q has one of the following forms.

1. Blocks $AB1_0$, $A0_1(-2)_2$, $B0_18_1$, $1_0(-2)_21_1$. Since $8_1 \neq 1_1$, these blocks cannot form a Pasch configuration.

2. Blocks $AB1_0$, $A0_1(-2)_2$, $B(-2)_22_2$, $1_00_1(-1)_2$. Since $2_2 \neq (-1)_2$, these blocks cannot form a Pasch configuration.

3. Blocks $AB1_0$, Ax_0y_0, Bx_0z_0, $1_0y_0z_0$. But these form a Pasch configuration in $\mathcal{S}_0$, a contradiction.

4. Blocks $AB1_0$, Ax_1y_1, Bx_1Z, $1_0y_1w_2$. The only possibility for these blocks to form a Pasch configuration is that $Z = w_2$ and consequently from Bx_1w_2 that $x_1 = 4_1$ and $w_2 = (-6)_2$. But then from Ax_1y_1, we find $y_1 = 8_1$, while from $1_0y_1w_2$ we find $y_1 = 5_1$, a contradiction.

5. Blocks $AB1_0$, Ax_2y_2, Bx_2Z, $1_0y_2w_1$. The only possibility for these blocks to form a Pasch configuration is that $Z = w_1$ and consequently from Bx_2w_1 that $x_2 = (-6)_2$ and $w_1 = 4_1$. But then from Ax_2y_2, we find $y_2 = 2_2$, while from $1_0y_2w_1$ we find $y_2 = (-5)_2$, a contradiction.

Finally suppose that $A0_01_0$ lies in a Pasch configuration Q with three blocks of $\mathcal{S}^*$. Then Q has one of the following forms.

1. Blocks $A0_01_0$, $A0_1(-2)_2$, $0_00_10_2$, $1_0(-2)_21_1$. Since $0_2 \neq 1_1$, these blocks cannot form a Pasch configuration.

2. Blocks $A0_01_0$, $A0_1(-2)_2$, $0_0(-2)_22_1$, $1_00_1(-1)_2$. Since $2_1 \neq (-1)_2$, these blocks cannot form a Pasch configuration.

3. Blocks $A0_01_0$, Ax_0y_0, 0_0x_0Z, 1_0y_0Z, where $Z = z_0$ or B (in fact Z cannot be B since $\mathcal{S}^*$ does not have a block containing the pair $B1_0$). But these four blocks form a Pasch configuration in $\mathcal{S}_0$, a contradiction.

4. Blocks $A0_01_0$, Ax_1y_1, $0_0x_1z_2$, $1_0y_1z_2$. Then $x+z \equiv 0$ and $1+y+z \equiv 0 \pmod{n}$, so $|x-y| \equiv 1 \pmod{n}$, giving $1 \in D_1^A$, a contradiction.

5. Blocks $A0_01_0$, Ax_2y_2, $0_0x_2z_1$, $1_0y_2z_1$. Then $x+z \equiv 0$ and $1+y+z \equiv 0 \pmod{n}$, so $|x-y| \equiv 1 \pmod{n}$, giving $1 \in D_2^A$, a contradiction.

It follows that the addition of the blocks $AB1$ and $A01$ to the relabelled system gives an APMMCT($3n+2$) and that the addition of a further block $AB0$ does not create any Pasch configurations. □

Remark It might be thought that we could add any triple on the set $\{A, B, 0_0, 1_0, 2_0\}$ to $\mathcal{S}^*$ without generating a Pasch configuration, but this is not so. Consider the triple $AB2_0$. This lies in a Pasch configuration with the blocks $A4_18_1, B0_18_1$ and $2_00_14_1$.

We now show how the results of Theorems 5.1 and 5.2 can be used to establish that for $s \geq 4$, there exists an APMMCT($6s+5$). The following lemma provides the inductive step.

Lemma 5.1 *For $s \geq 4$, there exists an APMMPTS($6s+5$) on the point set $V = \{A, B, 0, 1, \ldots, 6s+2\}$, with $\{A, B\}$-path $(1, 2, 0)$, generically labelled with no block $2w(w+4)$ (arithmetic modulo $6s+3$), such that adding the blocks $AB1, A01$ forms an APMMCT($6s+5$), to which a further block $AB0$ may be added without creating a Pasch configuration.*

Proof By computer searches we have constructed such designs for $s = 4, 5, 6, 7, 10$ and these are given in [5]. Theorem 5.1 establishes their existence for all $s \equiv 0$ or $2 \pmod 3$ with $s \geq 8$. So suppose (inductively) that such designs are also known for $s \equiv 1 \pmod 3$ when $4 \leq s \leq 3t+1$ and $t \geq 3$ (which is certainly true for $t = 3$). We show that this implies such designs also exist for $s = 3t+4$. We have $4 \leq t+1 \leq 3t+1$, so by the hypothesis or by Theorem 5.1 (depending on the value of $t+1$ modulo 3), there is such a design on $6(t+1)+5 = 6t+11$ points. Consequently, by Theorem 5.2 there is such a design on $3(6t+9)+2 = 18t+29$ points. But $18t+29 = 6(3t+4)+5$, and so the result follows by induction. □

We can now state the following definitive anti-Pasch result for optimal coverings in the case $v \equiv 5 \pmod 6$.

Theorem 5.3 *There exists an APMMCT(v) for all $v \equiv 5 \pmod 6$ with the single exception of $v = 11$.*

Proof The result for $v = 6s+5$ with $s \geq 4$ follows from Lemma 5.1. The case $s = 0$ is given by the blocks $012, 013, 014, 234$. The cases $s = 2$ and $s = 3$ follow from computer searches and the designs are given in [5]. Finally, since there is no APMMPTS(11) [4], there cannot be an APMMCT(11). □

By judicious relabelling of the systems produced by Theorem 5.1 and 5.2 we can obtain the following result mentioned in Section 4.

Theorem 5.4 *For $s \geq 2$ there exists an APPTS($6s+5$) on the point set $V = \{0, 1, \ldots, 6s+4\}$ with leave K_5 on the set $F = \{0, 1, 2, 3, 4\}$, containing blocks $\{0, 2i-1, 2i\}$ for $i = 3, 4, \ldots, 3s+2$, and having the additional property that adding the five blocks $012, 013, 014, 023$ and 234 does not generate a Pasch configuration.*

Proof For $s \geq 8$ and $s \neq 10$, take a system produced by Theorem 5.1 or Theorem 5.2 and form the associated APPTS($6s+5$) on the point set $W = \{A, B, 0, 1, \ldots, 6s+4\}$ with leave K_5 on the set $F = \{A, B, 0, 1, 2\}$. For each $s = 3, 4, 5, 6, 7, 10$ a computer search has produced an APPTS($6s+5$) on the point set W with leave K_5 on F (given in [5]). In all these cases, the generic labelling ensures that there are blocks $\{A, 2i-1, 2i\}$ for $i = 2, 3, \ldots, 3s+1$, and we can add the blocks $A12, B02, AB0, AB1$ and $A01$ without generating a Pasch configuration. So now apply the mapping $\chi : W \mapsto V$ given by

$$\chi(A) = 0,\ \chi(B) = 2,\ \chi(0) = 3,\ \chi(1) = 1,\ \chi(2) = 4,\ \text{and}$$
$$\chi(x) = x + 2 \text{ for } x = 3, 4, \ldots, 6s+2.$$

The resulting relabelled system satisfies the conditions required. This deals with every case except for $s = 2$. In this case an appropriate APPTS(17) has been found by a computer search and is given in [5]. □

Remark Here we list the small cases obtained by computer searches that are used in this Section, all of which are given in [5]. For $v = 23, 29, 35, 41, 47$ and 65 we give APPTS(v)s that yield APMMCT(v)s and associated APMMPTS(v)s with property G4m and the property that certain additional blocks can be added without creating a Pasch configuration. For $v = 17$ we give the APPTS(17) that gives the system required by Theorem 5.4 and which also yields an APMMCT(17).

6 APSTS(v) with property G4a

Our aim in this section is to prove that for every admissible v, apart from $v = 7, 13$, there is an APSTS(v) with property G4a. In fact the same is true with "G4a" replaced by "G4m" and this requires virtually no extra effort in the proof, so we prove the stronger version. However, the proof is lengthy and breaks into a large number of cases. We will make use of the fact that there is an APSTS(v) for all admissible v apart from $v = 7, 13$. We also use some small systems with property G4m found by computer searches as detailed below in Lemma 6.1.

Lemma 6.1 *There exists an APSTS(v) with property G4m for* $v = 19, 31, 37, 49, 61$ *and* 85.

Proof These systems are given in [5]. □

We split this section into two subsections, the first dealing with the case $v \equiv 3 \pmod 6$, and the second dealing with $v \equiv 1 \pmod 6$. In the second subsection we make use of some of the constructions presented in [7] and [11]. These are reviewed and then extended in our Theorems 6.2, 6.4 and 6.5. With the aid of these, the case $v \equiv 1 \pmod 6$ is completed in a sequence of lemmas. Table 2 should help to convince the reader that our results do indeed cover all admissible v apart from $v = 7$ and $v = 13$.

6.1 The case $v \equiv 3 \pmod 6$

Theorem 6.1 *For every* $v \equiv 3 \pmod 6$*, there exists an APSTS(v) on the point set* $V = \{A, B\} \cup \{0, 1, \ldots, v-3\}$*, with a block* $AB0$*, and that can be generically labelled with respect to* A, B*, so that there are no blocks* $0w(w+4)$ *(arithmetic modulo* $v-2$*). In other words, for every* $v \equiv 3 \pmod 6$*, there exists an APSTS(v) with property G4m.*

Proof Take $v = 6s + 3$. The proof breaks into two cases depending on whether or not $7 \mid (2s + 1)$. Provided $2s + 1$ is not divisible by 7, the Bose construction gives an APSTS(v) and we show it has the desired properties. When $2s + 1$ has a factor 7 we adopt a different approach.

Residue Class	Lemmas	Notes
$v \equiv 3 \pmod 6$	Theorem 6.1	
$v \equiv 7 \pmod{18}$	Lemma 6.2	Except $v = 7, 61, 97$
	Lemma 6.1	$v = 61$
	Lemma 6.3	$v = 97$
$v \equiv 1 \pmod{18}$	Lemma 6.1	$v = 19, 37$
	Lemma 6.4	$v \geq 55$, $v \not\equiv 91 \pmod{108}$
	Lemma 6.5	$v \equiv 91 \pmod{108}$
$v \equiv 13 \pmod{18}$	Lemma 6.1	$v = 31$
	Lemma 6.6	$v \equiv 31 \pmod{72}$, $v \geq 103$
	Lemma 6.1	$v = 85$
	Lemma 6.7	$v \equiv 13 \pmod{72}$, $v \geq 157$, $v \neq 13$.
	Lemma 6.8	$v \equiv 67 \pmod{72}$
	Lemma 6.1	$v = 49$
	Lemma 6.9	$v \equiv 49 \pmod{72}$, $v \geq 121$

Table 2: APSTS(v) with property G4m

The Bose construction for an STS($6s+3$) has point set $0_i, 1_i, \ldots, (2s)_i$ for $i = 0, 1, 2$. The blocks are of two types:

1. triples $x_i y_i (\frac{x+y}{2})_{i+1}$, $(x \neq y)$ where $(x+y)/2$ is taken modulo $2s+1$ and subscript arithmetic modulo 3,
2. triples $x_0 x_1 x_2$.

In particular, $0_0 0_1 0_2$ is a block. Provided $2s+1$ is not divisible by 7, the Bose system is anti-Pasch [3, Chapter 13]. The $\{0_2, 0_1\}$-cycles are all of length 6 and have the form

$$C_x = \left(x_1, (-x)_1, (-\frac{x}{2})_2, (-\frac{x}{4})_0, (\frac{x}{4})_0, (\frac{x}{2})_2\right), \quad x = 1, 2, \ldots, s.$$

We relabel using a mapping ϕ, first putting $\phi(0_2) = A$, $\phi(0_1) = B$ and $\phi(0_0) = 0$. Then we relabel all the remaining points with the integers $1, 2, \ldots, 6s$ by generically labelling each cycle C_x and doing it in such a way so as to avoid labelling any block $0w(w+4)$ (arithmetic modulo $6s+1$). For $s \neq 2$ this can be done as follows.

The *base* labelling of C_x is taken as $(3, 4, 1, 2, 5, 6)$ in the order shown and we add a constant λ_x to each of these base labels so, for example, $\phi(x_1) = 3 + \lambda_x$, $\phi((-x)_1) = 4 + \lambda_x$, etc. The values of λ_x are taken from $\{0, 6, 12, \ldots, 6(s-1)\}$ and $\lambda_x \neq \lambda_y$ unless $x = y$. This ensures that all the points of all the cycles are labelled uniquely and all the integers $1, 2, \ldots, 6s$ are used as labels. Thus ϕ provides a generic labelling. We show that for

$s \neq 2$ it is possible to choose the values of λ_x to avoid labelling any block $0w(w+4)$.

In the case $s = 1$ there is only one cycle and checking the required property is straightforward. The case $s = 2$ is treated separately. The case $s = 3$ gives $2s+1 = 7$ and the Bose system has Pasch configurations. So the first non-trivial case is $s = 4$ (giving $2s+1 = 9$) when we take $\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 12$ and $\lambda_4 = 18$. In other words, the relabelling is as follows.

$$\begin{array}{rcl} \phi : (1_1, 8_1, 4_2, 2_0, 7_0, 5_2) & \to & (3, 4, 1, 2, 5, 6) \\ \phi : (2_1, 7_1, 8_2, 4_0, 5_0, 1_2) & \to & (9, 10, 7, 8, 11, 12) \\ \phi : (3_1, 6_1, 3_2, 6_0, 3_0, 6_2) & \to & (15, 16, 13, 14, 17, 18) \\ \phi : (4_1, 5_1, 7_2, 8_0, 1_0, 2_2) & \to & (21, 22, 19, 20, 23, 24) \end{array}$$

The pairs occurring in blocks with 0_0 are $\{1_2, 8_2\} \to \{12, 7\}$, $\{2_2, 7_2\} \to \{24, 19\}$, $\{3_2, 6_2\} \to \{13, 18\}$, $\{4_2, 5_2\} \to \{1, 6\}$, $\{1_1, 2_0\} \to \{3, 2\}$, $\{2_1, 4_0\} \to \{9, 8\}$, $\{3_1, 6_0\} \to \{15, 14\}$, $\{4_1, 8_0\} \to \{21, 20\}$, $\{5_1, 1_0\} \to \{22, 23\}$, $\{6_1, 3_0\} \to \{16, 17\}$, $\{7_1, 5_0\} \to \{10, 11\}$, and $\{8_1, 7_0\} \to \{4, 5\}$. It is easily checked that in this generic labelling there are no blocks $0w(w+4)$ in the relabelled system.

For $s = 5$ (giving $2s+1 = 11$) the Bose system can be relabelled by taking $\lambda_1 = 0, \lambda_2 = 6, \lambda_3 = 18, \lambda_4 = 12$ and $\lambda_5 = 24$, and this provides a generically labelled system and avoids the creation of blocks $0w(w+4)$. Another way of expressing these choices of λ_x is to say that we take the cycles in the order C_1, C_2, C_4, C_3, C_5.

We next consider $s \geq 6$. Recall that $\phi(0_0) = 0$ and the blocks containing 0_0 (apart from the triple $0_0 0_1 0_2$) are of the forms

$$0_0 z_2 (-z)_2, \quad z = 1, 2, \ldots, s, \quad \text{and} \quad 0_0 z_1 (2z)_0, \quad z = 1, 2, \ldots, 2s.$$

A block of the form $0z_2(-z)_2$ arises from points within a single cycle, and in this cycle there is just one pair $z_2(-z)_2$, which is relabelled with difference $|\phi(z_2) - \phi((-z)_2)| = |1 - 6| = 5$. So whatever the choice of λ_x, there are no blocks $0_0 z_2(-z)_2$ relabelled as $0w(w+4)$.

Given a block of the form $0_0 z_1 (2z)_0$, the points z_1 and $(2z)_0$ could both occur in a single cycle, or occur in different cycles.

Within a single cycle C_x, two points are of the form z_1 and two are of the form w_0. Thus there are four pairs of points of the form $z_1 w_0$. In the relabelling, the differences $|\phi(z_1) - \phi(w_0)|$ are $|3-2| = 1, |3-5| = 2, |4-2| = 2$ and $|4-5| = 1$. So relabelling the points of C_x cannot relabel a block $0_0 z_1 (2z)_0$ as $0w(w+4)$ whatever the value of λ_x.

There remains the possibility that a block $0_0 z_1 (2z)_0$ might be relabelled as $0w(w+4)$ because z_1 and $(2z)_0$ are in different cycles, say C_x and C_y. Clearly this cannot happen if $|\lambda_x - \lambda_y| > 6$. So the problem reduces to choosing the values of λ_x to avoid this happening. In other words, in what order should the cycles be written? Consider the case when $\lambda_y = \lambda_x + 6 \pmod{6s}$ and suppose initially that $\lambda_x \neq 6(s-1)$. Then C_x is relabelled $\lambda_x + (3, 4, 1, 2, 5, 6)$ and C_y is relabelled $\lambda_x + (9, 10, 7, 8, 11, 12)$. The pairs $z_1 w_0$ that are relabelled with a difference of 4 are $(-x)_1(-\frac{y}{4})_0$ and $y_1(\frac{x}{4})_0$. If $(-x)_1(-\frac{y}{4})_0 = (-x)_1(-2x)_0$ then $y \equiv 8x \pmod{2s+1}$, and if $y_1(\frac{x}{4})_0 = y_1(2y)_0$ then $x \equiv 8y \pmod{2s+1}$. In the exceptional case when $\lambda_x = 6(s-1)$ and $\lambda_y = 0$, C_x is relabelled $(6s-3, 6s-2, 6s-5, 6s-4, 6s-1, 6s)$ and C_y is relabelled $(3, 4, 1, 2, 5, 6)$, and no $z_1 w_0$ pairs are relabelled with a difference of 4 (mod $6s+1$). It follows that we can avoid relabelling blocks as $0w(w+4)$ if we ensure that whenever $\lambda_y = \lambda_x + 6 \pmod{6s}$ then $y \not\equiv 8x$ and $x \not\equiv 8y \pmod{2s+1}$.

An easy way to check a potential sequencing $\ldots, C_x, C_y, \ldots$ is to write down the sequence $\ldots, x, y, \ldots$ and underneath each value z record $8z$ (mod $2s+1$). If we obtain the pattern

$$\begin{array}{cccc} \ldots, & x, & y, & \ldots \\ \ldots, & a, & b, & \ldots \end{array}$$

with either $x = b$ or $y = a$, then the proposed sequencing is invalid. If the sequence avoids this pattern, then it is valid.

To prove that this can be done for sufficiently large s, form a complete graph on vertices $1, 2, \ldots, s$. Delete each edge xy when either $y \equiv 8x$ or $x \equiv 8y \pmod{2s+1}$. Call the resulting graph G. At most 2 edges incident with each vertex have been deleted, so G has minimum degree at least $s-3$. If $s-3 \geq s/2$ then G has a Hamiltonian cycle $H = (\mu_1, \mu_2, \ldots, \mu_s)$. Finally, put $\lambda_{\mu_i} = 6(i-1)$, and we are done when $s \geq 6$.

There remains the exceptional case $s = 2$ ($2s+1 = 5$, $6s+1 = 13$) when we do not use the base labelling method. Instead we relabel C_1 and C_2 generically as follows.

$$\begin{array}{rcl} \phi : (1_1, 4_1, 2_2, 1_0, 4_0, 3_2) & \to & (1, 2, 5, 6, 3, 4) \\ \phi : (2_1, 3_1, 4_2, 2_0, 3_0, 1_2) & \to & (10, 9, 12, 11, 8, 7) \end{array}$$

The pairs occurring in blocks with 0_0 are $\{1_2, 4_2\} \to \{7, 12\}$, $\{2_2, 3_2\} \to \{5, 4\}$, $\{1_1, 2_0\} \to \{1, 11\}$, $\{2_1, 4_0\} \to \{10, 3\}$, $\{3_1, 1_0\} \to \{9, 6\}$, and $\{4_1, 3_0\} \to \{2, 8\}$. None of the differences equals 4 (mod 13).

So the conclusion is that the Bose construction produces an APSTS$(6s+3)$ when $2s+1$ is not divisible by 7, and this system can be represented with point set $V = \{A, B, 0, 1, \ldots, 6s\}$ in such a way that $AB0$ is a block, the $\{A, B\}$-cycles are generically labelled, and there are no blocks $0w(w+4)$ (arithmetic modulo $6s+1$). We also remark that any Bose system has a parallel class (i.e. a set of triples covering each point of the system precisely once) given by the triples $x_0 x_1 x_2$.

Next consider the case when $v \equiv 3 \pmod 6$ is divisible by 7. We can then write $v = 7u$ where $u \equiv 3 \pmod 6$. The construction of an APSTS(v) proceeds inductively, starting with the case $v = 21$. This follows the method given in [1]. A cyclic APSTS(21) is given by the starter blocks $\{0, 1, 3\}, \{0, 4, 12\}, \{0, 6, 11\}, \{0, 7, 14\}$. This gives rise to a 3-bipartite APSTS$(21, -3)$ with hole $\{0, 7, 14\}$ and bipartition $X = \{1, 2, 4, 8, 9, 11, 15, 16, 18\}$, $Y = \{3, 5, 6, 10, 12, 13, 17, 19, 20\}$. (The definition of m-bipartite systems was given in Section 2.) For later reference, note that there are three 6-cycles on the pair $\{0, 7\}$, namely $(1, 3, 15, 5, 4, 12)$, $(2, 20, 16, 10, 8, 17)$ and $(6, 11, 19, 18, 13, 9)$.

We relabel the hole as $\{A, B, 0\}$ and then generically label the cycles on A, B using $\{1, 2, \ldots, 18\}$. When this is done, the points of X will be labelled with odd integers and the points of Y with even integers (or vice-versa) and the blocks containing the point labelled 0 will contain (in addition to 0) one even integer and one odd integer, so no block will be of the form $0w(w+4)$ (absolute arithmetic). To give an explicit relabelling, denote this by ϕ, and put $\phi(0) = A, \phi(7) = B$ and $\phi(14) = 0$. In the original labelling the cycles on $\{0, 7\}$ are $(1, 3, 15, 5, 4, 12)$, $(2, 20, 16, 10, 8, 17)$ and $(9, 13, 18, 19, 11, 6)$. Relabel generically as follows

x	1	3	15	5	4	12	2	20	16	10	8	17	9	13	18	19	11	6
$\phi(x)$	1	2	5	6	3	4	7	8	11	12	9	10	13	14	17	18	15	16

The original point 14 appears with the original pairs $\{1, 10\}$, $\{2, 6\}$, $\{3, 9\}$, $\{4, 20\}$, $\{5, 18\}$, $\{8, 19\}$, $\{11, 12\}$, $\{13, 16\}$ and $\{15, 17\}$. These are relabelled under ϕ as $\{1, 12\}$, $\{7, 16\}$, $\{2, 13\}$, $\{3, 8\}$, $\{6, 17\}$, $\{9, 18\}$, $\{15, 4\}$, $\{14, 11\}$ and $\{5, 10\}$. None of these give rise to a difference of 4 in absolute arithmetic as all the differences are odd; in fact none give a difference of 4 in arithmetic modulo 19. It is important to note that this APSTS(21) also has a parallel class obtained from the starter block $\{0, 7, 14\}$ in its original presentation.

Now suppose (inductively) that $v = 7u$ where $u \equiv 3 \pmod 6$ and that we have an APSTS(u) with a parallel class P. We can assume $u \geq 9$. Denote the blocks of the parallel class P as $\{x_1, x_2, x_3\}$ for $x = 0, 1, \ldots, u/3 - 1$. All other blocks of the APSTS(u) will have the form $\{x_p, y_q, z_r\}$ where x, y, z are distinct from one another and where $p, q, r \in \{1, 2, 3\}$. We will call such a block *vertical* and order the points in such a block by requiring $x < y < z$.

Next we inflate by a factor 7, replacing each point x_p by seven points x_p^a for $a = 0, 1, \ldots, 6$. (We will refer to x_p as the projection of the points x_p^a, and that x_p^a projects to the point x_p.) Thus each block of the parallel class is replaced by a set of 21 points on which we will place a copy of our APSTS(21); the blocks from these will be called *horizontal*, and points x_p^a will be said to be at level x. For each vertical block $\{x_p, y_q, z_r\}$ (with $x < y < z$), form the blocks $\{x_p^a, y_q^b, z_r^{a+b}\}$ for $a, b \in \{0, 1, \ldots, 6\}$ with $a+b$ taken modulo 7. We claim that the resulting set of blocks forms an APSTS($7u$) having a parallel class. That it forms an STS($7u$) is fairly obvious since every pair of points lies in some block and the number of blocks is

$$70\frac{u}{3} + 49\left(\frac{u(u-1)}{6} - \frac{u}{3}\right) = \frac{7u(7u-1)}{6}.$$

It is also easy to see that there is a parallel class formed from the parallel classes of the STS(21)s. To see that there are no Pasch configurations, first observe that any Pasch configuration cannot contain two horizontal blocks because if these come from different STS(21)s then the blocks do not intersect, while if they are the same STS(21), then all six points of the Pasch configuration must come from that STS(21), but this is anti-Pasch. Next suppose that there is a Pasch configuration containing just one horizontal block $\{x_p^a, x_q^b, x_r^c\}$. Without loss of generality the other three blocks have the form $\{x_p^a, y_s^d, z_t^e\}$, $\{x_q^b, y_s^d, W\}$ and $\{x_r^c, z_t^e, W\}$, where x, y, z are distinct from one another. If $p = q$ then both x_p^a and x_q^b project to the same point x_p. So, from the second and third blocks, $W = z_t^f$ for some f, but this contradicts the fourth block since it cannot have two points at the same level z. A similar contradiction applies if $p = r$ or if $q = r$. So the only possibility for a Pasch configuration containing a horizontal block is if $\{p, q, r\} = \{1, 2, 3\}$. But then projecting each point x_ρ^α to x_ρ in the original APSTS(u) gives a Pasch configuration in that design, again a contradiction. Finally suppose that there are four vertical blocks forming a Pasch configuration: $\{x_p^a, y_q^b, z_r^c\}$, $\{x_p^a, i_s^d, j_t^e\}$, $\{y_q^b, i_s^d, k_w^f\}$, $\{z_r^c, j_t^e, k_w^f\}$ where without loss of generality, $x < y < z$, and x, i, j are distinct, y, i, k are distinct and z, j, k are distinct. If $i_s = z_r$ then $j_t = y_q$ and consequently $k_w = x_p$. But then $c \equiv a + b$, $d \equiv a + e$, $d \equiv b + f$ and $c \equiv e + f \pmod 7$. These give $a = f, b = e, c = d$ and the four blocks are identical and so do not form a Pasch configuration. If $i_s \neq z_r$ then $j_t \neq y_q$ and $k_w \neq x_p$, so the six points $x_p, y_q, z_r, i_s, j_t, k_w$ are all distinct and projecting each point x_ρ^α to x_ρ in the original APSTS(u) again gives a Pasch configuration in that design, a contradiction.

To show that the resulting APSTS(v) can be relabelled as required, first consider the blocks of the APSTS(u) that contain the point 0_1. Apart from the block $0_1 0_2 0_3$, these are all of the form $0_1 x_p y_q$ where we can assume $0 < x < y$. In the APSTS(v), consider the cycles on the pair $\{0_1^0, 0_1^1\}$. For $0 < x < y$ there are blocks $0_1^0 x_p^a y_q^b$ where $b \equiv a \pmod 7$, and there are also blocks $0_1^1 x_p^c y_q^d$ where $d \equiv c + 1 \pmod 7$. These result in a 14-cycle on the pair $\{0_1^0, 0_1^1\}$, namely $(x_p^0, y_q^0, x_p^6, y_q^6, x_p^5, y_q^5, \ldots, x_p^1, y_q^1)$. There are $(u-3)/2$

such 14-cycles. Assuming that the APSTS(21) described above is placed on the points 0_p^a ($a = 0, 1, \ldots, 6; p = 1, 2, 3$) in such a way that $A, B, 0$ correspond respectively to $0_1^0, 0_1^1, 0_1^2$ then, as explained earlier, within this APSTS(21) there are three 6-cycles on the pair $\{0_1^0, 0_1^1\}$. So altogether, the $(u-3)/2$ 14-cycles and the three 6-cycles comprise all the cycles on the pair $\{0_1^0, 0_1^1\}$.

We now relabel all the points of the APSTS(v) generically by the mapping ψ defined as follows. First put $\psi(0_1^0) = A, \psi(0_1^1) = B, \psi(0_1^2) = 0$, and then label the remaining 18 points 0_p^a by ψ so that this system is now identical with the APSTS(21) given above. This ensures that there is a block $AB0$ and that no block on the points $A, B, 0, 1, \ldots, 18$ of the form $0w(w+4)$ (arithmetic modulo $v-2$). Then define ψ on the remaining points by taking each 14-cycle separately and applying a generic labelling to it, obtained by adding an appropriate constant λ_{x_p} to the base labelling $(1, 2, 5, 6, 9, 10, 13, 14, 11, 12, 7, 8, 3, 4)$. The complete labelling is given in the following table

$z =$	x_p^0	y_q^0	x_p^6	y_q^6	x_p^5	y_q^5	x_p^4	y_q^4	x_p^3	y_q^3	x_p^2	y_q^2	x_p^1	y_q^1
$\psi(z) = \lambda_{x_p} +$	1	2	5	6	9	10	13	14	11	12	7	8	3	4

Now consider the blocks containing the point $0 = \psi(0_1^2)$ but excluding those containing points $A, B, 1, 2, \ldots, 18$. Before relabelling these had the form $0_1^2 x_p^a y_q^b$ where $b \equiv a + 2 \pmod 7$. But then $\psi(x_p^a), \psi(y_q^b)$ have opposite parities; indeed from the table above when $b \equiv a + 2 \pmod 7$, $|\psi(x_p^a) - \psi(y_q^b)| = 1, 7$ or 9. So in the relabelled system, there are no blocks of the form $0w(w+4)$ (arithmetic modulo $v-2$). $\square$

This completes the case $v \equiv 3 \pmod 6$.

6.2 The case $v \equiv 1 \pmod 6$

As previously indicated, to deal with $v \equiv 1 \pmod 6$ we make use of some of the constructions presented in [7] and [11]. We start with Theorem 2.3 of [11], which shows that from an APSTS(u) ($u \neq 3, 21, 33$), an APSTS($3(u-1)+1$) may be constructed. Briefly summarized, the proof relies on a construction involving three copies of an APSTS(u) intersecting in a common point with a special form of Latin square (a Kotzig square) placed across the three systems. We show how this construction can be used to produce systems having property G4m.

Theorem 6.2 *Suppose that there exists an APSTS(u) $= (U, \mathcal{B})$ ($u \neq 3, 21,$ 33) with the property that it can be generically labelled on the point set $\{A, B, 0, 1, \ldots, u-3\}$ with block $AB0$ and no block $0w(w+4)$ (absolute* *arithmetic), i.e. having property G4a. Then there exists an APSTS(v) with $v = 3(u-1)+1 = 3u-2$ that can be generically relabelled on the point set $V = \{A, B, 0, 1, \dots, v-3\}$ with block $AB0$ and no block $0w(w+4)$ (arithmetic modulo $v-2$), i.e. having property G4m.*

Proof Take three isomorphic copies of the APSTS(u), $\mathcal{S}_i = (U_i, \mathcal{B}_i)$ ($i = 0, 1, 2$), where $U_i \cap U_j = \{\alpha\}$ for $i \neq j$. Place the generically labelled copy of $\mathcal{S}_0$ so that the common point α of the three systems does not correspond to A, B or 0. The generic labelling partitions the points $1, 2, \dots, u-3$ into two sets $X_0 = \{1, 3, \dots, u-4\}$ and $Y_0 = \{2, 4, \dots, u-3\}$. The fact that all blocks $0wz$ with $w, z \in X_0 \cup Y_0$ have $|w-z| \neq 4$ implies that there are no such blocks with $|w-z| \equiv 4 \pmod{v-2}$.

Apart from blocks lying in $\mathcal{S}_0$, all the remaining blocks that contain A, B or 0 will contain a point from U_1 and a point from U_2. So the generic labelling may be extended to the APSTS($3(u-1)+1$) by labelling the points of U_1 (other than α) with $X_1 = \{u-2, u, \dots, v-4\}$ and the points of U_2 (other than α) with $Y_2 = \{u-1, u+1, \dots, v-3\}$. But then any block $0x_1y_2$ with $x_1 \in X_1$ and $y_2 \in Y_2$ will have $|x_1 - y_2| \equiv 1 \pmod 2$, so there are no blocks of the form $0w(w+4)$ in absolute arithmetic. Moreover, the greatest possible absolute value of $|x_1 - y_2|$ is $(v-3)-(u-2) = 2u-3 < v-6$, so we cannot have $|x_1 - y_2| \equiv 4 \pmod{v-2}$. Consequently there are no blocks of the form $0w(w+4)$ with arithmetic modulo $v-2$. □

Lemma 6.2 *If $v \equiv 7 \pmod{18}$ and $v \neq 7, 61, 97$ then there exists an APSTS(v) with property G4m.*

Proof For $s \geq 0$, Theorem 6.1 guarantees the existence of an APSTS($6s+3$) with property G4a. Then Theorem 6.2 gives an APSTS($3(6s+2)+1$) with property G4m, apart from $s = 0, 3, 5$. Thus we obtain an APSTS(v) for $v = 18s+7$ apart from $v = 7$, $v = 61$ and $v = 97$. □

Remark The exceptional case $v = 97$ will be dealt with in Lemma 6.3. The exceptional case $v = 61$ is covered by Lemma 6.1.

In [7] there is a construction involving m-bipartite systems. These were defined in Section 2 and they prove useful to us here, particularly when $m = 3$, but also in some other cases. We will make use of BAPSTS($u, -m$) designs given in [7] for $(u, -m) = (19, -3)$, $(31, -3)$, $(31, -7)$, $(43, -3)$, $(49, -7)$, $(49, -13)$, $(55, -13)$, $(61, -19)$, $(67, -7)$, $(67, -13)$, $(75, -19)$, $(85, -13)$, $(139, -19)$, and $(1260s + 901, -3)$ for $s \geq 0$. We also use $(u, -m) = (37, -3)$ and the corresponding design is given in [5].

The argument given in the proof of Theorem 6.1 that a BAPSTS$(21, -3)$ gives rise to an APSTS(21) with property G4a is easily generalized to show that any BAPSTS$(v, -3)$ gives rise to an APSTS(v) having property G4a.

Theorem 6.3 *Suppose that $\mathcal{S}$ is a 3-bipartite APSTS$(v, -3)$ on the point set $\{A, B, 0, 1, \ldots, v-3\}$ with hole $\{A, B, 0\}$. Then the resulting STS(v) with block $AB0$ is anti-Pasch and it can be generically labelled with respect to the pair $\{A, B\}$ such that it has no block $0w(w+4)$ (absolute arithmetic), i.e. it has property G4a.*

Proof Let X and Y denote the bipartition of $\{1, 2, \ldots, v-3\}$. Adding the block $AB0$ cannot produce a Pasch configuration because three blocks would then have the form $AB0$, Ax_1y_1 and Bx_1y_2, where $x_1 \in X$ and $y_1, y_2 \in Y$, thus forcing the fourth block $0y_1y_2$ and so contradicting the bipartition. So the resulting STS(v) is certainly anti-Pasch. Generically labelling the system with respect to $\{A, B\}$ gives even-length cycles of the form $(x_1, y_1, x_2, y_2, \ldots)$ (with $x_i \in X, y_i \in Y$) and we may take $X = \{1, 3, \ldots, v-4\}, Y = \{2, 4, \ldots, v-3\}$. But then any block $0x_iy_j$ has $|x_i - y_j| \equiv 1 \pmod 2$, so there are no blocks of the form $0w(w+4)$ with absolute arithmetic. $\square$

Remark The theorem above cannot be applied easily to Bose systems, hence the complicated argument of Theorem 6.1. For example the Bose STS(45) with the block $0_00_10_2$ removed is an APSTS$(45, -3)$, but it is not 3-bipartite. To see this note that (working modulo 15) there are blocks $0_23_112_1$, $0_112_16_2$, $0_26_23_0$, $0_13_012_0$, $0_212_09_2$, $0_19_23_1$. Consequently on the pair $\{0_2, 0_1\}$ there is a 6-cycle $(3_1, 12_1, 6_2, 3_0, 12_0, 9_2)$, and (multiplying by 2) another 6-cycle on the same pair, $(6_1, 9_1, 12_2, 6_0, 9_0, 3_2)$. There is also a block $0_03_16_0$, and if the system were 3-bipartite with bipartition $\{X, Y\}$ then we could take $3_1 \in X$ and $6_0 \in Y$. But then the two 6-cycles would give $\{3_1, 6_2, 12_0, 9_0, 6_1, 12_2\} \subseteq X$. However, there is a block $0_06_112_0$ that contradicts this possibility.

We need to discuss Theorem 2.1 of [7] in order to make further progress. For $n = 3$ or $n \geq 5$, an APSTS$(2n+m)$ and a BAPSTS$(2n+m, -m)$ are combined with an APSTS(u) to produce an APSTS$(n(u-1)+m)$. The result relies on the following construction that is explained in more detail in [7]. An $N_2 - TD(3, n)$ is a transversal design having three groups of size n and no Pasch configurations amongst its triples. Such a design is equivalent to a Latin square of order n that has no 2×2 subsquares, and these exist for $n = 3$ and $n \geq 5$. For other undefined terms, see [2].

Construction 3 Take an APSTS(u) and delete a point, say ∞, to obtain a 3-GDD of type $2^{(u-1)/2}$, say $\mathcal{G}$, with groups $\{x_i, y_i\}$ for $i = 1, 2, \ldots, (u-1)/2$. Replace each point x_i by n points $x_i^1, x_i^2, \ldots, x_i^n$, and do likewise for each y_i. Then use an N_2-$TD(3, n)$ to produce a 3-GDD of type $(2n)^{(u-1)/2}$, say $\mathcal{G}^*$. Denote the groups of this design by $G_1, G_2, \ldots, G_{(u-1)/2}$, so that $G_i = X_i \cup Y_i$, where $X_i = \{x_i^1, x_i^2, \ldots, x_i^n\}$ and $Y_i = \{y_i^1, y_i^2, \ldots, y_i^n\}$. Next, take m new points to form a set $M = \{\infty_1, \infty_2, \ldots, \infty_m\}$, say. Place a copy of the APSTS($2n + m$) onto the points of $M \cup X_1 \cup Y_1$. Then, for each $i = 2, 3, \ldots, (u-1)/2$, place a copy of the BAPSTS($2n+m, -m$) onto the points of $M \cup X_i \cup Y_i$ so that the labelling partition corresponds to M, X_i and Y_i. As shown in [7], the resulting set of blocks forms an APSTS($n(u-1) + m$) on the point set $M \cup (\bigcup_{i=1}^{(u-1)/2} G_i)$.

We will call the triples of $\mathcal{G}^*$ *vertical* triples, and the triples of the APSTS($2n+m$) and the BAPSTS($2n+m, -m$) *horizontal* triples. Points and triples from the same copy of the BAPSTS($2n+m, -m$) will be said to be on the same *level*; those from the APSTS($2n+m$) will be said to be from the *top level*. We will refer to a point z of $\mathcal{G}$ as being the projection of any one of the n points $z^1, z^2, \ldots, z^n$ which replaced it in forming $\mathcal{G}^*$.

Theorem 6.4 *If in Construction 3 (with $n \geq 3$ and $n \neq 4$) the APSTS($2n + m$) employed at the top level has property G4a, and if u is admissible with $u \neq 3, 7$ or 13, then the construction can be performed in such a manner that the resulting APSTS($n(u-1)+m$) has property G4m.*

Proof Place the APSTS($2n + m$) with property G4a onto $M \cup X_1 \cup Y_1$ so that the points $A, B, 0$ lie together in X_1. Then all the points of $M \cup X_1 \cup Y_1$ are labelled, and within the top level, there are no blocks $0w(w+4)$ with absolute arithmetic, and consequently none with arithmetic modulo $n(u-1)+m-2$. It remains to label the points at each level $i > 1$.

Consider a block of $\mathcal{G}$ that contains the point x_1. (Note that A, B and 0 correspond to x_1^p, x_1^q and x_1^r for some p, q, r.) Such a block will contain points from two different levels, say i, j, where $i, j > 1$. There are four different possible forms for such a block, namely $x_1x_ix_j$, $x_1x_iy_j$, $x_1y_ix_j$ and $x_1y_iy_j$. For $i > 1$, every x_i and every y_i will appear in one and only one block that contains x_1. If there is a block $x_1x_ix_j$ ($i, j > 1$ and $i \neq j$), then any cycle outside the top level on two points x_1^p, x_1^q that contains a point from X_i will alternate points of X_i and X_j. So the points of each such a cycle can be labelled generically in such a way that X_i is labelled with odd integers and X_j is labelled with even integers. A similar observation applies to the other three possible forms of vertical blocks containing x_1.

By taking in turn each block of $\mathcal{G}$ that contains the point x_1, all the points of $\bigcup_{i=2}^{(u-1)/2} G_i$ can be labelled with the integers $\{2n+m-2, 2n+m-1, \ldots, n(u-1)+m-3\}$. Because A, B and 0 correspond to some x_1^p, x_1^q and x_1^r, the labelling can be arranged to be generic with respect to A, B and any vertical triple $0wz$ will then have $|w-z| \equiv 1 \pmod 2$, so the design has property G4a. To see that it has property G4m, observe that for any vertical block $0wz$, the greatest possible absolute value of $|w-z|$ is

$$(n(u-1)+m-3)-(2n+m-2) < (n(u-1)+m-2)-4,$$

since $2n+m > 6$. □

Note In the many subsequent applications of Theorem 6.4 in Lemmas 6.3 to 6.9, the only requirements on the parameter u are that it be admissible and not equal to 3, 7 or 13. We do *not* require an APSTS(u) with any additional properties, such as G4a or G4m.

Lemma 6.1 deals with the exceptional case $v = 61$ of Lemma 6.2. We next deal with the remaining exception.

Lemma 6.3 *There exists an APSTS(v) with property G4m for $v = 97$.*

Proof Apply Theorem 6.4 with $n = 12, m = 1$ and $u = 9$. This requires an APSTS(25) with property G4a, which was given by Lemma 6.2 and which can be used at the top level in the construction. It also (trivially) gives a BAPSTS(25, −1) needed for the other levels. Thus we obtain an APSTS(97) with property G4m. □

Lemma 6.4 *If $v \equiv 1 \pmod{18}$ and $v \not\equiv 91 \pmod{108}$, then there exists an APSTS(v) with property G4m.*

Proof If $v \equiv 1 \pmod{18}$ then $v \equiv 1, 19$ or $37 \pmod{54}$.

An APSTS($54s+1$) with property G4m can be obtained for $s \geq 3$ by applying Theorem 6.4 with $n = 9, m = 1$ and $u = 6s+1$ ($s \geq 3$). This requires an APSTS(19) with property G4a, which was given by Lemma 6.1 and which can be used at the top level in the construction. It also (trivially) gives a BAPSTS(19, −1) needed for the other levels. This leaves $s = 1$ and $s = 2$, i.e. $v = 55$ and $v = 109$, unresolved.

The case $v = 55$ can be dealt with using Theorem 6.2 by taking $u = 19$. We have an APSTS(19) with property G4a, and thus an APSTS(55) is obtained having property G4m. The case $v = 109$ can also be dealt with using Theorem 6.2 by taking $u = 37$. We have an APSTS(37) with property

G4a from Lemma 6.1, and thus an APSTS(109) is obtained having property G4m.

An APSTS($54s + 19$) with property G4m for $s \geq 1$ can be obtained by applying Theorem 6.4 with $n = 9, m = 1$ and $u = 6s+3$ ($s \geq 1$). Again this requires an APSTS(19) with property G4a, and Lemma 6.1 not only gives this and a BAPSTS($19, -1$), but also covers the case $s = 0$ (i.e. $v = 19$).

Finally, if $v \equiv 37 \pmod{54}$, then $v \equiv 37$ or $91 \pmod{108}$. Again applying Theorem 6.4 with $n = 18, m = 1$ and $u = 6s + 3$ ($s \geq 1$), an APSTS($108s + 37$) with property G4m may be obtained for $s \geq 1$. This requires an APSTS(37) with property G4a, and Lemma 6.1 not only gives this and a BAPSTS($37, -1$), but also covers the case $s = 0$ (i.e. $v = 37$). $\square$

In Theorem 4.1 of [11] an APSTS($m + 2$) and an APSTS($n + 2$) are combined to produce an APSTS($mn + 2$). The associated construction is described below and was originally due to Lu [12].

Construction 4 Let $\mathcal{S}_m = (\mathbb{Z}_m \cup \{a, b\}, \mathcal{A})$ be an APSTS($m + 2$) with $\{a, b, 0\} \in \mathcal{A}$, and let $\mathcal{S}_n = (\mathbb{Z}_n \cup \{a, b\}, \mathcal{B})$ be a APSTS($n + 2$) with $\{a, b, 0\} \in \mathcal{B}$. To avoid trivialities we assume that $m, n \geq 7$. The double neighbourhood of a, b in $\mathcal{S}_m$ is given by

$$N(a, b) = \{\{x, y\} : axy \in \mathcal{A} \text{ or } bxy \in \mathcal{A}, \text{ and } \{x, y\} \cap \{a, b\} = \emptyset\}.$$

Thus $N(a, b)$ is a set of pairs on $\mathbb{Z}_m \backslash \{0\}$ with every element appearing in two pairs. Each pair can then be ordered so that each element is the first element of one pair, and the second element of another; call this set of ordered pairs $Q(a, b)$.

An APSTS($mn + 2$), $\mathcal{S}_{mn}$, is constructed on the point set $(\mathbb{Z}_m \times \mathbb{Z}_n) \cup \{a, b\}$ with triples of the following four types where $x_1, x_2, x_3 \in \mathbb{Z}_m$ and $y_1, y_2, y_3 \in \mathbb{Z}_n$.

(i) $\{(0, y_1), (0, y_2), (0, y_3)\}$ whenever $\{y_1, y_2, y_3\} \in \mathcal{B}$, and $\{\ell, (0, y_2), (0, y_3)\}$ whenever $\{\ell, y_2, y_3\} \in \mathcal{B}$ and $\ell \in \{a, b\}$, and $\{a, b, (0, 0)\}$.

(ii) $\{(x_1, y_1), (x_1, y_2), (x_2, y_3)\}$ where $(x_1, x_2) \in Q(a, b)$, $y_1 \neq y_2$, and $y_1 + y_2 \equiv 2y_3 \pmod{n}$.

(iii) $\{\ell, (x_1, y_1), (x_2, y_1)\}$ where $\ell \in \{a, b\}$ and $\{\ell, x_1, x_2\} \in \mathcal{A}$.

(iv) $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ where $\{x_1, x_2, x_3\} \in \mathcal{A}$, $x_1 < x_2 < x_3$ and $y_1 + y_2 + y_3 \equiv 0 \pmod{n}$.

Theorem 6.5 *If $m, n \geq 7$ and both the systems $\mathcal{S}_m$ and $\mathcal{S}_n$ have property G4a, then the system $\mathcal{S}_{mn}$ is an APSTS($mn+2$) with property G4m.*

Proof Note that the type (i) blocks form a copy of $\mathcal{S}_n$ on the point set $\{a, b, (0,0), (0,1), \ldots, (0, n-1)\}$.

We will assume that $\mathcal{S}_m$ is generically labelled with respect to the pair $\{a, b\}$ with property G4a and likewise that $\mathcal{S}_n$ is generically labelled with respect to the pair $\{a, b\}$ with property G4a. We relabel the system $\mathcal{S}_{mn}$ on the point set $\{A, B, 0, 1, \ldots, mn-1\}$ using a mapping $\phi : (\mathbb{Z}_m \times \mathbb{Z}_n) \cup \{a, b\} \mapsto \mathbb{Z}_{mn} \cup \{A, B\}$ that is defined as follows.

First define $\phi(a) = A, \phi(b) = B$ and $\phi((0, y)) = y$ for $y \in \mathbb{Z}_n$; in particular $\phi((0,0)) = 0$. Then any cycle of the relabelled system $\mathcal{S}_{mn}^{\phi}$ that contains a block Ay_1y_2 or By_1y_2 with $0 \leq y_1, y_2 \leq n-1$ will lie in a generically labelled $\{A, B\}$-cycle of that system. Moreover, because $\mathcal{S}_n$ has no blocks of the form $0w(w+4)$ (absolute arithmetic), there are no blocks of $\mathcal{S}_{mn}^{\phi}$ of the form $0w(w+4)$ with $1 \leq w < w+4 \leq n-1$.

We next define $\phi((x, y))$ for $1 \leq x \leq m-1$, $0 \leq y \leq n-1$. As a preliminary item, define $\psi : \mathbb{Z}_n \mapsto \mathbb{Z}_n$ by putting $\psi(y) = y$ except for the two cases $y = (n+1)/2$ or $y = (n+3)/2$, when we define $\psi((n+1)/2) = (n+3)/2$ and $\psi((n+3)/2) = (n+1)/2$. Then ψ is a bijection on $\mathbb{Z}_n$ and it has the key property that if $y \neq 0$, then $|\psi(y) - \psi(n-y)| \geq 2$. Now consider the generic labelling of $\mathcal{S}_m$, and suppose that $C = (x_1, x_2, \ldots, x_k)$ is any $\{a, b\}$-cycle in $\mathcal{S}_m$. Then for each $y \in \mathbb{Z}_n$ the type (iii) blocks give rise to an $\{a, b\}$-cycle C_y in $\mathcal{S}_{mn}$, given by $C_y = ((x_1, y), (x_2, y), \ldots, (x_k, y))$. For $x \in \mathbb{Z}_m \setminus \{0\}$ and $y \in \mathbb{Z}_n$, define

$$\phi((x, y)) = x + (n-1) + (m-1)\psi(y).$$

This provides a generic labelling for C_y and consequently ϕ provides a generic labelling for $\mathcal{S}_{mn}$: every point of $(\mathbb{Z}_m \times \mathbb{Z}_n) \cup \{a, b\}$ is labelled by ϕ and every $\{a, b\}$-cycle is generically labelled.

None of the type (ii) and type (iii) blocks contain the point $(0,0)$. So apart from type (i) blocks that have already been considered, the only other blocks that contain the point $(0,0)$ are type (iv) and these have the form $\{(0,0), (x_1, y), (x_2, n-y)\}$ where $\{0, x_1, x_2\} \in \mathcal{A}$ with $0 < x_1 < x_2$, and $y \in \mathbb{Z}_n$ (with the gloss that if $y = 0$ then $n - y$ is taken as 0). In such a block put $\delta = |\phi((x_1, y)) - \phi((x_2, n-y))| = |(x_1 - x_2) + (m-1)(\psi(y) - \psi(n-y))|$. If $y = 0$ then $\delta = |x_1 - x_2| = x_2 - x_1 \neq 4$. If $y \neq 0$ then $2 \leq |\psi(y) - \psi(n-y)| \leq n-2$, and $|x_1 - x_2| \leq m-2$. So $2(m-1) - (m-2) \leq \delta \leq (m-2) + (m-1)(n-2)$. Consequently $\delta \not\equiv 4$ (mod mn). It follows that the generic labelling of $\mathcal{S}_{mn}$ given by ϕ has property G4m. $\square$

Equipped with the preceding results we can now show that for $v \equiv 1$ (mod 6) with $v \geq 19$ there exits an APSTS(v) with property G4a. We first split the argument into cases modulo 18. We have already covered $v = 18s + 7$. In the case $v = 18s + 1$, all that remains is $v = 108s + 91$, so we deal with this next.

Lemma 6.5 *If* $v \equiv 91$ *(mod* 108*) then there exists an APSTS(v) with property G4m.*

Proof If $v \equiv 91 \pmod{108}$ then v has one of the following seven forms modulo 756. We explain in each case how an APSTS(v) with property G4m may be obtained.

1. $v = 756s + 91$. We proceed in two steps. First, in Theorem 6.4, take $n = 14, m = 3$ and $u = 18s + 3$ ($s \geq 1$) to produce an APSTS($14(18s + 2) + 3$) = APSTS($252s + 31$) with property G4m. This requires a BAPSTS($31, -3$) (given in [7]) and an APSTS(31) with property G4a (given by Theorem 6.3). So for $s \geq 0$ we have an APSTS($252s + 31$) with property G4a. Then apply Theorem 6.2 with $u = 252s + 31$ ($s \geq 0$) to produce an APSTS($3(252s + 30) + 1$) = APSTS($756s + 91$) with property G4m.

2. $v = 756s+199 = 14(54s+14)+3$. In Theorem 6.4, take $n = 14, m = 3$ and $u = 54s + 15$ ($s \geq 0$). This requires a BAPSTS($31, -3$) (given in [7]) and an APSTS(31) with property G4a (given by Theorem 6.3).

3. $v = 756s+307 = 21(36s+14)+13$. In Theorem 6.4, take $n = 21, m = 13$ and $u = 36s + 15$ ($s \geq 0$). This requires a BAPSTS($55, -13$) (given in [7]) and an APSTS(55) with property G4a. Theorem 6.2 with $u = 19$ gives an APSTS(55) with property G4m.

4. $v = 756s + 415 = 7(108s + 59) + 2$. Here we use Theorem 6.5 with $m = 7$ and $n = 108s + 59$. These choices require an APSTS(9) and an APSTS($108s + 61$) both with property G4a. The latter lies in the class $v \equiv 7 \pmod{18}$ and so by Lemmas 6.1 and 6.2 we have already established the existence of an APSTS($108s + 61$) with property G4a. The former is guaranteed by Theorem 6.1, but to give a specific example, here are the blocks of an APSTS(9) with property G4m:

$$AB0, A12, A34, A56, B14, B25, B36, 016, 023, 045, 135, 264.$$

   The cycle on $\{A, B\}$ is $(1, 2, 5, 6, 3, 4)$, and the pairs appearing in blocks also containing the point 0 have differences 1 or 2 (mod 7).

5. $v = 756s+523 = 21(36s+24)+19$. In Theorem 6.4, take $n = 21, m = 19$ and $u = 36s+25$ ($s \geq 0$). This requires a BAPSTS($61, -19$) (given in [7]) and an APSTS(61) with property G4a (given by Lemma 6.1).

6. $v = 756s+631 = 7(108s+90)+1$. In Theorem 6.4, take $n = 7, m = 1$ and $u = 108s+91$ ($s \geq 0$). This requires an APSTS(15) with property G4a, which is guaranteed by Theorem 6.1, and which automatically gives a BAPSTS$(15, -1)$.

7. $v = 756s + 739 = 3(252s + 246) + 1$. Here we use Theorem 6.2 with $u = 252s + 247$. This requires an APSTS$(252s + 247)$ with property G4a. To get this, note that $252s + 247 = 7(36s + 35) + 2$ and then apply Theorem 6.5 with $m = 7$ and $n = 36s + 35$. These choices require an APSTS(9) and an APSTS$(36s + 37)$ both with property G4a. The latter lies in one of the following classes modulo 108: $v \equiv 1$, $v \equiv 37$, $v \equiv 73$. So in Lemma 6.4 we have already established the existence of an APSTS$(36s+37)$ with property G4a. The APSTS(9) with property G4a is guaranteed by Theorem 6.1. $\square$

Next we turn our attention to the remaining case of $v \equiv 13 \pmod{18}$. We split the argument into four cases modulo 72, namely $v \equiv 13, 31, 49, 67$. We start with the easiest of these.

Lemma 6.6 *If $v \equiv 31$ (mod 72) then there exists an APSTS(v) with property G4m.*

Proof If $v = 72s + 31 = 12(6s + 2) + 7$, then apply Theorem 6.4 with $n = 12, m = 7$ and $u = 6s + 3$ ($s \geq 1$). This requires a BAPSTS$(31, -7)$ (given in [7]) and an APSTS(31) with property G4a. The latter is given by Lemma 6.1, which also deals with the case $s = 0$ (i.e. $v = 31$). $\square$

Lemma 6.7 *If $v \equiv 13$ (mod 72) and $v \neq 13$ then there exists an APSTS(v) with property G4m.*

Proof We start by recalling from Lemma 6.1 that there exist APSTS(v)s with property G4m for $v = 49$ and for $v = 85$.

If $v \equiv 13 \pmod{72}$ then v has one of the following three forms modulo 216.

1. $v = 216s + 13 = 18(12s) + 13$. In Theorem 6.4 take $n = 18, m = 13$ and $u = 12s + 1$ ($s \geq 2$). This requires a BAPSTS$(49, -13)$ (given in [7]), and an APSTS(49) with property G4a. The remaining case $s = 1$ corresponds to $v = 229$. To deal with this case, use Theorem 6.4 with $n = 27, m = 13$ and $u = 9$. This gives an APSTS(27(8) + 13)=APSTS(229) with property G4m provided we have a BAPSTS$(67, -13)$ and an APSTS(67) with property G4a. But a BAPSTS$(67, -13)$ is given in [7], and an APSTS(67) with property

G4m can be obtained from Theorem 6.4 by taking $n = 8, m = 3$ and $u = 9$ and using the BAPSTS$(19, -3)$ from [7] and the APSTS(19) with property G4a that follows from Theorem 6.3.

2. $v = 216s+85 = 36(6s+2)+13$. In Theorem 6.4 take $n = 36, m = 13$ and $u = 6s + 3$ $(s \geq 1)$. These choices require a BAPSTS$(85, -13)$ (given in [7]), and an APSTS(85) with property G4a. The case $s = 0$ corresponds to $v = 85$.

3. $v = 216s+157 = 18(12s+8)+13$. In Theorem 6.4 take $n = 18, m = 13$ and $u = 12s + 9$ $(s \geq 0)$. These choices require a BAPSTS$(49, -13)$ (given in [7]), and an APSTS(49) with property G4a. □

Lemma 6.8 *If $v \equiv 67$ (mod 72) then there exists an APSTS(v) with property G4m.*

Proof We start by recalling that an APSTS(49) with property G4m is given by Lemma 6.1 and a BAPSTS$(37, -3)$ is given in [5].

If $v \equiv 67$ (mod 72) then v has one of the following two forms modulo 144.

1. $v = 144s+67 = 8(18s+8)+3$. Apply Theorem 6.4 with $n = 8, m = 3$ and $u = 18s + 9$ $(s \geq 0)$. This requires a BAPSTS$(19, -3)$ (given in [7]) and an APSTS(19) with property G4a (given by Theorem 6.3).

2. $v = 144s + 139$. We split this case into five subcases modulo 720, as follows.

   (a) $v = 720s + 139 = 60(12s + 2) + 19$. Apply Theorem 6.4 with $n = 60, m = 19$ and $u = 12s + 3$ $(s \geq 1)$. This requires a BAPSTS$(139, -19)$ (given in [7]) and an APSTS(139) with property G4a. The latter can be formed (with property G4m) by applying Theorem 6.4 with $n = 17, m = 3$ and $u = 9$. This requires a BAPSTS$(37, -3)$, and an APSTS(37) with property G4a (given by Theorem 6.3). The case $s = 0$ corresponds to this APSTS(139).

   (b) $v = 720s + 283 = 20(36s + 14) + 3$. Apply Theorem 6.4 with $n = 20, m = 3$ and $u = 36s + 15$ $(s \geq 0)$. This requires a BAPSTS$(43, -3)$ (given in [7]) and an APSTS(43) with property G4a (given by Theorem 6.3).

   (c) $v = 720s + 427 = 30(24s + 14) + 7$. Apply Theorem 6.4 with $n = 30, m = 7$ and $u = 24s + 15$ $(s \geq 0)$. This requires a BAPSTS$(67, -7)$ (given in [7]) and an APSTS(67) with property G4a. The latter is given by case 1 above.

(d) $v = 720s + 571 = 15(48s + 38) + 1$. Apply Theorem 6.4 with $n = 15, m = 1$ and $u = 48s + 39$ $(s \geq 0)$. This requires an APSTS(31) with property G4a, which is given in Lemma 6.6, and which automatically gives a BAPSTS$(31, -1)$.

(e) $v = 720s + 715$. We split this subcase into seven subsubcases modulo 5040, as follows.

   i. $v = 5040s + 715 = 7(720s + 102) + 1$. Apply Theorem 6.4 with $n = 7, m = 1$ and $u = 720s + 103$ $(s \geq 0)$. This requires an APSTS(15) with property G4a, which is guaranteed by Theorem 6.1, and which automatically gives a BAPSTS$(15, -1)$.

   ii. $v = 5040s + 1435 = 21(240s + 68) + 7$. Apply Theorem 6.4 with $n = 21, m = 7$ and $u = 240s+69$ $(s \geq 0)$. This requires a BAPSTS$(49, -7)$ (given in [7]) and an APSTS(49) with property G4a.

   iii. $v = 5040s+2155 = 21(240s+102)+13$. Apply Theorem 6.4 with $n = 21, m = 13$ and $u = 240s + 103$ $(s \geq 0)$. This requires a BAPSTS$(55, -13)$ (given in [7]) and an APSTS(55) with property G4a (guaranteed by Lemma 6.4).

   iv. $v = 5040s+2875 = 28(180s+102)+19$. Apply Theorem 6.4 with $n = 28, m = 19$ and $u = 180s + 103$ $(s \geq 0)$. This requires a BAPSTS$(75, -19)$ (given in [7]) and an APSTS(75) with property G4a (guaranteed by Theorem 6.1).

   v. $v = 5040s + 3595 = (630s + 449)8 + 3$. Apply Theorem 6.4 with $n = 630s + 449, m = 3$ and $u = 9$ $(s \geq 0)$. This requires a BAPSTS$(1260s + 901, -3)$ (given in [7]) and an APSTS$(1260s + 901)$ with property G4a that follows from Theorem 6.3.

   vi. $v = 5040s + 4315 = 14(360s + 308) + 3$. Apply Theorem 6.4 with $n = 14, m = 3$ and $u = 360s + 309$ $(s \geq 0)$. This requires a BAPSTS$(31, -3)$ (given in [7]) and an APSTS(31) with property G4a that follows from Theorem 6.3.

   vii. $v = 5040s + 5035 = 7(720s + 719) + 2$. Apply Theorem 6.5 with $m = 7, n = 720s + 719$. These choices require an APSTS(9) and an APSTS$(720s + 721)$ both with property G4a. The latter lies in the class $v \equiv 1 \pmod{18}$ and we have already established the existence of an APSTS$(720s + 721)$ with property G4a in Lemma 6.4. The former is guaranteed by Theorem 6.1. □

The final case left to consider is $v \equiv 49 \pmod{72}$.

Lemma 6.9 *If $v \equiv 49 \pmod{72}$ then there exists an APSTS(v) with property G4m.*

Proof First recall that an APSTS(49) with property G4m is given by Lemma 6.1. If $v \equiv 49 \pmod{72}$ then v has one of the following two forms modulo 144.

1. $v = 144s+49 = 24(6s+2)+1$. Apply Theorem 6.4 with $n = 24, m = 1$ and $u = 6s + 3$ ($s \geq 1$). This requires an APSTS(49) with property G4a, which automatically gives a BAPSTS$(49, -1)$. The case $s = 0$ corresponds to $v = 49$.

2. $v = 144s + 121$. We split this case into two subcases modulo 288, as follows.

    (a) $v = 288s + 121 = (36s + 15)8 + 1$. Apply Theorem 6.4 with $n = 36s + 15, m = 1$ and $u = 9$ ($s \geq 0$). This requires an APSTS$(72s+31)$ with property G4a. This has been constructed in Lemma 6.6 and it gives a BAPSTS$(72s + 31, -1)$.

    (b) $v = 288s + 265 = (36s + 33)8 + 1$. Apply Theorem 6.4 with $n = 36s + 33, m = 1$ and $u = 9$ ($s \geq 0$). This requires an APSTS$(72s+67)$ with property G4a. This has been constructed in Lemma 6.8 and it gives a BAPSTS$(72s + 67, -1)$. □

Concluding remarks

1. First we list the small cases obtained by computer searches that are used in Section 6, all of which are given in [5]. These comprise APSTS(v)s with property G4m for $v = 19, 31, 37, 49, 61$ and 85, and a BAPSTS$(37, -3)$.

2. The covering designs in this paper are minimum coverings. It may be possible for larger (non-minimum) coverings to retain the anti-Pasch property, and some of our constructions may be of some use in this respect. An interesting question is how large a covering design, without repeated blocks, can be before a Pasch configuration is forced.

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