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Stochastic Maximum Principle for the System Governed by Backward Doubly Stochastic Differential Equations with Risk-Sensitive Control Problem and Applications

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To all my family

and my best friends

and my dear daughter

★★ AYA Elrahman ★★

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Contents

Abstract

This thesis based on the study of the stochastic maximum principle with risk-sensitive for two different systems. We obtain these systems by generalizing the results of Chala [10, 11], and by using the paper of Djehiche et al. in [13]: The first system is driven by a backward doubly stochastic differential equation. We use the risk-neutral model for which an optimal solution exists as a preliminary step, this is an extension of the initial control problem. Our goal is to establish necessary and sufficient optimality conditions for the risk-sensitive performance functional control problem. We show for the second system which is driven by a fully coupled forward-backward stochastic differential equation of mean-field type, by using the same technique as in the first case, we get the necessary and sufficient optimality conditions for the risk-sensitive, where the set of admissible controls is convex in all the cases. Finally, we illustrate our main results by giving applied examples of risk-sensitive control problems.

Key words: Backward doubly stochastic differential equation, fully coupled forward-backward stochastic differential equation of mean-field, risk-sensitive, stochastic maximum principle, variational principle, Logarithmic transformation.

Résumé

Cette thèse est intéressée par étude le principe du maximum stochastique avec sensible au risque pour deux systèmes différents. Nous obtenons ces systèmes par la généralisation des résultats de Chala [10, 11], et en utilisant le papier de Djehiche et al. dans [13]: Le premier système est basé sur une équation différentielle stochastique doublement rétrograde. Nous utilisons le modèle sans risque pour lequel une solution optimale existe comme une phase préliminaire, il s'agit d'une étape du système de contrôle initial pour ce type de problème. Notre objectif est d'établir les conditions d'optimalité nécessaires ainsi que suffisantes pour problème du contrôle fonctionnel de la performance sensible au risque. De plus, nous montrons que le deuxième système est basé sur une équation différentielle stochastique progressivement rétrograde totalement couplée de type champ moyen, en appliquant la même technique qui a utilisé dans le premier cas, nous obtenons les conditions d'optimalité nécessaires ainsi que suffisantes pour le sensible au risque, où un ensemble de contrôles admissibles est convexe dans les deux cas. Finalement, nous illustrons nos principaux résultats en donnant des exemples appliqués des problèmes de contrôle sensible au risque.

Mots clés: Équation différentielle stochastique doublement rétrograde, équation différentielle stochastique progressivement rétrograde totalement couplée de type champ moyen, sensible au risque, principe du maximum stochastique, transformation Logarithmique.

Symbols and Abbreviations

The different symbols and abbreviations used in this thesis

Symbols

$\mathbb{R}$	:	Real numbers.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$\{\mathcal{F}_t\}_{t\in[0,T]}$	:	Filtration.
$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$	:	Filtered probability space.
$\mathbb{P}$	:	Probability measure with respect to risk-neutral.
$\mathbb{P}^{\theta}$	:	Probability measure with respect to risk-sensitive.
$\mathbb{E}(\cdot \mid \mathcal{G}_t)$	:	Conditional expectation.
W, B	:	Brownian motion.
W^{θ}, B^{θ}	:	$\mathbb{P}^{\theta}-$ Brownian motion.
M	:	Vectoriel Brownian motion.
M^{θ}	:	$\mathbb{P}^{\theta}-$ Vectoriel Brownian motion.
$m(d\lambda)\,dt$	:	The compensator of N.
N	:	A Poisson random measure.
$\mathcal{N}$	:	The totality of the $\mathbb{P}-$null sets.
θ	:	Risk-sensitive index.

$\mathcal{U}$	:	The set of admissible strict controls.
U	:	The set of values taken by the strict control v.
v	:	Admissible control.
u	:	Optimal strict control.
$J^{\theta}(.)$	:	The cost function with risk-sensitive.
$\widetilde{H}^{\theta}$	:	Represent the risk-neutral Hamiltonian.
H^{θ}	:	Represent the risk-sensitive Hamiltonian.

Abbreviations

$SDEs$	:	Stochastic differential equations.
$BSDEs$	:	Backward stochastic differential equations.
$FBSDEs$	:	Forward-backward stochastic differential equations.
$BDSDEs$	:	Backward doubly stochastic differential equations.
$cadl\grave{a}g$	:	Right continuous with left limits
$HARA$	:	Hyperbolic absolute risk aversion.
SMP	:	Stochastic maximum principle.
$a.e$	:	Almost everywhere.
$a.s$	:	Almost surely.
$e.g$	:	For example.
$i.e$	:	It means.

General Introduction

The application of stochastic processes is mainly inspired by the subject of physics, economy, biology, games theory...

The history can be traced too early, in 1827 botanist Brown [5] published his observation about micro objects that pollen particles suspended on the surface of the water will traverse continuously in an unpredictable way.

After that, in 1905 Einstein [14] developed a physics model to support his statement that atoms exist, that means he used the notion of Brownian motion to describe the physics investigation and proved that the position of a particle can be followed by some normal distribution. Unfortunately, the mathematical description is not very correct given because of mathematicians.

Besides the works of Einstein, in 1923 Wiener [37] did provide a correct mathematical definition of the stochastic process observed by Brown and described by Einstein, which is the Brownian motion that we used.

In probability theory, in 1960 the Girsanov's Theorem (named after Igor Vladimirovich Girsanov) describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure see [16]. The theorem is especially important in the theory of financial mathematics as it tells how to convert from the physical measure, which describes the probability that an underlying instrument (such as a share price or interest rate) will take a particular value or values, to the risk-neutral measure which is a very useful tool for pricing derivatives on the underlying instrument.

Let U be a nonempty subset of $\mathbb{R}$. An admissible control v is a measurable process with values in U such that $\mathbb{E}\int_0^T |v_t|^2 \, dt < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.

The adapted solution for a linear backward stochastic differential equation (BSDE in short) which appears as the adjoint process for a stochastic control problem was first investigated by Bismut [4] in 1973, then by Bensoussan [3] in 1982, and others, while the first result for the existence of an adapted solution to a continuous nonlinear BSDE with Lipschitzian coefficient was obtained by Pardoux and Peng [30] in 1990. Later Peng and Pardoux developed the theory and applications of continuous BSDEs in a series of papers [31, 32, 33] under the assumption that the coefficients satisfy the globally or locally Lipschitzian condition but with some additional conditions.

Concerning mean-field backward stochastic differential equations (mean-field BSDEs), have been first studied by Buckdahn et al. in 2009, the interested reader is referred to [7, 8], the purpose of paper Carmona and Delarue [9] in 2013 was to provide an existence result for the solution of fully coupled FBSDE of the mean-field type. Mathematical mean-field approaches play a crucial role in diverse areas, such as physics, chemistry, economics, finance, and game theory, see for example Lasry and Lions [23] in 2007. Many papers have been studying the problem of mean-field and established the stochastic maximum principle, we can cite here some of them, the first work gave the necessary optimality conditions was Bukdahn et al. [6] in 2011, after this work many authors have generalized this problem into the other fields of applications, as the paper of Anderson et al. [1] in 2011 they have studied the problem of the mean-field type of SDE under the assumptions of convex action space. Besides, the problem of mean-field has been derived also via Malliavin calculus, by Meyer-Brandis et al. [26] in 2010 the stochastic maximum principle of mean-field have been obtained, also to the problem of singular mean-field with a good application to finance we can have the paper of Hu et al. [22] in 2014. The paper of Li [24] in 2012, she has investigated a large extension that is different from the classical ones to the mean-field system with an application to the linear quadratic problem.

In this thesis, we aim by using the Pontryagin's maximum principle to prove a necessary and sufficient optimality conditions for risk-sensitive control problem associated with dynamics driven by many systems. We solve these problems by using the approach developed by Djehiche et al. [13] in 2015, and the results of Chala [10, 11] in 2017. Our contribution can be summarized as follows, in the first paper they have established a stochastic maximum principle for a class of risk-sensitive mean-field type control problems, where the distribution enters only through the mean of state process, it means that the drift, diffusion, and terminal cost functions depend on the state, the control and the means of state process. Their work extends the results of Lim and Zhou [25] in 2005 to risk-sensitive control problems for dynamics that are non-Markovian and without mean-field term. An SMP for risk-sensitive optimal control problems for Markov diffusion processes with an exponential of integral performance functional was obtained by Lim and Zhou [25] in 2005, by making the relationship between the SMP and the Dynamic Programming Principle, the authors have used the first order adjoint process as the gradient of the value function of the control problem. This relationship holds only when the value function is smooth (see Lim and Zhou [25] in 2005 Assumption $B4$). By using the smoothness assumption (see the papers of [35, 36]), have been using the approach used above, but extended it into the jump diffusion.

In the first work published [19]: Nonlinear backward doubly stochastic differential equations (in short BDSDEs) has been introduced by Pardoux and Peng [31] in 1994, they have considered a new kind of BSDEs, that is a class of BDSDEs with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic Itô integral and a backward stochastic Itô integral.

About the system is governed by BDSDE, we will generalize the results obtained by Chala [10, 11] in 2017, to the BDSDE. The idea here is to reformulate in the first step the risk-sensitive control problem in terms of an augmented state process and terminal payoff problem. An intermediate stochastic maximum principle (SMP in short) is then obtained by applying the SMP of [2, 21] for loss functional without running cost, and for the same particular cases see [18] in 2019. Then, we transform the intermediate adjoint processes to a simpler form by using the fact that the set of controls is convex. Then, we establish necessary and sufficient optimality conditions see Chapter 2.

We note that necessary and sufficient optimality conditions for risk-sensitive controls, where the systems are governed by a stochastic differential equation (SDE in short), has been studied by Lim and Zhou [25] in 2005. We also note that necessary and sufficient optimality conditions for stochastic controls, where the systems are governed by nonlinear forward stochastic differential equation with jumps, have been studied by Shi and Wu [35] in 2011, in the case where the set of admissible controls is convex, and Shi and Wu [36] in 2012, in general case with application to finance. Furthermore, the systems are governed by a mean-field SDE, have been studied by Djehiche et al. [13] in 2015.

In the second work published [20]: We will generalize the results obtained by Chala [10, 11] in 2017, to the system governed by the fully coupled forward-backward stochastic differential equation of mean-field type (fully coupled FBSDE of mean-field type in short). The existence of an optimal solution for this problem has been solved to achieve the objective of this work and establish necessary and sufficient optimality conditions for this model.

Firstly, we give -without proof- the optimality conditions for risk-neutral controls as a preliminary step. The idea is to use an auxiliary state process which is a solution of some SDE, and we will transfer our system with two equations the first one is SDE, whereas the second is BSDE, into the system governed by three stochastic differential equations that the set of risk-neutral controls is convex. Then, the adjoint equations with respect to these three equations were given, the proof is a combination between the work of Min et al. [27] in 2014 and the work of Yong and Zhou [38, 39], the transformation of the adjoint equations will be use as the best approach, we suggest this transformation to remove the first adjoint equation. Necessary and sufficient optimality conditions have been established with respect only of the second and the third adjoint equations by using the classical way of the Logarithmic transformation method see Chapter 1, the necessary optimality conditions are obtained directly in the global form.

This thesis is organized as follows:

In the first chapter, the project of this chapter has been considered as a published paper by [12], we introduce basic notations of expected exponential utility and related field.

In the second chapter, the project of this chapter has been considered as a published paper by [19]. We establish necessary and sufficient optimality conditions where the system is given by a BDSDEs, to find necessary and sufficient optimality conditions for risk-sensitive. Finally, we improve the quality of the chapter by given two applications to linear quadratic stochastic control problem, the method which used the Riccati equations is applied in the second example.

In the third chapter, the project of this chapter has been considered as a published paper by [20]. We shall study our system of fully coupled FBSDEs of mean-field type, to find necessary and sufficient optimality conditions for risk-sensitive. We finish this chapter by given two applications, the linear quadratic stochastic control problem with risk-sensitive performance function is the first application a financial model of mean-variance with risk-sensitive performance functional is the best application for our problem.

The content of this thesis is the subject of the following works:

(1) A. Chala, D. Hafayed and R. Khallout, The use of Girsanov's theorem to describe the risk-sensitive problem and application to optimal control. In T. D. Deangelo, Stochastic differential equation-basics and applications-. Nova Science Publishers, Inc. (2018) , $111-142$.

(2) D. Hafayed and A. Chala, A general maximum principle for a mean-field forward-backward doubly stochastic differential equations with jumps processes, Random Operators and Stochastic Equations $27\,(1)$ (2019) , $9-25$.

(3) D. Hafayed and A. Chala, An optimal control of a risk-sensitive problem for backward doubly stochastic differential equations with applications, Random Operators and Stochastic Equations $28\,(1)$ (2020), $1-18$.

(4) D. Hafayed and A. Chala, On stochastic maximum principle for risk-sensitive of fully coupled forward-backward stochastic control of mean-field type with application, To be appeared in Journal of Evolution Equation and Control Theory (2020), doi:10.3934/eect.2020035.

(5) D. Hafayed and A. Chala, On the risk-sensitive stochastic maximum principle of backward with jump, Accepted to the Journal of Applied Pharmaceutical Science (2020).

CHAPTER 1

Basic Notations of Expected Exponential Utility and Related Field

CHAPTER 1

Basic Notations of Expected Exponential Utility and Related Field

In this chapter, we develop the general framework used in our papers [10, 11, 12, 19, 20]. We will demonstrate in detail our important lemma which explains the relation between the expected exponential utility and the quadratic backward stochastic differential equation, and this result plays an important role in my thesis. The next point for the discussion will be the standard risk-sensitive structures, and how constructions of this kind can be given a rigorous treatment. We investigate in this chapter the financial market of risk-sensitive for the dynamic diffusion, by using Girsanov's Theorems, and in virtue of Itô's formula.

1.1 Problem Formulation

Let T be a positive real number. Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\in[0,T]}, \mathbb{P}\right)$ be a probability space satisfying the usual conditions, in which a $one-$dimensional Brownian motion $W = (W_t, 0 \leq t \leq T)$ is defined. We assume that $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ is defined by $\forall t \geq 0,\ \mathcal{F}_t^W = \sigma\left(W_s,\ \text{for any } s \in [0,t]\right) \vee \mathcal{N}$, where $\mathcal{N}$ denote the totality of $\mathbb{P}-$null sets.

Let $\mathcal{M}^2\left([0,T], \mathbb{R}\right)$ denote the set of $one-$dimensional jointly measurable random processes $\{\varphi_t, t \in [0,T]\}$ which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{M}^2([0,T],\mathbb{R})} = \mathbb{E}\left[\int_0^T |\varphi_t|^2\, dt\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

We denote similarly by $\mathcal{S}^2\left([0,T], \mathbb{R}\right)$ the set of continuous $one-$dimensional random processes which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{S}^2([0,T],\mathbb{R})} = \mathbb{E}\left[\sup_{0\leq t\leq T} |\varphi_t|^2\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

Let U be a nonempty subset of $\mathbb{R}$.

1.2 Expected Exponential Utility

In this part, we want to prove the relationship between the expected exponential utility and the quadratic backward stochastic differential equation.

We require the following condition

$$A_{t,T}^{\theta} := \exp\theta\left\{\Psi\left(y_0^v\right) + \int_t^T l\left(s, y_s^v, v_s\right) ds\right\},$$

where $l : [0,T] \times \mathbb{R} \times U \to \mathbb{R}$, $\Psi : \mathbb{R} \to \mathbb{R}$.

We assume the following

$(\mathbf{N_1})$

$i)$ l and Ψ are continuously differentiable with respect to (y^v, v) .

$ii)$ The derivative of l is bounded by $C(1 + |y^v| + |v|)$.

$iii)$ The derivative of Ψ is bounded by $C(1 + |y^v|)$.

We denote by $l(t) := l(t, y_t^v, v_t)$.

First of all, it is very important to write the expected exponential utility under this form

$$\exp\left(\theta\Lambda_t^\theta\right) = \mathbb{E}\left[A_{t,T}^\theta \mid \mathcal{F}_t^W\right] = \mathbb{E}\left[\exp\theta\left\{\Psi\left(y_0^u\right) + \int_t^T l(s)\, ds\right\} \mid \mathcal{F}_t^W\right], \tag{1.1}$$

where θ is the risk-sensitive index, the process Λ^θ is the first component of the $\mathcal{F}_t^W-$adapted pair of processes $\left(\Lambda^\theta, \mathcal{D}\right)$ which is the unique solution to the following quadratic backward stochastic differential equation

$$\begin{cases} d\Lambda_t^\theta &= -\left(l(t) + \dfrac{\theta}{2}\left|\mathcal{D}(t)\right|^2\right) dt + \mathcal{D}(t)\, dW_t, \\ \Lambda_T^\theta &= \Psi\left(y_0^u\right), \end{cases} \tag{1.2}$$

where $\mathbb{E}\left[\int_0^T \left|\mathcal{D}(t)\right|^2 dt\right] < \infty.$

We also assume the following

$(\mathbf{N_2})$

$i)$ The process $\mathcal{D}(t)$ is $\mathcal{F}_t^W$ $-$ measurable with value in $\mathbb{R}$ such that $\mathbb{E}\left[\int_0^T \left|\mathcal{D}(t)\right|^2 dt\right] < \infty.$

$ii)$ The process $\left(\Lambda_t^\theta\right)_{t\geq 0}$ is $\mathbb{P}-$measurable uniformly bounded i.e. there exists a constant $C \geq 0$ such that $\mathbb{P} - a.s.$, $\mathbb{E}\left[\sup_{t\in[0,T]} \left|\Lambda_t^\theta\right|\right] \leq C.$

The following Lemma shows the relationship between the expected exponential utility and the quadratic backward stochastic differential equation.

Lemma 1.2.1 *We assume that* $\mathbf{N_1} - \mathbf{N_2}$ *hold. The necessary and sufficient condition for the expected exponential utility* (1.1) *to be hold, is the quadratic backward stochastic differential equation* (1.2).

Proof. We assume that (1.1) holds, then we have

$$\begin{aligned}
\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\} &= \mathbb{E}\left[\exp\theta\left\{\Psi\left(y_0^u\right)+\int_t^T l(s)\,ds+\int_0^t l(s)\,ds\right\}\mid\mathcal{F}_t^W\right]\\
&= \mathbb{E}\left[\exp\theta\left\{\Psi\left(y_0^u\right)+\int_0^T l(s)\,ds\right\}\mid\mathcal{F}_t^W\right]\\
&= \mathbb{E}\left[A_T^{\theta}\mid\mathcal{F}_t^W\right].
\end{aligned}$$

By Assumption ($\mathbf{N_1}$), we know that A_T^{θ} is the square integrable and $\mathbb{E}\left[A_T^{\theta}\mid\mathcal{F}_t^W\right]$ is a square integrable martingale, such that $\mathcal{F}_t^W=\sigma\left(W_s,\ \text{for any } s\in[0,t]\right)$, by using the martingale representation Theorem, there exist a unique square integrable process φ with respect to norm $\|.\|_{\mathcal{M}^2([0,T],\mathbb{R})}$ such that

$$\mathbb{E}\left[A_T^{\theta}\mid\mathcal{F}_t^W\right]-\mathbb{E}\left[A_T^{\theta}\right]=\int_0^t\varphi(s)\,dW_s.$$

Putting $\mathbb{E}\left[A_T^{\theta}\right]=\exp\theta\left\{\Lambda_0^{\theta}\right\}$, we get

$$\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}-\exp\theta\left\{\Lambda_0^{\theta}\right\}=\int_0^t\varphi(s)\,dW_s.$$

By applying Itô's formula to $\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}$, we obtain

$$\begin{aligned}
d\left(\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}\right) &= \theta l(t)\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}dt+\theta\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}d\Lambda_t^{\theta}\\
&\quad+\frac{\theta^2}{2}\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l(s)\,ds\right\}\left\langle d\Lambda_t^{\theta},d\Lambda_t^{\theta}\right\rangle\\
&= \varphi(t)\,dW_t.
\end{aligned}$$

Then

$$l(t)\,dt + d\Lambda_t^\theta + \frac{\theta}{2}\left\langle d\Lambda_t^\theta, d\Lambda_t^\theta\right\rangle = \frac{1}{\theta}\varphi(t)\exp\theta\left\{-\Lambda_t^\theta - \int_0^t l(s)\,ds\right\}dW_t. \tag{1.3}$$

Hence,

$$\left\langle d\Lambda_t^\theta, d\Lambda_t^\theta\right\rangle = \left[\frac{1}{\theta}\varphi(t)\exp\theta\left\{-\Lambda_t^\theta - \int_0^t l(s)\,ds\right\}\right]^2 dt := |\mathcal{D}(t)|^2\,dt. \tag{1.4}$$

Then, by replacing (1.4) in (1.3), we have the quadratic backward stochastic differential equation as the following expression

$$\begin{cases} d\Lambda_t^\theta &= -\left(l(t) + \dfrac{\theta}{2}|\mathcal{D}(t)|^2\right)dt + \mathcal{D}(t)\,dW_t, \\ \Lambda_T^\theta &= \Psi(y_0^u), \end{cases}$$

where

$$\mathcal{D}(t) = \frac{1}{\theta}\varphi(t)\exp\theta\left\{-\Lambda_t^\theta - \int_0^t l(s)\,ds\right\}.$$

On the other hand, we assume that (1.2) holds, and by applying Itô's formula to $\exp\left(\theta\Lambda_t^\theta\right)$, we get

$$d\left(\exp\theta\left\{\Lambda_t^\theta\right\}\right) + \theta l(t)\exp\theta\left\{\Lambda_t^\theta\right\}dt = \theta\mathcal{D}(t)\exp\theta\left\{\Lambda_t^\theta\right\}dW_t.$$

Multiply with $\exp\theta\left\{\int_0^t l(s)\,ds\right\}$ to both sides, we get

$$\begin{aligned}&\exp\theta\left\{\int_0^t l(s)\,ds\right\}d\left(\exp\theta\left\{\Lambda_t^\theta\right\}\right) + \theta l(t)\exp\theta\left\{\int_0^t l(s)\,ds\right\}\exp\theta\left\{\Lambda_t^\theta\right\}dt \\ &= \theta\mathcal{D}(t)\exp\theta\left\{\int_0^t l(s)\,ds\right\}\exp\theta\left\{\Lambda_t^\theta\right\}dW_t.\end{aligned}$$

The right side is the same as the $d\left(\exp\theta\left\{\Lambda_t^\theta + \int_0^t l(s)\,ds\right\}\right)$, then we have

$$d\left(\exp\theta\left\{\Lambda_t^\theta + \int_0^t l(s)\,ds\right\}\right) = \theta\mathcal{D}(t)\exp\theta\left\{\Lambda_t^\theta + \int_0^t l(s)\,ds\right\}dW_t.$$

By making the integral $\int_t^T .$ in both sides, we have

$$\int_t^T d\left(\exp\theta\left\{\Lambda_s^\theta + \int_0^s l(r)\,dr\right\}\right) = \theta\int_t^T \mathcal{D}(s)\exp\theta\left\{\Lambda_s^\theta + \int_0^s l(r)\,dr\right\}dW_s.$$

Then

$$\exp\theta\left\{\Lambda_T^\theta+\int_0^T l(r)\,dr\right\}=\exp\theta\left\{\Lambda_t^\theta+\theta\int_0^t l(r)\,dr\right\}+\theta\int_t^T \mathcal{D}(s)\exp\theta\left\{\Lambda_s^\theta+\int_0^s l(r)\,dr\right\}dW_s.$$

By taking conditional expectation in above equality, we have

$$\begin{aligned}\mathbb{E}\left[\exp\theta\left\{\Lambda_T^\theta+\int_0^T l(r)\,dr\right\}\mid\mathcal{F}_t^W\right]&=\mathbb{E}\left[\exp\theta\left\{\Lambda_t^\theta+\int_0^t l(r)\,dr\right\}\mid\mathcal{F}_t^W\right]\\&\quad+\theta\mathbb{E}\left[\int_t^T\mathcal{D}(s)\exp\theta\left\{\Lambda_s^\theta+\int_0^s l(r)\,dr\right\}dW_s\mid\mathcal{F}_t^W\right],\end{aligned}$$

such that $\mathbb{E}\left[\int_t^T\mathcal{D}(s)\exp\theta\left\{\Lambda_s^\theta+\int_0^s l(r)\,dr\right\}dW_s\mid\mathcal{F}_t^W\right]=0$, then

$$\mathbb{E}\left[\exp\theta\left\{\Lambda_T^\theta+\int_0^T l(r)\,dr\right\}\mid\mathcal{F}_t^W\right]=\exp\theta\left\{\Lambda_t^\theta+\int_0^t l(r)\,dr\right\}.$$

As we now that $\Lambda_T^\theta=\Psi(y_0^u)$, we can write

$$\mathbb{E}\left[\exp\theta\left\{\Psi(y_0^u)+\int_t^T l(s)\,ds\right\}\mid\mathcal{F}_t^W\right]=\exp\theta\left\{\Lambda_t^\theta\right\}.$$

■

1.3 Financial Market of the Risk-Sensitive

Next, we will discuss a result, which called Girsanov's Theorem, which plays an important role in the application especially in economics, and optimal control. In Girsanov's Theorem application, we can visit the papers [10, 13, 15, 19]. We can now show the versions of the Girsanov's Theorem. In the application of Itô calculus, Girsanov's Theorem get used frequently since it transforms a class of process to Brownian motion with an equivalent probability measure transformation see [16].

Definition 1.3.1 *Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\in[0,T]}, \mathbb{P}\right)$ be a probability space satisfying the usual conditions. Let $\mathbb{Q}$ be another probability measure on $\mathcal{F}_T$. We say that $\mathbb{Q}$ is equivalent to $\mathbb{P} \mid \mathcal{F}_T$ if $\mathbb{P} \mid \mathcal{F}_T \ll \mathbb{Q}$ and $\mathbb{Q} \ll \mathbb{P} \mid \mathcal{F}_T$, or equivalently, if $\mathbb{P}$ and $\mathbb{Q}$ have the same zero sets in $\mathcal{F}_T$.*

Remark 1.3.2 *By the Radon-Nikodym Theorem this is the case if and only if we have*

$$d\mathbb{Q}(w) = Z(T)\, d\mathbb{P}(w) \ \textit{on } \mathcal{F}_T, \textit{ and } d\mathbb{P}(w) = Z^{-1}(T)\, d\mathbb{Q}(w) \ \textit{on } \mathcal{F}_T.$$

Theorem 1.3.3 ***(Girsanov, 1960, [16]):*** *Assume that W_t is a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with underlying filtration $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$. Let f be a square integrable stochastic process adapts to $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$, such that*

$$\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T f^2(t)\, dt\right\}\right] < \infty, \tag{1.5}$$

for all $t \in [0,T]$, then $W_t^{\mathbb{Q}} = W_t - \int_0^t f(s)\, ds$ is a Brownian motion with respect to the equivalent probability measure $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(T) =: \exp\left\{\int_0^T f(t)\, dW_t - \frac{1}{2}\int_0^T f^2(t)\, dt\right\}.$$

Remark 1.3.4 *Using differential form, we can also say, if $dW_t^{\mathbb{Q}} = dW_t - f(t)\, dt$. Then $W_t^{\mathbb{Q}}$ is a Brownian motion with respect to the probability measure $\mathbb{Q}$.*

Remark 1.3.5 *The condition $\mathbb{E}\left[\exp\left\{\frac{1}{2}\int_0^T f^2(t)\, dt\right\}\right] < \infty$ is sufficient and not necessary, called the Novikov's condition.*

For more details the reader can see the Øksendal's book [29] pages $155 - 160$.

We modeled the dynamics of the investor with diffusion process as a following SDE

$$dx_t^v = b(t, x_t^v)\, dt + \Lambda dW_t, \ \text{ and } x_0^v = x. \tag{1.6}$$

We consider a financial market in which two assets (securities) can be investment choices, the first one is risk-free is called also bond (foreign currency deposit for example), whose price $S_0(t)$ at time t is given by

$$\frac{dS_0(t)}{S_0(t)} = r(t)\,dt \text{ or } \left(= r(t, x_t^v)\,dt\right).$$

The second risky asset is called stock, whose price $S_1(t)$ at time t is given by

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)\,dt + \sigma(t)\,dW_t \text{ or } \left(= \mu(t, x_t^v)\,dt + \sigma(t, x_t^v)\,dW_t\right),$$

where $r(t, x_t^v)$ is bond function interest rate, $\sigma(t, x_t^v)$ is function stock price volatility rate, and $\mu(t, x_t^v)$ is called the expected rate of return.

Now let us consider an investor who wants to invest in the risk-free (foreign currency deposit for example) and the stock, and whose decisions cannot affect the prices in the financial market.

We denote here that W_t is the Brownian motion given in measurable space $(\Omega, \mathcal{F})$.

Definition 1.3.6 *(Self-Financial) The market is called self-financial if there is no infusion or withdrawal of funds over* $[0, T]$.

We assume also that our market is to be self-financial, we denote by V_t the amount of the investor's wealth, and π_t is the proportion of the wealth invested in the stock at time t, then $v_t = \pi_t V_t$ is the amount stock and $(1 - \pi_t)V_t$ is the amount in the bond, that means the investor has $V_t - \pi_t V_t = V_t - v_t$ savings in bank.

Then wealth dynamics of the investor who wants invests in the financial market has the following form

$$\frac{dV_t}{V_t} = (V_t - v_t)\frac{dS_0(t)}{S_0(t)} + v_t\frac{dS_1(t)}{S_1(t)}.$$

Honestly, the wealth of the investor is described by

$$\begin{aligned}\frac{dV_t}{V_t} &= (V_t - v_t)\, r\,(t, x_t^v)\, dt + v_t\, (\mu\,(t, x_t^v)\, dt + \sigma\,(t, x_t^v)\, dW_t) \qquad (1.7)\\ &= (V_t - v_t)\, r\,(t, x_t^v)\, dt + v_t \mu\,(t, x_t^v)\, dt + v_t \sigma\,(t, x_t^v)\, dW_t\\ &= V_t r\,(t, x_t^v)\, dt - v_t r\,(t, x_t^v)\, dt + v_t \mu\,(t, x_t^v)\, dt + v_t \sigma\,(t, x_t^v)\, dW_t\\ &= \{V_t r\,(t, x_t^v) + (\mu\,(t, x_t^v) - r\,(t, x_t^v))\, v_t\}\, dt + v_t \sigma\,(t, x_t^v)\, dW_t.\end{aligned}$$

Definition 1.3.7 *An admissible strategy is an* $\left(\mathcal{F}_t^W\right)_{t\geq 0}$ *−adapted square integrable with* $\mathbb{R}-$*valued process* v *such that* (1.7) *has a strong solution* $(V_t)_{t\in[0,T]}$ *that satisfies* $\mathbb{E}\int_0^T |V_t|^2\, dt < \infty$, *the set of all the admissible strategies is denoted by* $\mathcal{U}$.

The investor wants to minimize his (or her) expected utility (HARA type) over the set $\mathcal{U}$ in some terminal time $T > 0$:

$$J^\theta\,(v\,(.)) = \frac{1}{\theta}\mathbb{E}\left(V_T^\theta\right). \qquad (1.8)$$

By choosing an appropriate portfolio choice strategy $v\,(.)$, where the exponent $\theta > 0$ is called risk-sensitive parameter. If we put $\theta = 1$ the utility (1.8) reduced to the usual risk-neutral case, the expectation under the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$.

Lemma 1.3.8 *We can rewrite the expectation* $\mathbb{E}\left(V_T^\theta\right)$ (1.8) *in term of the exponential expected of integral criterion as*

$$J^\theta\,(v\,(.)) = \frac{1}{\theta} V_0^\theta \mathbb{E}^\theta\left[\exp\left(\theta\int_0^T l\,(t, x_t^v, v_t)\, dt\right)\right],$$

where $\mathbb{E}^\theta$ *is the new expectation with respect to probability measure* $\mathbb{P}^\theta$, *and the function* l *is given by*

$$l\,(t, x_t^v, v_t) = \frac{1}{2}\,(\theta - 1)\, v_t^2 \sigma^2\,(t, x_t^v) + V_t r\,(t, x_t^v) + (\mu\,(t, x_t^v) - r\,(t, x_t^v))\, v_t.$$

Proof. Applying the Itô's formula to Logarithmic wealth value $\ln V_t^\theta = \theta \ln V_t = \theta f(t, V_t)$, we have

$$\begin{aligned}
\theta d\left(f\left(t,V_t\right)\right) &= \theta d\left(\ln V_t\right) \\
&= \theta \frac{\partial f}{\partial t}\left(t,V_t\right)dt + \theta \frac{\partial f}{\partial x}\left(t,V_t\right)dV_t + \theta \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(t,V_t\right)\langle dV_t, dV_t\rangle \\
&= \theta \frac{1}{V_t}dV_t + \theta\frac{1}{2}\left(-\frac{1}{V_t^2}\right)v_t^2\sigma^2\left(t,x_t^v\right)V_t^2 dt \\
&= \theta\left(\left\{V_t r\left(t,x_t^v\right) + \left(\mu\left(t,x_t^v\right) - r\left(t,x_t^v\right)\right)v_t\right\}dt + v_t\sigma\left(t,x_t^v\right)dW_t\right) - \frac{1}{2}\theta v_t^2\sigma^2\left(t,x_t^v\right)dt.
\end{aligned}$$

Then, by taking the integral from zero into T with respect to time, the exponential expectation gets the form

$$\begin{aligned}
J^\theta\left(v\left(.\right)\right) &= \frac{1}{\theta}\mathbb{E}\left(V_T^\theta\right) = \frac{1}{\theta}\mathbb{E}\left[\exp\left(\ln V_T^\theta\right)\right] = \frac{1}{\theta}\mathbb{E}\left[\exp\left(\theta \ln V_T\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[\exp\left(\theta f\left(V_0\right) + \theta\int_0^T \left\{V_t r\left(t,x_t^v\right) + \left(\mu\left(t,x_t^v\right) - r\left(t,x_t^v\right)\right)v_t\right\}dt\right.\right. \\
&\left.\left. + \ \theta\int_0^T v_t\sigma\left(t,x_t^v\right)dW_t - \frac{1}{2}\theta\int_0^T v_t^2\sigma^2\left(t,x_t^v\right)dt\right)\right] \\
&= \frac{1}{\theta}\mathbb{E}\left[\exp\left(\ln V_0^\theta + \theta\int_0^T \left\{V_t r\left(t,x_t^v\right) + \left(\mu\left(t,x_t^v\right) - r\left(t,x_t^v\right)\right)v_t\right\}dt\right.\right. \\
&\left.\left. + \ \theta\int_0^T v_t\sigma\left(t,x_t^v\right)dW_t - \frac{1}{2}\theta\int_0^T v_t^2\sigma^2\left(t,x_t^v\right)dt\right)\right] \\
&= \frac{1}{\theta}\exp\left(\ln V_0^\theta\right)\mathbb{E}\left[\exp\left(\theta\int_0^T \left\{V_t r\left(t,x_t^v\right) + \left(\mu\left(t,x_t^v\right) - r\left(t,x_t^v\right)\right)v_t\right\}dt\right.\right. \\
&\left.\left. + \ \theta\int_0^T v_t\sigma\left(t,x_t^v\right)dW_t - \frac{1}{2}\theta\int_0^T v_t^2\sigma^2\left(t,x_t^v\right)dt\right)\right].
\end{aligned}$$

Then, we get

$$\begin{aligned}
J^{\theta}(v(.)) &= \frac{1}{\theta}V_0^{\theta}\mathbb{E}\left[\exp\left(\theta\int_0^T\{V_t r(t,x_t^v)+(\mu(t,x_t^v)-r(t,x_t^v))v_t\}\,dt\right.\right.\\
&+\theta\int_0^T v_t\sigma(t,x_t^v)\,dW_t-\frac{1}{2}\theta\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt\\
&\left.\left.-\ \frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt+\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt\right)\right]\\
&=\frac{1}{\theta}V_0^{\theta}\mathbb{E}\left[\exp\left\{\left(-\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt+\theta\int_0^T v_t\sigma(t,x_t^v)\,dW_t\right)\right.\right.\\
&-\frac{1}{2}\theta\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt+\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt\\
&\left.\left.+\ \theta\int_0^T\{V_t r(t,x_t^v)+(\mu(t,x_t^v)-r(t,x_t^v))v_t\}\,dt\right\}\right]\\
&=\frac{1}{\theta}V_0^{\theta}\mathbb{E}\left[I_1\times I_2\right],
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \exp\left(-\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt+\theta\int_0^T v_t\sigma(t,x_t^v)\,dW_t\right),\\
I_2 &= \exp\left(-\frac{1}{2}\theta\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt+\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2(t,x_t^v)\,dt\right.\\
&\left.+\ \theta\int_0^T\{V_t r(t,x_t^v)+(\mu(t,x_t^v)-r(t,x_t^v))v_t\}\,dt\right)\\
&=\exp\left(\theta\int_0^T\frac{1}{2}(\theta-1)v_t^2\sigma^2(t,x_t^v)\,dt+\theta\int_0^T\{V_t r(t,x_t^v)+(\mu(t,x_t^v)-r(t,x_t^v))v_t\}\,dt\right)\\
&=\exp\left(\theta\int_0^T l(t,x_t^v,v_t)\,dt\right),
\end{aligned}$$

where

$$l(t,x_t^v,v_t)=\frac{1}{2}(\theta-1)v_t^2\sigma^2(t,x_t^v)+V_t r(t,x_t^v)+(\mu(t,x_t^v)-r(t,x_t^v))v_t.$$

In virtue of Noviko's condition (1.5) from Girsanov's Theorem 1.3.3, we get

$$\mathbb{E}\left(\exp\alpha v_t^2\right)\le C, \tag{1.9}$$

where some constants α, C are positive.

By applying Girsanov's transformation (see the Theorem $1.3.3$), the stochastic integral term can be deleted, and according to the condition (1.9) , we get

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\theta^2\int_0^T v_t^2\sigma^2\left(t,x_t^v\right)dt + \theta\int_0^T v_t\sigma\left(t,x_t^v\right)dW_t\right).$$

Hence

$$\begin{aligned} J^\theta\left(v\left(.\right)\right) &= \frac{1}{\theta}\mathbb{E}\left(V_T^\theta\right) = \frac{1}{\theta}V_0^\theta\mathbb{E}\left[I_1\times I_2\right] \\ &= \frac{1}{\theta}V_0^\theta\mathbb{E}\left[\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\times\exp\left(\theta\int_0^T l\left(t,x_t^v,v_t\right)dt\right)\right] \\ &= \frac{1}{\theta}V_0^\theta\mathbb{E}^\theta\left[\exp\left(\theta\int_0^T l\left(t,x_t^v,v_t\right)dt\right)\right], \end{aligned}$$

we denote by

$$W_t^\theta = W_t - \theta\int_0^t v_s\sigma\left(s,x_s^v\right)ds,$$

is a standard Brownian motion under the probability measure $\mathbb{P}^\theta$.

As a conclusion, for every $0\leq s\leq t\leq T$, our dynamics (1.6) satisfies the SDE

$$\begin{aligned} dx_t^v = b\left(t,x_t^v\right)dt + \Lambda dW_t &= b\left(t,x_t^v\right)dt + \Lambda d\left(W_t^\theta + \theta\int_0^t v_s\sigma\left(s,x_s^v\right)ds\right) \\ &= b\left(t,x_t^v\right)dt + \Lambda dW_t^\theta + \Lambda\theta v_t\sigma\left(t,x_t^v\right)dt \\ &= \left(b\left(t,x_t^v\right) + \Lambda\theta v_t\sigma\left(t,x_t^v\right)\right)dt + \Lambda dW_t^\theta. \end{aligned}$$

An auxiliary criterion function of the expected utility, whose wants the investor minimized, is given by

$$J^\theta\left(v\left(.\right)\right) = \frac{1}{\theta}V_0^\theta\mathbb{E}^\theta\left[\exp\left(\theta\int_0^T l\left(t,x_t^v,v_t\right)dt\right)\right].$$

The proof is completed. ■

1.4 Mean-Variance of Loss Functional

We require the following condition

$$A_T^\theta := \exp \theta \left\{ \int_0^T l\left(t, x_t^v, v_t\right) dt \right\},$$

and we can put also

$$\Psi\left(T\right) := \int_0^T l\left(t, x_t^v, v_t\right) dt, \tag{1.10}$$

the risk-sensitive loss functional is given by

$$\Phi\left(\theta\right) := \frac{1}{\theta} \log \left[\mathbb{E}\left(\exp \theta \left\{ \int_0^T l\left(t, x_t^v, v_t\right) dt \right\} \right) \right] = \frac{1}{\theta} \log \left[\mathbb{E}\left(\exp \theta \Psi\left(T\right) \right) \right]. \tag{1.11}$$

Lemma 1.4.1 *Let $\Phi\left(\theta\right)$ be the loss functional has written as (1.11), where $\Psi\left(T\right)$ is given by (1.10). Then, if the risk-sensitive index θ is small, the loss functional $\Phi\left(\theta\right)$ can be expanded as*

$$\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta}{2} \mathbb{V}ar\left(\Psi\left(T\right)\right) + O\left(\theta^2\right).$$

Proof. The limited development of the function $f\left(x\right) = \exp\left(\theta x\right)$ with rang two in the neighborhood of zero is given by

$$f\left(x\right) = \exp\left(\theta x\right) = \textstyle\sum_{k=0}^{2} \frac{\left(\theta x\right)^k}{k!} = 1 + \theta x + \frac{1}{2}\left(\theta x\right)^2 + O\left(\theta^2\right).$$

Then, by replacing x by $\Psi\left(T\right)$, we get

$$\exp\left(\theta \Psi\left(T\right)\right) = 1 + \theta \Psi\left(T\right) + \frac{1}{2}\left(\theta \Psi\left(T\right)\right)^2 + O\left(\theta^2\right).$$

By taking expectation, we have

$$\begin{aligned} \mathbb{E}\left(\exp\left(\theta \Psi\left(T\right)\right)\right) &= \mathbb{E}\left[1 + \theta \Psi\left(T\right) + \frac{1}{2}\left(\theta \Psi\left(T\right)\right)^2 + O\left(\theta^2\right) \right] \\ &= 1 + \theta \mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2} \mathbb{E}\left(\Psi^2\left(T\right)\right) + O\left(\theta^2\right). \end{aligned}$$

Then

$$\log \mathbb{E}\left(\exp\left(\theta\Psi\left(T\right)\right)\right) = \log\left(1 + \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{E}\left(\Psi^2\left(T\right)\right) + O\left(\theta^2\right)\right).$$

If we take $X = \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{E}\left(\Psi^2\left(T\right)\right) + O\left(\theta^2\right)$, and by using the limited development of the function $g\left(X\right) = \ln(1 + X)$, with rang two in neighborhood of zero

$$g\left(X\right) = \ln(1 + X) = \sum_{k=1}^{2} \frac{(-1)^{k-1}}{k} X^k.$$

Then

$$\begin{aligned}
\log \mathbb{E}\left(\exp\left(\theta\Psi\left(T\right)\right)\right) &= \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{E}\left(\Psi^2\left(T\right)\right) + O\left(\theta^2\right) \\
&+ \left(-1\right)\frac{1}{2}\left[\theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{E}\left(\Psi^2\left(T\right)\right) + O\left(\theta^2\right)\right]^2 + O(\theta^2) \\
&= \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{E}\left(\Psi^2\left(T\right)\right) - \frac{\theta^2}{2}\left(\mathbb{E}\left(\Psi\left(T\right)\right)\right)^2 - \frac{\theta^4}{4}\left(\mathbb{E}\left(\Psi^2\left(T\right)\right)\right)^2 + ... + O\left(\theta^2\right) \\
&= \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\left[\mathbb{E}\left(\Psi^2\left(T\right)\right) - \left(\mathbb{E}\left(\Psi\left(T\right)\right)\right)^2\right] + O\left(\theta^2\right) \\
&= \theta\mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta^2}{2}\mathbb{V}ar\left(\Psi\left(T\right)\right) + O\left(\theta^2\right).
\end{aligned}$$

This implies that

$$\Phi\left(\theta\right) = \frac{1}{\theta}\log \mathbb{E}\left(\exp\left(\theta\Psi\left(T\right)\right)\right) = \mathbb{E}\left(\Psi\left(T\right)\right) + \frac{\theta}{2}\mathbb{V}ar\left(\Psi\left(T\right)\right) + O\left(\theta^2\right).$$

The proof is complete. ■

CHAPTER 2

A Risk-Sensitive Stochastic Maximum Principle for Backward Doubly Stochastic Differential Equations with Application

CHAPTER 2

A Risk-Sensitive Stochastic Maximum Principle for Backward Doubly Stochastic Differential Equations with Application

In this chapter, we concern on an optimal control problem where the system is driven by a backward doubly stochastic differential equation with risk-sensitive performance functional. We generalized the result of Chala [10] to a backward doubly stochastic differential equation by using the same contribution of Djehiche et al. in [13]. We use the risk-neutral model for which an optimal solution exists as a preliminary step. This is an extension of an initial control system to this type of problem, where the admissible controls set is convex. We establish necessary and sufficient optimality conditions for the risk-sensitive performance functional control problem. We illustrate the chapter by given two different examples for a linear quadratic system.

2.1 Formulation of the Problem

Let T be a positive real number. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space in which $one-$dimensional Brownian motions

$W = (W_t, 0 \leq t \leq T)$ and $B = (B_t, 0 \leq t \leq T)$ are defined, where W and B are two mutually independent standard Brownian motions processes. Let $\mathcal{N}$ denote the class of $\mathbb{P}-$null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t^{(W,B)} = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, where for any process $\{L_t\}_{t\in[0,T]}$, one has $\mathcal{F}_{s,t}^L = \sigma \{L_r - L_s, s \leq r \leq t\} \vee \mathcal{N}$ and $\mathcal{F}_t^L = \mathcal{F}_{0,t}^L$.

Note that the collection $\left\{\mathcal{F}_t^{(W,B)}, t \in [0, T]\right\}$ is neither increasing nor decreasing, and it does not constitute a filtration. We may define the subfiltration $(\mathcal{G}_t)_{t\in[0,T]}$ such as $\mathcal{G}_t \subset \mathcal{F}_t^{(W,B)}$, $\forall t \in [0, T]$.

Let $\mathcal{M}^2([0, T], \mathbb{R})$ denote the set of $one-$dimensional jointly measurable random processes $\{\varphi_t, t \in [0, T]\}$ which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{M}^2([0,T],\mathbb{R})} = \mathbb{E}\left[\int_0^T |\varphi_t|^2 dt\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^{(W,B)}-$measurable for any $t \in [0, T]$.

Similarly, we denote by $\mathcal{S}^2([0, T], \mathbb{R})$ the set of $one-$dimensional continuous random processes which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{S}^2([0,T],\mathbb{R})} = \mathbb{E}\left[\sup_{t\in[0,T]} |\varphi_t|^2\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^{(W,B)}-$measurable for any $t \in [0, T]$.

Let U be a nonempty subset of $\mathbb{R}$.

Definition 2.1.1 *An admissible control v is a measurable process with values in U such that $\mathbb{E}\int_0^T |v_t|^2 dt < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.*

For any $v \in \mathcal{U}$, we consider the following BDSDE system

$$\begin{cases} dy_t^v &= -f\left(t, y_t^v, z_t^v, v_t\right) dt - g\left(t, y_t^v, z_t^v, v_t\right) \overleftarrow{dB}_t + z_t^v dW_t, \\ y_T^v &= \xi, \end{cases} \tag{2.1}$$

where $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \rightarrow \mathbb{R}$, $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \rightarrow \mathbb{R}$ are jointly measurable and such that for any $(y^v, z^v, v) \in \mathbb{R} \times \mathbb{R} \times U$ one has $f\left(., y^v, z^v, v\right) \in \mathcal{M}^2\left([0,T], \mathbb{R}\right)$, $g\left(., y^v, z^v, v\right) \in \mathcal{M}^2\left([0,T], \mathbb{R}\right)$, and z_t^v is square integrable and the terminal condition ξ is a $\mathcal{F}_T^W-$measurable and square integrable random variable.

Note that the integral with respect to $(B_t)_{t\in[0,T]}$ is a "backward" Itô integral, while the integral with respect to $(W_t)_{t\in[0,T]}$ is a standard forward Itô integral. These two types of integrals are particular cases of the Itô-Skorohod integral; for more details we refer to [28].

We define the criterion to be minimized, with initial risk-sensitive performance functional cost, as follows

$$J^\theta\left(v\left(.\right)\right) = \mathbb{E}\left[\exp\theta\left\{\Psi\left(y_0^v\right) + \int_0^T l\left(t, y_t^v, z_t^v, v_t\right) dt\right\}\right], \tag{2.2}$$

where $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ and $l : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \rightarrow \mathbb{R}$ are jointly measurable and θ is the risk-sensitive index.

The optimal control problem is to minimize the functional J^θ over $\mathcal{U}$ if $u \in \mathcal{U}$ is an optimal control (solution), that is,

$$J^\theta\left(u\right) = \inf_{v\in\mathcal{U}} J^\theta\left(v\right). \tag{2.3}$$

We assume the following

$(\mathbf{H_1})$ There exist constants $c > 0$ and $0 \leq \lambda < 1$, such that for any

$(w, t) \in \Omega \times [0,T]$, and $(y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}$, we have

$$|f(t,y_1,z_1)-f(t,y_2,z_2)|^2 \leq c\left(|y_1-y_2|^2+|z_1-z_2|^2\right),$$

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|^2 \leq c\,|y_1-y_2|^2+\lambda\,|z_1-z_2|^2.$$

Theorem 2.1.2 *For any given admissible control $v(.)$, suppose that Assumption $(\mathbf{H_1})$ holds. Then the BDSDE* (2.1) *has a unique solution $(y_t^v, z_t^v) \in \mathcal{S}^2([0,T],\mathbb{R}) \times \mathcal{M}^2([0,T],\mathbb{R})$.*

Proof. See [31] (Theorem 1.1 page 212). ■

A control that solves the problem $\{(2.1),(2.2),(2.3)\}$ is called optimal. Our objective is to establish risk-sensitive necessary and sufficient optimality conditions, satisfied by a given optimal control, in the form of risk-sensitive SMP.

We also assume the following

$(\mathbf{H_2})$

- $i)$ $f,\ g,\ l$ and Ψ are continuously differentiable with respect to (y^v, z^v, v).
- $ii)$ All the derivatives of $f,\ g$ and l are bounded by $C(1+|y^v|+|z^v|+|v|)$.
- $iii)$ The derivative of Ψ is bounded by $C(1+|y^v|)$.

Under the above Assumptions $(\mathbf{H_1})-(\mathbf{H_2})$, for each $v \in \mathcal{U}$ equation (2.1) has a unique strong solution, and the cost function J^θ is well defined from $\mathcal{U}$ into $\mathbb{R}$.

For more details the reader can see the paper of Han et al. [21].

Remark 2.1.3 *We use the Euclidean norm $|.|$ in $\mathbb{R}$, $\top$ is a matrix transpose and Tr is the trace of a matrix. All the equalities and inequalities mentioned in this chapter are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0,T] \times \Omega$.*

2.2 Risk-Sensitive Stochastic Maximum Principle of Backward Doubly Type Control

The proof of our risk-sensitive stochastic maximum principle necessitates a certain auxiliary state process x_t^v, which is the solution of the following forward SDE

$$dx_t^v = l\left(t, y_t^v, z_t^v, v_t\right) dt, \quad x_0^v = 0.$$

Our control problem of $\{(2.1), (2.2), (2.3)\}$ is equivalent to

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\Psi\left(y_0^v\right) + x_T^v\right\}\right] = \inf_{v\in\mathcal{U}} \mathbb{E}\left[\varphi\left(y_0^v, x_T^v\right)\right], \\ \text{subject to} \\ dx_t^v = l\left(t, y_t^v, z_t^v, v_t\right) dt, \\ dy_t^v = -f\left(t, y_t^v, z_t^v, v_t\right) dt - g\left(t, y_t^v, z_t^v, v_t\right) \overleftarrow{dB}_t + z_t^v dW_t, \\ x_0^v = 0, \ y_T^v = \xi. \end{cases} \tag{2.4}$$

We require the following notation $A_T^\theta := \exp\theta\left\{\Psi\left(y_0^v\right) + \int_0^T l\left(t, y_t^v, z_t^v, v_t\right) dt\right\}$. If we put $\Theta_T = \Psi\left(y_0^v\right) + \int_0^T l\left(t, y_t^v, z_t^v, v_t\right) dt$, then the risk-sensitive loss functional is given by

$$\mathcal{H}\left(\theta, v\right) := \frac{1}{\theta}\log\left[\mathbb{E}\left(\exp\theta\left\{\Psi\left(y_0^v\right) + \int_0^T l\left(t, y_t^v, z_t^v, v_t\right) dt\right\}\right)\right] = \frac{1}{\theta}\log\left[\mathbb{E}\left(\exp\theta\Theta_T\right)\right].$$

When the risk-sensitive index θ is small, by Lemma 1.4.1 the loss functional $\mathcal{H}\left(\theta, u\right)$ can be expanded as

$$\mathbb{E}\left(\Theta_T\right) + \frac{\theta}{2} Var\left(\Theta_T\right) + O\left(\theta^2\right),$$

where $Var\left(\Theta_T\right)$ denotes the variance of Θ_T. If $\theta < 0$, the variance of Θ_T, as a measure of risk, improves the performance $\mathcal{H}\left(\theta, v\right)$, in this case the optimizer called risk seeker. But, when $\theta > 0$, the variance of Θ_T worsens the performance $\mathcal{H}(\theta, v)$, in this case the optimizer called risk averse. The risk-neutral loss functional $\mathbb{E}(\Theta_T)$ can be seen as a limit of the risk-sensitive functional $\mathcal{H}(\theta, v)$ when $\theta \to 0$.

Next, let us introduce the following notations

Notation 2.2.1 *We denote by* $X^v := \begin{pmatrix} x^v \\ y^v \end{pmatrix}$, $M_t := \begin{pmatrix} W_t \\ B_t \end{pmatrix}$, $d\overrightarrow{p}(t) := \begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix}$, $F(t, y_t^v, z_t^v, v_t) := \begin{pmatrix} l(t, y_t^v, z_t^v, v_t) \\ -f(t, y_t^v, z_t^v, v_t) \end{pmatrix}$ *and* $G(t, y_t^v, z_t^v, v_t) := \begin{pmatrix} 0 & 0 \\ z_t^v & -g(t, y_t^v, z_t^v, v_t) \end{pmatrix}$, *with these notations the problem* (2.4) *can be rewritten in the following compact SDE form*

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\Psi(y_0^v) + x_T^v\right\}\right] = \inf_{v\in\mathcal{U}} \mathbb{E}\left[\varphi(x_t^v, y_t^v)\right], \\ \textit{subject to} \\ dX_t^v = F(t, y_t^v, z_t^v, v_t)\,dt + G(t, y_t^v, z_t^v, v_t)\,dM_t, \\ X^v\begin{pmatrix} 0 \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}. \end{cases} \tag{2.5}$$

For convenience, we will use the following notations throughout this chapter. For $\phi \in \{f, g, l\}$, *we define*

$$\begin{cases} \phi(t) = \phi(t, y_t^v, z_t^v, v_t), \\ \partial\phi(t) = \phi(t, y_t^v, z_t^v, v_t) - \phi(t, y_t^u, z_t^u, u_t), \\ \phi_\zeta(t) = \dfrac{\partial\phi}{\partial\zeta}(t, y_t^v, z_t^v, v_t), \quad \zeta = y, z, v, \end{cases}$$

where v_t *is an admissible control from* $\mathcal{U}$.

We suppose that Assumptions $(\mathbf{H_1}) - (\mathbf{H_2})$ hold. We may combine the SMP for a risk-neutral controlled BDSDE type from [2, 21] with the result of Yong [38] and with augmented state dynamics (x^u, y^u, z^u) to derive the adjoint equation. There exist unique $\mathcal{G}_t-$adapted pairs of processes

(p_1, q_1) and (p_2, q_2) that solve the following matrix system of BSDEs:

$$\begin{cases} d\overrightarrow{p}(t) & = -A(t)\,dt + R(t)\,dM_t, \\ \begin{pmatrix} p_1(T) \\ p_2(0) \end{pmatrix} & = \theta A_T^\theta \begin{pmatrix} 1 \\ -\Psi_y(y_0^u) \end{pmatrix}, \end{cases} \tag{2.6}$$

with

$$\mathbb{E}\left[\sum_{i=1}^{2} \sup_{t\in[0,T]} |p_i(t)|^2 + \sum_{i=1}^{2} \int_0^T |q_i(t)|^2\,dt\right] < \infty,$$

where

$$A(t) = \begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix},$$

and

$$R(t) = \begin{pmatrix} q_1(t) & 0 \\ p_3(t) & -q_2(t) \end{pmatrix},$$

such that

$$p_3(t) = -Tr\left[\begin{pmatrix} l_z(t) & -f_z(t) \\ 0 & -g_z(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_1(t) \\ p_2(t) & q_2(t) \end{pmatrix}\right].$$

Let $\widetilde{H}^\theta$ be the Hamiltonian associated with the optimal state dynamics (x^u, y^u, z^u), and let the two pairs of adjoint process $((p_1, q_1), (p_2, q_2))$ be given by

$$\widetilde{H}^\theta(t) := \widetilde{H}^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)) = l(t)\,p_1(t) - f(t)\,p_2(t) - g(t)\,q_2(t). \tag{2.7}$$

Theorem 2.2.2 *We suppose that Assumptions $(\mathbf{H_1}) - (\mathbf{H_2})$ hold. If (x^u, y^u, z^u) is an optimal solution of the risk-neutral control problem (2.5), then there exist two pairs of $\mathcal{G}_t-$adapted processes $((p_1, q_1), (p_2, q_2))$ that satisfy (2.6) such that*

$$\widetilde{H}_v^\theta(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t))(u_t - v_t) \le 0, \tag{2.8}$$

for all $u \in U$, almost every $t \in [0, T]$ and $\mathbb{P}-$almost surely, where $\widetilde{H}_v^\theta(t)$ is defined in Notation 2.2.1.

Proof. We suppose that the Assumptions $(\mathbf{H_1}) - (\mathbf{H_2})$ hold, we may combine the SMP for risk-neutral of controlled BDSDE type from [2, 21], with the result of Yong [38]. ■

2.3 New Adjoint Equations and Risk-sensitive Necessary Optimality Conditions

Mentioned Theorem 2.2.2 is a good SMP for the risk-neutral control problem of forward-backward doubly type. We follow the same approach used in [10, 11, 13], and suggest a transformation of the adjoint processes (p_1, q_1) and (p_2, q_2) in such a way that we can omit the first component (p_1, q_1) in (2.6) and express the SMP in terms of only one adjoint process which we denote by $(\widetilde{p}_2, \widetilde{q}_2)$.

From (2.6), we note that $dp_1(t) = q_1(t)\, dW_t$ and $p_1(T) = \theta A_T^{\theta}$, the explicit solution of this BSDE is

$$p_1(t) = \theta \mathbb{E}\left[A_T^{\theta} \mid \mathcal{G}_t\right] = \theta V_t^{\theta}, \tag{2.9}$$

where $V_t^{\theta} := \mathbb{E}\left[A_T^{\theta} \mid \mathcal{G}_t\right], \ \ 0 \leq t \leq T.$

In view of (2.9), it would be natural to choose a transformation of $(\overrightarrow{p}, \overrightarrow{q})$ into an adjoint process $(\widetilde{p}, \widetilde{q})$, where $\widetilde{p}_1(t) = \dfrac{1}{\theta V_t^{\theta}} p_1(t) = 1.$

We consider the following transform

$$\widetilde{p}(t) = \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \end{pmatrix} := \frac{1}{\theta V_t^{\theta}} \overrightarrow{p}(t), \ \ 0 \leq t \leq T. \tag{2.10}$$

By using (2.6) and (2.10), we have

$$\widetilde{p}(.) := \begin{pmatrix} \widetilde{p}_1(T) \\ \\ \widetilde{p}_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ \\ -\Psi_y(y_0^u) \end{pmatrix}.$$

The following properties of the generic martingale V^θ are essential in order to investigate the properties of these new process $(\widetilde{p}(t), \widetilde{q}(t))$.

In this part, we want to prove the relationship between the expected exponential utility and the quadratic backward stochastic differential equation.

First of all, it's very important to write the expected exponential utility under this form

$$\exp\left(\theta\Lambda_t^\theta\right) = \mathbb{E}\left[A_{t,T}^\theta \mid \mathcal{G}_t\right] = \mathbb{E}\left[\exp\theta\left\{\Psi(y_0^u) + \int_t^T l(s)\,ds\right\} \mid \mathcal{G}_t\right]. \tag{2.11}$$

Lemma 2.3.1 *The necessary and sufficient condition for the expected exponential utility* (2.11) *to be hold, is the following quadratic backward stochastic differential equation*

$$\begin{cases} d\Lambda_t^\theta &= -\left(l(t) + \dfrac{\theta}{2}|\mathcal{D}(t)|^2\right)dt + \mathcal{D}(t)\,dW_t, \\ \Lambda_T^\theta &= \Psi(y_0^u), \end{cases} \tag{2.12}$$

where $\mathbb{E}\left[\int_0^T |\mathcal{D}(t)|^2\,dt\right] < \infty.$

Proof. By the same technique in Lemma 1.2.1, we can prove the Lemma 2.3.1. ∎

The process Λ^θ is the first component of the $\mathcal{G}_t-$adapted pair of processes $\left(\Lambda^\theta, \mathcal{D}\right)$ which is the unique solution to the quadratic backward stochastic differential equation (2.12).

Next, we will state and prove the necessary optimality conditions for the system is driven by a BDSDE with a risk-sensitive performance functional type.

To this end, let us summarize and prove some Lemmas that we will use thereafter.

Lemma 2.3.2 *Suppose that Assumption* $(\mathbf{H_2})$ *holds. Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Lambda_t^{\theta}\right|\right]\leq C_T, \tag{2.13}$$

where C_T *is a positive constant that depends only on* T *and the boundedness of* l *and* Ψ.

In particular, V^{θ} *solves the following linear BSDE:*

$$dV_t^{\theta}=\theta\mathcal{D}\left(t\right)V_t^{\theta}dM_t,\ \ V_T^{\theta}=A_T^{\theta}. \tag{2.14}$$

Hence, the process defined on $\left(\Omega,\mathcal{F},\left(\mathcal{G}_t\right)_{t\geq 0},\mathbb{P}\right)$ *by* L_t^{θ}, *where*

$$L_t^{\theta}:=\frac{V_t^{\theta}}{V_0^{\theta}}=\exp\left(\theta\int_0^t\mathcal{D}\left(s\right)dM_s-\frac{\theta^2}{2}\int_0^t\left|\mathcal{D}\left(s\right)\right|^2ds\right),\ \ 0\leq t\leq T, \tag{2.15}$$

is a uniformly bounded $\mathcal{G}_t-$*martingale.*

Proof. First, we prove (2.13). By Assumption $(\mathbf{H_2})$, l and Ψ are bounded by a constant $C>0$. We have

$$0<e^{-(1+T)C\theta}\leq A_T^{\theta}\leq e^{(1+T)C\theta}. \tag{2.16}$$

Therefore, V^{θ} is a uniformly bounded $\mathcal{G}_t-$martingale satisfying

$$0<e^{-(1+T)C\theta}\leq V_t^{\theta}\leq e^{(1+T)C\theta},\ \ 0\leq t\leq T. \tag{2.17}$$

The sufficient conditions of the Logarithmic transform established in ([13], Proposition 3.1), can be applied in the martingale V^{θ} as follows:

$$V_t^{\theta}=\exp\theta\left\{\Lambda_t^{\theta}+\int_0^t l\left(s\right)ds\right\},\ \ 0\leq t\leq T,$$

and $V_0^{\theta}=\exp\theta\left\{\Lambda_0^{\theta}\right\}=\mathbb{E}\left[A_T^{\theta}\right]$. It is very easy to see from (2.17) and the boundedness of l that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Lambda_t^{\theta}\right|\right]\leq C_T,$$

where C_T is a positive constant that depends only on T and the boundedness of l and Ψ.

Second, we find the explicit form of (2.14). Using the second Itô's formula to $V_t^\theta = \exp\theta\left\{\Lambda_t^\theta + \int_0^t l(s)\,ds\right\}$, we get

$$dV_t^\theta = \theta\mathcal{D}(t)\,V_t^\theta dM_t.$$

Now, we can prove (2.15) by starting from the integral form of (2.14) such that

$$dV_t^\theta = \theta\mathcal{D}(t)\,V_t^\theta dM_t,\ \ V_T^\theta = A_T^\theta.$$

On the other hand, we have

$$V_t^\theta = \exp\theta\left\{\Lambda_t^\theta + \int_0^t l(s)\,ds\right\}.$$

Using expression (1.2), we can write

$$V_t^\theta = \exp\left(\theta\int_0^t \mathcal{D}(s)\,dM_s - \frac{\theta^2}{2}\int_0^t |\mathcal{D}(s)|^2\,ds + \theta\Lambda_0^\theta\right).$$

Then

$$L_t^\theta := \frac{V_t^\theta}{V_0^\theta} = \exp\left(\theta\int_0^t \mathcal{D}(s)\,dM_s - \frac{\theta^2}{2}\int_0^t |\mathcal{D}(s)|^2\,ds\right),\ \ 0\le t\le T.$$

In view of (2.13) the above equality is a uniformly bounded $\mathcal{G}_t-$martingale. ■

Proposition 2.3.3 *The main risk-sensitive of second adjoint equation for* $(\widetilde{p}_2, \widetilde{q}_2)$ *and* $(V^\theta, \mathcal{D})$ *becomes*

$$\begin{cases} d\widetilde{p}_2(t) &= -H_y^\theta(t)\,dt - H_z^\theta(t)\,dW_t^\theta - \{\widetilde{q}_2(t) + \theta\mathcal{D}_2(t)\,\widetilde{p}_2(t)\}\,\overleftarrow{dB}_t^\theta, \\ dV_t^\theta &= \theta\mathcal{D}(t)\,V_t^\theta dM_t, \\ \widetilde{p}_2(T) &= -\Psi_y(y_0^v),\ \ V_T^\theta = A_T^\theta. \end{cases} \tag{2.18}$$

The system (2.18) *admits a unique* $\mathcal{G}_t-$*adapted solution* $(\widetilde{p}, \widetilde{q}, V^\theta, \mathcal{D})$, *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\widetilde{p}(t)|^2 + \sup_{t\in[0,T]}\left|V^\theta(t)\right|^2 + \int_0^T\left(|\widetilde{q}(t)|^2 + |\mathcal{D}(t)|^2\right)dt\right] < \infty, \tag{2.19}$$

where

$$H^{\theta}(t) := H^{\theta}\left(t, y_t^v, z_t^v, v_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\theta}(t), \mathcal{D}_1(t)\right) = l(t) - \left(f(t) - \theta \mathcal{D}_1(t) z_t^v\right) \widetilde{p}_2(t) - g(t) \widetilde{q}_2(t). \tag{2.20}$$

Proof. We wish to identify the processes $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$d\widetilde{p}(t) = -\widetilde{\alpha}(t)\, dt + \widetilde{\beta}(t)\, dM_t, \tag{2.21}$$

where

$$\widetilde{\beta}(t) =: \begin{pmatrix} \widetilde{\beta}_1(t) \\ \widetilde{\beta}_2(t) \end{pmatrix} =: \begin{pmatrix} \widetilde{\beta}_{11}(t) & \widetilde{\beta}_{12}(t) \\ \widetilde{\beta}_{21}(t) & \widetilde{\beta}_{22}(t) \end{pmatrix}.$$

By applying Itô's formula to the process $\overrightarrow{p}(t) = \theta V_t^{\theta} \widetilde{p}(t)$ and using the expression of V^{θ} in (2.14), we obtain

$$\begin{aligned} d\widetilde{p}(t) = &-\frac{1}{\theta V_t^{\theta}} \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right] dt \\ &- \theta \mathcal{D}(t) \widetilde{\beta}(t)\, dt + \frac{1}{\theta V_t^{\theta}} \begin{pmatrix} q_1(t) & 0 \\ p_3(t) & -q_2(t) \end{pmatrix} dM_t - \theta \widetilde{p}(t) \mathcal{D}(t)^{\top} dM_t. \end{aligned}$$

By identifying the coefficients of above equation to (2.21), we get the drift term of the process $\widetilde{p}(t)$:

$$\widetilde{\alpha}(t) = \frac{1}{\theta V_t^{\theta}} \left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} \right] + \theta \mathcal{D}(t) \widetilde{\beta}(t),$$

and the diffusion coefficient of the process $\widetilde{p}(t)$:

$$\widetilde{\beta}(t) = \frac{1}{\theta V_t^{\theta}} \begin{pmatrix} q_1(t) & 0 \\ p_3(t) & -q_2(t) \end{pmatrix} - \theta \widetilde{p}(t) \mathcal{D}(t)^{\top}.$$

Using the relation $\widetilde{p}(t) = \dfrac{1}{\theta V_t^\theta}\overrightarrow{p}(t)$, the drift term $\widetilde{\alpha}(t)$ it will be written as:

$$\widetilde{\alpha}(t) = \begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{pmatrix} + \theta\mathcal{D}(t)\widetilde{\beta}(t),$$

and the diffusion coefficient of the process $\widetilde{p}(t)$:

$$\widetilde{\beta}(t) = \begin{pmatrix} \widetilde{q}_1(t) & 0 \\ \widetilde{p}_3(t) & -\widetilde{q}_2(t) \end{pmatrix} - \theta\widetilde{p}(t)\mathcal{D}(t)^\top. \tag{2.22}$$

Finally, we obtain

$$d\widetilde{p}(t) = -\left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{pmatrix}\right]dt$$
$$+ \widetilde{\beta}(t)\left[dM_t - \theta\mathcal{D}(t)\,dt\right].$$

It is easily verified that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\left[dM_t - \theta\mathcal{D}(t)\,dt\right], \;\; \widetilde{p}_1(T) = 1.$$

In view of (2.15), we may use Girsanov's Theorem (see [12], Theorem 2.1 page 115) to claim that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\,dM_t^\theta, \;\; \mathbb{P}^\theta - a.s, \;\; \widetilde{p}_1(T) = 1,$$

where $dM_t^\theta = dM_t - \theta\mathcal{D}(t)\,dt$. By using Notation 2.2.1, dM_t^θ can be written as

$$dM_t^\theta = \begin{pmatrix} dW_t \\ dB_t \end{pmatrix} - \theta\begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \end{pmatrix}dt,$$

which is a $\mathbb{P}^\theta-$Brownian motion, where

$$\left.\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\right|_{\mathcal{G}_t} := L_t^\theta = \exp\left(\theta\int_0^t \mathcal{D}(s)\,dM_s - \frac{\theta^2}{2}\int_0^t |\mathcal{D}(s)|^2\,ds\right), \;\; 0 \le t \le T.$$

But according to (2.15) and (2.16), the probability measures $\mathbb{P}^{\theta}$ and $\mathbb{P}$ are in fact equivalent. Hence, noting that $\widetilde{p}_1(t) := \dfrac{1}{\theta V_t^{\theta}} p_1(t)$ is square integrable, we get that $\widetilde{p}_1(t) = \mathbb{E}^{\theta}[\widetilde{p}_1(T) \mid \mathcal{G}_t] = 1$. Thus, its quadratic variation $\int_0^T |\widetilde{q}_1(t)|^2 dt = 0$. This implies that, for almost every $0 \le t \le T$, $\widetilde{q}_1(t) = 0$, $\mathbb{P}^{\theta}$ and $\mathbb{P}-$a.s, we have

$$d\widetilde{p}(t) = -\left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{pmatrix}\right] dt + \widetilde{\beta}(t)\, dM_t^{\theta}. \tag{2.23}$$

Now replacing (2.22) in (2.23), to obtain

$$\begin{aligned} d\widetilde{p}(t) = &-\left[\begin{pmatrix} 0 & 0 \\ l_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \end{pmatrix}\right] dt \\ &+ \begin{pmatrix} \widetilde{\beta}_{11}(t) & \widetilde{\beta}_{12}(t) \\ \widetilde{\beta}_{21}(t) & \widetilde{\beta}_{22}(t) \end{pmatrix}\begin{pmatrix} dW_t^{\theta} \\ \overleftarrow{dB}_t^{\theta} \end{pmatrix}, \end{aligned}$$

where

$$\widetilde{\beta}_{11}(t) = \widetilde{q}_1(t) - \theta\mathcal{D}_1(t)\widetilde{p}_1(t),\ \widetilde{\beta}_{12}(t) = -\theta\mathcal{D}_2(t)\widetilde{p}_1(t),\ \widetilde{\beta}_{21}(t) = \widetilde{p}_3(t) - \theta\mathcal{D}_1(t)\widetilde{p}_2(t),$$
$$\widetilde{\beta}_{22}(t) = -\widetilde{q}_2(t) - \theta\mathcal{D}_2(t)\widetilde{p}_2(t),\ \widetilde{p}_3(t) = -Tr\left[\begin{pmatrix} l_z(t) & -f_z(t) \\ 0 & -g_z(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) & \widetilde{q}_1(t) \\ \widetilde{p}_2(t) & \widetilde{q}_2(t) \end{pmatrix}\right].$$

From (2.23), we get

$$\begin{aligned} d\widetilde{p}_2(t) &= -\{l_y(t)\widetilde{p}_1(t) - f_y(t)\widetilde{p}_2(t) - g_y(t)\widetilde{q}_2(t)\}\, dt \\ &- \{l_z(t)\widetilde{p}_1(t) - (f_z(t) - \theta\mathcal{D}_1(t))\widetilde{p}_2(t) - g_z(t)\widetilde{q}_2(t)\}\, dW_t^{\theta} - \{\widetilde{q}_2(t) + \theta\mathcal{D}_2(t)\widetilde{p}_2(t)\}\overleftarrow{dB}_t^{\theta}, \end{aligned} \tag{2.24}$$

We can rewrite (2.14) and (2.24) as the system below

$$\begin{cases} d\widetilde{p}_2(t) &= -H_y^\theta(t)\,dt - H_z^\theta(t)\,dW_t^\theta - \{\widetilde{q}_2(t) + \theta\mathcal{D}_2(t)\,\widetilde{p}_2(t)\}\,\overleftarrow{dB}_t^\theta, \\ dV_t^\theta &= \theta\mathcal{D}(t)\,V_t^\theta dM_t, \\ \widetilde{p}_2(T) &= -\Psi_y(y_0^v)\,,\ \ V_T^\theta = A_T^\theta. \end{cases}$$

The system (2.18) admits a unique $\mathcal{G}_t-$adapted solution $\left(\widetilde{p}, \widetilde{q}, V^\theta, \mathcal{D}\right)$, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\widetilde{p}(t)|^2 + \sup_{t\in[0,T]}\left|V^\theta(t)\right|^2 + \int_0^T \left(|\widetilde{q}(t)|^2 + |\mathcal{D}(t)|^2\right)dt\right] < \infty,$$

where

$$H^\theta(t) := H^\theta\left(t, y_t^v, z_t^v, v_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^\theta(t), \mathcal{D}_1(t)\right) = l(t) - (f(t) - \theta\mathcal{D}_1(t)\,z_t^v)\,\widetilde{p}_2(t) - g(t)\,\widetilde{q}_2(t)\,.$$

The proof of this Proposition 2.3.3 is completed. ■

Theorem 2.3.4 ***(Risk-sensitive necessary optimality conditions)***

Assume that Assumptions $(\mathbf{H_1}) - (\mathbf{H_2})$ hold. If (y^u, z^u, u) is an optimal solution of the risk-sensitive control problem $\{(2.1), (2.2), (2.3)\}$. Then there exist two pairs of $\mathcal{G}_t-$adapted processes $\left(V^\theta, \mathcal{D}\right)$, $(\widetilde{p}, \widetilde{q})$ which satisfy (2.18) and (2.19) such that

$$H_v^\theta\left(t, y_t^u, z_t^u, u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^\theta(t), \mathcal{D}_1(t)\right)(u_t - v_t) \leq 0, \tag{2.25}$$

for all $u \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}$-almost surely.

Proof. We arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\widetilde{p}_2, \widetilde{q}_2)$ and $\left(V^\theta, \mathcal{D}\right)$ which solve (2.18), where the Hamiltonian $\widetilde{H}^\theta$ associated with (2.4), given by (2.7) satisfies

$$\widetilde{H}^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) = \left\{\theta V_t^\theta\right\} H^\theta\left(t, y_t^u, z_t^u, u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^\theta(t), \mathcal{D}_1(t)\right), \tag{2.26}$$

and H^{θ} is the risk-sensitive Hamiltonian given by (2.20). Hence, since $V^{\theta} > 0$, the variational inequality (2.8) translates into

$$H_v^{\theta}\left(t, y_t^u, z_t^u, u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\theta}(t), \mathcal{D}_1(t)\right)(u_t - v_t) \leq 0,$$

for all $u \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}-$almost surely. This finishes the proof of Theorem 2.3.4. ■

2.4 Risk-Sensitive Sufficient Optimality Conditions

In this section, we study when the necessary optimality conditions for risk-sensitive (2.8) become sufficient.

Theorem 2.4.1 ***(Risk-sensitive sufficient optimality conditions)***

Assume that the functions Ψ and $(x_t^v, y_t^v, z_t^v, v_t) \to \widetilde{H}^{\theta}(t, x_t^v, y_t^v, z_t^v, v_t, \overrightarrow{p}(t), \overrightarrow{q}(t))$ are convex and for any $v_t \in \mathcal{U}$, $y_T^v = \xi$ is one dimensional $\mathcal{F}_T^W-$measurable random variable such that $\mathbb{E}|\xi|^2 < \infty$. Then u is an optimal solution of the control problem $\{(2.1), (2.2), (2.3)\}$ if it satisfies (2.8).

Proof. Let u be an arbitrary element of $\mathcal{U}$ (candidate to be optimal). For any $v \in \mathcal{U}$, we have

$$J^{\theta}(v) - J^{\theta}(u) = \mathbb{E}\left[\exp\left(\theta\left\{\Psi(y_0^v) + x_T^v\right\}\right)\right] - \mathbb{E}\left[\exp\left(\theta\left\{\Psi(y_0^u) + x_T^u\right\}\right)\right].$$

By applying the Taylor's expansion and the convexity of Ψ, we get

$$\begin{aligned} J^{\theta}(v) - J^{\theta}(u) \geq\ & \mathbb{E}\left[\theta \exp\left(\theta\left\{\Psi(y_0^u) + x_T^u\right\}\right)(x_T^v - x_T^u)\right] \\ & + \mathbb{E}\left[\theta \exp\left(\theta\left\{\Psi(y_0^u) + x_T^u\right\}\right)\Psi_y(y_0^u)(y_0^v - y_0^u)\right]. \end{aligned}$$

It follows from (2.6), that $p_1(T) = \theta A_T^\theta$, $p_2(0) = -\theta A_T^\theta \Psi_y(y_0^u)$, then we have

$$J^\theta(v) - J^\theta(u) \geq \mathbb{E}\left[p_1(T)\left(x_T^v - x_T^u\right)\right] - \mathbb{E}\left[p_2(0)\left(y_0^v - y_0^u\right)\right]. \tag{2.27}$$

Applying Itô's formula and taking expectation to $p_1(t)(x_t^v - x_t^u)$ and $p_2(t)(y_t^v - y_t^u)$, leads to

$$\mathbb{E}\left[p_1(T)\left(x_T^v - x_T^u\right)\right] = \mathbb{E}\left[\int_0^T \left(l\left(t, y_t^v, z_t^v, v_t\right) - l\left(t, y_t^u, z_t^u, u_t\right)\right) p_1(t)\, dt\right],$$

and

$$\begin{aligned}
-\mathbb{E}\left[p_2(0)\left(y_0^v - y_0^u\right)\right] = &-\mathbb{E}\left[\int_0^T \widetilde{H}_y^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(y_t^v - y_t^u\right) dt\right] \\
&- \mathbb{E}\left[\int_0^T \widetilde{H}_z^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(z_t^v - z_t^u\right) dt\right] \\
&- \mathbb{E}\left[\int_0^T \left(f\left(t, y_t^v, z_t^v, v_t\right) - f\left(t, y_t^u, z_t^u, u_t\right)\right) p_2(t)\, dt\right] \\
&- \mathbb{E}\left[\int_0^T \left(g\left(t, y_t^v, z_t^v, v_t\right) - g\left(t, y_t^u, z_t^u, u_t\right)\right) q_2(t)\, dt\right].
\end{aligned}$$

Putting the two above formulas into (2.27), we get

$$\begin{aligned}
J^\theta(v) - J^\theta(u) \geq\; &\mathbb{E}\left[\int_0^T \widetilde{H}^\theta\left(t, x_t^v, y_t^v, z_t^v, v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
&- \mathbb{E}\left[\int_0^T \widetilde{H}^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
&- \mathbb{E}\left[\int_0^T \widetilde{H}_y^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(y_t^v - y_t^u\right) dt\right] \\
&- \mathbb{E}\left[\int_0^T \widetilde{H}_z^\theta\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(z_t^v - z_t^u\right) dt\right].
\end{aligned} \tag{2.28}$$

Since the Hamiltonian $\widetilde{H}^{\theta}$ is convex with respect to (y^v, z^v, v), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \widetilde{H}^{\theta}\left(t, x_t^v, y_t^v, z_t^v, v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] - \mathbb{E}\left[\int_0^T \widetilde{H}^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
&\geq \mathbb{E}\left[\int_0^T \widetilde{H}_y^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(y_t^v - y_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T \widetilde{H}_z^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(z_t^v - z_t^u\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(v_t - u_t\right) dt\right],
\end{aligned}$$

Then, by using above inequality in (2.28), we obtain

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t, x_t^u, y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(v_t - u_t\right) dt\right] \geq 0.$$

In virtue of the necessary optimality conditions (2.8), the last inequality implies that $J^{\theta}(v) - J^{\theta}(u) \geq 0$. Thus the theorem is proved. ■

Remark 2.4.2 *In virtue of* (2.26) *there is a relationship between the Hamiltonian with respect to risk-neutral and the Hamiltonian with respect to risk-sensitive. In fact, we have*

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T \theta V_t^{\theta} H_v^{\theta}\left(t, y_t^u, z_t^u, u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\theta}(t), \mathcal{D}_1(t)\right)\left(v_t - u_t\right) dt\right] \geq 0,$$

we know that $\theta V_t^{\theta} > 0$. *Then the above inequality can be rewritten as*

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T H_v^{\theta}\left(t, y_t^u, z_t^u, u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), V^{\theta}(t), \mathcal{D}_1(t)\right)\left(v_t - u_t\right) dt\right] \geq 0.$$

In virtue of the necessary optimality conditions (2.25), *the last inequality implies that* $J^{\theta}(v) - J^{\theta}(u) \geq 0.$

2.5 Applications: A Linear Quadratic Risk-Sensitive Control Problem

We illustrate the chapter by given two different examples for the linear quadratic system.

2.5.1 Example 01

We provide a concrete example of a risk-sensitive backward doubly stochastic LQ problem, give the explicit optimal control and validate our major theoretical results in Theorem 2.4.1 (Risk-sensitive sufficient optimality conditions). First, let the control domain be $U = [-1, 1]$. Consider the following linear quadratic risk-sensitive control problem

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\frac{1}{2}\int_0^T v_t^2 dt + \frac{1}{2}\left(y_0^v\right)^2\right\}\right], \text{ subject to} \\ dy_t = -\left(Ay_t^v + Bz_t^v + Cv_t + D\right)dt - \left(A^{'}y_t^v + B^{'}z_t^v + C^{'}v_t + D^{'}\right)\overleftarrow{dB}_t + z_t^v dW_t, \\ y_T = \xi, \end{cases} \tag{2.29}$$

where A, B, C, D, $A^{'}$, $B^{'}$, $C^{'}$ and $D^{'}$ are positive real constants.

Let (y_t^v, z_t^v) be a solution of (2.29) associated with v_t. Then there exist unique $\mathcal{G}_t-$adapted two pairs of processes $(p_1, p_2), (q_1, q_2)$ of the following forward-backward doubly stochastic differential equations system (in short FBDSDEs) (called adjoint equation), according to equation (2.6):

$$\begin{cases} dp_1(t) = q_1(t)\, dW_t, \\ dp_2(t) = \left[Ap_2(t) + A^{'}q_2(t)\right]dt + \left[Bp_2(t) + B^{'}q_2(t)\right]dW_t - q_2(t)\overleftarrow{dB}_t, \\ p_1(T) = \theta A_T^\theta, \;\; p_2(0) = -\theta y_0^v A_T^\theta, \end{cases} \tag{2.30}$$

where

$$A_T^\theta := \exp\theta\left\{\frac{1}{2}\int_0^T v_t^2 dt + (y_0^v)^2\right\}.$$

We give the Hamiltonian $\widetilde{H}^\theta$ defined by

$$\begin{aligned}\widetilde{H}^\theta(t) &:= \widetilde{H}^\theta\left(t, x_t^v, y_t^v, z_t^v, v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\\ &= \frac{1}{2}v_t^2 p_1(t) - \left(Ay_t^v + Bz_t^v + Cv_t + D\right)p_2(t) - \left(A^{'}y_t^v + B^{'}z_t^v + C^{'}v_t + D^{'}\right)q_2(t).\end{aligned}$$

We have $\widetilde{H}_v^\theta(t) = \left[v_t p_1(t) - Cp_2(t) - C^{'}q_2(t)\right]$. Minimizing the Hamiltonian yields

$$u_t = \frac{1}{p_1^u(t)}\left(Cp_2^u(t) + C^{'}q_2^u(t)\right). \tag{2.31}$$

We only need to prove that u_t is an optimal control of (2.29).

Theorem 2.5.1 *(Risk-sensitive sufficient optimality conditions for a linear quadratic control problem). Assume that* $\theta > 0$ *and suppose that* u_t *satisfies* (2.31)*, where* $(\overrightarrow{p}, \overrightarrow{q})$ *satisfy* (2.30)*. Then* u_t *is the unique optimal control of the above BDSDE of the linear quadratic problem* (2.29).

Proof. From the definition of the functional cost J^θ, we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E}\left[\exp\theta\left\{\frac{1}{2}\int_0^T v_t^2 dt + \frac{1}{2}(y_0^v)^2\right\}\right] - \mathbb{E}\left[\exp\theta\left\{\frac{1}{2}\int_0^T u_t^2 dt + \frac{1}{2}(y_0^u)^2\right\}\right].$$

We put $x_T^v = \frac{1}{2}\int_0^T v_t^2 dt$, and by applying the Taylor's expansion, we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E}\left[\theta\exp\theta\left\{x_T^u + \frac{1}{2}(y_0^u)^2\right\}(x_T^v - x_T^u)\right] + \mathbb{E}\left[\theta y_0^u\exp\theta\left\{x_T^u + \frac{1}{2}(y_0^u)^2\right\}(y_0^v - y_0^u)\right].$$

It follows from (2.6) that $p_1^u(T) = \theta A_T^\theta$ and $p_2^u(0) = -\theta y_0^u A_T^\theta$. Then we have

$$J^\theta(v_t) - J^\theta(u_t) = \mathbb{E}\left[p_1^u(T)(x_T^v - x_T^u)\right] - \mathbb{E}\left[p_2^u(0)(y_0^v - y_0^u)\right]. \tag{2.32}$$

By applying Itô's formula to $p_1^u(t)(x_t^v - x_t^u)$ and $p_2^u(t)(y_t^v - y_t^u)$ that lead to

$$\mathbb{E}\left[p_1^u(T)(x_T^v - x_T^u)\right] = \mathbb{E}\left[\int_0^T \frac{1}{2}\left(v_t^2 - u_t^2\right)p_1^u(t)\,dt\right],$$

and

$$-\mathbb{E}\left[p_2^u(0)(y_0^v - y_0^u)\right] = -\mathbb{E}\left[\int_0^T C(v_t - u_t)\,p_2^u(t)\,dt\right] - \mathbb{E}\left[\int_0^T C^{'}(v_t - u_t)\,q_2^u(t)\,dt\right].$$

By replacing the two above formulas into (2.32), then we get

$$\begin{aligned} J^\theta(v_t) - J^\theta(u_t) = &\ \mathbb{E}\left[\int_0^T v_t(v_t - u_t)\,p_1^u(t)\,dt\right] + \mathbb{E}\left[\int_0^T u_t(v_t - u_t)\,p_1^u(t)\,dt\right] \\ &- \mathbb{E}\left[\int_0^T C(v_t - u_t)\,p_2^u(t)\,dt\right] - \mathbb{E}\left[\int_0^T C^{'}(v_t - u_t)\,q_2^u(t)\,dt\right]. \end{aligned}$$

Because of $\theta > 0$, we have $(v_t - u_t) > 0$. Thus we get the following result:

$$\begin{aligned} J^\theta(v_t) - J^\theta(u_t) \geq &\ \mathbb{E}\left[\int_0^T u_t(v_t - u_t)\,p_1^u(t)\,dt\right] - \mathbb{E}\left[\int_0^T C(v_t - u_t)\,p_2^u(t)\,dt\right] \\ &- \mathbb{E}\left[\int_0^T C^{'}(v_t - u_t)\,q_2^u(t)\,dt\right]. \end{aligned}$$

Then

$$J^\theta(v_t) - J^\theta(u_t) \geq \mathbb{E}\left[\int_0^T \left(u_t p_1^u(t) - C p_2^u(t) - C^{'} q_2^u(t)\right)(v_t - u_t)\,dt\right].$$

By replacing u_t with its value in (2.31), we obtain $J^\theta(v_t) \geq J^\theta(u_t)$, i.e. u_t is optimal.

This proof is finished. ■

2.5.2 Example 02

In this section, we apply the risk-sensitive maximum principles obtained in the previous section (Theorem 2.3.4) to deal with the linear-quadratic risk-sensitive stochastic optimal control problem

$\{(2.1),(2.2),(2.3)\}$ mentioned in Section 2.1. Our state dynamics is

$$\begin{cases} dy_t^v &= -\left(y_t^v + v_t\right)dt - \sigma v_t \overleftarrow{dB}_t + z_t^v dW_t, \\ y_T^v &= \xi, \end{cases} \tag{2.33}$$

and our functional cost is the following expected exponential-of-integral form:

$$J^\theta\left(v\left(.\right)\right) = \mathbb{E}\left[\exp\theta\left\{\int_0^T l\left(t, y_t^v, z_t^v, v_t\right)dt\right\}\right], \tag{2.34}$$

where

$$\theta > 0,\ \theta \neq 1,\ l\left(t, y_t^v, z_t^v, v_t\right) = \frac{1}{2}\left(v_t^2 + \left(y_t^v\right)^2\right).$$

We want to minimize (2.34) subject to (2.33) by choosing v over $\mathcal{U}$. Hence, we may apply Theorem 2.3.4 to solve our linear-quadratic risk-sensitive stochastic optimal control problem $\{(2.33),(2.34)\}$. The Hamiltonian function (2.20) is defined by

$$\begin{aligned} H^\theta\left(t\right) &:= H^\theta\left(t, y_t^v, z_t^v, v_t, \widetilde{p}_2\left(t\right), \widetilde{q}_2\left(t\right), V^\theta\left(t\right), \mathcal{D}_1\left(t\right)\right) \\ &= \frac{1}{2}\left(v_t^2 + \left(y_t^v\right)^2\right) - \left(y_t^v + v_t - \theta\mathcal{D}_1\left(t\right) z_t^v\right)\widetilde{p}_2\left(t\right) - \sigma v_t \widetilde{q}_2\left(t\right). \end{aligned} \tag{2.35}$$

Let $\left(y_t^u, z_t^u, u_t\right)$ be an optimal solution. The adjoint equation (2.18) can be written by

$$\begin{cases} d\widetilde{p}_2\left(t\right) &= \left[-y_t^v + \widetilde{p}_2\left(t\right)\right]dt - \theta\mathcal{D}_1\left(t\right)\widetilde{p}_2\left(t\right)dW_t^\theta - \left[\widetilde{q}_2\left(t\right) + \theta\mathcal{D}_2\left(t\right)\widetilde{p}_2\left(t\right)\right]\overleftarrow{dB}_t^\theta, \\ \widetilde{p}_2\left(T\right) &= 0. \end{cases} \tag{2.36}$$

Minimizing the Hamiltonian (2.35), we obtain

$$u_t = \widetilde{p}_2^u\left(t\right) + \sigma\widetilde{q}_2^u\left(t\right). \tag{2.37}$$

By substituting (2.37) into the BDSDE (2.33), we obtain

$$\begin{cases} dy_t^u &= -\left[y_t^u + \widetilde{p}_2^u\left(t\right) + \sigma\widetilde{q}_2^u\left(t\right)\right]dt - \left[\sigma\left(\widetilde{p}_2^u\left(t\right) + \sigma\widetilde{q}_2^u\left(t\right)\right)\right]\overleftarrow{dB}_t + z_t^u dW_t, \\ y_T^u &= \xi. \end{cases}$$

Similarly, by substituting equation (2.37) into the BDSDE (2.36), gives

$$\begin{cases} d\widetilde{p}_2^u(t) &= [-y_t^u + \widetilde{p}_2^u(t)]\,dt - \theta\mathcal{D}_1(t)\,\widetilde{p}_2^u(t)\,dW_t^\theta - [\widetilde{q}_2^u(t) + \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t)]\,\overleftarrow{dB}_t^\theta, \\ \widetilde{p}_2^u(T) &= 0. \end{cases} \tag{2.38}$$

Replacing $dW_t^\theta = dW_t - \theta\mathcal{D}_1(t)\,dt$ and $\overleftarrow{dB}_t^\theta = \overleftarrow{dB}_t - \theta\mathcal{D}_2(t)\,dt$ in (2.38), we get

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left[-y_t^u + \left(1+\theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right)\widetilde{p}_2^u(t) + \theta\mathcal{D}_2(t)\,\widetilde{q}_2^u(t)\right]dt, \\ &- [\widetilde{q}_2^u(t) + \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t)]\,\overleftarrow{dB}_t - \theta\mathcal{D}_1(t)\,\widetilde{p}_2^u(t)\,dW_t, \\ \widetilde{p}_2^u(T) &= 0. \end{cases} \tag{2.39}$$

Peng and Shi [34] introduced a type of time-symmetric forward-backward stochastic differential equations (SFBSDE in short), i.e., so-called fully coupled FBDSDEs. Therefore, an optimal solution $(\widetilde{p}_2^u, y^u, u)$ can be obtained by solving the following type of SFBSDE

$$\begin{cases} dy_t^u &= -[y_t^u + \widetilde{p}_2^u(t) + \sigma\widetilde{q}_2^u(t)]\,dt - [\sigma(\widetilde{p}_2^u(t) + \sigma\widetilde{q}_2^u(t))]\,\overleftarrow{dB}_t + z_t^u dW_t, \\ d\widetilde{p}_2^u(t) &= \left[-y_t^u + \left(1+\theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right)\widetilde{p}_2^u(t) + \theta\mathcal{D}_2(t)\,\widetilde{q}_2^u(t)\right]dt \\ &- [\widetilde{q}_2^u(t) + \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t)]\,\overleftarrow{dB}_t - \theta\mathcal{D}_1(t)\,\widetilde{p}_2^u(t)\,dW_t, \\ y_T^u &= \xi,\ \ \widetilde{p}_2^u(T) = 0. \end{cases} \tag{2.40}$$

Unfortunately, in such a system it is difficult to find the explicit solution. To solve this type of SFBSDE (2.40), we use a technique similar to the one used by Yong and Zhou [39]. We conjecture that the solution to (2.40) is related by

$$\widetilde{p}_2^u(t) = \varphi(t)\,y_t^u + \chi(t), \tag{2.41}$$

for some deterministic differentiable functions $\varphi(t)$ and $\chi(t)$. Application Itô's formula to (2.41) gives

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left[\overset{\bullet}{\varphi}(t)\,y_t^u - \varphi(t)\,y_t^u - \varphi(t)\,(\widetilde{p}_2^u(t) + \sigma\widetilde{q}_2^u(t)) + \overset{\bullet}{\chi}(t)\right]dt \\ &- \varphi(t)\,\sigma\,(\widetilde{p}_2^u(t) + \sigma\widetilde{q}_2^u(t))\,\overleftarrow{dB}_t + \varphi(t)\,z_t^u dW_t, \\ \widetilde{p}_2^u(T) &= 0. \end{cases} \tag{2.42}$$

Putting (2.41) into (2.42), we get

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left[\left(\overset{\bullet}{\varphi}(t) - \varphi^2(t) - \varphi(t)\right) y_t^u - \varphi(t)\chi(t) - \varphi(t)\sigma\widetilde{q}_2^u(t) + \overset{\bullet}{\chi}(t)\right] dt \\ &- \left[\varphi^2(t)\sigma y_t^u + \varphi(t)\chi(t)\sigma + \varphi(t)\sigma^2\widetilde{q}_2^u(t)\right] \overleftarrow{dB}_t + \varphi(t) z_t^u dW_t, \\ \widetilde{p}_2^u(T) &= 0. \end{cases} \tag{2.43}$$

On the other hand, after substituting (2.41) into (2.39), we arrive at

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left[\left(\left(1 + \theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right)\varphi(t) - 1\right) y_t^u \right. \\ &+ \left. \left(1 + \theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right)\chi(t) + \theta\mathcal{D}_2(t)\widetilde{q}_2^u(t)\right] dt \\ &- \left[\widetilde{q}_2^u(t) + \theta\mathcal{D}_2(t)\varphi(t) y_t^u + \theta\mathcal{D}_2(t)\chi(t)\right]\overleftarrow{dB}_t - \left[\theta\mathcal{D}_1(t)\varphi(t) y_t^u + \theta\mathcal{D}_1(t)\chi(t)\right] dW_t, \\ \widetilde{p}_2^u(T) &= 0. \end{cases} \tag{2.44}$$

Equating the coefficients of (2.43) and (2.44), we have

$$\left(\widetilde{p}_2^u(t), \widetilde{q}_2^u(t)\right) = \left(\varphi(t) y_t^u + \chi(t), \frac{\varphi(t)\left(\sigma\varphi(t) - \theta\mathcal{D}_2(t)\right) y_t^u + \left(\varphi(t)\sigma - \theta\mathcal{D}_2(t)\right)\chi(t)}{1 - \varphi(t)\sigma^2}\right), \tag{2.45}$$

where $\varphi(t)$ is the solution to the following Riccati type equation:

$$\begin{cases} \overset{\bullet}{\varphi}(t) - \varphi^2(t) - 2\varphi(t)\left(1 + \frac{1}{2}\theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right) + 1 = 0, \\ \varphi(T) = 0, \end{cases} \tag{2.46}$$

and $\chi(t)$ is a solution to the following ordinary differential equation:

$$\begin{cases} \overset{\bullet}{\chi}(t) - \left(\varphi(t) + 1 + \theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right)\chi(t) - \left(\theta\mathcal{D}_2(t) + \varphi(t)\sigma\right)\widetilde{q}_2^u(t) = 0, \\ \chi(T) = 0. \end{cases} \tag{2.47}$$

By using the same identification, we get

$$\chi(t) = -\frac{1}{\theta\mathcal{D}_1(t)}\varphi(t) z_t^u - \varphi(t) y_t^u. \tag{2.48}$$

Finally, by (2.37) and (2.45), we can get the optimal control in the following state in feedback form:

$$u_t = \frac{(1-\sigma\theta\mathcal{D}_2(t))}{1-\varphi(t)\sigma^2}\varphi(t)\,y_t^u + \frac{(1-\sigma\theta\mathcal{D}_2(t))}{1-\varphi(t)\sigma^2}\chi(t). \tag{2.49}$$

Putting (2.48) in (2.49) we get

$$u_t = -\frac{(1-\sigma\theta\mathcal{D}_2(t))}{(1-\varphi(t)\sigma^2)\,\theta\mathcal{D}_1(t)}\varphi(t)\,z_t^u, \tag{2.50}$$

where $\varphi(t)$ is determined by (2.46).

Theorem 2.5.2 *We assume that the pair* $(\varphi(t), \chi(t))$ *has the solution of system* (2.46) *and* (2.47)*. Then the optimal control of our linear-quadratic risk-sensitive stochastic optimal control problem* $\{(2.33), (2.34)\}$ *has the state feedback from* (2.50).

2.5.2.1 Solution of the Deterministic Functions $\varphi(t)$ and $\chi(t)$ via Riccati Equation

In the best our knowledge, it is very hard to find the explicit solution to Riccati equation in general. But in our case, we can found the explicit solution of

$$\begin{cases} \overset{\bullet}{\varphi}(t) - \varphi^2(t) + 2\varphi(t)K(t) + 1 = 0, \\ \varphi(T) = 0, \end{cases} \tag{2.51}$$

where

$$K(t) = -\left(1 + \frac{1}{2}\theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right). \tag{2.52}$$

We denote $\varphi(T) = s_1(T)$ the solution of (2.51), then the general solution is the form suit

$$\varphi(t) = s_1(t) + n_1(t).$$

By replacing $\varphi(t)$ by $s_1(t)+n_1(t)$ on (2.51), on obtain

$$\begin{cases} \overset{\bullet}{s}_1(t)+\overset{\bullet}{n}_1(t)=s_1(t)^2+n_1(t)^2+2s_1(t)\,n_1(t)-2K(t)\,s_1(t)-2K(t)\,n_1(t)-1, \\ s_1(T)+n_1(T)=0. \end{cases}$$

And because

$$\overset{\bullet}{s}_1(t)=s_1(t)^2-2K(t)\,s_1(t)-1.$$

Then

$$\overset{\bullet}{n}_1(t)=n_1(t)^2+2s_1(t)\,n_1(t)-2K(t)\,n_1(t).$$

Or

$$\overset{\bullet}{n}_1(t)-2s_1(t)\,n_1(t)+2K(t)\,n_1(t)=n_1(t)^2.$$

Let $s_1(t)=0$, then

$$\overset{\bullet}{n}_1(t)+2K(t)\,n_1(t)=n_1(t)^2, \tag{2.53}$$

is a Bernoulli's equation. The substitution necessary for the solution of this Bernoulli's equation (2.53) is then

$$o(t)=n_1^{1-2}(t)=\frac{1}{n_1(t)}.$$

It leads to the linear equation

$$\overset{\bullet}{o}(t)-2K(t)\,o(t)=-1. \tag{2.54}$$

Then the homogeneous solution $h(t)$ of (2.54) such that

$$\overset{\bullet}{h}(t)-2K(t)\,h(t)=0,$$

is given by

$$h(t)=I\exp\left(2\int_t^T K(s)\,ds\right).$$

Now we put

$$M(t) = I(t)\exp\left(2\int_t^T K(s)\,ds\right), \tag{2.55}$$

$M(t)$ is the particular solution of equation (2.54) i.e.,

$$\overset{\bullet}{I}(t)\exp\left(2\int_t^T K(s)\,ds\right)+2I(t)\,K(t)\exp\left(2\int_t^T K(s)\,ds\right)-2I(t)\,K(t)\exp\left(2\int_t^T K(s)\,ds\right) = -1$$

Hence,

$$\overset{\bullet}{I}(t) = -\exp\left(-2\int_t^T K(s)\,ds\right).$$

Then,

$$I(t) = \frac{1}{2K(t)}\exp\left(-2\int_t^T K(s)\,ds\right).$$

Then the equation (2.55) rewrite as follow

$$\begin{aligned} M(t) &= I(t)\exp\left(2\int_t^T K(s)\,ds\right) \\ &= \frac{1}{2K(t)}\exp\left(-2\int_t^T K(s)\,ds\right)\exp\left(2\int_t^T K(s)\,ds\right) \\ &= \frac{1}{2K(t)}. \end{aligned}$$

This concludes to that the general solution $o(t)$ of (2.54) is

$$\begin{aligned} o(t) &= h(t) + M(t) \\ &= I\exp\left(2\int_t^T K(s)\,ds\right) + \frac{1}{2K(t)}. \end{aligned}$$

Then the general solution of our problem (2.51) is given by

$$\varphi(t) = \frac{1}{I\exp\left(2\int_t^T K(s)\,ds\right) + \frac{1}{2K(t)}}. \tag{2.56}$$

We put

$$\alpha(t) = -\left(\varphi(t) + 1 + \theta^2\left(\mathcal{D}_1^2(t) + \mathcal{D}_2^2(t)\right)\right), \beta(t) = -\left(\theta\mathcal{D}_2(t) + \varphi(t)\sigma\right)\widetilde{q}_2(t), \tag{2.57}$$

we rewrite equation (2.47) as follows

$$\begin{cases} \overset{\bullet}{\chi}(t) + \alpha(t)\chi(t) + \beta(t) = 0, \\ \chi(T) = 0. \end{cases} \tag{2.58}$$

The explicit solution to equation (2.58) is

$$\chi(t) = \left[\exp\left(\int_t^T \alpha(s)\,ds\right)\right]\left[\int_t^T -\beta(s)\exp\left(\int_t^T \alpha(r)\,dr\right)ds\right], \tag{2.59}$$

where $\alpha(t)$, $\beta(t)$ are determined by (2.57).

Corollary 2.5.3 *The explicit solution of the Riccati equation* (2.51) *is given by* (2.56) *and equation* (2.58) *has an explicit solution given by* (2.59)*, where the determined* $K(t)$ *and* $\alpha(t)$*,* $\beta(t)$ *are given by* (2.52) *and* (2.57) *respectively.*

Corollary 2.5.4 *We assume that the pair* $(\varphi(t), \chi(t))$ *has the unique solution given by* (2.56), (2.59)*. Then the optimal control of the problem* $\{(2.33), (2.34)\}$ *has the state feedback from* (2.50)*, where the determined* $K(t)$ *and* $\alpha(t)$*,* $\beta(t)$ *are given by* (2.52) *and* (2.57) *respectively.*

CHAPTER 3

A Risk-Sensitive Stochastic Maximum Principle for Fully Coupled Forward-Backward Stochastic Differential Equations of Mean-Field Type with Application

CHAPTER 3

A Risk-Sensitive Stochastic Maximum Principle for Fully Coupled Forward-Backward Stochastic Differential Equations of Mean-Field Type with Application

In this chapter, we focus on an optimal control problem where the system is driven by a fully coupled forward-backward stochastic differential equation of mean-field type with risk-sensitive performance functional. We study the risk-neutral model for which an optimal solution exists as a preliminary step. This is an extension of an initial stochastic control problem to this type of risk-sensitive performance problem, where the admissible set of controls is convex. We establish necessary and sufficient optimality conditions for the risk-sensitive performance functional control problem. Finally, we illustrate our main result by giving two examples of risk-sensitive control problem under linear stochastic dynamics with an exponential quadratic cost function, the second example will be a mean-variance portfolio with a recursive utility functional optimization problem involving optimal control. The explicit expression of the optimal portfolio selection strategy is obtained in the state feedback.

3.1 Problem Formulation and Assumptions

Let T be a positive real number. Let $\left(\Omega, \mathcal{F}, \left(\mathcal{F}_t^W\right)_{t\geq 0}, \mathbb{P}\right)$ be a probability space filtered satisfying the usual conditions, in which $one-$dimensional standard Brownian motion $W = (W_t, 0 \leq t \leq T)$ is given. We assume that $\mathcal{F}_t^W$ is defined by $\forall t \geq 0, \mathcal{F}_t^W = \sigma\left(W_r, 0 \leq r \leq t\right) \vee \mathcal{N}$, where $\mathcal{N}$ denote the totality of $\mathbb{P}-$null sets of $\mathcal{F}$.

Let $\mathcal{M}^2\left([0,T], \mathbb{R}\right)$ denote the set of $one-$dimensional jointly measurable random process $\{\varphi_t, t \in [0,T]\}$ which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{M}^2([0,T],\mathbb{R})} =: \mathbb{E}\left[\int_0^T |\varphi_t|^2 \, dt\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

Similarly, we denote by $\mathcal{S}^2\left([0,T], \mathbb{R}\right)$ the set of continuous $one-$dimensional random process which satisfy the following conditions:

$(i) : \|\varphi\|_{\mathcal{S}^2([0,T],\mathbb{R})} =: \mathbb{E}\left[\sup_{t\in[0,T]} |\varphi_t|^2\right] < \infty$, $(ii) : \varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

Let U be a nonempty subset of $\mathbb{R}$.

Definition 3.1.1 *An admissible control v is a process with values in U such that $\mathbb{E}\left[\int_0^T |v_t|^2 \, dt\right] < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.*

For any $v \in \mathcal{U}$, we consider the following fully coupled forward-backward stochastic differential equation of mean-field type control system:

$$\begin{cases} dx_t^v &= b\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt \\ &+ \sigma\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dW_t, \\ x_0^v &= a, \\ dy_t^v &= -f\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt + z_t^v dW_t, \\ y_T^v &= \xi, \end{cases} \tag{3.1}$$

where $b : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are jointly measurable, and z_t^v is square integrable and the terminal condition ξ is a $\mathcal{F}_T^W-$measurable and square integrable random variable.

We defined the criterion to be minimized, with initial and terminal risk-sensitive performance functional cost, as follows

$$\begin{aligned} J^{\theta}\left(v\left(.\right)\right) = \mathbb{E}\Big[\exp\theta\Big\{\Phi\left(x_T^v, \mathbb{E}^{'}\left(x_T^v\right)\right) + \Psi\left(y_0^v, \mathbb{E}^{'}\left(y_0^v\right)\right) \\ + \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt\Big\}\Big], \end{aligned} \tag{3.2}$$

where $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $l : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are jointly measurable and θ is the risk-sensitive index.

The control problem is to minimize the functional J^{θ} over $\mathcal{U}$ if $u \in \mathcal{U}$ is an optimal control solution, that is

$$J^{\theta}\left(u\right) = \inf_{v \in \mathcal{U}} J^{\theta}\left(v\right). \tag{3.3}$$

Remark 3.1.2 *We use the Euclidean norm $|.|$ in $\mathbb{R}$, $\top$ is a transpose and Tr is the trace of a matrix. All the equalities and inequalities mentioned in this chapter are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0,T] \times \Omega$.*

Notation 3.1.3 *We use the following notations:*

$$\Upsilon = \begin{pmatrix} x^v \\ y^v \\ z^v \end{pmatrix}, \Upsilon^{'} = \begin{pmatrix} \mathbb{E}^{'}(x^v) \\ \mathbb{E}^{'}(y^v) \\ \mathbb{E}^{'}(z^v) \end{pmatrix} \text{ and } D\left(t, \Upsilon, \Upsilon^{'}\right) = \begin{pmatrix} b \\ \sigma \\ -f \end{pmatrix} \left(t, \Upsilon, \Upsilon^{'}\right),$$

where $\mathbb{E}^{'}$ *is the expected value with respect to the measure probability* $\mathbb{P}\left(w^{'}\right)$, *and* $\mathbb{E}$ *is the expected value with respect to the measure probability* $\mathbb{P}(w)$.

We assume the following assumptions

$(\mathbf{A_1})$ For each $\Upsilon, \Upsilon^{'} \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $D\left(t, \Upsilon, \Upsilon^{'}\right)$ is in $\mathcal{M}^2([0,T], \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ i.e. $D\left(t, \Upsilon, \Upsilon^{'}\right)$ is an $\mathcal{F}_t^W-$measurable process defined on $[0,T]$.

$(\mathbf{A_2})$ $D\left(t, \Upsilon, \Upsilon^{'}\right)$ is uniformly Lipschitz with respect to $\left(\Upsilon, \Upsilon^{'}\right)$. There exists a constant $k > 0$, such that $\left|D\left(t, \Upsilon_1, \Upsilon^{'}\right) - D\left(t, \Upsilon_2, \Upsilon^{'}\right)\right| \leq k\left|\Upsilon_1 - \Upsilon_2\right|, \forall \Upsilon_1, \Upsilon_2, \Upsilon^{'} \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \forall t \in [0,T]$.

We also need the following monotonic conditions introduced by Min et al. [27], which are the main assumptions in this chapter.

$(\mathbf{A_3})$ $\left\langle D\left(t, \Upsilon_1, \Upsilon^{'}\right) - D\left(t, \Upsilon_2, \Upsilon^{'}\right), \Upsilon_1 - \Upsilon_2\right\rangle \leq -\alpha_1 \left|x_1 - x_2\right|^2 - \alpha_2\left(\left|y_1 - y_2\right|^2 + \left|z_1 - z_2\right|^2\right)$, $\forall\ \Upsilon_1 = (x_1, y_1, z_1)^{\top}, \Upsilon_2 = (x_2, y_2, z_2)^{\top}, \Upsilon^{'} = \left(\mathbb{E}^{'}(x), \mathbb{E}^{'}(y), \mathbb{E}^{'}(z)\right)^{\top}, \forall t \in [0,T]$ where α_1 and α_2 are a positive constants.

Or we need the following

$(\mathbf{A_4})$ $\left\langle D\left(t, \Upsilon_1, \Upsilon^{'}\right) - D\left(t, \Upsilon_2, \Upsilon^{'}\right), \Upsilon_1 - \Upsilon_2\right\rangle \geq \alpha_1 \left|x_1 - x_2\right|^2 + \alpha_2\left(\left|y_1 - y_2\right|^2 + \left|z_1 - z_2\right|^2\right)$, $\forall\ \Upsilon_1 = (x_1, y_1, z_1)^{\top}, \Upsilon_2 = (x_2, y_2, z_2)^{\top}, \Upsilon^{'} = \left(\mathbb{E}^{'}(x), \mathbb{E}^{'}(y), \mathbb{E}^{'}(z)\right)^{\top}, \forall t \in [0,T]$ where α_1 and α_2 are a positive constants.

Theorem 3.1.4 *For any given admissible control* $v(.)$, *and under the above Assumptions* $(\mathbf{A_1}) - (\mathbf{A_3})$. *Then the fully coupled FBSDE of mean-field type control* (3.1) *has a unique solution*

$(x_t^v, y_t^v, z_t^v) \in \mathcal{S}^2([0,T], \mathbb{R}) \times \mathcal{S}^2([0,T], \mathbb{R}) \times \mathcal{M}^2([0,T], \mathbb{R})$.

Proof. See [27] Theorem 6 page 3. ∎

A control that solves the problem $\{(3.1), (3.2), (3.3)\}$ is called optimal. Our goal is to establish risk-sensitive necessary and sufficient optimality conditions, satisfied by a given optimal control, in the form of mean-field stochastic maximum principle with a risk-sensitive performance functional type.

We also assume the following

$(\mathbf{A_5})$

i) b, σ, f, l, Φ and Ψ are continuously differentiable with respect to

$\left(x^v, y^v, z^v, \mathbb{E}'(x^v), \mathbb{E}'(y^v), \mathbb{E}'(z^v), v\right)$.

ii) All the derivatives of b, σ, f and l are bounded by

$C\left(1 + |x^v| + |y^v| + |z^v| + \left|\mathbb{E}'(x^v)\right| + \left|\mathbb{E}'(y^v)\right| + \left|\mathbb{E}'(z^v)\right| + |v|\right)$.

iii) All the derivatives of Φ and Ψ are bounded by $C\left(1 + |x^v| + \left|\mathbb{E}'(x^v)\right|\right)$ and $C\left(1 + |y^v| + \left|\mathbb{E}'(y^v)\right|\right)$ respectively.

Under the above assumptions, for every $v \in \mathcal{U}$ equation (3.1) has a unique strong solution, and the cost function J^θ is well defined from $\mathcal{U}$ into $\mathbb{R}$.

For more details in this kind of problem the reader can see the paper of Min et al. [27].

3.2 Relation between the Risk-Neutral and Risk-Sensitive Stochastic Maximum Principle

The proof of our risk-sensitive stochastic maximum principle necessitates a certain an auxiliary state process m_t^v, which is the solution of the following stochastic differential equation of mean-field type control (SDE of mean-field type control):

$$dm_t^v = l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt, \quad m_0^v = 0.$$

Our control problem of $\{(3.1), (3.2), (3.3)\}$ and from the above auxiliary process, new control problem translated is equivalent to

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[\exp\theta\left\{\Phi\left(x_T^v, \mathbb{E}^{'}\left(x_T^v\right)\right) + \Psi\left(y_0^v, \mathbb{E}^{'}\left(y_0^v\right)\right) + m_T^v\right\}\right] = \inf_{v\in\mathcal{U}} \mathbb{E}\left[\varphi\left(x_T^v, y_0^v, m_T^v\right)\right], \\ \text{subject to} \\ \begin{aligned} dm_t^v &= l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt, \\ dx_t^v &= b\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt \\ &+ \sigma\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dW_t, \\ dy_t^v &= -f\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt + z_t^v dW_t, \\ m_0^v &= 0, \ \ x_0^v = a, \ \ y_T^v = \xi. \end{aligned} \end{cases} \tag{3.4}$$

We require the following notation

$$\begin{aligned} A_T^\theta := \exp\theta\Big\{&\Phi\left(x_T^v, \mathbb{E}^{'}\left(x_T^v\right)\right) + \Psi\left(y_0^v, \mathbb{E}^{'}\left(y_0^v\right)\right) \\ &+ \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt\Big\}, \end{aligned}$$

and we put also

$$\Theta_T := \Phi\left(x_T^v, \mathbb{E}^{'}\left(x_T^v\right)\right) + \Psi\left(y_0^v, \mathbb{E}^{'}\left(y_0^v\right)\right) + \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt.$$

Then risk-sensitive loss of functional is given by

$$\begin{aligned}\mathcal{H}(\theta, v) &:= \frac{1}{\theta}\log\left[\mathbb{E}\left(\exp\theta\left\{\Phi\left(x_T^v, \mathbb{E}^{'}\left(x_T^v\right)\right) + \Psi\left(y_0^v, \mathbb{E}^{'}\left(y_0^v\right)\right)\right.\right.\right.\\ &\left.\left.\left.+ \int_0^T l\left(t, x_t^v, y_t^v, z_t^v, \mathbb{E}^{'}\left(x_t^v\right), \mathbb{E}^{'}\left(y_t^v\right), \mathbb{E}^{'}\left(z_t^v\right), v_t\right) dt\right\}\right)\right]\\ &= \frac{1}{\theta}\log\left[\mathbb{E}\left(\exp\theta\Theta_T\right)\right].\end{aligned}$$

When the risk-sensitive index θ is small, by Lemma 1.4.1 the loss functional $\mathcal{H}(\theta, v)$ can be expanded as

$$\mathbb{E}\left(\Theta_T\right) + \frac{\theta}{2}Var\left(\Theta_T\right) + O\left(\theta^2\right),$$

where, $Var\left(\Theta_T\right)$ denotes the variance of Θ_T. If $\theta < 0$, the variance of Θ_T, as a measure of risk, improves the performance $\mathcal{H}(\theta, v)$, in this case the optimizer called risk seeker. But, when $\theta > 0$, the variance of Θ_T worsens the performance $\mathcal{H}(\theta, v)$, in this case the optimizer called risk averse. The risk-neutral loss functional $\mathbb{E}\left(\Theta_T\right)$ can be seen as a limit of risk-sensitive functional $\mathcal{H}(\theta, v)$ when $\theta \to 0$.

Next, let us introduce the following notations.

Notation 3.2.1 *For convenience, we will use the following notations throughout this chapter. For* $\phi \in \{b, \sigma, f, l\}$, *we define*

$$\begin{cases}\phi(t) &= \phi\left(t, \mathcal{O}^v(t), v_t\right),\\ \partial\phi(t) &= \phi\left(t, \mathcal{O}^v(t), v_t\right) - \phi\left(t, \mathcal{O}^u(t), u_t\right),\\ \phi_\zeta(t) &= \dfrac{\partial}{\partial\zeta}\phi\left(t, \mathcal{O}^v(t), v_t\right),\end{cases}$$

where

$$\mathcal{O}^u(t) = x_t^u, y_t^u, z_t^u, \mathbb{E}'(x_t^u), \mathbb{E}'(y_t^u), \mathbb{E}'(z_t^u),\ \mathcal{O}^v(t) = x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v),$$

$$\zeta = x, y, z, \bar{x}, \bar{y}, \bar{z}, v, \text{ and } v_t \text{ is an admissible control from } \mathcal{U}.$$

We assume that Assumptions $(\mathbf{A_1}) - (\mathbf{A_5})$ hold. We may apply the SMP for a risk-neutral of fully coupled forward-backward of mean-field type control from Min et al. [27] and with augmented state dynamics (m^u, x^u, y^u, z^u) to derive the adjoint equation. There exist unique $\mathcal{F}_t^W-$adapted three pairs of processes $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3) solve the following matrix system of BSDEs:

$$\begin{cases} d\overrightarrow{p}(t) = -A(t)\,dt + R(t)\,dW_t, \\ \begin{pmatrix} p_1(T) \\ p_2(T) \\ p_3(0) \end{pmatrix} = \theta A_T^\theta \begin{pmatrix} 1 \\ \Phi_x\left(x_T^u, \mathbb{E}'(x_T^u)\right) \\ -\Psi_y\left(y_0^u, \mathbb{E}'(y_0^u)\right) \end{pmatrix} + \theta \mathbb{E}' \left(A_T^\theta \begin{pmatrix} 0 \\ \Phi_{\bar{x}}\left(x_T^u, \mathbb{E}'(x_T^u)\right) \\ -\Psi_{\bar{y}}\left(y_0^u, \mathbb{E}'(y_0^u)\right) \end{pmatrix} \right), \end{cases} \tag{3.5}$$

with

$$\mathbb{E}\left[\sum_{i=1}^{3} \sup_{t\in[0,T]} |p_i(t)|^2 + \sum_{i=1}^{2} \int_0^T |q_i(t)|^2\,dt\right] < \infty,$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}$$
$$+ \mathbb{E}'\left[\begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}\right] + \mathbb{E}'\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}\right],$$

and

$$R(t)=\begin{pmatrix} q_1(t) \\ q_2(t) \\ \delta^\theta(t) \end{pmatrix},$$

such that

$$\delta^\theta(t)=-Tr\left[\begin{pmatrix} l_z(t) & b_z(t) \\ \sigma_z(t) & -f_z(t) \end{pmatrix}\begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix}\right] \\ -Tr\left[\mathbb{E}^{'}\left[\begin{pmatrix} l_{\bar{z}}(t) & b_{\bar{z}}(t) \\ \sigma_{\bar{z}}(t) & -f_{\bar{z}}(t) \end{pmatrix}\begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix}\right]\right].$$

We suppose here that $\widetilde{H}^\theta$ be the Hamiltonian associated with the optimal state dynamics (m^u, x^u, y^u, z^u) and let the three pairs of adjoint processes $(\overrightarrow{p}(t), \overrightarrow{q}(t))$ be given by

$$\widetilde{H}^\theta(t) := \widetilde{H}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) = \begin{pmatrix} l(t) \\ b(t) \\ -f(t) \end{pmatrix}\left(\overrightarrow{p}(t)\right)^\top + \begin{pmatrix} 0 \\ \sigma(t) \\ 0 \end{pmatrix}\left(\overrightarrow{q}(t)\right)^\top. \quad (3.6)$$

Theorem 3.2.2 *Assume that* $(\mathbf{A_1})-(\mathbf{A_5})$ *hold. If* (m^u, x^u, y^u, z^u) *is an optimal solution of the risk-neutral control problem* (3.4) *, then there exist three pairs of* $\mathcal{F}_t^W-$*adapted processes* (p_1, q_1) , (p_2, q_2) *and* (p_3, q_3) *that satisfy* (3.5) *, such that*

$$\widetilde{H}_v^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(u_t - v_t) \leq 0, \quad (3.7)$$

for all $u \in \mathcal{U}$*, almost every t and* $\mathbb{P}-$*almost surely, where* $\widetilde{H}_v^\theta(t)$ *is defined in Notation* 3.2.1.

Proof. We suppose that the Assumptions $(\mathbf{A_1})-(\mathbf{A_5})$ hold, we may combine the SMP for risk-neutral of controlled fully coupled FBSDE of mean-field type from [27] with the results of Yong [38, 39]. ■

3.3 New Adjoint Equations and Risk-Sensitive Necessary Optimality Conditions

To the best of our acknowledge, the Theorem 3.2.2 is the SMP for the risk-neutral of fully coupled forward-backward of mean-field type control problem. We follow the new approach has been used in [10, 11, 13], a transformation of the adjoint processes (p_1, q_1), (p_2, q_2) and (p_3, q_3) has been suggested in such a way to the first component (p_1, q_1) in (3.5) has been omitted, and express the SMP in terms of only the last two adjoint processes, that we denote them by $(\widetilde{p}_2, \widetilde{q}_2)$ and $(\widetilde{p}_3, \widetilde{q}_3)$.

Noting that $dp_1(t) = q_1(t)\, dW_t$ and $p_1(T) = \theta A_T^\theta$, the explicit solution of this BSDE is

$$p_1(t) = \theta \mathbb{E}\left[A_T^\theta \mid \mathcal{F}_t^W\right] = \theta V_t^\theta, \tag{3.8}$$

where $V_t^\theta := \mathbb{E}\left[A_T^\theta \mid \mathcal{F}_t^W\right], \forall\, 0 \le t \le T$.

In view of (3.8), it would be natural to choose a transformation of $(\overrightarrow{p}, \overrightarrow{q})$ into an adjoint process $(\widetilde{p}, \widetilde{q})$, where $\widetilde{p}_1(t) = \dfrac{1}{\theta V_t^\theta} p_1(t) = 1$.

We consider the following transform:

$$\widetilde{p}(t) = \begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} := \frac{1}{\theta V_t^\theta} \overrightarrow{p}(t), \quad 0 \le t \le T. \tag{3.9}$$

By using (3.5) and (3.9), we have

$$\widetilde{p}(.) := \begin{pmatrix} \widetilde{p}_1(T) \\ \widetilde{p}_2(T) \\ \widetilde{p}_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ \Phi_x\left(x_T^u, \mathbb{E}'(x_T^u)\right) \\ -\Psi_y\left(y_0^u, \mathbb{E}'(y_0^u)\right) \end{pmatrix} + \frac{1}{V_T^\theta}\mathbb{E}'\left(V_T^\theta \begin{pmatrix} 0 \\ \Phi_{\bar{x}}\left(x_T^u, \mathbb{E}'(x_T^u)\right) \\ -\Psi_{\bar{y}}\left(y_0^u, \mathbb{E}'(y_0^u)\right) \end{pmatrix}\right).$$

The following properties of the generic martingale V^{θ} are essential in order to investigate the properties of these new process $(\widetilde{p}(t), \widetilde{q}(t))$.

In this part, we want to prove the relationship between the expected exponential utility and the quadratic backward stochastic differential equation.

First of all, it's very important to write the expected exponential utility under this form

$$\exp\left(\theta\Lambda_t^{\theta}\right) = \mathbb{E}\left[A_{t,T}^{\theta} \mid \mathcal{F}_t^W\right] = \mathbb{E}\left[\exp\theta\left\{\Phi\left(x_T^u, \mathbb{E}'\left(x_T^u\right)\right) + \Psi\left(y_0^u, \mathbb{E}'\left(y_0^u\right)\right) + \int_t^T l(s)\,ds\right\} \mid \mathcal{F}_t^W\right]. \tag{3.10}$$

Lemma 3.3.1 *The necessary and sufficient condition for the expected exponential utility* (3.10) *to be hold, is the following quadratic backward stochastic differential equation*

$$\begin{cases} d\Lambda_t^{\theta} &= -\left(l(t) + \dfrac{\theta}{2}\left|\mathcal{D}(t)\right|^2\right)dt + \mathcal{D}(t)\,dW_t, \\ \Lambda_T^{\theta} &= \Phi\left(x_T^u, \mathbb{E}'\left(x_T^u\right)\right) + \Psi\left(y_0^u, \mathbb{E}'\left(y_0^u\right)\right), \end{cases} \tag{3.11}$$

where $\mathbb{E}\left[\int_0^T \left|\mathcal{D}(t)\right|^2 dt\right] < \infty$.

Proof. By the same technique in Lemma 1.2.1, we can prove Lemma 3.3.1. ■

As is proved in Lemma 3.3.1, the process Λ^{θ} is the first component of the $\mathcal{F}_t^W-$adapted pair of processes $\left(\Lambda^{\theta}, \mathcal{D}\right)$ which is the unique solution to the quadratic backward stochastic differential equation (3.11).

Next, we will state and prove the necessary optimality conditions for the system driven by a fully coupled FBSDE of mean-field type control with a risk-sensitive performance functional kind.

To this end, let us summarize and prove some Lemmas that we will use thereafter.

Lemma 3.3.2 *Suppose that Assumption* $(\mathbf{A_5})$ *holds. Then*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Lambda_t^{\theta}\right|\right]\leq C_T, \tag{3.12}$$

where, C_T *is a positive constant that depends only on* T *and the boundedness of* l, Φ *and* Ψ.

In particular, V^{θ} *solves the following linear BSDE*

$$dV_t^{\theta}=\theta\mathcal{D}\left(t\right)V_t^{\theta}dW_t,\quad V_T^{\theta}=A_T^{\theta}. \tag{3.13}$$

Hence, the process defined on $\left(\Omega,\mathcal{F},\left(\mathcal{F}_t^W\right)_{t\geq0},\mathbb{P}\right)$ *by* L_t^{θ}, *where*

$$L_t^{\theta}:=\frac{V_t^{\theta}}{V_0^{\theta}}=\exp\left(-\frac{\theta^2}{2}\int_0^t\left|\mathcal{D}\left(s\right)\right|^2ds+\theta\int_0^t\mathcal{D}\left(s\right)dW_s\right),\quad\forall\,0\leq t\leq T, \tag{3.14}$$

is a uniformly bounded $\mathcal{F}_t^W-$*martingale.*

Proof. The proof is similar to Lemma 2.3.2, by using the expression of (3.13), we can write

$$V_t^{\theta}=\exp\left(-\frac{\theta^2}{2}\int_0^t\mathcal{D}^2\left(s\right)ds+\theta\int_0^t\mathcal{D}\left(s\right)dW_s+\theta\Lambda_0^{\theta}\right).$$

Then,

$$L_t^{\theta}:=\frac{V_t^{\theta}}{V_0^{\theta}}=\exp\left(-\frac{\theta^2}{2}\int_0^t\mathcal{D}^2\left(s\right)ds+\theta\int_0^t\mathcal{D}\left(s\right)dW_s\right).\quad0\leq t\leq T.$$

In view of (3.12), the above equality is a uniformly bounded $\mathcal{F}_t^W-$martingale. ■

Proposition 3.3.3 *The second and the third risk-sensitive adjoint equations for* $(\widetilde{p}_2,\widetilde{q}_2)$, $(\widetilde{p}_3,\widetilde{q}_3)$ *and*

$\left(V^{\theta},\mathcal{D}\right)$ *becomes*

$$\begin{cases} d\widetilde{p}_2(t) &= -\left[H_x^{\theta}(t)+\frac{1}{V_t^{\theta}}\mathbb{E}'\left[V_t^{\theta}H_{\bar{x}}^{\theta}(t)\right]\right]dt+\left[\widetilde{q}_2(t)-\theta\mathcal{D}_2(t)\widetilde{p}_2(t)\right]dW_t^{\theta}, \\ \widetilde{p}_2(T) &= \Phi_x\left(x_T^v,\mathbb{E}'\left(x_T^v\right)\right)+\frac{1}{V_T^{\theta}}\mathbb{E}'\left[V_T^{\theta}\Phi_{\bar{x}}\left(x_T^v,\mathbb{E}'\left(x_T^v\right)\right)\right], \\ d\widetilde{p}_3(t) &= -\left[H_y^{\theta}(t)+\frac{1}{V_t^{\theta}}\mathbb{E}'\left[V_t^{\theta}H_{\bar{y}}^{\theta}(t)\right]\right]dt-\left[H_z^{\theta}(t)+\frac{1}{V_t^{\theta}}\mathbb{E}'\left[V_t^{\theta}H_{\bar{z}}^{\theta}(t)\right]\right]dW_t^{\theta}, \\ \widetilde{p}_3(0) &= -\Psi_y\left(y_0^v,\mathbb{E}'\left(y_0^v\right)\right)-\frac{1}{V_T^{\theta}}\mathbb{E}'\left[V_T^{\theta}\Psi_{\bar{y}}\left(y_0^v,\mathbb{E}'\left(y_0^v\right)\right)\right], \\ dV_t^{\theta} &= \theta\mathcal{D}(t)V_t^{\theta}dW_t, \\ V_T^{\theta} &= A_T^{\theta}. \end{cases} \tag{3.15}$$

The system (3.15) *admits a unique* $\mathcal{F}_t^W$*−adapted solution* $\left(\widetilde{p},\widetilde{q},V^{\theta},\mathcal{D}\right)$*, such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\widetilde{p}(t)\right|^2+\sup_{t\in[0,T]}\left|V^{\theta}(t)\right|^2+\int_0^T\left(\left|\widetilde{q}(t)\right|^2+\left|\mathcal{D}(t)\right|^2\right)dt\right]<\infty, \tag{3.16}$$

where

$$H^{\theta}(t):=H^{\theta}\left(t,\mathcal{O}^v(t),v_t,\widetilde{p}_2(t),\widetilde{q}_2(t),\widetilde{p}_3(t),V^{\theta}(t),\mathcal{D}_3(t)\right) \tag{3.17}$$

$$=l(t)+b(t)\widetilde{p}_2(t)+\sigma(t)\widetilde{q}_2(t)-\left(f(t)-\theta\mathcal{D}_3(t)z_t^v\right)\widetilde{p}_3(t).$$

Proof. We hope to identify the processes $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that

$$d\widetilde{p}(t)=-\widetilde{\alpha}(t)dt+\widetilde{\beta}(t)dW_t. \tag{3.18}$$

By applying Itô's formula to the processes $\overrightarrow{p}(t)=\theta V_t^{\theta}\widetilde{p}(t)$ and using the expression of V^{θ} in

(3.13) , we obtain

$$d\widetilde{p}(t) = -\frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} dt - \frac{1}{\theta V_t^\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} dt$$
$$-\frac{1}{\theta V_t^\theta}\mathbb{E}'\left[\begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix}\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}\right] dt$$
$$-\frac{1}{\theta V_t^\theta}\mathbb{E}'\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix}\begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}\right] dt - \theta\begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix}\widetilde{\beta}(t)\, dt$$
$$+\frac{1}{\theta V_t^\theta}\begin{pmatrix} q_1(t) \\ q_2(t) \\ \delta^\theta(t) \end{pmatrix} dW_t - \theta\begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix}\widetilde{p}(t)\, dW_t.$$

By identifying the coefficients of above equation to (3.18) , and using the relation $\widetilde{p}(t) = \dfrac{1}{\theta V_t^\theta}\overrightarrow{p}(t)$, the diffusion coefficient $\widetilde{\beta}(t)$ it will be written as

$$\widetilde{\beta}(t) = \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{\delta}^\theta(t) \end{pmatrix} - \theta\begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix}\widetilde{p}(t), \tag{3.19}$$

and the drift term of the process $\widetilde{p}(t)$

$$\begin{aligned}\widetilde{\alpha}(t) = &\begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} \\ &+ \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right] \\ &+ \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix}\right] + \theta \begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix}\widetilde{\beta}(t).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}d\widetilde{p}(t) = &-\begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt \\ &- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right] dt \\ &- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix}\right] dt - \theta \begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix}\widetilde{\beta}(t)\, dt + \widetilde{\beta}(t)\, dW_t.\end{aligned}$$

It is easily verified that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\left[dW_t - \theta\mathcal{D}_1(t)\, dt\right], \quad \widetilde{p}_1(T) = 1.$$

In view of (3.14) , we may use Girsanov's Theorem (see [12], Theorem 2.1 page 115) to claim that

$$d\widetilde{p}_1(t) = \widetilde{\beta}_1(t)\, dW_t^{\theta}, \quad \mathbb{P}^{\theta} - a.s, \quad \widetilde{p}_1(T) = 1,$$

where $dW_t^{\theta} = dW_t - \theta \mathcal{D}(t)\, dt$ is a $\mathbb{P}^{\theta}-$Brownian motion, where

$$\left.\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}}\right|_{\mathcal{F}_t^W} := L_t^{\theta} = \exp\left(\theta \int_0^t \mathcal{D}(s)\, dW_s - \frac{\theta^2}{2}\int_0^t |\mathcal{D}(s)|^2\, ds\right), \quad 0 \le t \le T.$$

In view of (3.14), the probability measures $\mathbb{P}^{\theta}$ and $\mathbb{P}$ are in fact equivalent. Hence, noting that $\widetilde{p}_1(t) := \frac{1}{\theta V_t^{\theta}} p_1(t)$ is square integrable, we get that $\widetilde{p}_1(t) = \mathbb{E}^{\theta}\left[\widetilde{p}_1(T) \mid \mathcal{F}_t^W\right] = 1$. Thus, its quadratic variation $\int_0^T |\widetilde{q}_1(t)|^2\, dt = 0$. This implies that, for almost every $0 \le t \le T$, $\widetilde{q}_1(t) = 0$, $\mathbb{P}^{\theta}$ and $\mathbb{P}-a.s$, we have

$$\begin{aligned} d\widetilde{p}(t) = &-\begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt \qquad (3.20) \\ &- \frac{1}{V_t^{\theta}}\mathbb{E}'\left[V_t^{\theta}\begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right] dt \\ &- \frac{1}{V_t^{\theta}}\mathbb{E}'\left[V_t^{\theta}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix}\right] dt + \widetilde{\beta}(t)\, dW_t^{\theta}. \end{aligned}$$

Now replacing (3.19) in (3.20) , to obtain

$$
\begin{aligned}
d\widetilde{p}(t) =& -\begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix} dt - \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix} dt \\
&- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) \\ \widetilde{p}_2(t) \\ \widetilde{p}_3(t) \end{pmatrix}\right] dt \\
&- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{\bar{x}}(t) & 0 \\ 0 & \sigma_{\bar{y}}(t) & 0 \end{pmatrix}\begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{q}_3(t) \end{pmatrix}\right] dt + \begin{pmatrix} \widetilde{q}_1(t) \\ \widetilde{q}_2(t) \\ \widetilde{\delta}^\theta(t) \end{pmatrix} dW_t^\theta - \theta \begin{pmatrix} \mathcal{D}_1(t) \\ \mathcal{D}_2(t) \\ \mathcal{D}_3(t) \end{pmatrix} \widetilde{p}(t)\, dW_t^\theta,
\end{aligned} \tag{3.21}
$$

where

$$
\begin{aligned}
\widetilde{\delta}^\theta(t) =& -Tr\left[\begin{pmatrix} l_z(t) & b_z(t) \\ \sigma_z(t) & -f_z(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) & \widetilde{q}_2(t) \\ \widetilde{p}_2(t) & \widetilde{p}_3(t) \end{pmatrix}\right] \\
&- Tr\left[\frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta \begin{pmatrix} l_{\bar{z}}(t) & b_{\bar{z}}(t) \\ \sigma_{\bar{z}}(t) & -f_{\bar{z}}(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1(t) & \widetilde{q}_2(t) \\ \widetilde{p}_2(t) & \widetilde{p}_3(t) \end{pmatrix}\right]\right].
\end{aligned}
$$

From (3.21), we get

$$
\begin{aligned}
d\widetilde{p}_2(t) =& -\left[l_x(t) + b_x(t)\widetilde{p}_2(t) - f_x(t)\widetilde{p}_3(t) + \sigma_x(t)\widetilde{q}_2(t)\right] dt \\
&- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta\left[l_{\bar{x}}(t) + b_{\bar{x}}(t)\widetilde{p}_2(t) - f_{\bar{x}}(t)\widetilde{p}_3(t) + \sigma_{\bar{x}}(t)\widetilde{q}_2(t)\right]\right] dt \\
&+ \left[\widetilde{q}_2(t) - \theta\mathcal{D}_2(t)\widetilde{p}_2(t)\right] dW_t^\theta,
\end{aligned} \tag{3.22}
$$

$$
\begin{aligned}
d\widetilde{p}_3(t) =& -\left[l_y(t) + b_y(t)\widetilde{p}_2(t) - f_y(t)\widetilde{p}_3(t) + \sigma_y(t)\widetilde{q}_2(t)\right] dt \\
&- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta\left[l_{\bar{y}}(t) + b_{\bar{y}}(t)\widetilde{p}_2(t) - f_{\bar{y}}(t)\widetilde{p}_3(t) + \sigma_{\bar{y}}(t)\widetilde{q}_2(t)\right]\right] dt \\
&- \left[l_z(t) + b_z(t)\widetilde{p}_2(t) - \left(f_z(t) - \theta\mathcal{D}_3(t)\right)\widetilde{p}_3(t) + \sigma_z(t)\widetilde{q}_2(t)\right] dW_t^\theta \\
&- \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta\left[l_{\bar{z}}(t) + b_{\bar{z}}(t)\widetilde{p}_2(t) - f_{\bar{z}}(t)\widetilde{p}_3(t) + \sigma_{\bar{z}}(t)\widetilde{q}_2(t)\right]\right] dW_t^\theta.
\end{aligned} \tag{3.23}
$$

We can rewrite (3.13), (3.22) and (3.23) as the system below

$$\begin{cases} d\widetilde{p}_2(t) &= -\left[H_x^\theta(t) + \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta H_{\bar{x}}^\theta(t)\right]\right]dt + \left[\widetilde{q}_2(t) - \theta\mathcal{D}_2(t)\widetilde{p}_2(t)\right]dW_t^\theta, \\ \widetilde{p}_2(T) &= \Phi_x\left(x_T^v, \mathbb{E}'(x_T^v)\right) + \frac{1}{V_T^\theta}\mathbb{E}'\left[V_T^\theta\Phi_{\bar{x}}\left(x_T^v, \mathbb{E}'(x_T^v)\right)\right], \\ d\widetilde{p}_3(t) &= -\left[H_y^\theta(t) + \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta H_{\bar{y}}^\theta(t)\right]\right]dt - \left[H_z^\theta(t) + \frac{1}{V_t^\theta}\mathbb{E}'\left[V_t^\theta H_{\bar{z}}^\theta(t)\right]\right]dW_t^\theta, \\ \widetilde{p}_3(0) &= -\Psi_y\left(y_0^v, \mathbb{E}'(y_0^v)\right) - \frac{1}{V_T^\theta}\mathbb{E}'\left[V_T^\theta\Psi_{\bar{y}}\left(y_0^v, \mathbb{E}'(y_0^v)\right)\right], \\ dV_t^\theta &= \theta\mathcal{D}(t)V_t^\theta dW_t, \\ V_T^\theta &= A_T^\theta. \end{cases}$$

The system (3.15) admits a unique $\mathcal{F}_t^W-$adapted solution $\left(\widetilde{p}, \widetilde{q}, V^\theta, \mathcal{D}\right)$, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\widetilde{p}(t)|^2 + \sup_{t\in[0,T]}\left|V^\theta(t)\right|^2 + \int_0^T\left(|\widetilde{q}(t)|^2 + |\mathcal{D}(t)|^2\right)dt\right] < \infty,$$

where

$$H^\theta(t) := H^\theta\left(t, \mathcal{O}^v(t), v_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^\theta(t), \mathcal{D}_3(t)\right)$$

$$= l(t) + b(t)\widetilde{p}_2(t) + \sigma(t)\widetilde{q}_2(t) - \left(f(t) - \theta\mathcal{D}_3(t)z_t^v\right)\widetilde{p}_3(t).$$

This finished the proof of Proposition 3.3.3. ■

Theorem 3.3.4 ***(Risk-sensitive necessary optimality conditions)***

We assume that $(\mathbf{A_1}) - (\mathbf{A_5})$ *hold. If* (x^u, y^u, z^u, u) *is an optimal solution of the risk-sensitive control problem* $\{(3.1), (3.2), (3.3)\}$*, then there exist pairs of* $\mathcal{F}_t^W-$*adapted processes* $\left(V^\theta, \mathcal{D}\right)$, $(\widetilde{p}, \widetilde{q})$ *that satisfy* (3.15) *and* (3.16)*, such that*

$$H_v^\theta\left(t, \mathcal{O}^u(t), u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^\theta(t), \mathcal{D}_3(t)\right)(u_t - v_t) \le 0,$$

for all $u \in \mathcal{U}$*, almost every* $0 \le t \le T$ *and* $\mathbb{P}-$*almost surely.*

Proof. We arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes $(\widetilde{p}_2, \widetilde{q}_2)$, $(\widetilde{p}_3, \widetilde{q}_3)$ and $\left(V^\theta, \mathcal{D}\right)$ which solve (3.15), where the Hamiltonian $\widetilde{H}^\theta$ associated with (3.4), given by (3.6) satisfies

$$\widetilde{H}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) = \left\{\theta V_t^\theta\right\} H^\theta\left(t, \mathcal{O}^u(t), u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^\theta(t), \mathcal{D}_3(t)\right), \tag{3.24}$$

and H^θ is the risk-sensitive Hamiltonian given by (3.17). Hence, since $V^\theta > 0$, the variational inequality (3.7) translates into

$$H_v^\theta\left(t, \mathcal{O}^u(t), u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^\theta(t), \mathcal{D}_3(t)\right)(u_t - v_t) \leq 0, \tag{3.25}$$

for all $u \in \mathcal{U}$, almost every $0 \leq t \leq T$ and $\mathbb{P}-$almost surely. This completed the proof of Theorem 3.3.4. ■

3.4 Risk-Sensitive Sufficient Optimality Conditions

In this section, we study when the necessary optimality conditions (3.7) become sufficient. For any $v \in \mathcal{U}$, we denote by (x^v, y^v, z^v) the solution of equation (3.1) controlled by v to state the following result.

Theorem 3.4.1 ***(Risk-sensitive sufficient optimality conditions)***

Assume that the functions Φ, Ψ and $(m^v, \mathcal{O}^v, v) \to \widetilde{H}^\theta\left(t, m_t^v, \mathcal{O}^v(t), v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)$ are convex and for any $v \in \mathcal{U}$ such that $\mathbb{E}\left[\int_0^T |v|^2 dt\right] < \infty$. Then u is an optimal solution of the control problem $\{(3.1), (3.2), (3.3)\}$ if it satisfies (3.7).

Proof. Let u be an arbitrary element of $\mathcal{U}$ (candidate to be optimal). For any $v \in \mathcal{U}$, we have

$$J^{\theta}(v) - J^{\theta}(u) = \mathbb{E}\left[\exp\theta\left\{\Phi\left(x_T^v, \mathbb{E}'(x_T^v)\right) + \Psi\left(y_0^v, \mathbb{E}'(y_0^v)\right) + m_T^v\right\}\right] - \mathbb{E}\left[\exp\theta\left\{\Phi\left(x_T^u, \mathbb{E}'(x_T^u)\right) + \Psi\left(y_0^u, \mathbb{E}'(y_0^u)\right) + m_T^u\right\}\right].$$

By applying the Taylor's expansion and since Φ and Ψ are convex, we get

$$\begin{aligned} J^{\theta}(v) - J^{\theta}(u) \geq{} & \mathbb{E}\left[\theta A_T^{\theta}(m_T^v - m_T^u)\right] \\ & + \mathbb{E}\left[\theta\left[A_T^{\theta}\Phi_x\left(x_T^u, \mathbb{E}'(x_T^u)\right) + \mathbb{E}'\left[A_T^{\theta}\Phi_{\bar{x}}\left(x_T^u, \mathbb{E}'(x_T^u)\right)\right]\right](x_T^v - x_T^u)\right] \\ & + \mathbb{E}\left[\theta\left[A_T^{\theta}\Psi_y\left(y_0^u, \mathbb{E}'(y_0^u)\right) + \mathbb{E}'\left[A_T^{\theta}\Psi_{\bar{y}}\left(y_0^u, \mathbb{E}'(y_0^u)\right)\right]\right](y_0^v - y_0^u)\right]. \end{aligned}$$

It follows from (3.5), we remark that $p_1(T) = \theta A_T^{\theta}$,

$p_2(T) = \theta\left[A_T^{\theta}\Phi_x\left(x_T^u, \mathbb{E}'(x_T^u)\right) + \mathbb{E}'\left[A_T^{\theta}\Phi_{\bar{x}}\left(x_T^u, \mathbb{E}'(x_T^u)\right)\right]\right]$, and

$p_3(0) = -\theta\left[A_T^{\theta}\Psi_y\left(y_0^u, \mathbb{E}'(y_0^u)\right) + \mathbb{E}'\left[A_T^{\theta}\Psi_{\bar{y}}\left(y_0^u, \mathbb{E}'(y_0^u)\right)\right]\right]$, then we have

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[p_1(T)(m_T^v - m_T^u)\right] + \mathbb{E}\left[p_2(T)(x_T^v - x_T^u)\right] - \mathbb{E}\left[p_3(0)(y_0^v - y_0^u)\right]. \tag{3.26}$$

By applying Itô's formula to $p_1(t)(m_t^v - m_t^u)$, $p_2(t)(x_t^v - x_t^u)$ and $p_3(t)(y_t^v - y_t^u)$, that lead to

$$\begin{aligned} \mathbb{E}\left[p_2(T)(x_T^v - x_T^u)\right] ={} & -\mathbb{E}\left[\int_0^T \widetilde{H}_x^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(x_t^v - x_t^u)\,dt\right] \\ & - \mathbb{E}\left[\int_0^T \mathbb{E}'\left[\widetilde{H}_{\bar{x}}^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right](x_t^v - x_t^u)\,dt\right] \\ & + \mathbb{E}\left[\int_0^T \left(b(t, \mathcal{O}^v(t), v_t) - b(t, \mathcal{O}^u(t), u_t)\right)p_2(t)\,dt\right] \\ & + \mathbb{E}\left[\int_0^T \left(\sigma(t, \mathcal{O}^v(t), v_t) - \sigma(t, \mathcal{O}^u(t), u_t)\right)q_2(t)\,dt\right], \end{aligned} \tag{3.27}$$

and

$$\mathbb{E}\left[p_1(T)(m_T^v - m_T^u)\right] = \mathbb{E}\left[\int_0^T \left(l(t, \mathcal{O}^v(t), v_t) - l(t, \mathcal{O}^u(t), u_t)\right)p_1(t)\,dt\right], \tag{3.28}$$

and

$$\begin{aligned}
-\mathbb{E}\left[p_3(0)\left(y_0^v-y_0^u\right)\right] = & -\mathbb{E}\left[\int_0^T \widetilde{H}_y^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(y_t^v-y_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \mathbb{E}^{'}\left[\widetilde{H}_{\bar{y}}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right]\left(y_t^v-y_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \widetilde{H}_z^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(z_t^v-z_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \mathbb{E}^{'}\left[\widetilde{H}_{\bar{z}}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right]\left(z_t^v-z_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \left(f\left(t, \mathcal{O}^v(t), v_t\right)-f\left(t, \mathcal{O}^u(t), u_t\right)\right) p_3(t)\, dt\right].
\end{aligned} \tag{3.29}$$

By replacing (3.27) , (3.28) and (3.29) into (3.26), we get

$$\begin{aligned}
J^\theta(v)-J^\theta(u) \geq & \ \mathbb{E}\left[\int_0^T \widetilde{H}^\theta\left(t, m_t^v, \mathcal{O}^v(t), v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \widetilde{H}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \widetilde{H}_x^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(x_t^v-x_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \mathbb{E}^{'}\left[\widetilde{H}_{\bar{x}}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right]\left(x_t^v-x_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \widetilde{H}_y^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(y_t^v-y_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \mathbb{E}^{'}\left[\widetilde{H}_{\bar{y}}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right]\left(y_t^v-y_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \widetilde{H}_z^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\left(z_t^v-z_t^u\right) dt\right] \\
& -\mathbb{E}\left[\int_0^T \mathbb{E}^{'}\left[\widetilde{H}_{\bar{z}}^\theta\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right]\left(z_t^v-z_t^u\right) dt\right].
\end{aligned} \tag{3.30}$$

Since the Hamiltonian $\widetilde{H}^{\theta}$ is convex with respect to $(x, \bar{x}, y, \bar{y}, z, \bar{z}, v)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \widetilde{H}^{\theta}\left(t, m_t^v, \mathcal{O}^v(t), v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] - \mathbb{E}\left[\int_0^T \widetilde{H}^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right) dt\right] \\
&\geq \mathbb{E}\left[\int_0^T \widetilde{H}_x^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(x_t^v - x_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \mathbb{E}'\left[\widetilde{H}_{\bar{x}}^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right](x_t^v - x_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \widetilde{H}_y^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(y_t^v - y_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \mathbb{E}'\left[\widetilde{H}_{\bar{y}}^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right](y_t^v - y_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \widetilde{H}_z^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(z_t^v - z_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \mathbb{E}'\left[\widetilde{H}_{\bar{z}}^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)\right](z_t^v - z_t^u)\, dt\right] \\
&+ \mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(v_t - u_t)\, dt\right].
\end{aligned}$$

Then, by using above inequality in (3.30), we obtain

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T \widetilde{H}_v^{\theta}\left(t, m_t^u, \mathcal{O}^u(t), u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\right)(v_t - u_t)\, dt\right].$$

In virtue of the necessary optimality conditions (3.7), then the last inequality implies that

$J^{\theta}(v) - J^{\theta}(u) \geq 0$. Then the Theorem 3.4.1 is proved. ■

Remark 3.4.2 *In the last step of proof, and according to* (3.24), *we have*

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T \theta V_t^{\theta} H_v^{\theta}\left(t, \mathcal{O}^u(t), u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^{\theta}(t), \mathcal{D}_3(t)\right)(v_t - u_t)\, dt\right],$$

we know that $\theta V_t^{\theta} > 0$. *Then the above equation can be rewritten as*

$$J^{\theta}(v) - J^{\theta}(u) \geq \mathbb{E}\left[\int_0^T H_v^{\theta}\left(t, \mathcal{O}^u(t), u_t, \widetilde{p}_2(t), \widetilde{q}_2(t), \widetilde{p}_3(t), V^{\theta}(t), \mathcal{D}_3(t)\right)(v_t - u_t)\, dt\right].$$

In virtue of the necessary optimality conditions (3.25) , *then the last inequality implies that*

$J^{\theta}(v) - J^{\theta}(u) \geq 0.$

3.5 Applications

3.5.1 Example 1: Risk-Sensitive Control Applied to the Mean-Field Linear-Quadratic

We provide a concrete example of a the mean-field risk-sensitive forward-backward stochastic LQ problem and we give the explicit optimal control and validate our major theoretical results in Theorem 3.4.1 (Sufficient optimality conditions for risk-sensitive). First let the control domain be $U = [-1, 1]$. Consider the following mean-field linear quadratic risk-sensitive control problem

$$\begin{cases} \inf_{v \in \mathcal{U}} \mathbb{E}\left[\exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 dt + \frac{1}{2}(x_T^v)^2 + \frac{1}{2}(y_0^v)^2 \right\}\right], \\ \text{subject to} \\ dx_t^v = \left(A_1 x_t^v + A_2 \mathbb{E}'(x_t^v) + A_3 v_t\right) dt + \left(B_1 x_t^v + B_2 \mathbb{E}'(x_t^v) + B_3 v_t\right) dW_t, \\ dy_t^v = -\left(C_1 x_t^v + C_2 \mathbb{E}'(x_t^v) + C_3 y_t^v + C_4 \mathbb{E}'(y_t^v) + C_5 z_t^v + C_6 \mathbb{E}'(z_t^v) + C_7 v_t\right) dt + z_t^v dW_t, \\ x_0^v = a, \ y_T^v = \xi. \end{cases} \tag{3.31}$$

where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, C_4, C_5, C_6$ and C_7 are positive real constants.

Let (x_t^v, y_t^v, z_t^v) be a solution of (3.31) associated with v_t. Then, there exist unique $\mathcal{F}_t^W-$adapted pairs of processes $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3) of following FBSDE of mean-field type system (called adjoint equations), according to the equations (3.5)

$$\begin{cases} dp_1(t) &= q_1(t)\, dW_t, \\ dp_2(t) &= -\Big(A_1 p_2(t) + B_1 q_2(t) - C_1 p_3(t) + A_2 \mathbb{E}^{'}(p_2(t)) + B_2 \mathbb{E}^{'}(q_2(t)) \\ &- \quad C_2 \mathbb{E}^{'}(p_3(t))\Big)\, dt + q_2(t)\, dW_t, \\ dp_3(t) &= \Big(C_3 p_3(t) + C_4 \mathbb{E}^{'}(p_3(t))\Big)\, dt + \Big(C_5 p_3(t) + C_6 \mathbb{E}^{'}(p_3(t))\Big)\, dW_t, \\ p_1(T) &= \theta A_T^{\theta}, \ \ p_2(T) = \theta x_T^v A_T^{\theta}, \ \ p_3(0) = -\theta y_0^v A_T^{\theta}, \end{cases} \tag{3.32}$$

where

$$A_T^{\theta} := \exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 dt + \frac{1}{2} (x_T^v)^2 + \frac{1}{2} (y_0^v)^2 \right\}.$$

We give the Hamiltonian $\widetilde{H}^{\theta}$ defined by

$$\begin{aligned} \widetilde{H}^{\theta}(t) &:= \widetilde{H}^{\theta}\Big(t, m_t^v, \mathcal{O}^v(t), v_t, \overrightarrow{p}(t), \overrightarrow{q}(t)\Big) \\ &= \frac{1}{2} v_t^2 p_1(t) + \Big(A_1 x_t^v + A_2 \mathbb{E}^{'}(x_t^v) + A_3 v_t\Big) p_2(t) + \Big(B_1 x_t^v + B_2 \mathbb{E}^{'}(x_t^v) + B_3 v_t\Big) q_2(t) \\ &- \Big(C_1 x_t^v + C_2 \mathbb{E}^{'}(x_t^v) + C_3 y_t^v + C_4 \mathbb{E}^{'}(y_t^v) + C_5 z_t^v + C_6 \mathbb{E}^{'}(z_t^v) + C_7 v_t\Big) p_3(t). \end{aligned}$$

We have $\widetilde{H}_v^{\theta}(t) = v_t p_1(t) + A_3 p_2(t) + B_3 q_2(t) - C_7 p_3(t)$. Minimizing the Hamiltonian yields

$$u_t = (C_7 p_3(t) - A_3 p_2(t) - B_3 q_2(t))\, p_1^{-1}(t). \tag{3.33}$$

We only need to prove that u_t is an optimal control of (3.31).

Theorem 3.5.1 *(Risk-sensitive sufficient optimality conditions for a linear quadratic control problem). Suppose that u_t satisfies* (3.33)*, where $\left(\overrightarrow{p}, \overrightarrow{q}\right)$ satisfy* (3.32)*. Then u_t is the unique optimal control of the above mean-field FBSDE of linear quadratic problem* (3.31)*.*

Proof. From the definition of the cost functional J^{θ}, we have

$$J^{\theta}(v_t) - J^{\theta}(u_t) = \mathbb{E}\left[\exp\theta\left\{\frac{1}{2}\int_0^T v_t^2 dt + \frac{1}{2}(x_T^v)^2 + \frac{1}{2}(y_0^v)^2\right\}\right] - \mathbb{E}\left[\exp\theta\left\{\frac{1}{2}\int_0^T u_t^2 dt + \frac{1}{2}(x_T^u)^2 + \frac{1}{2}(y_0^u)^2\right\}\right].$$

We put $m_T^v = \frac{1}{2}\int_0^T v_t^2 dt$, and by applying the Taylor's expansion, we have

$$J^{\theta}(v_t) - J^{\theta}(u_t) = \mathbb{E}\left[p_1(T)(m_T^v - m_T^u)\right] + \mathbb{E}\left[p_2(T)(x_T^v - x_T^u)\right] - \mathbb{E}\left[p_3(0)(y_0^v - y_0^u)\right], \tag{3.34}$$

where $p_1(T) = \theta A_T^{\theta}$, $p_2(T) = \theta x_T^u A_T^{\theta}$ and $p_3(0) = -\theta y_0^u A_T^{\theta}$.

By applying Itô's formula to $p_1(t)(m_t^v - m_t^u)$, $p_2(t)(x_t^v - x_t^u)$ and $p_3(t)\,(y_t^v - y_t^u)$, and used the explicit forms of the adjoint equations (3.32), that lead to

$$\mathbb{E}\left[p_1(T)(m_T^v - m_T^u)\right] = \mathbb{E}\left[\int_0^T \frac{1}{2}\left(v_t^2 - u_t^2\right) p_1(t)\, dt\right],$$

and

$$\mathbb{E}\left[p_2(T)(x_T^v - x_T^u)\right] = \mathbb{E}\left[\int_0^T C_1 p_3(t)(x_t^v - x_t^u)\, dt\right] + \mathbb{E}\left[\int_0^T C_2 p_3(t)(x_t^v - x_t^u)\, dt\right] + \mathbb{E}\left[\int_0^T A_3 p_2(t)(v_t - u_t)\, dt\right] + \mathbb{E}\left[\int_0^T B_3 q_2(t)(v_t - u_t)\, dt\right],$$

and

$$-\mathbb{E}\left[p_3(0)(y_0^v - y_0^u)\right] = -\mathbb{E}\left[\int_0^T C_1 p_3(t)(x_t^v - x_t^u)\, dt\right] - \mathbb{E}\left[\int_0^T C_2 p_3(t)(x_t^v - x_t^u)\, dt\right] - \mathbb{E}\left[\int_0^T C_7 p_3(t)(v_t - u_t)\, dt\right].$$

By replacing the three above formulas into (3.34), then we get

$$J^{\theta}(v_t) - J^{\theta}(u_t) = \mathbb{E}\left[\int_0^T \frac{1}{2}(v_t - u_t)(v_t - u_t) p_1(t)\, dt\right] + \mathbb{E}\left[\int_0^T u_t(v_t - u_t) p_1(t)\, dt\right] - \mathbb{E}\left[\int_0^T C_7 p_3(t)(v_t - u_t)\, dt\right] + \mathbb{E}\left[\int_0^T A_3 p_2(t)(v_t - u_t)\, dt\right] + \mathbb{E}\left[\int_0^T B_3 q_2(t)(v_t - u_t)\, dt\right].$$

Because $(v_t - u_t)$ being nonnegative. Then we have the following result:

$$\begin{aligned} J^{\theta}(v_t) - J^{\theta}(u_t) \geq \mathbb{E}\left[\int_0^T u_t (v_t - u_t) p_1(t)\, dt\right] + \mathbb{E}\left[\int_0^T A_3 p_2(t)(v_t - u_t)\, dt\right] \\ + \mathbb{E}\left[\int_0^T B_3 q_2(t)(v_t - u_t)\, dt\right] - \mathbb{E}\left[\int_0^T C_7 p_3(t)(v_t - u_t)\, dt\right]. \end{aligned}$$

Then

$$J^{\theta}(v_t) - J^{\theta}(u_t) \geq \mathbb{E}\left[\int_0^T (u_t p_1(t) + A_3 p_2(t) + B_3 q_2(t) - C_7 p_3(t))(v_t - u_t)\, dt\right]. \tag{3.35}$$

By replacing u_t with its value in (3.35), we obtain $J^{\theta}(v_t) \geq J^{\theta}(u_t)$, i.e. u_t is optimal. This proof is finished. ■

3.5.2 Example 2: Financial Application: Mean-Variance Risk-Sensitive Stochastic Optimal Portfolio Problem

We deal with the mean-variance risk-sensitive stochastic optimal control problem, and apply the risk-sensitive necessary optimality conditions (Theorem 3.3.4). Our state dynamics is

$$\begin{cases} dx_t^v &= (\rho v_t + r x_t^v)\, dt + \sigma v_t dW_t, \\ x_0^v &= m_0, \end{cases} \tag{3.36}$$

and

$$\begin{cases} dy_t^v &= -(c x_t^v + \rho v_t - \lambda y_t^v)\, dt + a dW_t, \\ y_T^v &= 0. \end{cases} \tag{3.37}$$

According to by Lemma 1.4.1, we conclude

$$\mathcal{J}^{\theta}(v(.)) = \frac{1}{\theta}\log\left[J^{\theta}(v(.))\right] = \mathbb{E}(\Theta_T) + \frac{\theta}{2}Var(\Theta_T) + O(\theta^2),$$

where $\Theta_T := \Phi\left(x_T^v, \mathbb{E}'(x_T^v)\right) + \Psi\left(y_0^v, \mathbb{E}'(y_0^v)\right) + \int_0^T l(t, \mathcal{O}^v(t), v_t)\, dt.$

We put $l\left(t,\mathcal{O}^{v}\left(t\right),v_{t}\right)=0$, $\Phi\left(x_{T}^{v},\mathbb{E}^{'}\left(x_{T}^{v}\right)\right)=x_{T}^{v}$ and $\Psi\left(y_{0}^{v},\mathbb{E}^{'}\left(y_{0}^{v}\right)\right)=-y_{0}^{v}$, we get $\Theta_{T}:=x_{T}^{v}-y_{0}^{v}$.

Then, the cost functional be the following

$$\mathcal{J}^{\theta}\left(v\left(.\right)\right)=\frac{1}{\theta}\log\left[J^{\theta}\left(v\left(.\right)\right)\right]=\mathbb{E}\left[x_{T}^{v}-y_{0}^{v}+\frac{\theta}{2}\left[x_{T}^{v}-y_{0}^{v}-\vartheta\right]^{2}\right], \tag{3.38}$$

where $\theta>0,\ \theta\neq1,\ \vartheta=\mathbb{E}\left(x_{T}^{v}-y_{0}^{v}\right).$

The investor wants to minimize (3.38) subject to (3.36) and (3.37), by taking $v\left(.\right)$ over $\mathcal{U}$.

The Hamiltonian function (3.17) gets the form

$$H^{\theta}\left(t\right):=H^{\theta}\left(t,\mathcal{O}^{v}\left(t\right),v_{t},\widetilde{p}_{2}\left(t\right),\widetilde{q}_{2}\left(t\right),\widetilde{p}_{3}\left(t\right),\mathcal{D}_{3}\left(t\right)\right) \tag{3.39}$$

$$=\rho v_{t}\left(\widetilde{p}_{2}\left(t\right)-\widetilde{p}_{3}\left(t\right)\right)+\sigma v_{t}\widetilde{q}_{2}\left(t\right)+rx_{t}^{v}\widetilde{p}_{2}\left(t\right)+\left(\lambda y_{t}^{v}-cx_{t}^{v}+\theta\mathcal{D}_{3}\left(t\right)a\right)\widetilde{p}_{3}\left(t\right).$$

Let $\left(x_{t}^{u},y_{t}^{u},z_{t}^{u}\right)$ be an optimal triplet of the system $\left\{(3.36),(3.37)\right\}$. The adjoint equations (3.15) reduces to

$$\begin{cases} d\widetilde{p}_{2}\left(t\right) &= c\widetilde{p}_{3}\left(t\right)-r\widetilde{p}_{2}\left(t\right)dt+\left[\widetilde{q}_{2}\left(t\right)-\theta\mathcal{D}_{2}\left(t\right)\widetilde{p}_{2}\left(t\right)\right]dW_{t}^{\theta}, \\ \widetilde{p}_{2}\left(T\right) &= 1+\theta\left[x_{T}^{v}-y_{0}^{v}-\vartheta\right], \end{cases} \tag{3.40}$$

and

$$\begin{cases} d\widetilde{p}_{3}\left(t\right) &= -\lambda\widetilde{p}_{3}\left(t\right)dt, \\ \widetilde{p}_{3}\left(0\right) &= 1+\theta\left[x_{T}^{v}-y_{0}^{v}-\vartheta\right]. \end{cases}$$

Minimizing the Hamiltonian (3.39), we obtain the following result

$$\rho\left(\widetilde{p}_{2}\left(t\right)-\widetilde{p}_{3}\left(t\right)\right)+\sigma\widetilde{q}_{2}\left(t\right)=0. \tag{3.41}$$

The SDE (3.36), and the adjoint equation (3.40) with respect to optimal control, being

$$\begin{cases} dx_{t}^{u} &= \left(\rho u_{t}+rx_{t}^{u}\right)dt+\sigma u_{t}dW_{t}, \\ x_{0}^{u} &= m_{0}, \end{cases} \tag{3.42}$$

and

$$\begin{cases} d\widetilde{p}_2^u(t) &= c\widetilde{p}_3^u(t) - r\widetilde{p}_2^u(t)\,dt + \left[\widetilde{q}_2^u(t) - \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t)\right]dW_t^\theta, \\ \widetilde{p}_2^u(T) &= 1 + \theta\left[x_T^u - y_0^u - \vartheta\right]. \end{cases} \tag{3.43}$$

Replacing $dW_t^\theta = dW_t - \theta\mathcal{D}_2(t)\,dt$ in (3.43), we get

$$\begin{cases} d\widetilde{p}_2^u(t) &= \left[c\widetilde{p}_3^u(t) - r\widetilde{p}_2^u(t) - \theta\mathcal{D}_2(t)\,\widetilde{q}_2^u(t) + \theta^2\mathcal{D}_2^2(t)\,\widetilde{p}_2^u(t)\right]dt + \left[\widetilde{q}_2^u(t) - \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t)\right]dW_t, \\ \widetilde{p}_2^u(T) &= 1 + \theta\left[x_T^u - y_0^u - \vartheta\right]. \end{cases} \tag{3.44}$$

Therefore, an optimal solution $(\widetilde{p}_2^u(t), x_t^u, u_t)$ can be obtained by solving the system of FBSDE with mean-field type control (3.42) and (3.44). To solve the FBSDE $\{(3.42), (3.44)\}$, we conjecture the solution to (3.42) and (3.44) is related by

$$\widetilde{p}_2^u(t) = \varpi(t)\,x_t^u + \varsigma(t)\,\mathbb{E}'(x_t^u) + \gamma(t), \tag{3.45}$$

for some deterministic differentiable functions $\varpi(t)$, $\varsigma(t)$ and $\gamma(t)$, as the best of our acknowledge the term $\sigma u_t dW_t$ is called stochastic integral, so it goes to zero with respect to $\mathbb{E}'$, we have

$$\begin{cases} d\mathbb{E}'(x_t^u) &= \left(\rho\mathbb{E}'(u_t) + r\mathbb{E}'(x_t^u)\right)dt, \\ \mathbb{E}'(x_0^u) &= m_0. \end{cases}$$

By applying Itô's formula to (3.45), we get

$$\begin{cases} d\widetilde{p}_2^u(t) &= \Big[\left(\overset{\bullet}{\varpi}(t) + \varpi(t)\,r\right)x_t^u + \left(\overset{\bullet}{\varsigma}(t) + \varsigma(t)\,r\right)\mathbb{E}'(x_t^u) \\ &\quad + \overset{\bullet}{\gamma}(t) + \varpi(t)\,\rho u_t + \varsigma(t)\,\rho\mathbb{E}'(u_t)\Big]dt + \varpi(t)\,\sigma u_t dW_t, \\ \widetilde{p}_2^u(T) &= \varpi(T)\,x_T^u + \varsigma(T)\,\mathbb{E}'(x_T^u) + \gamma(T). \end{cases} \tag{3.46}$$

By equating the coefficients and the terminal conditions of (3.44) and (3.46), we have

$$\widetilde{q}_2^u(t) = \varpi(t)\,\sigma u_t + \theta\mathcal{D}_2(t)\,\widetilde{p}_2^u(t),\ \varpi(T) = \theta,\ \varsigma(T) = 0,\ \gamma(T) = 1 - \theta y_0^u - \theta\vartheta, \tag{3.47}$$

and

$$0=\left(\overset{\bullet}{\varpi}(t)+\varpi(t)r\right)x_t^u+\left(\overset{\bullet}{\varsigma}(t)+\varsigma(t)r\right)\mathbb{E}^{'}(x_t^u)+\overset{\bullet}{\gamma}(t)+\varpi(t)\rho u_t \tag{3.48}$$
$$+\varsigma(t)\rho\mathbb{E}^{'}(u_t)-c\widetilde{p}_3^u(t)+r\widetilde{p}_2^u(t)+\theta\mathcal{D}_2(t)\widetilde{q}_2^u(t)-\theta^2\mathcal{D}_2^2(t)\widetilde{p}_2^u(t).$$

By substituting (3.47) into (3.48), and by using (3.45), we obtain

$$0=\left(\overset{\bullet}{\varpi}(t)+2\varpi(t)r\right)x_t^u+\left(\overset{\bullet}{\varsigma}(t)+2\varsigma(t)r\right)\mathbb{E}^{'}(x_t^u)+\overset{\bullet}{\gamma}(t)+r\gamma(t) \tag{3.49}$$
$$+\varpi(t)\rho u_t+\varsigma(t)\rho\mathbb{E}^{'}(u_t)-c\widetilde{p}_3^u(t)+\theta\mathcal{D}_2(t)\varpi(t)\sigma u_t.$$

By (3.49) , we deduce that $\varpi(t)$, $\varsigma(t)$ and $\gamma(t)$ satisfying the following ordinary differential equations (in short ODEs)

$$\begin{cases}\overset{\bullet}{\varpi}(t)+2\varpi(t)r=0,\\ \varpi(T)=\theta,\\ \overset{\bullet}{\varsigma}(t)+2\varsigma(t)r=0,\\ \varsigma(T)=0,\\ \overset{\bullet}{\gamma}(t)+r\gamma(t)+\varpi(t)\rho u_t+\varsigma(t)\rho\mathbb{E}^{'}(u_t)-c\widetilde{p}_3^u(t)+\theta\mathcal{D}_2(t)\varpi(t)\sigma u_t=0,\\ \gamma(T)=1-\theta y_0^u-\theta\vartheta.\end{cases} \tag{3.50}$$

By solving the first and second ODEs in (3.50) , we get

$$\varpi(t)=\theta\exp\left(-2\int_t^T rds\right), \tag{3.51}$$
$$\varsigma(t)=0\exp\left(-2\int_t^T rds\right). \tag{3.52}$$

Using integrating factor method, to solve the third ODE in (3.50), we know that

$$\begin{cases}\overset{\bullet}{\gamma}(t)+r\gamma(t)+\varpi(t)\rho u_t+\varsigma(t)\rho\mathbb{E}^{'}(u_t)-c\widetilde{p}_3^u(t)+\theta\mathcal{D}_2(t)\varpi(t)\sigma u_t=0,\\ \gamma(T)=1-\theta y_0^u-\theta\vartheta.\end{cases} \tag{3.53}$$

We put

$$\delta(t) = \varpi(t)\rho u_t + \varsigma(t)\rho\mathbb{E}^{'}(u_t) - c\widetilde{p}_3^u(t) + \theta\mathcal{D}_2(t)\varpi(t)\sigma u_t. \tag{3.54}$$

We rewrite (3.53) as follows

$$\begin{cases} \overset{\bullet}{\gamma}(t) + r\gamma(t) + \delta(t) = 0, \\ \gamma(T) = 1 - \theta y_0^u - \theta\vartheta. \end{cases} \tag{3.55}$$

The explicit solution of the equation (3.55) is

$$\gamma(t) = \left[1 - \theta y_0^u - \theta\vartheta - \int_t^T \delta(s)\exp\left(\int_t^s r do\right) ds\right]\exp\left(-\int_t^T r ds\right), \tag{3.56}$$

where $\delta(t)$ is determined by (3.54).

Finally, we can have the optimal control in the following state feedback form by using (3.47), we have $u_t = \frac{1}{\varpi(t)\sigma}\widetilde{q}_2^u(t) - \frac{1}{\varpi(t)\sigma}\theta\mathcal{D}_2(t)\widetilde{p}_2^u(t)$, then by replacing the value of $\widetilde{q}_2^u(t)$ from (3.41), and $\widetilde{p}_2^u(t)$ from (3.45) into the last expression of u_t above, we have

$$\begin{aligned} u_t = &-(\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\varpi(t)x_t^u - (\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\varsigma(t)\mathbb{E}^{'}(x_t^u) \\ &-(\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\gamma(t) + \frac{\rho}{\sigma^2\varpi(t)}\widetilde{p}_3^u(t), \end{aligned} \tag{3.57}$$

where $\varpi(t)$, $\varsigma(t)$ and $\gamma(t)$ are determined by (3.51), (3.52) and (3.56) respectively.

Theorem 3.5.2 *We assume that* $\varpi(t)$, $\varsigma(t)$ *and* $\gamma(t)$ *have the unique solution given by* (3.51), (3.52) *and* (3.56) *respectively. Then the optimal control of the problem* $\{(3.36), (3.38)\}$ *has the state feedback from* (3.57).

It's very important to remark that the solution of the function $\gamma(t)$ in the expression (3.53) is depend to the solution of $\widetilde{p}_3^u(t)$. If we put $\widetilde{p}_3^u(t) = \mathcal{E}(t)y_t^u + \mathcal{B}(t)\mathbb{E}^{'}(y_t^u) + \kappa(t)$, for smooth deterministic functions $\mathcal{E}(t)$, $\mathcal{B}(t)$ and $\kappa(t)$. By using the similar technique as an optimal solution in the last paragraph, to the optimal solution of $(\widetilde{p}_3^u(t), y_t^u, u_t)$, then the solutions of functions $\mathcal{E}(t), \mathcal{B}(t)$ and $\kappa(t)$ yield respectively the equations

$$\begin{cases} \overset{\bullet}{\mathcal{E}}(t) + 2\lambda\mathcal{E}(t) = 0, \\ \mathcal{E}(0) = -\theta, \\ \overset{\bullet}{\mathcal{B}}(t) + 2\lambda\mathcal{B}(t) = 0, \\ \mathcal{B}(0) = 0, \\ \overset{\bullet}{\kappa}(t) + \lambda\kappa(t) - \mathcal{E}(t)\, c x_t^u - \mathcal{E}(t)\,\rho u_t - \mathcal{B}(t)\, c\mathbb{E}'(x_t^u) - \mathcal{B}(t)\,\rho\mathbb{E}'(u_t) = 0, \\ \kappa(0) = 1 + \theta x_0^u - \theta\vartheta. \end{cases} \tag{3.58}$$

By solving the first and second ODEs in (3.58), we have

$$\mathcal{E}(t) = -\theta \exp\left(-2\int_0^t \lambda ds\right), \tag{3.59}$$

$$\mathcal{B}(t) = 0 \exp\left(-2\int_0^t \lambda ds\right). \tag{3.60}$$

Using the integrating factor method, to solve the third ODE in (3.58), we know that

$$\begin{cases} \overset{\bullet}{\kappa}(t) + \lambda\kappa(t) - \mathcal{E}(t)\, c x_t^u - \mathcal{E}(t)\,\rho u_t - \mathcal{B}(t)\, c\mathbb{E}'(x_t^u) - \mathcal{B}(t)\,\rho\mathbb{E}'(u_t) = 0, \\ \kappa(0) = 1 - \theta x_0^u - \theta\vartheta. \end{cases} \tag{3.61}$$

We put

$$\psi(t) = -\mathcal{E}(t)\, c x_t^u - \mathcal{E}(t)\,\rho u_t - \mathcal{B}(t)\, c\mathbb{E}'(x_t^u) - \mathcal{B}(t)\,\rho\mathbb{E}'(u_t). \tag{3.62}$$

We rewrite (3.61) as follows

$$\begin{cases} \overset{\bullet}{\kappa}(t) + \lambda\kappa(t) + \psi(t) = 0, \\ \kappa(0) = 1 + \theta x_0^u - \theta\vartheta. \end{cases} \tag{3.63}$$

The explicit solution of equation (3.63) is

$$\kappa(t) = \left[1 + \theta x_0^u - \theta\vartheta - \int_0^t \psi(s) \exp\left(\int_0^s \lambda dr\right) ds\right] \exp\left(\int_0^t -\lambda ds\right), \tag{3.64}$$

where $\psi(t)$ is determined by (3.62).

Then by using the expression of $\widetilde{p}_3^u(t)$, the feedback form of the control in (3.57) can be rewritten as

$$u_t = -(\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\varpi(t)x_t^u - (\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\varsigma(t)\mathbb{E}'(x_t^u) \tag{3.65}$$
$$-(\rho + \sigma\theta\mathcal{D}_2(t))\frac{1}{\sigma^2\varpi(t)}\gamma(t) + \frac{\rho}{\sigma^2\varpi(t)}\mathcal{E}(t)y_t^u + \frac{\rho}{\sigma^2\varpi(t)}\mathcal{B}(t)\mathbb{E}'(y_t^u) + \frac{\rho}{\sigma^2\varpi(t)}\kappa(t).$$

Corollary 3.5.3 *The explicit solution of the first and second ODEs in* (3.58) *are given by* (3.59), (3.60) *and the third ODE in* (3.58) *has an explicit solution given by* (3.64)*, where* $\varrho(t)$ *and* $\psi(t)$ *are determined functions given by* (3.62).

At the end, we can sum up the problem of portfolio $\{(3.36), (3.37), (3.38)\}$ for mean-variance with risk-sensitive performance, in the next Theorem 3.5.4, as the main result.

Theorem 3.5.4 *We assume that* $\varpi(t)$, $\varsigma(t)$ *and* $\gamma(t)$ *have the unique solution given by* (3.51), (3.52) *and* (3.56) *respectively,* $\mathcal{E}(t)$, $\mathcal{B}(t)$ *and* $\kappa(t)$ *have the explicit solution given by* (3.59), (3.60) *and* (3.64). *Then the optimal control of the problem* $\{(3.36), (3.37), (3.38)\}$ *has the state feedback from* (3.65)*, where* $\delta(t)$ *is determined by* (3.54), $\varrho(t)$ *and* $\psi(t)$ *are given by* (3.62).

General Conclusion

This thesis contains two main results in every chapter. The first result is Theorems 2.3.4 and 3.3.4, establishes the necessary optimality conditions for the system of BDSDE with risk-sensitive performance and the system is governed by fully coupled FBSDE of mean-field type control given in form of risk-sensitive performance respectively, using an almost similar scheme as in Chala [10]. The second main result, Theorems 2.4.1 and 3.4.1, suggests sufficient optimality conditions of BDSDE given in form of risk-sensitive performance and fully coupled FBSDE of mean-field type control given in form of risk-sensitive performance respectively, as our best acknowledge that these results are a good extension of the result established by Chala in [11]. The proof is based on the convexity conditions of the Hamiltonian function, the initial and terminal terms of the performance function. Note that the risk-sensitive control problems studied by Lim and Zhou in [25] are different from ours. Remarkably, the maximum principle of risk-neutral for the system BDSDE obtained by [2, 21], and Yong [38] are similar to our Theorem 2.2.2, but the adjoint equations and maximum conditions heavily depend on the risk-sensitive parameter. The maximum principle of risk-neutral for the system obtained by Min et al. [27], is similar to (Theorem 3.2.2), but the adjoint equations and maximum conditions heavily depend on the risk-sensitive parameter. If we put $\theta = \mathcal{E} = \mathcal{B} = \kappa = 0$, we can compare our feedback control of (3.65) with such control obtained by Hafayed [17].

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