Revising the observable consequences of slow-roll inflation

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We study the generation of primordial perturbations in a (single-field) slow-roll inflationary Universe. In momentum space, these (Gaussian) perturbations are characterized by a zero mean and a nonzero variance $\Delta^2(k, t)$. However, in position space the variance diverges in the ultraviolet. The requirement of a finite variance in position space forces one to regularize $\Delta^2(k, t)$. This can (and should) be achieved by proper renormalization in an expanding Universe in a unique way. This affects the predicted scalar and tensorial power spectra (evaluated when the modes acquire classical properties) for wavelengths that today are at observable scales. As a consequence, the imprint of slow-roll inflation on the cosmic microwave background anisotropies is significantly altered. We find a nontrivial change in the consistency condition that relates the tensor-to-scalar ratio r to the spectral indices. For instance, an exact scale-invariant tensorial power spectrum, $n_t = 0$, is now compatible with a nonzero ratio $r \approx 0.12 \pm 0.06$, which is forbidden by the standard prediction ($r = -8n_t$). The influence of relic gravitational waves on the cosmic microwave background may soon come within the range of planned measurements, offering a nontrivial test of the new predictions.

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I. INTRODUCTION AND SUMMARY

Inflation [1] provides a natural solution to the horizon and flatness problems of the hot big-bang cosmology. A sufficiently long period of rapid expansion can explain the large scale homogeneity, isotropy, and flatness of our visible Universe. Inflation also provides a quantitative explanation [2] to account for the origin of small inhomogeneities in the early Universe. These inhomogeneities are responsible for the structure formation in the Universe and for the anisotropies present in the cosmic microwave background (CMB), which were first detected by the COBE satellite and further analyzed by the WMAP satellite [3]. The potential-energy density of the inflaton field is assumed to cause the inflationary accelerated expansion, and the amplification of its quantum fluctuations and those of the metric are inevitable consequences in an expanding Universe [4]. These fluctuations acquire classical properties in the inflationary period and provide the initial conditions for classical cosmological perturbations after the big bang. The detection of the effects of primordial tensorial metric fluctuations (gravitational waves) in future high-precision measurements of the CMB anisotropies will serve as a highly nontrivial test of the inflationary paradigm and to constrain specific models. Therefore, it is particularly important to scrutinize the predictions of inflation for the tensorial and scalar power spectra. In this

respect, it was pointed out in [5] (see also [6]) that quantum field renormalization significantly modifies the amplitude of quantum fluctuations, and hence the corresponding power spectra, in de Sitter inflation. The analysis was further improved in [7] (see also the essay [8]) to understand how the basic testable predictions of (single-field) slow-roll inflation could be affected by quantum field renormalization. In this work we further study this issue, improve the technical analysis, and provide a more complete and robust discussion of how the observable consequences of inflation are altered when quantum field renormalization is taken into account.

Let us briefly summarize the logic of our approach. Let us assume that $\varphi(\vec{x}, t)$ represents a perturbation obeying a free field wave-equation on the inflationary background $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$, where $a(t)$ is a quasi-exponential expansion factor ($a(t) \sim e^{Ht}$). At the quantum level, this field is expanded as

$$\varphi(\vec{x}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k [\varphi_k(t)a_{\vec{k}}e^{i\vec{k}\vec{x}} + \varphi_k^*(t)a_{\vec{k}}^\dagger e^{-i\vec{k}\vec{x}}], \tag{1.1}$$

where the creation and annihilation operators satisfy the canonical commutation relation $[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta^3(\vec{k} - \vec{k}')$. The mode functions $\varphi_k(t)$ are required to satisfy the adiabatic condition (see, for instance, [9]). The power spectrum for this perturbation, $\Delta_\varphi^2(k, t)$, is usually defined in terms of the Fourier transform of the variance of the field [10,11]

$$\langle \hat{\varphi}_{\vec{k}}(t)\hat{\varphi}_{\vec{k}'}^\dagger(t) \rangle = \delta^3(\vec{k} - \vec{k}')\frac{2\pi^2}{k^3}\Delta_\varphi^2(k, t), \tag{1.2}$$

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where $\hat{\varphi}_{\vec{k}}(t) \equiv \varphi_k(t)a_{\vec{k}}$. These modes describe a perturbation field characterized, in momentum space, by a zero mean $\langle \hat{\varphi}_{\vec{k}}(t) \rangle = 0$ and the variance (1.2). The advantage of working in momentum space resides in the fact that different modes fluctuate independently of each other, as explicitly displayed by the presence of the delta function in (1.2). This way, the quantum field is regarded as an infinite collection of oscillators, each with a different value of $\vec{k}$. In position space the perturbation is also characterized by a zero mean $\langle \varphi(\vec{x}, t) \rangle = 0$ and a variance (or dispersion)

$$\langle \varphi^2(\vec{x}, t) \rangle = \frac{1}{(2\pi)^3} \int d^3k d^3k' \langle \hat{\varphi}_{\vec{k}}(t)\hat{\varphi}_{\vec{k}'}^\dagger(t) \rangle e^{i(\vec{k}-\vec{k}')\vec{x}}, \quad (1.3)$$

which, due to spatial homogeneity, turns out to be independent of $\vec{x}$. This variance is formally related to the power spectrum by

$$\langle \varphi^2(\vec{x}, t) \rangle = \int_0^\infty \frac{dk}{k} \Delta_\varphi^2(k, t). \quad (1.4)$$

As is well known in quantum field theory, the above expectation value quadratic in the field φ is divergent. It suffers from quadratic and logarithmic ultraviolet divergences

$$\langle \varphi^2(\vec{x}, t) \rangle \sim \frac{1}{4\pi^2} \int_0^\infty dk \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2 k} + \dots \right). \quad (1.5)$$

The first term corresponds to the usual contribution from vacuum fluctuations in Minkowski space. This contribution can be eliminated by renormalization as claimed, for instance, in [12]. However, the second term is also characteristic of vacuum fluctuations in a curved background. Because the different k modes fluctuate independently of each other, one could be tempted to get rid of this logarithmic ultraviolet divergence by simply eliminating the modes with $k > aH$ and leaving the rest unaffected (see, for instance, [13]). If one eliminates this divergence using a window function in this way, as is usual for random fields, then one obtains $\Delta_\varphi^2(k) \approx H^2/4\pi^2$, where $\Delta_\varphi^2(k)$ is defined by the quantity $\Delta_\varphi^2(k, t)$ evaluated a few Hubble times after the "horizon crossing time" t_k, since this is the time scale at which the modes behave as classical perturbations.[1] However, one should take into account that the field fluctuations are quantum in nature and, therefore, one should consider the subtle points of quantum field theory (QFT) regarding the ultraviolet divergences.

Even though free quantum field theory is usually regarded as an infinite set of independent harmonic oscillators (one for each k mode), there are fundamental holistic aspects of QFT that can not be properly understood in terms of independent modes. Renormalization is the hallmark of the holistic aspect of QFT. This is clear in the fact

that, although the renormalization schemes in QFT in curved spacetimes are based on the ultraviolet behavior of the theory, the infrared sector is also affected by renormalization, leading potentially to observable consequences. This can be explicitly displayed by considering, for instance, the Casimir effect. The energy density between the two conducting plates obtained by proper renormalization provides the well-known and experimentally tested expression. However, a naive subtraction obtained by introducing a high-frequency cutoff in the integrals in momentum space (i.e., treating the k modes as being independent) produces a quite different result (see the discussion of [14]).

Taking this into account, we see that the logarithmic divergence in (1.5) should be dealt with by renormalization and one cannot rule out, *a priori*, the possibility that the treatment of the divergences at very high values of k may produce some impact at lower momenta. Therefore, we propose that in the standard definitions of the spectrum $\Delta_\varphi^2(k, t)$, as given in (1.2) and (1.4), one should replace the unrenormalized $\langle \varphi^2(\vec{x}, t) \rangle$ by the renormalized variance, $\langle \varphi^2(\vec{x}, t) \rangle_{\text{ren}}$. Writing $\tilde{\Delta}_\varphi^2(k, t)$ for the spectrum defined in this way, the definition in (1.4) (and similarly in (1.2)) is replaced by the corresponding renormalized expression

$$\langle \varphi^2(\vec{x}, t) \rangle_{\text{ren}} = \int_0^\infty \frac{dk}{k} \tilde{\Delta}_\varphi^2(k, t). \quad (1.6)$$

This way, the physical variance $\langle \varphi^2(\vec{x}, t) \rangle_{\text{ren}}$ remains a well-defined quantity, in the same way as one could obtain a finite expression for the expectation values of the quantum stress-energy tensor. To complete the physical consistency of this approach, it would be desirable to define a unique expression for the necessary subtractions required to produce a consistent $\langle \varphi^2(\vec{x}, t) \rangle_{\text{ren}}$. Since the power spectrum is defined in momentum space, the natural scheme is renormalization in momentum space, so we define

$$\langle \varphi^2(\vec{x}, t) \rangle_{\text{ren}} = \frac{4\pi}{(2\pi)^3} \int_0^\infty k^2 dk (|\varphi_k(t)|^2 - C_k(t)), \quad (1.7)$$

where $C_k(t)$ represents the expected counterterms. As we will see, adiabatic renormalization [9,15,16], which works by subtracting a set of counterterms "mode-by-mode," provides a natural expression for the counterterms encoded in $C_k(t)$. Moreover, the DeWitt-Schwinger renormalization, originally defined in position space (see, for instance, [9,16]), can be nicely translated to momentum space [17], thus providing another answer for $C_k(t)$. When these two schemes are applied to the field perturbations arising from inflation, the resulting expressions for $C_k(t)$ coincide, thus defining a unique expression for the spectrum $\tilde{\Delta}_\varphi^2(k, t)$.

The holistic nature of QFT is then explicitly realized through (1.6). Although the counterterms are fully determined by the ultraviolet behavior of the modes, the long wavelength sector, and hence the new $\tilde{\Delta}_\varphi^2(k, t)$, is significantly affected by the subtractions. In the slow-roll sce-

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[1]The time t_k is defined by $a(t_k)/k = H(t_k)$, where $a(t)$ is the expansion factor and $H = \dot{a}/a$ is the Hubble rate.

nario, when H slowly decreases with time, the effects of renormalization have a nontrivial impact on $\tilde{\Delta}^2_\varphi(k, t)$ when this quantity is evaluated a few Hubble times after the time of horizon crossing t_k. For instance, for the tensorial modes we obtain

$$\tilde{\Delta}^2(k) \sim \left(\frac{H}{2\pi}\right)^2 \epsilon, \qquad (1.8)$$

where ϵ is the usual slow-roll parameter. A similar expression is obtained for the scalar perturbations (involving now the slow-roll ϵ and η parameters), with the corresponding changes in the tensor-to-scalar ratio, the spectral indices and the consistency relation. The new predictions remain in agreement with observation for the simplest forms of inflation (ϕ^2 and ϕ^4 potentials). It is worth pointing out that the new consistency condition for single-field inflation is predicted to be

$$r = 4(1 - n_s - n_t) + \frac{4n'_t}{n_t^2 - 2n'_t}$$
$$\times (1 - n_s - \sqrt{2n'_t + (1 - n_s)^2 - n_t^2}), \qquad (1.9)$$

instead of the standard prediction $r = -8n_t$. The tensor-to-scalar ratio is now related with the spectral indices n_t, $(1 - n_s)$, and also $n'_t \equiv dn_t/d\ln k$. This modification has far reaching consequences. For instance, since the observations from the 5-year WMAP [3](with BAO+SN) strongly suggest that $(1 - n_s) \approx 0.030 \pm 0.015$ (with $r < 0.22$), expression (1.9) allows for an exact scale-invariant tensorial power spectrum, $n_t = 0$, while being compatible with a nonzero ratio $r \approx 0.12 \pm 0.06$. We will comment on this further later on.

The paper is organized as follows. In Sec. II, we briefly review the standard ways to derive the expressions for the tensor and scalar power spectra of single-field slow-roll inflation. In Sec. III, we work out the new definition of the power spectra and give the technical details involving the required renormalization in momentum space. In Sec. IV, we provide, as a consequence of the new power spectra, the corresponding expression for the tensor-to-scalar ratio r, the tensorial and scalar spectral indices, and the slow-roll parameters. This leads, in particular, to a change in the consistency condition that relates the tensor-to-scalar amplitude ratio to the spectral indices. In Sec. V, we summarize our results and conclusions. We use natural units $\hbar = 1 = c$.

II. SPECTRUM OF FLUCTUATIONS FROM INFLATION

We will now proceed to briefly review the standard predictions for the power spectra in single-field slow-roll inflation. In obtaining these predictions, quantum field renormalization in the curved spacetime of the expanding Universe is not taken into account.

A. Tensorial spectrum

Let us focus on the production of relic gravitational waves by considering fluctuating tensorial modes $h_{ij}(\vec{x}, t)$ in an expanding, spatially flat Universe

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j. \qquad (2.1)$$

The wave equation obeyed by these modes comes from the linearized Einstein equations and is given by

$$- a^2 \ddot{h}_{ij} - 3a\dot{a}\dot{h}_{ij} + \nabla^2 h_{ij} = 0. \qquad (2.2)$$

Expanding the fluctuating fields h_{ij} in plane wave modes $h_k(t)e_{ij}e^{i\vec{k}\vec{x}}$, where e_{ij} is a constant polarization tensor obeying the conditions $e_{ij} = e_{ji}$, $e_{ii} = 0$ and $k_i e_{ij} = 0$, we obtain the equation

$$\ddot{h}_k + 3H\dot{h}_k + \frac{k^2}{a^2}h_k = 0, \qquad (2.3)$$

with $k \equiv |\vec{k}|$. The conditions for the polarization tensor imply that the perturbation field h_{ij} can be decomposed into two polarization states described by a couple of massless scalar fields $h_{ij} = h_+ e^+_{ij} + h_\times e^\times_{ij}$, where $e^s_{ij}e^{s'}_{ij} = 2\delta_{ss'}$ ($s = +, \times$ stands for the two independent polarizations), both obeying the above wave equation [18] (see also, for instance, [19]). Note that, since tensor perturbations are gauge-invariant at linear order [10], the fields $h^{+,\times} e^{+,\times}_{ij}$ represent the physical degrees of freedom (from now on we omit the subindex $+$ or $\times$). On scales larger than the Hubble radius, the damping term $3H\dot{h}_k$ dominates. However, on scales smaller than the Hubble radius it is the spatial gradient term that dominates over the damping term, thus leading to the conventional flat-space oscillatory behavior of modes. To constrain the form of the modes defining the quantization, it is natural to impose the adiabatic asymptotic condition [4,9] for large k

$$\frac{h_k(t)}{\sqrt{16\pi G}} \sim (2(2\pi)^3 w(t)a^3(t))^{-1/2}e^{-i\int^t w(t')dt'}, \qquad (2.4)$$

with $w(t) = k/a(t)$, where the factor $\sqrt{16\pi G}$ (G is the Newton constant) has to be introduced to get a canonically normalized variable. This condition does not uniquely fix the form of the modes. For instance, in an exact de Sitter background ($a(t) = e^{Ht}$, with H a strict constant) one can invoke de Sitter invariance to uniquely determine the modes [20], and one obtains

$$h_k(t) = \sqrt{\frac{16\pi G}{2(2\pi)^3 k^3}}(H - ike^{-Ht})e^{i(kH^{-1}e^{-Ht})}. \qquad (2.5)$$

These modes oscillate until the physical wave length reaches the Hubble length H^{-1}. The amplitude of the modes at this time, usually called the "horizon exit" time t_k, defined by $k/a(t_k) = H$, is then $|h_k|^2 = \frac{2GH^2}{\pi^2 k^3}$. A few Hubble times after horizon exit, the modes get frozen

as classical perturbations [10,13,19] (see also [21]) with constant amplitude $|h_k|^2 = \frac{GH^2}{\pi^2 k^3}$. The freezing amplitude is usually codified through the unrenormalized quantity $\Delta_h^2(k) = 4\pi k^3 |h_k|^2$. Taking into account the two polarizations $\langle h_{ij}(x) h^{ij}(x) \rangle = \int_0^\infty k^{-1} dk 4\Delta_h^2(k)$, one easily gets the standard scale-free tensorial power spectrum $P_t(k) \equiv 4\Delta_h^2(k) = \frac{8}{M_P^2} (\frac{H}{2\pi})^2$, where $M_P = 1/\sqrt{8\pi G}$ is the reduced Planck mass in natural units. The appearance of such frozen fluctuations converts the modes into classical perturbations with wavelengths that are still stretched by the rapid expansion to reach astronomical scales. This is essentially what happens in the inflationary era.

1. Slow-roll inflation

To take into account that inflation lasts for a finite period of time, one usually considers the so-called slow-roll scenario [10,13,19]. The homogeneous part of the inflaton field $\phi_0(t)$ rolls slowly down its potential $V(\phi)$ toward a minimum. Both ϕ_0 and $H \approx \sqrt{\frac{8\pi G}{3} V(\phi_0)}$ are changing very gradually and this change is parameterized by the slow-roll parameters ϵ, η, where $\epsilon = -\dot{H}/H^2$, and $\eta - \epsilon = \ddot{\phi}_0/(H\dot{\phi}_0)$. In the slow-roll approximation, defined when the parameters are small $\epsilon, |\eta| \ll 1$, one can relate them to the derivatives of the inflaton potential $\epsilon = (M_P^2/2)(V'/V)^2$, $\eta = M_P^2(V''/V)$. To generate an approximate form for the modes, it is convenient to introduce the conformal time variable $\tau \equiv \int dt/a(t)$. In terms of τ the wave equation turns out to be of the form

$$\frac{d^2 h_k}{d\tau^2} + 2Ha \frac{dh_k}{d\tau} + k^2 h_k = 0, \qquad (2.6)$$

and taking into account that in the slow-roll approximation

$$(1 - \epsilon)\tau = -\frac{1}{aH}, \qquad (2.7)$$

we get

$$\frac{d^2 h_k}{d\tau^2} - \frac{2(1 + \epsilon)}{\tau} \frac{dh_k}{d\tau} + k^2 h_k = 0. \qquad (2.8)$$

Within this approximation, and treating now the parameter ϵ as a constant, one can exactly solve the above equation as follows:

$$h_k(t) = (-16\pi G \tau \pi/4(2\pi)^3 a^2)^{1/2}$$
$$\times [E(k)H_\nu^{(1)}(-k\tau) + F(k)H_\nu^{(2)}(-k\tau)], \qquad (2.9)$$

where the index of the Bessel function is $\nu = 3/2 + \epsilon$, and the complex coefficients $E(k)$ and $F(k)$ obey the normalization requirement

$$|E(k)|^2 - |F(k)|^2 = 1. \qquad (2.10)$$

The adiabaticity condition (2.4) implies that

$$\lim_{k \to \infty} E(k) = 1 \qquad \lim_{k \to \infty} F(k) = 0. \qquad (2.11)$$

The simplest way to choose $E(k)$ and $F(k)$ would be to require that, for $\epsilon \to 0$, we recover the exact de Sitter form of the modes. This would mean that $E(k) = 1$ and $F(k) = 0$ for every value of k.

2. Infrared divergences

The choice $E(k) = 1$ and $F(k) = 0$ is, however, an idealized situation which assumes that inflation started at an infinite time in the past. A consequence of this assumption is that the two-point function

$$\langle 0|h(x)h(x')|0 \rangle = \int d^3 k e^{i\vec{k}(\vec{x} - \vec{x}')} h_k(\tau) h_k^*(\tau') \qquad (2.12)$$

is ill-defined due to an infrared divergence. This is so because in the limit $k \to 0$ the integrand in (2.12) behaves as

$$\frac{dk}{k} \frac{1}{k^{2\epsilon}}, \qquad (2.13)$$

and $\epsilon > 0$. For general $E(k)$ and $F(k)$ we have, instead,

$$\frac{dk}{k} \frac{|E(k) - F(k)|^2}{k^{2\epsilon}}. \qquad (2.14)$$

The infrared divergence is avoided if $|E(k) - F(k)|^2 \to 0$, as $k \to 0$. This happens naturally if one assumes that inflation started smoothly at some early, but finite, time (see, for instance, [22]). Assuming that the initial vacuum is well-defined, and since the dynamical evolution cannot generate infrared divergences [23,24], one should get $E(k) \to F(k)$, when $k \to 0$ (as explicitly obtained in [22]), and a finite contribution to the two-point function in the infrared end.

The scale of this infrared behavior of the functions $E(k)$ and $F(k)$ is given by the Hubble radius at the beginning of inflation. Only those wavelengths that were already outside the inflationary Hubble radius

$$k/a(t) \ll H(t) \qquad (2.15)$$

when inflation began can see this infrared behavior. However, only the wavelengths that crossed the Hubble radius

$$k/a(t_k) = H(t_k) \qquad (2.16)$$

at 50–60 e folds before the end of the inflationary epoch have cosmological observable consequences today. For these wavelengths the functions $E(k)$ and $F(k)$ behave as in the Bunch-Davies vacuum ($E(k) \sim 1$, $F(k) \sim 0$). Despite having no direct observable consequences, the behavior of $E(k)$ and $F(k)$ for very large wavelengths is important for the consistency of the theoretical framework because it avoids an infrared divergence in the two-point function.

3. Nearly scale-invariant spectrum

The Hubble exit time t_k can be rewritten, in terms of the conformal time, as

$$-k\tau(t_k) = (1 + \epsilon). \qquad (2.17)$$

Therefore, at the Hubble exit time t_k the amplitude of the modes h_k given in (2.9) takes the value $|h_k|^2 = \frac{2GH^2(t_k)}{\pi^2 k^3}$, and then $\Delta_h^2(k) = \frac{8GH^2(t_k)}{\pi}$. This provides a nearly scale-invariant spectrum of fluctuations. The k dependence of $H^2(t_k)$

$$\frac{d\ln H(t_k)}{d\ln k} = -\epsilon \qquad (2.18)$$

leads to

$$\Delta_h^2(k) = \Delta_h^2(k_0)\left(\frac{k}{k_0}\right)^{-2\epsilon}, \qquad (2.19)$$

where k_0 is a pivot scale. One defines the tensorial spectral index n_t as the exponent in the above expression. So

$$n_t = -2\epsilon(t_k). \qquad (2.20)$$

In the latter formula we have explicitly considered that the constant parameter ϵ in the previous calculation is given by its value around the Hubble exit for the k mode.

The same conclusion can be achieved by evaluating the power spectrum a few e foldings after crossing the Hubble radius. One can then consider the approximation $-k\tau \ll 1$, which implies

$$|H_\nu^{(1)}(-k\tau)|^2 \sim \frac{2}{\pi}(-k\tau)^{-2\nu}. \qquad (2.21)$$

Furthermore, from the slow-roll Eq. (2.7) we obtain

$$a = a(\tau_0)\left(\frac{\tau}{\tau_0}\right)^{1/(\epsilon-1)}, \qquad (2.22)$$

where τ_0 is an arbitrary reference time. Therefore, we can write

$$|h_k(t)|^2 \sim \frac{16\pi G}{4(2\pi)^3}\frac{(-\pi\tau)}{a^2}\frac{2}{\pi}(-k\tau)^{-2\nu}$$

$$\sim \frac{G}{\pi^2 k a^2(\tau_0)}(-k\tau_0)^{-2-2\epsilon}, \qquad (2.23)$$

which explicitly shows the dependence on $k^{-2\epsilon}$ of $\Delta_h^2(k) \propto k^3|h_k(t)|^2$. Now, taking for convenience the reference time τ_0 in (2.22) as $\tau_0 = \tau(t_k)$ and using that $a(t_k) = k/H(t_k)$ and $\tau(t_k) = -\frac{1}{(1-\epsilon)k}$, one can find immediately that

$$|h_k(t)|^2 \sim \frac{G}{\pi^2 k^3}H^2(t_k)\left(\frac{1}{1-\epsilon}\right)^{-2-2\epsilon} \approx \frac{G}{\pi^2 k^3}H^2(t_k). \qquad (2.24)$$

Taking into account the two polarizations one finally gets the same unrenormalized power spectrum $P_t(k) \equiv 4\Delta_h^2(k) = 16\pi k^3|h_k|^2$ (up to a factor $1/2$)

$$P_t(k) = \frac{8}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2. \qquad (2.25)$$

B. Scalar spectrum

We now proceed to reproduce the standard results for the unrenormalized power spectrum of scalar perturbations. We first consider, for simplicity, the inflaton field in an unperturbed background and compute the Δ_φ^2 evaluated at t_k. In order to evaluate Δ_φ^2 at later times (a few e folds after t_k), we must improve the calculation by taking into account the fluctuation of the background metric using the spatially flat slicing. We also show how to reproduce this result using a more rigorous approach in terms of gauge-invariant quantities.

Consider the inflaton field $\phi(t, \vec{x})$ made out of a homogeneous part $\phi_0(t)$ and a small fluctuating part $\delta\phi(\vec{x}, t)$. The fluctuation $\delta\phi(\vec{x}, t)$ satisfies, in the approximation of considering an unperturbed background spacetime, the wave equation

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - a^{-2}\nabla^2\delta\phi + V''(\phi_0)\delta\phi = 0. \qquad (2.26)$$

Because of the slow rolling of $\phi_0(t)$ from the hill potential, the term V'' is very small, which allows one to estimate the amplitude of quantum fluctuations Δ_ϕ^2 in a way similar to Δ_h^2. Inflaton fluctuations translate into curvature perturbations, which constitute the "seeds" for structure formation and are characterized by their scalar power spectrum $P_{\mathcal{R}}(k) = (H/\dot{\phi}_0)^2\Delta_\phi^2$. Evaluating Δ_ϕ^2 and $(H/\dot{\phi}_0)^2$ at the time of horizon exit t_k [10,19] one gets $\Delta_\phi^2 = H^2(t_k)/2\pi^2$ and $H^2(t_k)/\dot{\phi}_0^2(t_k) = \frac{1}{2M_P^2\epsilon(t_k)}$. Therefore, $P_{\mathcal{R}}(k) \sim \frac{1}{M_P^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2$.

One can improve this approximation by taking into account the effect of the metric perturbation in (2.26). To properly define the wave equation for $\delta\phi(\vec{x}, t)$ we have to specify a slicing. Choosing for convenience the spatially flat slicing, the equation for the field perturbation $\delta\phi(\vec{x}, t)$ is (see, for instance, [13])

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - a^{-2}\nabla^2\delta\phi + V''(\phi_0)\delta\phi$$
$$+ \frac{1}{a^3}\frac{d}{dt}\left(\frac{2a^3\dot{H}}{H}\right)\delta\phi = 0. \qquad (2.27)$$

In the slow-roll approximation, the corresponding Fourier components of $\delta\phi$ obey

$$\ddot{\delta\phi}_k + 3H\dot{\delta\phi}_k + a^{-2}k^2\delta\phi_k + H^2(3\eta - 6\epsilon)\delta\phi_k = 0. \qquad (2.28)$$

To deal with this equation, one usually considers that the ϵ and η parameters are constant, while $H = \dot{a}/a$ remains a time-dependent function. So we have

$$\delta\ddot{\phi}_k + 3\frac{\dot{a}}{a}\delta\dot{\phi}_k + a^{-2}k^2\delta\phi_k + \frac{\dot{a}^2}{a^2}(3\eta - 6\epsilon)\delta\phi_k = 0. \tag{2.29}$$

With this assumption one can exactly solve the equation in terms of the conformal time τ. The form of the modes $\delta\phi_k$ is the same as for the tensorial ones, up to the coefficient $\sqrt{16\pi G}$,

$$\delta\phi_k(t) = (-\tau\pi/4(2\pi)^3 a^2)^{1/2}H_\mu^{(1)}(-k\tau), \tag{2.30}$$

where the index of the Bessel function is now $\mu = 3/2 + 3\epsilon - \eta$. Evaluating this perturbation a few Hubble times after the horizon crossing time, one obtains the standard result

$$P_{\mathcal{R}}(k) = (H/\dot{\phi}_0)^2 4\pi k^3|\delta\phi_k|^2 = \frac{1}{2M_P^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2, \tag{2.31}$$

where the constant parameter ϵ is given at the Hubble radius time scale t_k. Because of the k dependence of $H(t_k)$ and $\epsilon(t_k)$, one also gets a nearly scale-free spectrum

$$P_{\mathcal{R}}(k) = P_{\mathcal{R}}(k_0)\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{2.32}$$

where the so-called scalar spectral index n_s is found to be

$$n_s = 1 - 6\epsilon + 2\eta. \tag{2.33}$$

To complete our presentation, we will explain how the above result can be recovered without specifying any particular slicing and threading of spacetime. These scalar perturbations are commonly studied through the gauge-invariant quantity $\mathcal{R}$ (the comoving curvature perturbation)

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}_0}\delta\phi, \tag{2.34}$$

where Ψ is the curvature perturbation $(R^{(3)} = 4\nabla^2\Psi/a^2)$ of the spatial metric $g_{ij} = a^2[(1 - 2\Psi)\delta_{ij} + 2\partial_{ij}E]$. In momentum space, $\mathcal{R}$ obeys the equation [25,26]

$$\frac{d^2\mathcal{R}_k}{d\tau^2} + \frac{2}{z}\frac{dz}{d\tau}\frac{d\mathcal{R}_k}{d\tau} + k^2\mathcal{R}_k = 0, \tag{2.35}$$

where $z \equiv a\dot{\phi}_0/H$. In the slow-roll approximation $z^{-1}dz/d\tau = aH(1 + 2\epsilon - \eta)$ and hence

$$z(\tau) = z(\tau_0)\left(\frac{\tau}{\tau_0}\right)^{-1-3\epsilon+\eta}, \tag{2.36}$$

where τ_0 is again a reference instant of time. Furthermore, the above wave equation simplifies to

$$\frac{d^2\mathcal{R}_k}{d\tau^2} - \frac{2(1 + 3\epsilon - \eta)}{\tau}\frac{d\mathcal{R}_k}{d\tau} + k^2\mathcal{R}_k = 0. \tag{2.37}$$

The solution obeying the adiabatic condition (and the de Sitter symmetry for H constant) is

$$\mathcal{R}_k(t) = (-\pi\tau/4(2\pi)^3 z^2)^{1/2}H_\mu^{(1)}(-\tau k), \tag{2.38}$$

where $\mu = 3/2 + 3\epsilon - \eta$. Evaluating $\mathcal{R}_k(t)$ at $t = t_k$ one obtains the power spectrum estimated above in terms of the fluctuations of the inflaton field. We can also evaluate the amplitude $|\mathcal{R}_k|^2$ a few Hubble times after t_k. Using $\tau(t_k)$ as reference time in (2.36), it is easy to get $(P_{\mathcal{R}}(k) = \Delta_{\mathcal{R}}^2(k) = 4\pi k^3|\mathcal{R}_k|^2)$

$$P_{\mathcal{R}}(k) = \frac{1}{2M_P^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2. \tag{2.39}$$

C. Tensor-to-scalar amplitude ratio and spectral indices

We have seen that the concrete value of the amplitude of the power spectra depends on the time at which they are evaluated. In the literature, one find different times that can be parameterized by λ, according to the condition $k/a(t) = \lambda H$. The power spectra are then modified by the factor $(1 + \lambda^2)$ (see for instance [27]), which goes to unit exponentially fast. This ambiguity, however, is irrelevant in the evaluation of physical observables such as the tensor-to-scalar ratio r and the spectral indices. In fact, irrespective of the evaluation time (t_k or a few e foldings after it), the ratio

$$r = \frac{P_t(k)}{P_{\mathcal{R}}(k)} \tag{2.40}$$

gives a constant quantity

$$r = 16\epsilon(t_k). \tag{2.41}$$

The other two physical observables are neither affected by the ambiguity in the evaluation time of the power spectra. The spectral indices are unambiguously given by

$$n_t = -2\epsilon(t_k), \tag{2.42}$$

and

$$1 - n_s = 6\epsilon(t_k) - 2\eta(t_k). \tag{2.43}$$

From the above formulas one infers a necessary relation between measurable quantities

$$r = -8n_t, \tag{2.44}$$

which, as claimed many times in the literature, should be verified by any single-field slow-roll inflationary model irrespective of the form of the potential. For this reason, the future experimental checking of this condition is usually regarded as an important test of the simplest forms of inflation.

This concludes our review of the standard derivation of the unrenormalized predictions of single-field slow-roll inflation.

III. THE ROLE OF ULTRAVIOLET DIVERGENCES: RENORMALIZATION OF THE SPECTRUM OF FLUCTUATIONS

In this section we reexamine the derivation of the power spectra taking into account the effects of renormalization.

A. Ultraviolet divergences and momentum-space renormalization

It is easy to see that $\Delta_h^2(k, t)$ gives the formal contribution, per $d \ln k$, to the variance of the gravitational wave fields $h_{+,\times}$

$$\langle h^2 \rangle = \int_0^\infty k^2 dk \int d\Omega |h_k|^2 = \int_0^\infty \frac{dk}{k} \Delta_h^2(k, t), \quad (3.1)$$

since, as already defined, $\Delta_h^2(k, t) \equiv 4\pi k^3 |h_k|^2$. Although, as noted previously, the small k behavior of the functions $E(k)$ and $F(k)$ cures the potential infrared divergence in the above integral, the large k behavior of the modes makes the integral divergent

$$\langle h^2 \rangle = \int_0^\infty \frac{dk}{k} \frac{16\pi G k^3}{4\pi^2 a^3} \left[\frac{a}{k} \left[1 + \frac{(2 + 3\epsilon)}{2k^2 \tau^2} \right] + \ldots \right], \quad (3.2)$$

It is a common view to bypass this point by regarding $h(\vec{x}, t)$ as a classical random field. One then introduces a window function $W(kR)$ in the integral to smooth out the field on a certain scale R and to remove the Fourier modes with $k^{-1} < R$. However, as explained in the introduction, it is our view to regard the variance as a basic physical object and treat h and $\mathcal{R}$ (or $\delta \phi$ in the flat-slicing gauge) as a proper quantum field. Renormalization is then the natural solution to eliminate the ultraviolet divergences and keep the variance in position space finite and well-defined. Since the physically relevant quantity (power spectrum) is expressed in momentum space, the natural renormalization scheme to apply is the so-called adiabatic subtraction [5], as it renormalizes the theory in momentum space. Adiabatic renormalization [9,15,16] removes the divergences present in the formal expression (3.1) by subtracting counterterms mode-by-mode in the integral (3.1)

$$\langle h^2 \rangle_{\mathrm{ren}} = \int_0^\infty \frac{dk}{k} \left[4\pi k^3 |h_{\vec{k}}|^2 - \frac{16\pi G k^3}{4\pi^2 a^3} (w_k^{-1} + (W_k^{-1})^{(2)}) \right], \quad (3.3)$$

with $w_k = k/a(t)$. The subtraction of the first term $(16\pi G k^3 / 4\pi^2 a^3 w_k)$ cancels the typical flat-space vacuum fluctuations, which are responsible for the quadratic divergence in the integral (3.2). The additional term, proportional to $(W_k^{-1})^{(2)}$ and which involves $\dot{a}^2$ and $\ddot{a}$, is necessary to properly perform the renormalization in an expanding Universe. It cancels the logarithmic divergence in (3.2).

However, one can legitimately ask if the momentum-space counterterms are uniquely fixed. In other words, can a different renormalization scheme, as the DeWitt-Schwinger subtraction prescription in momentum space [17], lead to a different expression for the counterterms? Before going further in our analysis, let us briefly summarize the main steps of momentum-space renormalization (for an extensive exposition see [9]).

B. Momentum-space renormalization

1. Adiabatic renormalization

Let us consider a generic free scalar field φ in our spatially flat cosmological metric obeying the field equation

$$\ddot{\varphi} + 3H\dot{\varphi} - a^{-2}\nabla^2 \varphi + (m^2 + \xi R)\varphi = 0, \quad (3.4)$$

where R is the four-dimensional scalar curvature

$$R = 6 \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right], \quad (3.5)$$

and ξ is the curvature coupling. We include this term to be general and for illustrative purposes. The Fourier components of φ obey the equation

$$\ddot{\varphi}_k + 3\frac{\dot{a}}{a} \dot{\varphi}_k + a^{-2} k^2 \varphi_k + \left[6 \left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} \right) \xi + m^2 \right] \varphi_k = 0. \quad (3.6)$$

Note that the role of the dimensionless parameter ξ here is similar to the role of the ϵ and η parameters in the wave equation (2.29) for the perturbation.

Let us assume that the form of the normalized modes is $(2Va^3(t))^{-1/2} e^{i\vec{k}\cdot\vec{x}} \bar{\varphi}_k(t)$, where $V \equiv L^3$ is the volume of a box of coordinate length L. The continuous limit is obtained by replacing V by $(2\pi)^3$ and $\sum_{\vec{k}}$ by $\int d^3 \vec{k}$. The function $\bar{\varphi}_k(t)$ obeys the equation

$$\frac{d^2}{dt^2} \bar{\varphi}_k + \Omega_k^2 \bar{\varphi}_k = 0, \quad (3.7)$$

where

$$\Omega_k^2 = w_k^2 + \sigma, \quad (3.8)$$

with $w_k = \sqrt{k^2/a^2 + m^2}$ and

$$\sigma = \left(6\xi - \frac{3}{4} \right) \left(\frac{\dot{a}}{a} \right)^2 + \left(6\xi - \frac{3}{2} \right) \frac{\ddot{a}}{a}. \quad (3.9)$$

The function $\bar{\varphi}_k(t)$ is assumed to obey also the adiabatic condition $\bar{\varphi}_k(t) \sim w_k^{-1/2} e^{-i \int w_k(t') dt'}$ in the large k regime. This condition does not uniquely determine the form of $\bar{\varphi}_k(t)$ (different solutions lead to different sets of modes and, therefore, to different vacuum states) but it uniquely determines an asymptotic expansion for all possible solutions. This expansion is characterized by

$$\bar{\varphi}_k(t) \sim W_k^{-1/2} e^{-i \int W_k(t') dt'}, \quad (3.10)$$

with a recursive expansion

$$W_k = w_k + w_k^{(2)} + w_k^{(4)} + \dots \qquad (3.11)$$

where

$$w_k^{(2)} \equiv \left(\frac{1}{2} w_k^{-1/2} \frac{d^2}{dt^2} w_k^{-1/2} + \frac{1}{2} w_k^{-1} \sigma\right). \qquad (3.12)$$

Similar expressions for $w_k^{(4)}$ and all the other higher-order terms can be found. Each order in the (adiabatic) expansion is characterized by the number of time derivatives of $a(t)$ appearing in a term. While w_k is of zero order, $w_k^{(2)}$ is of second order, as one can trivially verify by simple counting. Note that σ is of second adiabatic order. One should have in mind that the adiabatic expansion is an asymptotic series and does not converge in general. This is why the form of the modes is not uniquely defined by the adiabatic condition, thus allowing to have different solutions and hence different vacuum states.

The evaluation of the variance of the field φ as a sum in modes leads to a divergent expression. In the continuous limit it is given by

$$\langle \varphi^2 \rangle = (4\pi^2 a^3)^{-1} \int_0^\infty dk k^2 |\bar{\varphi}_k|^2. \qquad (3.13)$$

As is evident from the asymptotic expansion (3.10), one necessarily encounters ultraviolet divergences in the above quantity. Generically one encounters quadratic and logarithmic divergences. In the adiabatic renormalization, the physically relevant finite expression is obtained from the formal one by subtracting mode-by-mode each term in the adiabatic expansion of the integrand that contains at least one ultraviolet divergent part for arbitrary values of the parameters (m and ξ) of the theory [9]. Applying this to the particular case of the variance of the field φ one gets

$$\langle \varphi^2 \rangle_{\text{ren}} = (4\pi^2 a^3)^{-1} \int_0^\infty dk k^2 (|\bar{\varphi}_k|^2 - w_k^{-1} - (W_k^{-1})^{(2)}), \qquad (3.14)$$

where w_k^{-1} and $(W_k^{-1})^{(2)}$ are the zeroth and second-order terms, respectively, in the adiabatic expansion of W_k^{-1}. Generically, the second adiabatic counterterm $(W^{-1})^{(2)}$ is given by

$$(W_k^{-1})^{(2)} = -\frac{1}{w_k^2}\left[\frac{1}{2} w_k^{-1/2} \frac{d^2}{dt^2} w_k^{-1/2} + \frac{1}{2} w_k^{-1} \sigma\right]. \qquad (3.15)$$

Note that, in the flat-space limit $\bar{\varphi}_k$ goes to the usual Minkowski form of the modes, $(W_k^{-1})^{(2)}$ goes to zero, and $\langle \varphi^2 \rangle_{\text{ren}}$ gives a vanishing result, in agreement with normal ordering. In a dynamical Universe, $(W_k^{-1})^{(2)}$ is generically nonzero and the renormalization produces a nontrivial, well-defined result for the physical variance.

For its physical relevance in our reevaluation of the inflationary power spectra, it is specially interesting to analyze the massless limit of the above result. When $m \to 0$, the subtraction counterterms take the form

$$(4\pi^2 a^3)^{-1} k^2 (w_k^{-1} + (W_k^{-1})^{(2)})$$

$$= (4\pi^2 a^3)^{-1} k^2 \left(\frac{a}{k} + (1 - 6\xi)\frac{a^3}{2k^3}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]\right). \qquad (3.16)$$

2. Comparison with the momentum-space representation of the DeWitt-Schwinger subtraction terms

Using a local momentum representation in a normal neighborhood admitting Riemann normal coordinates, one can express in momentum space the DeWitt-Schwinger proper time representation of the Green functions [17] (see also [9,16]). Up to second adiabatic order, we have

$$G_{DS}^{(2)}(P, Q) = -i\frac{|g(P)|^{1/4}}{(2\pi)^4} \int d^4 k e^{iky}$$

$$\times \left[\frac{1}{(k^2 + m^2)} + \frac{(\frac{1}{6} - \xi)R}{(k^2 + m^2)^2}\right], \qquad (3.17)$$

where $g(P)$ is the determinant of the metric at the point P in the so-called Riemann normal coordinates y^μ, $ky \equiv -k_0 y^0 + \vec{k}\,\vec{y}$, and the contour of integration in the k_0 plane is assumed to be the usual contour defining the particular two-point function. The point Q has been taken as the reference point for constructing the Riemann coordinates. These coordinates are constructed by considering the unique geodesic that joins the reference point with an arbitrary point P in a normal neighborhood of Q. The Riemann coordinates y^μ of P are given by

$$y^\mu = \lambda \xi^\mu, \qquad (3.18)$$

where λ is the value at P of an affine parameter of the geodesic joining Q, at $\lambda = 0$, to P. The vector ξ^μ is the tangent to the geodesic at the point Q

$$\xi^\nu = \frac{dx^\mu}{d\lambda}\bigg|_Q. \qquad (3.19)$$

Although in these coordinates the form of the geodesic equations is trivial, the form of the metric at Q is not necessarily Minkowskian. When we impose, additionally, that $ds^2|_Q = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2$, we have then the so-called Riemann normal coordinates. In our spatially flat Universe $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$ and for our problem, we take the initial and final points ($Q \equiv (t_0, \vec{x}')$, $P \equiv (t_0, \vec{x})$) at the constant time hypersurface $t = t_0$. We then have $y^0 = 0$ and

$$\vec{y} = a(t_0)(\vec{x} - \vec{x}'). \qquad (3.20)$$

Moreover, $g(P) = g(Q) = 1$.

The explicit form of $G_{DS}^{(2)}(P, Q)$ in our cosmological scenario reduces, after performing the k_0 integration, to

$$G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0) = I_1 - (\frac{1}{6} - \xi)RI_2, \qquad (3.21)$$

where R is the scalar curvature and

$$I_1 = \frac{1}{2(2\pi)^3} \int d^3\vec{k} \frac{e^{i\vec{k}(\vec{y}-\vec{y}')}}{w_{ka}}, \qquad (3.22)$$

$$I_2 = \frac{1}{2(2\pi)^3} \int d^3\vec{k} \frac{e^{i\vec{k}(\vec{y}-\vec{y}')}}{2w_{ka}^3}, \qquad (3.23)$$

with $w_{ka}^2 = k^2 + m^2$. Finally, performing the change of variables $\vec{k} \to \vec{k}/a(t_0)$ we get

$$G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0) = \frac{1}{2(2\pi)^3 a^3} \int d^3\vec{k}$$
$$\times \left[\frac{1}{w_k} + \frac{(\frac{1}{6} - \xi)R}{2w_k^3} \right] e^{i\vec{k}(\vec{x}-\vec{x}')}. \qquad (3.24)$$

In the DeWitt-Schwinger point-splitting framework, the renormalization of the variance $\langle \varphi^2(\vec{x}, t_0) \rangle$ proceeds by subtracting $G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0)$ to the two-point function and taking the coincidence-point limit $\vec{x}' \to \vec{x}$

$$\langle \varphi^2(\vec{x}, t_0) \rangle_{\text{ren}} = \lim_{\vec{x}' \to \vec{x}} [\langle \varphi(\vec{x}, t_0) \varphi(\vec{x}', t_0) \rangle - G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0)]. \qquad (3.25)$$

In the massless limit, $m \to 0$, the integrand of the momentum-space representation of $G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0)$ is

$$G_{DS}^{(2)}(\vec{x}, t_0; \vec{x}', t_0) = \frac{1}{2(2\pi)^3 a^3} \int d^3\vec{k} \left(\frac{a}{k} + (1 - 6\xi) \frac{a^3}{2k^3} \right)$$
$$\times \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] e^{i\vec{k}(\vec{x}-\vec{x}')}$$
$$= \frac{1}{4\pi^2 a^3} \int_0^\infty dk k^2 \left(\frac{a}{k} + (1 - 6\xi) \frac{a^3}{2k^3} \right)$$
$$\times \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right] \frac{\sin k|\vec{x} - \vec{x}'|}{k|\vec{x} - \vec{x}'|}. \qquad (3.26)$$

Taking $\vec{x} \to \vec{x}'$, the integrand of the above expression coincides exactly with the momentum-space counterterms obtained previously via the adiabatic renormalization. Therefore, the question ending Sec. III A is answered. The adiabatic subtraction coincides with the DeWitt-Schwinger subtraction prescription in momentum space in the massless limit, irrespective of the value of the dimensionless parameter ξ.

C. Tensorial spectrum

Let us apply the above scheme to the tensorial fluctuations of the metric. To determine the counterterm $(W_k^{-1})^{(2)}$ one should rewrite the wave equation (2.3) in the form (3.7). This can be easily obtained by performing the

change $\bar{h}_k = a^{3/2} h_k$. One obtains $w_k = k/a$ (i.e., $m = 0$) and a second-order adiabatic term σ of the form (3.9) with $\xi = 0$

$$\sigma = -\frac{3}{4} \left(\frac{\dot{a}}{a} \right)^2 - \frac{3}{2} \frac{\ddot{a}}{a}. \qquad (3.27)$$

A straightforward calculation gives

$$(W_k^{-1})^{(2)} = \frac{\dot{a}^2}{2a^2 w_k^3} + \frac{\ddot{a}}{2a w_k^3}. \qquad (3.28)$$

To perform the explicit computation it is useful to take into account the slow-roll relations

$$\frac{da}{d\tau} = \frac{(1 + 2\epsilon)}{(H\tau^2)}, \qquad (3.29)$$

$$\frac{d^2 a}{d\tau^2} = -\frac{(2 + 5\epsilon)}{(H\tau^3)}. \qquad (3.30)$$

These derivatives are related with the usual dotted derivatives as follows:

$$\dot{a} = a^{-1} \frac{da}{d\tau}, \qquad (3.31)$$

$$\ddot{a} = -a^{-3} \left(\frac{da}{d\tau} \right)^2 + a^{-2} \frac{d^2 a}{d\tau^2}. \qquad (3.32)$$

The final expression for zeroth and second-order adiabatic counterterms is then[2]

$$w_k^{-1} + (W_k^{-1})^{(2)} = \frac{a}{k} \left[1 + \frac{(2 + 3\epsilon)}{2k^2\tau^2} \right]. \qquad (3.33)$$

The subtraction of these counterterms produces the expression (assuming that $E(k) = 1$ and $F(k) = 0$ for every k)[3]

$$\langle h^2 \rangle_{\text{ren}} = \int_0^\infty \frac{dk}{k} \frac{16\pi G k^3 (-\tau\pi)}{4\pi^2 2a^2} \left(|H_\nu^{(1)}(-k\tau)|^2 \right.$$
$$\left. - \frac{2}{\pi(-k\tau)} \left[1 + \frac{(2 + 3\epsilon)}{2k^2\tau^2} \right] \right). \qquad (3.34)$$

Therefore, the renormalized expression for $\Delta_h^2(k, t)$ is

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[2]In a similar way, the DeWitt-Schwinger subtraction terms in momentum space lead to the same result. Note in passing that neither of the proposed (adiabatic) counterterms defined in [28] for the tensorial and scalar power spectra seem to agree with those obtained here by adiabatic regularization.

[3]Note that the second adiabatic counterterm introduces an infrared divergence in the continuum limit. For $k \to 0$ the integrand in $\langle h^2 \rangle_{\text{ren}}$ is proportional to dk/k. This logarithmic divergence disappears in the finite box formulation and has no impact on the observable power spectrum, in contrast to the ultraviolet divergences. Furthermore, the calculation of the renormalized stress-energy tensor $\langle T_{\mu\nu} \rangle$, and hence the evolution of the Universe, is insensitive to the infrared cutoff L when $L \to \infty$.

$$\tilde{\Delta}_h^2(k, t) = \frac{16\pi G k^3(-\tau\pi)}{4\pi^2 2a^2}\left(|H_\nu^{(1)}(-k\tau)|^2 - \frac{2}{\pi(-k\tau)}\right.$$

$$\left.\times\left[1 + \frac{(2 + 3\epsilon)}{2k^2\tau^2}\right]\right). \tag{3.35}$$

Note that the asymptotic behavior for large k of the Hankel function is

$$|H_\nu^{(1)}(-k\tau)|^2 \sim \frac{2}{\pi(-k\tau)}\left[1 + \frac{(4\nu^2 - 1)}{8(-k\tau)^2}\right] + \ldots. \tag{3.36}$$

The first two terms for $\nu = 3/2 + \epsilon$ are exactly the adiabatic counterterms subtracted above.

It is useful to remark that, in adiabatic renormalization (or, equivalently, in the DeWitt-Schwinger subtraction algorithm), the number of terms that should be subtracted is determined by the degree of divergence of the expectation value under consideration. In our case, this degree of divergence is of second order. Therefore, one must stop necessarily at second order if the purpose is to make finite the expectation value $\langle h^2 \rangle$. The relevant subtractions are carried out mode-by-mode and result in a sum-over-modes that has no ultraviolet divergence. In addition, as we have already stressed, it is important to point out that it is necessary to subtract the relevant adiabatic counterterms for all modes, including those with large wavelength, even if the expansion of the Universe is not slow. Note that the reason for the name "adiabatic" regularization is a result of the fact that the adiabatic (or arbitrarily slow) limit is used to identify the necessary subtractions. Therefore, it is not correct to argue, as in [28], that the adiabatic renormalization terms are invalid when the expansion is rapid, i.e., in the nonadiabatic regime. In fact, as noted before, the adiabatic counterterms coincide with those obtained in the DeWitt-Schwinger renormalization in momentum space (which identifies the counterterms in a different way), irrespective of the rate of the cosmological expansion.

D. Scalar spectrum

We will work out the (renormalized) spectrum of scalar perturbations using the gauge-invariant formalism. To convert Eq. (2.37) into one of the form (3.7) we have to perform the change $\bar{\mathcal{R}}_k = a^{1/2}z\mathcal{R}_k$. We then obtain that $w_k = k/a$, and that the second-order adiabatic function σ is given by

$$\sigma = -\frac{3}{4}\left(\frac{\dot{a}}{a}\right)^2 - \frac{3}{2}\frac{\ddot{a}}{a} + (3\eta - 6\epsilon)\left(\frac{\dot{a}}{a}\right)^2. \tag{3.37}$$

Note that this σ is different from that of (3.9). The reason for this can be easily seen by looking at the wave equation in the flat-slicing gauge (2.29). That equation does not have a constant massive term, and the second adiabatic order term is $H^2(3\eta - 6\epsilon)$, which is just the last term in (3.37). This implies that the scalar perturbation (the inflaton in the flat-slicing gauge or $\mathcal{R}$ in the gauge-invariant approach)

should be regarded as a massless field.[4] The corresponding second-order counterterm is thus of the form

$$(W_k^{-1})^{(2)} = \frac{\dot{a}^2}{2a^2 w_k^3} + \frac{\ddot{a}}{2a w_k^3} - \frac{1}{2w_k^3}(3\eta - 6\epsilon)\left(\frac{\dot{a}}{a}\right)^2. \tag{3.38}$$

Therefore, the final expression for the counterterms is

$$w_k^{-1} + (W_k^{-1})^{(2)} = \frac{a}{k}\left[1 + \frac{(2 + 3(3\epsilon - \eta))}{2(-k\tau)^2}\right]. \tag{3.39}$$

Proceeding in a parallel way as for the tensorial case, we get a renormalized value for $\langle\mathcal{R}^2\rangle$

$$\langle\mathcal{R}^2\rangle_{\text{ren}} = \int_0^\infty \frac{dk}{k}\left[4\pi k^3|\mathcal{R}_k|^2\right.$$

$$\left. - \frac{k^3}{4\pi^2 z^2 a}(w_k^{-1} + (W_k^{-1})^{(2)})\right]. \tag{3.40}$$

Therefore,

$$\langle\mathcal{R}^2\rangle_{\text{ren}} = \int_0^\infty \frac{dk}{k}4\pi k^3\frac{-\pi\tau}{4(2\pi)^3 z^2}\left(|H_\mu^{(1)}(-\tau k)|^2\right.$$

$$\left. - \frac{2}{\pi(-k\tau)}\left[1 + \frac{(2 + 3(3\epsilon - \eta))}{2(-k\tau)^2}\right]\right), \tag{3.41}$$

where the index μ is given by $\mu = 3/2 + 3\epsilon - \eta$. The renormalized expression for $\Delta_\mathcal{R}^2(k, t)$ is

$$\tilde{\Delta}_\mathcal{R}^2(k, t) = \frac{4\pi k^3(-\pi\tau)}{4(2\pi)^3 z^2}\left(|H_\mu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}\right.$$

$$\left.\times\left[1 + \frac{(2 + 3(3\epsilon - \eta))}{2(-k\tau)^2}\right]\right). \tag{3.42}$$

Note finally that, as expected, the same result is obtained working in the flat-slicing gauge. One then obtains

$$\tilde{\Delta}_\phi^2(k, t) = \frac{k^3(-\tau\pi)}{4\pi^2 2a^2}\left(|H_\mu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}\right.$$

$$\left.\times\left[1 + \frac{(2 + 3(3\epsilon - \eta))}{2(-k\tau)^2}\right]\right), \tag{3.43}$$

where $\mu = 3/2 + 3\epsilon - \eta$. Taking into account that $\tilde{\Delta}_\mathcal{R}^2(k, t) = (a^2/z^2)\tilde{\Delta}_\phi^2(k, t)$ one recovers the expression (3.42) for $\tilde{\Delta}_\mathcal{R}^2(k, t)$.

IV. TESTABLE CONSEQUENCES

In this section we work out new expressions for the observable magnitudes that follow from the renormalized power spectra of the previous section.

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[4]This was first taken into account in the calculation of the renormalized power spectra in [7,29] in terms of the gauge-invariant quantity $\mathcal{R}$. This quantity was also quantized in [30], but there the V'' term in (2.27) was treated as zeroth adiabatic order, unlike the present treatment, in the evaluation of the counterterms.

A. Tensor-to-scalar ratio

Let us now consider the tensor-to-scalar ratio

$$r \equiv 4 \frac{\tilde{\Delta}_h^2(k, t)}{\tilde{\Delta}_\mathcal{R}^2(k, t)}$$

$$= 4 \frac{16\pi G k^3 (-\tau\pi)}{4\pi^2 2a^2} \frac{4(2\pi)^3 z^2}{4\pi k^3 (-\pi\tau)}$$

$$\times \frac{(|H_\nu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}[1 + \frac{(2+3\epsilon)}{2(-k\tau)^2}])}{(|H_\mu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}[1 + \frac{(2+3(3\epsilon - \eta))}{2(-k\tau)^2}])}. \quad (4.1)$$

Since at the Hubble exit time t_k we have $z^2(t_k) = 2M_P^2\epsilon(t_k)a^2(t_k)$, it follows that

$$4 \frac{16\pi G k^3 (-\tau\pi)}{4\pi^2 2a^2} \frac{4(2\pi)^3 z^2}{4\pi k^3 (-\pi\tau)} \bigg|_{t=t_k} = 16\epsilon(t_k). \quad (4.2)$$

We also find that

$$\left(|H_\nu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}\left[1 + \frac{(2+3\epsilon)}{2(-k\tau)^2}\right]\right)\bigg|_{t=t_k}$$

$$= \frac{2}{\pi}\alpha\epsilon(t_k), \quad (4.3)$$

where α is a numerical constant of order unity, $\alpha \approx 0.904$. A similar estimation is obtained for the factor coming from the scalar spectrum (up to the substitution $\nu \to \mu$, i.e., $\epsilon \to 3\epsilon - \eta$)

$$\left(|H_\mu^{(1)}(-\tau k)|^2 - \frac{2}{\pi(-k\tau)}\left[1 + \frac{(2+3(3\epsilon - \eta))}{2(-k\tau)^2}\right]\right)\bigg|_{t=t_k}$$

$$= \frac{2}{\pi}\alpha(3\epsilon(t_k) - \eta(t_k)). \quad (4.4)$$

Therefore, the tensor-to-scalar ratio r, evaluated at the Hubble exit time, is

$$r = 16\epsilon(t_k) \frac{\epsilon(t_k)}{3\epsilon(t_k) - \eta(t_k)}. \quad (4.5)$$

An important comment is now in order. To obtain this result we have evaluated the power spectra at the Hubble exit time. Nevertheless, as we will show in the next paragraph, this estimate does not depend critically on the precise time at which the counterterms are evaluated. During slow-roll inflation the counterterms decay as $|\tau|^{2\epsilon}$ (tensorial), $|\tau|^{2(3\epsilon - \eta)}$ (scalar), in the late-time limit as $|\tau| \to 0$. This means that they decay very slowly and are in fact constant in exact de Sitter inflation. After reheating, the counterterms decay more rapidly, as is obvious from (3.16). One would then recover the standard prediction $r = 16\epsilon(t_k)$. However, since the modes acquire classical properties soon after exiting the Hubble sphere, the relevant time to evaluate these magnitudes falls in the interval between t_k and a few e foldings after it.

Let us analyze in detail the time dependence of $\tilde{\Delta}_h^2(k, t)$, $\tilde{\Delta}_\mathcal{R}^2(k, t)$, and r in terms of the number n of e folds after

crossing the Hubble radius. Since $-k\tau(t_k) = 1 + \epsilon$, we can write $-k\tau$ as

$$-k\tau = (1 + \epsilon)\frac{\tau}{\tau(t_k)} = (1 + \epsilon)\left(\frac{a(t_k)}{a}\right)^{1/(1+\epsilon)}$$

$$= (1 + \epsilon)e^{-(n/(1+\epsilon))}. \quad (4.6)$$

The expression for $\tilde{\Delta}_h^2(k, t)$ when $-k\tau \ll 1$ leads to

$$\tilde{\Delta}_h^2(k, t) \approx \frac{16\pi G k^3 (-\tau\pi)}{4\pi^2 2a^2} \frac{2}{\pi}(-k\tau)^{-2\nu}$$

$$\times \left[1 - (-k\tau)^{2\nu}\frac{(2+3\epsilon)}{2(-k\tau)^3}\right]$$

$$\approx \frac{2}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2\left[1 - (-k\tau)^{2\nu}\frac{(2+3\epsilon)}{2(-k\tau)^3}\right], \quad (4.7)$$

and using (4.6), we find

$$\tilde{\Delta}_h^2(k, n) \approx \frac{2}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2\left[1 - \frac{(2+3\epsilon)}{2}e^{-2\epsilon n}\right]. \quad (4.8)$$

Assuming that we are just a few e folds after the Hubble exit but before the end of inflation, i.e., $n > 1$ but $n\epsilon \ll 1$, so that $e^{-2\epsilon n} \approx 1 - 2\epsilon n$, we obtain[5]

$$\tilde{\Delta}_h^2(k, n) \approx \frac{2}{M_P^2}\left(\frac{H(t_k)}{2\pi}\right)^2\epsilon(t_k)(2n - 3/2). \quad (4.9)$$

A similar estimation can be obtained for the scalar power spectrum

$$\tilde{\Delta}_\mathcal{R}^2(k, n) \approx \frac{1}{2M_P^2\epsilon(t_k)}\left(\frac{H(t_k)}{2\pi}\right)^2(3\epsilon(t_k) - \eta(t_k))(2n - 3/2). \quad (4.10)$$

Note that the parameter n enters in the power spectra parameterizing the (unknown) time at which the modes exhibit classical behavior. However, since both tensorial and scalar spectra have the same dependence on n, the tensor-to-scalar ratio is not sensitive to the unknown parameter n, in the same way as it is insensitive to the scale of inflation $H(t_k)$, and it is essentially given by (4.5). The same conclusion can be drawn if one estimates the time derivative of r. One then obtains that

$$\frac{dr}{dn} \approx O(\epsilon^2). \quad (4.11)$$

This means that the renormalized value for r is changing very slowly, in agreement with the previous evaluation. In other words, the value of r evaluated at the Hubble radius crossing time remains nearly constant during this period of inflation. Obviously it is not strictly constant, as in the

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[5]We note that if one evaluates the power spectra at the end of the slow-roll era (where $n\epsilon \sim 1$) the contribution of the counterterms is still significant. However, we find it more natural to evaluate the spectra soon after Hubble exit, when the modes have already acquired classical properties.

computation without renormalization, but its change is slow.[6] We regard this result as a signal of the robustness of renormalization in determining the spectra of inflationary perturbations.

B. Spectral indices

The above calculations have another important consequence: the spectral indices remain unchanged when they are evaluated at t_k or a few Hubble times after it. By definition, we have

$$n_t \equiv \frac{d \ln P_t}{d \ln k} = 2 \frac{d \ln H(t_k)}{d \ln k} + \frac{d \ln \epsilon(t_k)}{d \ln k}, \qquad (4.12)$$

which, according to (4.9), turns out to be independent of n. The result is

$$n_t = 2(\epsilon - \eta), \qquad (4.13)$$

where ϵ and η are evaluated at t_k. For the scalar index we have

$$\begin{aligned} n_s - 1 &\equiv \frac{d \ln P_{\mathcal{R}}}{d \ln k} \\ &= 2 \frac{d \ln H(t_k)}{d \ln k} - \frac{d \ln \epsilon(t_k)}{d \ln k} + \frac{d \ln(3\epsilon(t_k) - \eta(t_k))}{d \ln k}. \end{aligned}$$
$$(4.14)$$

Explicit evaluation gives

$$n_s - 1 = -6\epsilon + 2\eta + \frac{(12\epsilon^2 - 8\epsilon\eta + \xi)}{3\epsilon - \eta}, \qquad (4.15)$$

where ξ is another slow-roll parameter [10]: $\xi \equiv M_P^4 (V'V'''/V^2)$. This parameter can be re-expressed in terms of ϵ, η and the running of the tensorial index $n_t' \equiv dn_t/d \ln k$ as follows:

$$n_t' = 8\epsilon(\epsilon - \eta) + 2\xi. \qquad (4.16)$$

C. Consistency condition

The formulas (4.5), (4.13), (4.14), (4.15), and (4.16) provide an algebraic relation between the tensor-to-scalar ratio and the spectral indices, $r = r(n_t, n_s, n_t')$, that takes the following form:

$$\begin{aligned} r = 4(1 - n_s - n_t) &+ \frac{4n_t'}{n_t^2 - 2n_t'} \\ &\times (1 - n_s - \sqrt{2n_t' + (1 - n_s)^2 - n_t^2}). \end{aligned} \qquad (4.17)$$

Positivity of the argument of the square root in this expression imposes the following constraint

$$(1 - n_s)^2 - n_t^2 + 2n_t' \geq 0. \qquad (4.18)$$

The consistency relation (4.17), contrasts with that obtained without invoking renormalization, namely, $r = -8n_t$. Since r must be positive, according to the standard prediction one would expect n_t to be negative. This restriction, however, does not follow either from (4.17) nor from (4.18). Even more, from (4.18) it is easy to see that our prediction allows for $n_t \geq 0$, in sharp contrast with the standard prediction, which necessarily requires $n_t = -2\epsilon < 0$. We also remark that, according to the standard derivation, the running of n_t is fully determined by the values of n_s and n_t, since then one finds $n_t' = -n_t(1 - n_s + n_t)$. On the contrary, the manipulations that led to (4.17) indicate that n_t' is now an independent quantity that needs to be measured in order to check the new consistency relation (4.17). This aspect could make more challenging the experimental verification of the consistency condition of (single-field) slow-roll inflation.

We stress again that the new consistency relation allows for a null tensorial tilt $n_t = 0$ while being compatible with a nonzero ratio $r \approx 4(1 - n_s)$. Since the observations from the 5-year WMAP [3](with BAO+SN) strongly suggest that $(1 - n_s) \approx 0.030 \pm 0.015$ (with $r < 0.22$)[7] it follows that, for the case of an exact scale-invariant tensorial power spectrum, $n_t = 0$, our consistency relation leads to $r \approx 0.12 \pm 0.06$. These possibilities may soon come within the measurement range of forthcoming CMB polarization experiments such as the PLANCK satellite, the QUIJOTE CMB experiment, SPIDER, Polar BEAR, EBEX, BICEP, SPUD, and the future CMBPol mission [31].

D. Comparison between the standard and the new predictions

We can compare the differences between the standard and the new predictions in terms of the representative set of chaotic potential models $V(\phi) = \lambda M_P^{(4-p)} \phi^p$. The standard prediction is

$$\begin{aligned} r = \frac{4p}{N}, &\qquad 1 - n_s = \frac{(p + 2)}{2N}, \\ n_t = -\frac{p}{2N}, &\qquad n_t' = \frac{p}{2N^2}, \end{aligned} \qquad (4.19)$$

where $N \equiv \ln a_{\text{end}}/a(t_k)$ is the number of e folds of inflation between the horizon crossing time t_k of cosmological wavelengths and the end of inflation. If we invoke renormalization we get, instead,

$$\begin{aligned} r = \frac{4p^2}{(p + 2)N}, &\qquad 1 - n_s = \frac{p}{2N}, \\ n_t = \frac{(2 - p)}{2N}, &\qquad n_t' = \frac{(2 - p)}{2N^2}. \end{aligned} \qquad (4.20)$$

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[6]To be precise, $d(\ln r)/dn = 2\epsilon - \eta$. Note that for the exponential potential model $V(\phi) \propto \exp[-(\phi/M_P)\sqrt{2/p}]$ we have $\eta = 2\epsilon$. Then the new predictions coincide, by accident, with the standard ones. So in this case $\dot{r} = 0$.

[7]Marginalized over all other parameters of a flat ΛCDM model.

Note that now the quadratic potential is characterized by having an exact scale-invariant behavior for the tensorial spectrum.

E. Comparison with WMAP data

Although a definite test of the new predictions, like the above proposed consistency condition, requires more accurate data on tensor fluctuations (so far we have only upper limits on r) we can compare the new predictions with the standard ones on the basis of the 5 yr WMAP data. We can contrast the predictions (4.20) for r and n_s, for the representative values $p = 2$ and $p = 4$, with the WMAP 5-year data (see Fig. 5 of Ref. [3]). We find that both models are compatible with the experimental data for the reasonable range of N between 50 and 60. This is in sharp contrast with the prediction of the standard approach, where the monomial potential with $p = 4$ is excluded convincingly.

V. CONCLUSIONS AND FINAL COMMENTS

Inflationary cosmology predicts that, due to quantum effects, small density perturbations are generated in the very early Universe with a nearly "scale-free" spectrum. The detection and analysis of anisotropies in the cosmic microwave background has confirmed this prediction. Moreover, inflation also predicts the creation of primordial gravitational waves, which still remain undetectable. Forthcoming high-precision measurements of the cosmic microwave background [31] may measure effects of relic gravitational waves, and this will be crucial to test the inflationary paradigm and strongly constrain inflationary models. Therefore, it is of crucial importance to scrutinize, from all points of view, the quantitative predictions of inflation.

In this work we have pointed out that, if quantum field renormalization is taken into account, the predictions of (single-field) slow-roll inflation for both the scalar and tensorial power spectra change significantly. In our physical context, renormalization is naturally implemented in a mode-by-mode subtraction scheme and uniquely defines the momentum-space counterterms. These counterterms are evaluated in the period when the perturbations acquire classical properties and this leads to testable predictions that differ significantly from the standard ones. Because of the question of the underlying quantum nature of the gravitational field, there are several ways one could think about the process by which the dispersion spectrum of the inflaton field influences the spectrum of the scalar perturbations of the metric. One way is to think of the gravitational field as making a measurement of each mode of the inflaton fluctuation field within a few Hubble times of its exit from the Hubble sphere. If this measurement of the inflaton dispersion spectrum is similar to standard measurements in quantum mechanics, then it should measure the renormalized value of the inflaton spectrum at the time when the measurement is carried out, which we take to be the time when the perturbations acquire significant classical properties. The time t at which this occurs affects both the scalar and tensorial power spectra, but for a large range of values of t, the effect of renormalization remains significant for wavelengths that today are at observable scales. We have shown that if the power spectra are evaluated n e folds after the Hubble radius exit time, where $n \ll 1/\epsilon$, then the observable parameters r and n_s, n_t, n_t' are insensitive to the value of n. If the power spectra were evaluated at a time t well after the end of inflation, one would recover the standard predictions. With the present understanding of the nonlinear aspects of quantum gravity, it is difficult to reach a definitive answer regarding the value of n, so the fact that we find observable differences offers a deep way to experimentally probe this question.

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