Separating expansion from contraction: generalized TOV condition, LTB models with pressure and ΛCDM

Morgan Le Delliou*,†, Filipe C. Mena** and José P. Mimoso‡,†

***Speaker**;*Instituto de Física Teórica UAM/CSIC, Facultad de Ciencias, C-XI, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid SPAIN. Email: Morgan.LeDelliou@uam.es*
†*Centro de Física Teórica e Computacional, Universidade de Lisboa, Av. Gama Pinto 2, 1649-003 Lisboa, Portugal.*
****Centro de Matemática, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal Email: fmena@math.uminho.pt*
‡*Departamento de Física, Faculdade de Ciências, Edifício C8, Campo Grande, P-1749-016 Lisboa, Portugal Email: jpmimoso@cii.fc.ul.pt*

Abstract. We discuss the existence of a dividing shell separating expanding and collapsing regions in spherically symmetric solutions with pressure. We obtain gauge invariant conditions relating not only the intrinsic spatial curvature of the shells to the ADM mass, but also a function of the pressure which we introduce that generalises the Tolman-Oppenheimer-Volkoff equilibrium condition, in the framework of a 3+1 spacetime splitting. We consider the particular case of a Lemaître-Tolman-Bondi dust models with a cosmological constant (a Λ-CDM model) as an example of our results.

INTRODUCTION

Cosmological structure formation assumes the collapse of inhomogeneities, via gravitational instability, into "bound" structures, with the underlying idea that they depart from the cosmological expansion. This approach usually models overdensities within closed patches embedded in Friedman backgrounds, in particular in the spherical collapse included in the Press & Schechter scheme [1]. Birkhoff's theorem is often invoked to claim that the evolution of the overdensities is independent [2], while, rigorously, it only applies to asymptotically flat spacetimes from the cosmic expansion [3].

In this work, we define general conditions for the existence of a shell separating collapse from expansion, and will illustrate our results with a simple example of inhomogeneous Λ-CDM models. Our work differs from previous approaches (see e.g. [4, 5]) since it does not involve spacetime matchings or metric perturbations. Instead, we adopt the Generalised Painlevé-Gullstrand (hereafter GPG) formalism used in Lasky & Lun [6], which involves a $3+1$ splitting (ADM) and the consideration of gauge invariants kinematic quantities [7]. We then define general conditions for the existence of a shell separating contraction from expansion, before proposing particular examples of inhomogeneous Λ-CDM models implemented as Lemaître-Tolman-Bondi models with dust and a cosmological constant. Finally, we perform a dynamical study of such models.

ADM APPROACH TO LTB MODELS IN GPG SYSTEM

We consider a spherically symmetric Generalised Lemaître-Tolman-Bondi metric to include pressure. Performing an ADM 3+1 splitting in the GPG coordinates [6] , the metric reads

$$ds^2 = -\alpha(t,r)^2 dt^2 + \frac{1}{1+E(t,r)}\left(\beta(t,r)dt+dr\right)^2 + r^2 d\Omega^2 \ , \tag{1}$$

where α is a lapse function, β is a shift, and $E > -1$ is a curvature-energy characterising the curvature of the spatial surfaces orthogonal to the direction $n_a = (-\alpha,0,0,0)$ of the flow. For a perfect fluid, the projected Bianchi identities $T^a_{b;a} = 0$ yield the energy density conservation equation after projecting along the flow n^b, and the Euler equation projecting orthogonally to n^b. Using the projection h^b_a, and the Lie derivative of the density ρ along the flow $\mathscr{L}_n\rho$, we
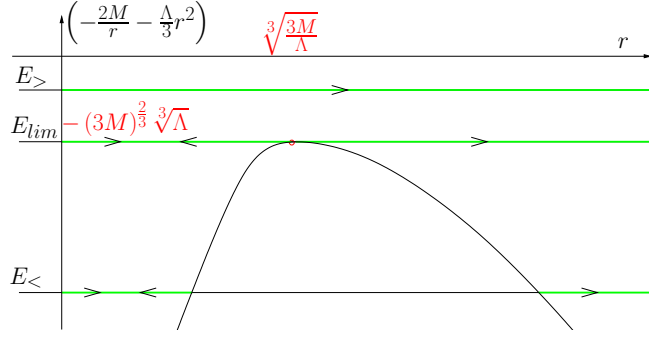
FIGURE 1. Kinematic analysis of motion in the pseudo-potential V. Depending on E relative to E_{lim}, for a given shell of constant M and E, the fate of the shell is either to remain bound ($E_< < E_{lim}$) or to escape and cosmologically expand ($E_> > E_{lim}$). There exists a critical behaviour where the shell will forever expand, but within a finite, bound radius ($E = E_{lim}$, $r \leq r_{lim}$).

have:

$$n^b T^a_{b;a} = -\mathscr{L}_n \rho - (\rho + P)\Theta = 0, \qquad h^b_a T^c_{b;c} = 0 \Rightarrow P' = -(\rho + P)\frac{\alpha'}{\alpha}, \tag{2}$$

where P is the pressure, the radial derivatives are denoted by a prime, $'$, the time derivatives are represented by a dot, $\dot{}$, and the expansion is Θ.

Introducing the ADM (also called Misner-Sharp) mass

$$M = r^2 (1+E)(\ln\alpha)' - 4\pi P r^3 + \frac{1}{3}\Lambda r^3 + r^2 \mathscr{L}_n\left(\frac{\beta}{\alpha}\right), \tag{3}$$

where $\Lambda > 0$ is the cosmological constant, we write Einstein's field equations (EFEs) as Lie derivatives along the flow and the radial evolution

$$\mathscr{L}_n E = 2\left(\frac{\beta}{\alpha}\right)\frac{1+E}{\rho + P}P', \qquad\qquad \Rightarrow \dot{E} = \beta\left(E' + 2\frac{1+E}{\rho + P}P'\right), \tag{4}$$

$$\mathscr{L}_n M = 4\pi P r^2 \left(\frac{\beta}{\alpha}\right), \qquad\qquad \Rightarrow \dot{M} = \beta\left(M' + 4\pi P r^2\right), \tag{5}$$

$$\left(\frac{\beta}{\alpha}\right)^2 = E + 2\frac{M}{r} + \frac{1}{3}\Lambda r^2. \tag{6}$$

The system becomes then closed when an equation of state is supplied. The Λ term can be absorbed in M, E and P, and many fluids mass equations written with each component using the $\frac{\beta}{\alpha}$ term for the overall sum of the masses.

SEPARATING COLLAPSE FROM EXPANSION

To characterise the separation of expansion from collapse, we shall use GLTB coordinates in which the flow direction reduces to ∂_T by choosing $\beta = -\dot{r}$. We thus have

$$ds^2 = -\alpha(T,R)^2 (\partial_T t)^2 dT^2 + \frac{(\partial_R r)^2}{1 + E(T,R)} dR^2 + r^2 d\Omega^2, \tag{7}$$

$$\dot{M} = \beta 4\pi P r^2, \quad \dot{E} r' = 2\beta\frac{1+E}{\rho + P}P', \quad \left(-\frac{\dot{r}}{\alpha}\right)^2 = E + 2\frac{M}{r} + \frac{1}{3}\Lambda r^2. \tag{8}$$

where t and r are functions of the coordinates T and R. There are two situations one should consider in parallel. On the one hand, we look for the gauge invariant expansion Θ, defined as $n^a_{;a}$, since our goal is to separate an inner collapsing, spherical region from the outer expanding universe. On the other hand, the total ADM mass of the spherical
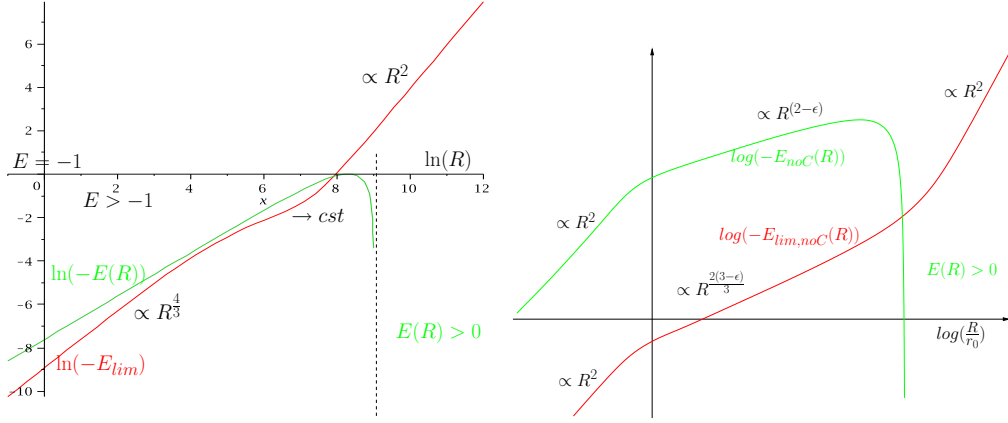
FIGURE 2. Plots of $\log(-E_{lim}) - \log(R)$ and $\log(-E) - \log(R)$ for a) NFW [8] with background and an E profile given by $E(R) = -4E_{min}\left(\frac{R}{r_1}\right)\left(1 - \frac{R}{r_1}\right)$, for $E_{min} = -1 + e^{-10}$ and $r_1 = e^9$, and for b)power law density without cusp + background and Hubble initial flow.

region that departs from the expansion flow should be conserved. This is indeed suggested by the dust case where that happens for every shell.

Denoting with $\star$ an evaluation at the dividing shell, we find the following relation between the expansion Θ and the shear a, in that at the $\star$ shell

$$r\left(\frac{\Theta}{3} + a\right) = -\frac{\beta}{\alpha} = \mathscr{L}_n r \Rightarrow \qquad \Theta_\star + 3a_\star = 0 \Leftrightarrow \left.\frac{\beta}{\alpha}\right|_\star = 0 \Leftrightarrow \mathscr{L}_n r|_\star = 0 \text{ when } \mathscr{L}_n M(t, r_\star(t)) = 0. \tag{9}$$

Moreover the generalized Friedman constraint

$$^{(3)}R + \frac{2}{3}\Theta^2 = 6a^2 + 16\pi\rho + 2\Lambda \tag{10}$$

tells us that the vanishing of Θ only happens in regions of positive 3-curvature $^{(3)}R$. On the other hand, if we demand that the separating shell has a dust-like vanishing mass/energy flow, i.e., has a conserved ADM mass along n^a:

$$\forall t, \mathscr{L}_n M(t, r_\star(t)) = 0 \qquad\qquad \Rightarrow \forall t, E = -2\frac{M}{r_\star} - \frac{1}{3}\Lambda r_\star^2 < 0. \tag{11}$$

We remark that M refers to the total ADM mass, thus including a cosmological constant.

Taking into account that the equilibrium of static spherical configurations requires the satisfaction of the Tolman-Oppenheimer-Volkoff equation of state [9], we are led to define a generalized gTOV function

$$\text{gTOV} = \left[\frac{1+E}{\rho+P}P' + 4\pi Pr + \frac{M}{r^2} - \frac{1}{3}\Lambda r\right] = \mathscr{L}_n\left(\frac{\beta}{\alpha}\right) = -\mathscr{L}_n^2 r, \tag{12}$$

which reduces to the usual TOV equation when it vanishes on the $\star$-shell.

The radial behaviour of the $\star$-shell is then similar to a turnaround shell. Indeed $r_\star = -\frac{2M_\star}{E_\star}$ leads to a null radial velocity $\dot{r}_\star = 0$ while its acceleration reveals the importance of the gTOV parameter

$$\ddot{r}_\star = -\frac{\alpha^2}{1 + \frac{\Lambda r_\star^2}{E_\star}} \frac{1 - \frac{\Lambda}{3M_\star}r_\star^3}{1 + \frac{\Lambda}{6M_\star}r_\star^3}\left[\text{gTOV}_\star - r_\star^2\frac{\text{gTOV}_\star^2}{M_\star - \frac{\Lambda}{3}r_\star^3}\right]; \qquad \ddot{r}_{GLTB,\star} = -\alpha^2\text{gTOV}_\star. \tag{13}$$

The local staticity of the $\star$-shell is then shown to be equivalent to having a local TOV equation on this limit shell

$$\text{gTOV}_\star = 0 \Leftrightarrow \mathscr{L}_n^2 r_\star = 0 \Leftrightarrow \mathscr{L}_n(\Theta + 3a)_\star = 0 \qquad\qquad \Leftrightarrow -\frac{1}{\rho+P}P' = \left[\frac{4\pi Pr + \frac{M}{r^2} - \frac{1}{3}\Lambda r}{1 - \frac{2M}{r} - \frac{1}{3}\Lambda r}\right]_\star. \tag{14}$$
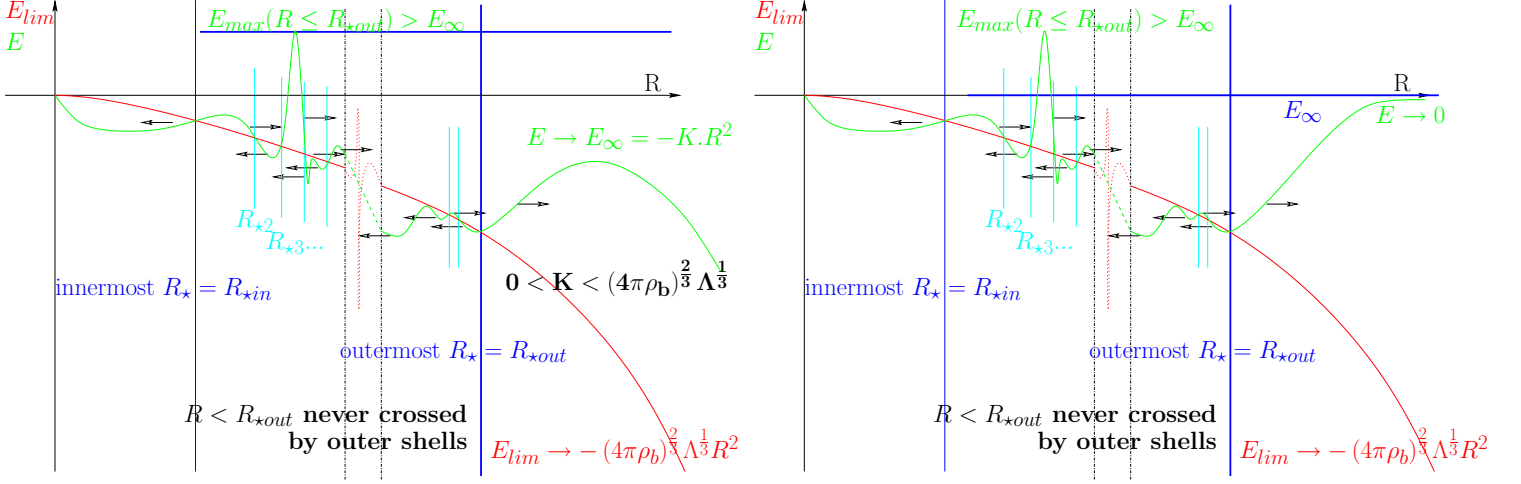
FIGURE 3. Closed and flat backgrounds with outwards escaping shells: protected inner shells.

AN EXAMPLE: ΛCDM

A simple illustration of our result is given by the case of dust with a Λ. There is then no pressure gradients and M_{dust} and E are conserved, i.e. $\alpha = 1$, which simplifies the analysis and allow us to perform a kinematic study (see Fig. 1) per shell of Eq. (8-c). An effective potential is defined by $V(r) \equiv -\frac{2M}{r} - \frac{\Lambda}{3}r^2$, so that when $\dot{r} = 0$ we have indeed $V(r) = E$:

$$\dot{r}^2 = 2\frac{M}{r} + \frac{1}{3}\Lambda r^2 + E, \qquad \text{with } \ddot{r} = -\frac{M}{r^2} + \frac{\Lambda}{3}r. \qquad (15)$$

For each shell there is a virtual static state at $r_{lim} = \sqrt[3]{\frac{3M}{\Lambda}}$, $E_{lim} = -(3M)^{\frac{2}{3}}\Lambda^{\frac{1}{3}}$, which only depends on Λ and $M(R)$. We also recall that, in this model, gTOV $= \frac{M}{r^2} - \frac{\Lambda}{3}r = -\ddot{r}$.

We then need to choose initial conditions ρ_i, which set the E_{lim} profile, and v_i, which set the E_i profile. The intersection of E_{lim} with E_i will be static at turnaround, hence defining our limit shell. We use two sets of cosmologically motivated initial conditions: one with ρ_i as a well known "universal" CDM halo density profile from simulations, refered to as NFW profile [8] and E_i as a parabola (Fig. 2-a), and the second, with a cuspless power law as ρ_i and a Hubble flow for v_i (Fig. 2-b). Explorations of all general cases for initial conditions yields a split between the separation of inner shells from outer influence, which is guaranteed for ever expanding backgrounds (thus unless the background recollapses, see Figs. 3 and 4) and that of outer shells from inner influence, which occur for open and some flat backgrounds (see Figs. 4).

CONCLUSIONS

Using non-singular, generalized Painlevé-Gullstrand coordinate formulation of the ADM spherically symmetric, perfect fluid system [6] we have shown [10, 11] that the existence of shells locally separating between inner collapsing and outer expanding regions, is governed by the condition that the combination of expansion scalar and shear $\theta + 3a$ should vanish on the shell. The ADM mass of the shell is then conserved. This condition requires that the separating shell must be located in an elliptic ($E < 0$) region. Moreover, for that shell to exist over time, we have shown that the TOV equation must be locally satisfied. We argue in some cases that this local condition is global in a cosmological context (FLRW match at radial asymptote). Given appropriate initial conditions, this translates into global separations between an expanding outer region and an eventually collapsing inner region. We present simple but physically interesting illustrations of the results, a model of Lemaître-Tolman-Bondi dust with Λ representing spherical perturbations in a ΛCDM model with two different initial sets of cosmologically interesting conditions consistent with known phenomenological constraints [10, 11, and Refs. therein]: an NFW density profile with a simple curvature
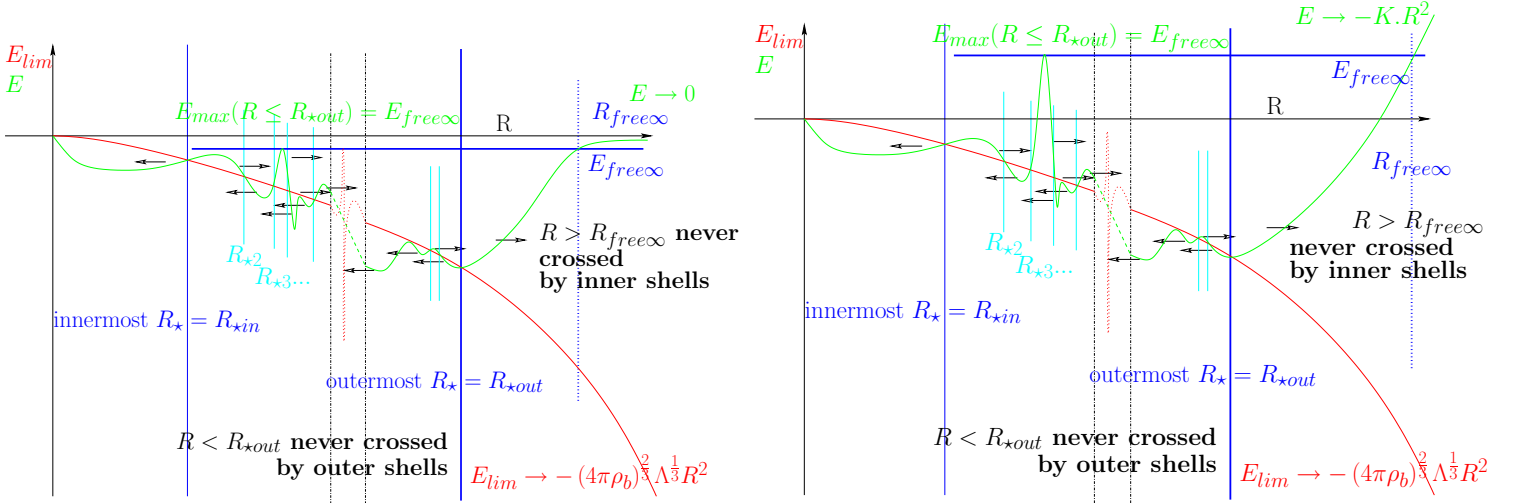
FIGURE 4. Flat, inner shells free, and open backgrounds: two limit shells

profile going from bound to unbound conditions, and a non cuspy power law fluctuation with initial Hubble flow. We show, for these models, the existence of a global separation. We argue that these shells are trapped matter surfaces [10, 11] and that, therefore they separate domains of influence of cosmic expansion from local domains of matter dynamics and can hold the place, in structure formation studies, of the incorrect invocation of Birkhoff's theorem.

ACKNOWLEDGMENTS

REFERENCES

1. W. H. Press and P. Schechter, *ApJ.* **187**, 425 (1974).
2. P. J. E. Peebles, *The large-scale structure of the universe*, Princeton U. Press, Princeton, N.J., 1980; T. Padmanabhan, *Structure Formation in the Universe*, Cambridge U. Press, Cambridge, UK, 1993.
3. G. D. Birkhoff, *Relativity and Modern Physics*, Harvard U. Press, Cambridge, MA, 1923.
4. W. B. Bonnor, Mon. Not. R. Astr. Soc. 282 (1996) 1467-1469.
5. A. Abreu, H. Hernandez & L. A. Nunez, *Class. Quant. Grav.* **24** (2007) 4631-4645 ; Di Prisco, L. Herrera & V. Varela, *Gen. Rel. Grav.*, **29** (1997) 1239
6. P. D. Lasky and A. W. C. Lun, 2006, *PRD* **74**, 084013 (2006).
7. G. F. R. Ellis and H. van Elst, NATO Adv. Study I.S.C.M.P.S. **541** (1999) 1 [arXiv:gr-qc/9812046].
8. J. F. Navarro, C. S. Frenk and S. D. M. White, *ApJ* **462**, 563 (1996).
9. R. C. Tolman, *PR* **55**, 364 (1939). ; J. R. Oppenheimer and G. M. Volkoff, *PR* **55**, 374 (1939).
10. J. P. Mimoso, M. Le Delliou and F. C. Mena, Submitted to PRD, 2009 [arXiv:0910.5755]
11. M. L. Delliou and J. P. Mimoso, AIP Conf. Proc. **1122**, 316 (2009) [arXiv:0903.4651 [gr-qc]].