

IFT-UAM/CSIC-09-44
December 13th, 2009

Domain Walls and Instantons

in

$N = 1, d = 4$ Supergravity

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Abstract

We study the supersymmetric sources of (multi-) domain-wall and (multi-) instanton solutions of generic $N = 1, d = 4$ supergravities, that is: the worldvolume effective actions for these supersymmetric topological defects.

The domain-wall solutions naturally couple to the two 3-forms recently found as part of the $N = 1, d = 4$ tensor hierarchy (i.e. they have two charges in general) and their tension is the absolute value of the superpotential section \mathcal{L} . The introduction of sources (we study sources with finite and vanishing thickness) is equivalent to the introduction of local coupling constants and results in dramatic changes of the solutions. Our results call for a democratic reformulation of $N = 1, d = 4$ supergravity in which coupling constants are, off-shell, scalar fields.

The effective actions for the instantons are always proportional to a null coordinate (in the Wick-rotated scalar manifold) which is constant over the whole space in the instanton solution. We show their supersymmetry and find the associated supersymmetric (multi-) instanton solutions.

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1 Introduction

Over the last decades, the effective actions of supersymmetric extended objects (*branes*) have played a crucial role in many developments. First of all, as source terms for the supergravity solutions that describe the branes, they confirm the relation between $(p+1)$ -form potentials and p -branes. Often, the requirements of κ -symmetry and gauge invariance of the worldvolume effective action of a given p -brane plus the relation via duality with the effective actions of other branes lead to the addition of worldvolume fields different from the embedding coordinates which can be associated to the dynamics of the boundaries of other branes ending on the p -brane's worldvolume. They also lead to the presence of the potentials associated to other branes in the Wess-Zumino term. These non-trivial elements of the action give a great deal of information about possible intersections of branes and have led, for instance to the discoveries of the Myers and enhançon effects.

One of the fundamental ingredients of the p -branes worldvolume effective actions is the knowledge of the $(p+1)$ -form potentials of the supergravity theory (their own existence and gauge and supersymmetry transformations). While the lower-rank potentials are present from the onset in the standard formulations of the supergravity theories, the higher-rank potentials have to be found. In some cases this can be done using via Hodge-duality, but this procedure becomes very complicated for $p = d - 2$ (forms coupling to strings, which are dual to scalars) and $p = d - 1$ (forms coupling to domain walls, which are dual to coupling constants, not all of which may be known) and impossible for $p = d$ (forms coupling to spacetime-filling branes). Therefore, the *democratic* formulation of the $d = 10$ type II supergravity theories [1] is necessarily incomplete, although self-consistent and sufficient if one is mainly interested in D-branes. This has motivated the systematic search for all the higher-rank potentials of (in particular) the $d = 10$ type II supergravity theories [2, 3] which can later be used for the construction of the worldvolume effective theories of the corresponding branes [4].

It is, therefore, very interesting and useful to find the effective actions of as many supersymmetric extended objects as possible. Since much less is known about the effective actions of the p -branes of lower-dimensional supergravities, in this article we will focus on the supersymmetric objects of $N = 1$ $d = 4$ supergravity, which correspond to the supersymmetric solutions recently classified and characterized in Refs. [5, 6], plus the supersymmetric instantons which, being solutions of the Wick-rotated theory, were not studied in those references. In particular, we will focus on the effective actions of supersymmetric domain walls and instantons, since in $N = 1, d = 4$ supergravity there are no supersymmetric black holes (0-branes) nor spacetime-filling branes (3-branes) and the supersymmetric string solutions are essentially identical to those of the $N = 2$ case, treated in [8] and their worldsheet effective actions should not be too different; in particular, their tensions should be given by the momentum maps associated to the symmetries involved.

Domain-wall solutions in $N = 1$ $d = 4$ sugra were first found in Ref. [9], and extensively discussed in Ref. [10], but the 3-forms to which they must couple are lacking from the ordinary formulation of $N = 1, d = 4$ supergravity. Recently, the consistent addition of 2-, 3- and 4-form potentials to $N = 1, d = 4$ supergravity was systematically investigated in

Ref. [11] using the general form of the 4-dimensional tensor hierarchy [12, 13, 14] determined in Ref. [15] and supersymmetrizing it.

Two types of 3-form potentials were found: the 3-form potentials associated to the possible gaugings of isometries of the scalar manifold (and, therefore, associated or “dual”¹ to the corresponding components of the embedding tensor) on the one hand, and two 3-forms not related to such gaugings. The interesting difference between these two sets of 3-forms is that the supersymmetry variation of the latter contain the gravitino, whereas the former do not. This difference is enough to discard the first set as possible 3-forms to couple electrically to dynamical domain walls as the kinetic term of the domain-wall effective action should contain a standard Nambu-Goto term, whose supersymmetry variation always contains the gravitino. The standard Bose-Fermi matching arguments [16] predict that a κ -symmetric 3-brane action can be constructed without additional worldvolume fields using the second set of 3-forms.

These last two 3-form potentials, not predicted by the bosonic tensor hierarchy, denoted by C and C' in Ref. [11] are distinguished by the fact that C , must have a vanishing field strength ($dC = 0$, i.e. it is dual to nothing), whereas the field strength of C' is dual to the part of the scalar potential of $N = 1, d = 4$ supergravity that depends on the superpotential (and not on the gaugings). These conditions, required by the closure of the local supersymmetry algebra on the 3-forms, were interpreted in Ref. [11] as follows: C' must be associated to a deformation parameter g that can be made manifest by rescaling the superpotential with g . This coupling constant g , then, is associated to the presence of the superpotential in the theory: when $g = 0$ there is no superpotential. As for C , it was conjectured that $N = 1, d = 4$ supergravity may admit another, yet unknown, deformation to which C would be associated.

As we are going to show in this paper, both 3-forms play indeed a very similar role: if combine them into a complex 3-form $\mathcal{C} = C^{(1)} + iC^{(2)}$, then \mathcal{C} is associated to a complex deformation parameter $g^{(1)} + ig^{(2)}$ that can be made manifest by rescaling with it the superpotential. The field strength $d\mathcal{C}$ has to be the dual of the the part of the scalar potential of $N = 1, d = 4$ supergravity that depends on the superpotential, but multiplied by $g^{(1)} + ig^{(2)}$, so its real or imaginary part can be made dual to nothing if the real or imaginary parts of the complex coupling constant are set to zero. Then, the reason why C and C' in Ref. [11] seemed to play a very different role was due to the fact that the standard coupling of the superpotential to the rest of the supergravity theory is not the most general one. The most general one is obtained by multiplying everywhere the superpotential by a complex phase. This generalization does not modify the bosonic part of the theory, but it is noticeable in the couplings to fermions. This freedom is reflected in the existence of another deformation parameter which justifies the existence of a second 3-form (the old C). In practice, this freedom may not have important physical effects because it can always be

¹To be precise, 3-forms are not dual to the deformation parameters (coupling constants, mass parameters etc) themselves. As shown in Ref. [15] the Hodge dual of each 3-form field strength is the derivative of the scalar potential with respect to the associated deformation parameter. Only when the potential is just a cosmological constant (the square of a deformation parameter), 3-forms are dual in a strict sense to deformation parameters.

absorbed in redefinitions of the superpotential but confirms the standard lore that there must be a deformation parameter for every $(d - 1)$ -form potential.

The construction of an effective action for a domain-wall charged with respect to both 3-forms is straightforward. As mentioned before, no additional worldvolume fields are needed for Bose-Fermi matching which implies, in particular, that there is no Born-Infeld vector associated to strings ending on the domain wall. This agrees with the absence of supersymmetric solutions describing them [5, 6]: in the only supersymmetric solutions describing strings and domain walls, these are parallel [7]. The worldvolume 2-forms that may describe them are non-dynamical².

A problem arises when we couple said domain-wall effective action to the (bulk) $N = 1, d = 4$ supergravity action to use it as a source, as we lack a fully democratic formulation of the action including the 3-forms. However, on general grounds [18] the coupling of 3-forms to the rest of the supergravity action can always be constructed as follows: promoting the coupling constant g to a coupling *function* $g(x)$ and adding a Lagrange-multiplier term enforcing the constraint $dg = 0$. This Lagrange-multiplier term is of the form $C \wedge dg$ where C is the $(d - 1)$ -form associated to the deformation parameter g . The promotion of g to a function $g(x)$ breaks gauge and supersymmetry invariance by terms of the form $\Delta \wedge dg$, but this can be compensated by assigning to C the transformation rules $\delta C = -\Delta$. In our case, the coupling constant was introduced in the action by rescaling with it the superpotential and, thus, appears multiplying the standard scalar potential $V_{\text{new}} = g^2(x)V_{\text{old}}$. Then, the coupling constant modulates the scalar potential which has profound implications for the solutions.

As usual with supersymmetric configurations, there are first-order equations (Bogomol'nyi equations, flow equations etc.) which imply some or all the (sourceless) equations of motion so, in order to find supersymmetric solutions, it is enough to solve these first-order equations and then just a few (or no) equations of motion. This is guaranteed by the Killing spinor identities [19, 20] or, alternatively, by the integrability conditions of the Killing spinor equations. A remarkable fact of our construction is that the modified (with a spacetime-dependent coupling $g(x)$) first-order equations now imply the same equations of motion *with sources*. This fact supports the consistency of our model and calls for the construction of a fully supersymmetric and democratic action for $N = 1, d = 4$ supergravity.

The instantons we are after are modelled on the type IIB D-instanton [21]:³ they are solutions to Wick rotated $N = 1, d = 4$ sugra with a flat metric.

As is discussed in Ref. [21], and more recently in Ref. [22], the effect of the Wick rotation

²In $N = 2, d = 4$ supergravity the situation is completely different: a worldvolume vector field is necessary for Bose-fermi matching [17], so strings can end on $N = 2, d = 4$ domain walls. Since there are several kinds of $N = 2, d = 4$ strings and probably also of domain walls (this has not been studied yet because the 3-forms of $N = 2, d = 4$ supergravity and their supergravity transformations are not known) the construction of the effective actions is not trivial. The spacetime-filling branes of $N = 2, d = 4$ supergravity may also have interesting interactions with other $N = 2, d = 4$ branes for similar reasons.

³ It would be more consistent to talk about D-instanton-like solutions or σ -model instantons, but we shall refer to them plainly as instantons as no confusion should arise.

on the scalar manifold is that it is no longer Kähler, but rather para-Kähler, which is a real manifold of split signature; such instanton solutions can however be discussed for any theory of gravity coupled to a non-linear σ -model as long as the metric of the σ -model is pseudo-Riemannian. As is well-known, these σ -model instantons correspond to null-geodesics on the scalar manifold [23, 24, 25]. An effective action, or a source, for these instantons can be found by observing that the effective action should just be some function of the scalars evaluated at the location of the instanton. This leads to the conclusion that the source term is the coordinate orthonormal to the twistfree congruence of the null-geodesic.

This discussion is applicable to all instantons, but we are interested in supersymmetric instantons and we will show that this kind of effective action is also invariant under properly Wick-rotated supersymmetry transformation rules.

The outline of this paper is as follows: in Section 2 we will give a brief outline of $N = 1, d = 4$ supergravity coupled to chiral supermultiplets. In Section 3 we will treat the domain walls, first from the bulk perspective in Section 3.1. Section 3.2 will be dedicated to the construction of the worldvolume action which will be used as the source term; Section 3.3 will then be dedicated to the discussion of the change brought about by the introduction of the coupling function, needed for the consistency of the construct. In Section 3.4, then, these changes will be illustrated by means of a simple example.

Section 4 is dedicated to the instantons and starts off by a general discussion of instantons and source-terms in Section 4.1. Supersymmetric instantons and their sources are treated in Section 4.2, followed by some examples in Sections 4.2.1 and 4.2.2. Section 5 contains our conclusions and outlook for future work.

2 $N = 1, d = 4$ Supergravity coupled to n_C chiral multiplets

The theory we are going to work with consists of the supergravity multiplet with one graviton $e^a{}_\mu$ and one chiral gravitino ψ_μ and n_C chiral multiplets with as many chiral dilatini χ^i and complex scalars Z^i , $i = 1, \dots, n_C$ parametrizing a Kähler-Hodge manifold with Kähler potential $\mathcal{K}(Z, Z^*)$. The couplings are dictated by \mathcal{K} and by the holomorphic superpotential $W(Z)$, which appears in the theory through the covariantly holomorphic⁴ section $\mathcal{L}(Z, Z^*)$ of Kähler weight $(1, -1)$ defined by

$$\mathcal{L}(Z, Z^*) \equiv W(Z)e^{\mathcal{K}/2}. \quad (2.3)$$

⁴ The Kähler connection 1-form Q is defined by

$$Q \equiv \frac{1}{2i} dZ^i \partial_i \mathcal{K} + \text{c.c.} \quad (2.1)$$

and, therefore, we have

$$\mathcal{D}_{i^*} \mathcal{L} = (\partial_{i^*} + iQ_{i^*}) \mathcal{L} = e^{\mathcal{K}/2} \partial_{i^*} (e^{-\mathcal{K}/2} \mathcal{L}) = e^{\mathcal{K}/2} \partial_{i^*} W = 0. \quad (2.2)$$

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|} [R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*j^*} - V(Z, Z^*)] , \quad (2.4)$$

where the scalar potential $V_u(Z, Z^*)$ is entirely determined by the superpotential \mathcal{L} through the expression⁵

$$V(Z, Z^*) = -24|\mathcal{L}|^2 + 8\mathcal{G}^{ij^*} \mathcal{D}_i \mathcal{L} \mathcal{D}_{j^*} \mathcal{L}^* , \quad (2.5)$$

where

$$\mathcal{D}_i \mathcal{L} = (\partial_i + i\mathcal{Q}_i) \mathcal{L} = e^{-\mathcal{K}/2} \partial_i (e^{\mathcal{K}/2} \mathcal{L}) = e^{-\mathcal{K}/2} \partial_i (e^{\mathcal{K}} W) . \quad (2.6)$$

We will also use the ‘‘fermion shift’’

$$\mathcal{N}^i \equiv 2\mathcal{G}^{ij^*} \mathcal{D}_{j^*} \mathcal{L}^* , \quad (2.7)$$

that appear in the chiralino supersymmetry transformations, in terms of which the scalar potential takes the form

$$V(Z, Z^*) = -24|\mathcal{L}|^2 + 2\mathcal{G}_{ij^*} \mathcal{N}^i \mathcal{N}^{*j^*} . \quad (2.8)$$

For vanishing fermions, the *standard* fermionic supersymmetry transformations take the form

$$\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon + i\mathcal{L} \gamma_\mu \epsilon^* = [\nabla_\mu + \frac{i}{2} \mathcal{Q}_\mu] \epsilon + i\mathcal{L} \gamma_\mu \epsilon^* , \quad (2.9)$$

$$\delta_\epsilon \chi^i = i \not{\partial} Z^i \epsilon^* + \mathcal{N}^i \epsilon , \quad (2.10)$$

where \mathcal{Q}_μ is the pullback of the Kähler connection 1-form

$$\mathcal{Q}_\mu = \partial_\mu Z^i \mathcal{Q}_i + \text{c.c.} \quad (2.11)$$

Observe that replacing \mathcal{L} by, for instance, $\frac{1}{\sqrt{2}}(1+i)\mathcal{L}$, leaves the bosonic action invariant but modifies the fermion shifts in the fermion supersymmetry transformations. This change can be seen as a field redefinition since the phase $\frac{1}{\sqrt{2}}(1+i)$ can always be reabsorbed into \mathcal{L} , bringing the supersymmetry transformations back to the *standard* form. This will not be possible after the introduction of a local coupling constant in the coming sections and, therefore, it is important to notice this possibility.

The supersymmetry transformation rules for the bosonic fields for vanishing fermions are

⁵It is customary to introduce a coupling constant, g , into the potential, or in the superpotential W , but as it can be reinstated trivially and not really needed in this section, we will obviate it for the moment; it will, however, be reinstated in csection 3.3.

$$\delta_\epsilon e^a{}_\mu = -\frac{i}{4}\bar{\psi}_\mu\gamma^a\epsilon^* + \text{c.c.}, \quad (2.12)$$

$$\delta_\epsilon Z^i = \frac{1}{4}\bar{\chi}^i\epsilon. \quad (2.13)$$

We denote (the l.h.s. of) the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}, \quad \mathcal{E}^i \equiv -\frac{\mathcal{G}^{ij^*}}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^{*j^*}}, \quad (2.14)$$

and they take the form

$$\mathcal{E}_{\mu\nu} = G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\partial_\mu Z^i\partial_\nu Z^{*j^*} - \frac{1}{2}g_{\mu\nu}\partial_\rho Z^i\partial^\rho Z^{*j^*}] + \frac{1}{2}g_{\mu\nu}V, \quad (2.15)$$

$$\mathcal{E}_i = \mathcal{G}_{ij^*}\nabla^2 Z^{*j^*} + \frac{1}{2}\partial_i V. \quad (2.16)$$

A compact expression for the derivative of the potential is

$$\frac{1}{2}\mathcal{G}^{ij^*}\partial_{j^*}V = -4\mathcal{L}\mathcal{N}^i + \mathcal{N}^{*j^*}\mathcal{D}_{j^*}\mathcal{N}^i. \quad (2.17)$$

3 Supersymmetric domain walls

3.1 Sourceless supersymmetric domain-wall solutions

In this section we are going to review the standard supersymmetric domain-wall solutions of $N = 1, d = 4$ supergravity that can be found in the literature. They solve the equations of motion derived from the supergravity action alone, without any additional sources and can be thought of as describing the gravitational and scalar fields far from where the possible sources are placed.

The metric of a 4-dimensional domain-wall solution can always be brought into the form

$$ds^2 = H\eta_{\mu\nu}dx^\mu dx^\nu = H[\eta_{mn}dx^m dx^n - dy^2], \quad m, n = 0, 1, 2, \quad (3.1)$$

where H is a function of the transverse coordinate $x^3 \equiv y$ only.

In the Vierbein basis

$$e^a{}_\mu = H^{1/2}\delta^a{}_\mu, \quad e_a{}^\mu = H^{-1/2}\delta_a{}^\mu, \quad (3.2)$$

the components of the spin connection are

$$\omega_\mu{}^{bc} = \eta^{3[b}e^c]{}_\mu\partial_y \log H. \quad (3.3)$$

Using these in the world-volume components of the gravitino supersymmetry transformation rule ($\mu = m = 0, 1, 2$) Eq. (2.9) and assuming that both the complex scalars Z^i and the Killing spinors ϵ only depend on y , we find the unbroken supersymmetry condition

$$i\gamma^3 \partial_{\underline{y}} H^{-1/2} \epsilon + 2\mathcal{L}\epsilon^* = 0, \quad (3.4)$$

which can only be consistently imposed if the metric function H satisfies the flow equation

$$\partial_{\underline{y}} H^{-1/2} = \pm 2|\mathcal{L}|. \quad (3.5)$$

When this equation is satisfied, the unbroken supersymmetry condition becomes the 1/2-supersymmetry-preserving projector⁶

$$i\gamma^3 (e^{-i\alpha/2} \epsilon) \pm (e^{-i\alpha/2} \epsilon)^* = 0, \quad (3.6)$$

where we have defined the phase

$$e^{i\alpha} \equiv \mathcal{L}/|\mathcal{L}|. \quad (3.7)$$

Multiplying the above projector by $-\gamma^{012}$ and using the chirality of the spinors $-i\gamma^{0123}\epsilon = \gamma_5\epsilon = -\epsilon$ we can rewrite in the characteristic form of a domain-wall supersymmetry projector:

$$(e^{-i\alpha/2} \epsilon) \pm i\gamma^{012} (e^{-i\alpha/2} \epsilon)^* = 0. \quad (3.8)$$

Using now the above projector into the chiralino supersymmetry transformation rule Eq. (2.10) we find a second flow equation for the complex scalars [27]:

$$\partial_{\underline{y}} Z^i = \pm e^{i\alpha} \mathcal{N}^i H^{1/2}. \quad (3.9)$$

It is enough to impose the first-order flow equations (3.5) and (3.9) on H, Z^i to have a solution of the second-order supergravity equations of motion, as can be deduced from the corresponding Killing spinor identities [19, 20]. In particular, the worldvolume components of the Einstein equations⁷ (2.15) can be written in the form

$$\mathcal{E}_{\underline{mn}} \sim \eta_{\underline{mn}} \partial_{\underline{y}} [\partial_{\underline{y}} H^{-1/2} \mp 2|\mathcal{L}|] = 0, \quad (3.11)$$

the transverse components in the form

⁶The Killing spinor equation associated to the transverse direction has a complicated form, but can always be solved without requiring any further conditions.

⁷The non-vanishing components of the Einstein tensor of the domain-wall metric Eq. (3.1) are

$$\begin{aligned} G_{\underline{mn}} &= \eta_{\underline{mn}} [H^{-1} \partial_{\underline{y}}^2 H - \frac{3}{4} H^{-2} (\partial_{\underline{y}} H)^2], \\ G_{\underline{yy}} &= -\frac{3}{4} H^{-2} (\partial_{\underline{y}} H)^2. \end{aligned} \quad (3.10)$$

$$\mathcal{E}_{yy} \sim -3H[\partial_y H^{-1/2} \mp 2|\mathcal{L}|^2 + \mathcal{G}_{ij^*}[\partial_y Z^i \mp e^{i\alpha} \mathcal{N}^i H^{1/2}][\partial_y Z^{*j^*} \mp e^{-i\alpha} \mathcal{N}^{*i^*} H^{1/2}]] = 0, \quad (3.12)$$

and the equations of motion of the scalars (2.16) can be written in a similar, more complicated form, proportional to the flow equations as well.

3.2 Supersymmetric sources: world-volume effective actions

Charged 4-dimensional domain walls must couple to 3-forms. If their effective action (our candidate for a source term for domain-wall solutions) is going to be κ -symmetric it must be invariant under supersymmetry transformations and this requires that it consists of a kinetic Nambu-Goto-like term and a Wess-Zumino-like term containing the 3-form. As discussed in the introduction, no worldvolume fields should be needed in addition to the embedding coordinates and fermions. Furthermore, the supersymmetry transformations of the 3-form must contain the gravitino in order to have a chance to cancel the supersymmetry transformations of the metric in the kinetic term. And, *vice versa*, if the supersymmetry transformations of the 3-forms contain chiralinos, then the Nambu-Goto term must contain a function of the complex scalar fields to cancel it.

In Ref. [11] consistent on-shell supersymmetry transformation rules for two 3-forms transforming into the gravitino were found. The on-shell condition is quite different for both in spite of the fact that their supersymmetry transformations are the real and imaginary part of the same expression: one of them says that the field strength is the dual of the scalar potential (the part that depends on the superpotential) and the other says that the field strength must vanish identically. The interpretation is that one of them is associated to a coupling constant/deformation parameter of the theory and the other is not.

This asymmetry is a bit surprising and a first hint that it is simply the result of an asymmetrical description of the theory comes from the observation (related to a similar observation made in Section 2) that, if we replace everywhere (except in the supersymmetry transformations of these 3-forms) \mathcal{L} by $i\mathcal{L}$, a redefinition that does not change the bosonic Lagrangian, the roles of the two 3-forms and their on-shell conditions are interchanged. Replacing \mathcal{L} by $\frac{1}{\sqrt{2}}(1+i)\mathcal{L}$ instead, we find that the two 3-forms play entirely analogous roles.

This suggests that the coupling constant/deformation parameter associated to the superpotential is complex and the 3-forms are associated to the its real and imaginary parts.

To make this relation explicit

1. We replace everywhere \mathcal{L} by $(g^1 + ig^2)\mathcal{L}$ where g^1 and g^2 will be the two coupling constants.
2. This means that, in the bosonic Lagrangian, the part of the scalar potential that depends on the superpotential is rescaled by a factor $(g^1)^2 + (g^2)^2$. In the supersymmetry transformation rules, only those of the gravitino and chiralino are modified

by this rescaling. Finally, the only parameter in the local supersymmetry algebra in Ref. [11] that is modified is $\Lambda_{\mu\nu}$ which becomes

$$\Lambda_{\mu\nu} = -C_{\mu\nu\rho}\xi^\rho + 2\Re[(g^1 + ig^2)\mathcal{L}\phi_{\mu\nu}], \quad \phi_{\mu\nu} = \bar{\epsilon}^*\gamma_{\mu\nu}\eta^*, \quad (3.13)$$

where $C_{\mu\nu\rho}$ is a 3-form to be determined that appears in the 2-forms field strengths.

3. We observe that there is a complex 3-form $\mathcal{C}_{\mu\nu\rho} = C^1_{\mu\nu\rho} + iC^2_{\mu\nu\rho}$ with supersymmetry transformation rules

$$\delta_\epsilon\mathcal{C}_{\mu\nu\rho} = 12i\mathcal{L}^*\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} + 2\mathcal{D}_{i^*}\mathcal{L}^*\bar{\epsilon}\gamma_{\mu\nu\rho}\chi^{*i^*}, \quad (3.14)$$

such that

$$[\delta_\eta, \delta_\epsilon]\mathcal{C}_{\mu\nu\rho} = \mathcal{L}_\xi\mathcal{C}_{\mu\nu\rho} + 2\partial_{[\mu}\tilde{\Lambda}_{\nu\rho]} + [\mathcal{G} - (g^1 + ig^2)\star V(\mathcal{L})]_{\mu\nu\rho\sigma}\xi^\sigma, \quad (3.15)$$

where

$$\mathcal{G} \equiv d\mathcal{C}, \quad \tilde{\Lambda}_{\mu\nu} \equiv -C_{\mu\nu\rho}\xi^\rho + 4i\frac{(g^1 + ig^2)}{|g^1 + ig^2|^2}[(g^1 - ig^2)\mathcal{L}^*\phi_{\mu\nu}^*]. \quad (3.16)$$

4. The supersymmetry algebra closes if

$$\mathcal{G} = (g^1 + ig^2)\star V(\mathcal{L}), \quad (3.17)$$

which we can rewrite in components

$$G^i = \frac{1}{2}\star\frac{\partial V}{\partial g^i}, \quad G^i \equiv dC^i, \quad i = 1, 2, \quad (3.18)$$

so each of the real 2-forms is associated to a real coupling constant/deformation parameter, as expected on general grounds.

5. Comparing the gauge parameter 2-form $\tilde{\Lambda}_{\mu\nu}$ with $\Lambda_{\mu\nu}$ above, we conclude that the 3-form that appears in the 2-form field strengths is

$$C_{\mu\nu\rho} = \frac{1}{2}\Im[(g^1 - ig^2)\mathcal{C}] = \frac{1}{2}(g^1C^2_{\mu\nu\rho} - g^2C^1_{\mu\nu\rho}), \quad (3.19)$$

while

$$C'_{\mu\nu\rho} = \frac{1}{2}\Re[(g^1 - ig^2)\mathcal{C}] = \frac{1}{2}[g^1C^1_{\mu\nu\rho} + g^2C^2_{\mu\nu\rho}], \quad (3.20)$$

decouples. For a single coupling constant ($g^2 = 0$) taking the value $g^1 = 1$ we recover the two 3-forms of Ref. [11]:

$$C = \frac{1}{2}C^2, \quad C' = \frac{1}{2}C^1. \quad (3.21)$$

6. Finally, we can construct a supersymmetric worldvolume action

$$S_{\text{DW}} = |q^1 + iq^2| \int d^3\xi \left\{ |\mathcal{L}| \sqrt{|g_{(3)}|} + \frac{1}{8 \cdot 3!} q^i \epsilon^{mnp} C^i{}_{mnp} \right\}, \quad (3.22)$$

where $|g_3|$ the determinant of the pullback $g_{(3)mn}$ of the spacetime metric to the 3-dimensional worldvolume and C_{mnp} is the pullback of the 3-form to the worldvolume:

$$g_{(3)mn} \equiv g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^m} \frac{\partial X^\nu}{\partial \xi^n}, \quad C_{mnp} \equiv C_{\mu\nu\rho} \frac{\partial X^\mu}{\partial \xi^m} \frac{\partial X^\nu}{\partial \xi^n} \frac{\partial X^\rho}{\partial \xi^p}. \quad (3.23)$$

It is convenient to work in the static gauge in which we identify the worldvolume coordinates ξ^m with the first three spacetime coordinates X^m , so

$$\frac{\partial X^\mu}{\partial \xi^m} = \delta^\mu{}_m, \quad (3.24)$$

and

$$g_{(3)mn} = g_{mn}, \quad C_{mnp} = C_{mnp}. \quad (3.25)$$

It is then straightforward to see that the above action is invariant to lowest order in fermions under the supersymmetry transformations Eqs. (2.12, 2.13) and (3.14) iff the spinors satisfy the condition

$$e^{-i(\alpha+q)/2}\epsilon + \gamma^{012}(e^{-i(\alpha+q)/2}\epsilon)^* = 0, \quad e^{iq} \equiv \frac{q^1 + iq^2}{|q^1 + iq^2|}, \quad e^{i\alpha} \equiv \frac{\mathcal{L}}{|\mathcal{L}|}. \quad (3.26)$$

which generalizes Eq. (3.8).

3.3 Sourceful supersymmetric domain-wall solutions

Now we are going to couple the action Eq. (3.22) found in the previous section to the bulk $N =, d = 4$ action to use it as a source term for domain-wall solutions. We will only consider the $q^2 = C^2 = 0$ case for the sake of simplicity, renaming $C \equiv C^1/2$.

However, we cannot couple it to the bulk supergravity action, Eq. (2.4), by simply adding them up because the 3-form only occurs in the source. As discussed before, we must promote the coupling constant $g \equiv g^1$ to a scalar field $g(x)$ and add to the bulk supergravity action a Lagrange-multiplier term containing the 3-form as to enforce the constancy of g . Thus, we are led to consider the bulk supergravity action,

$$S_{\text{bulk}} = \frac{1}{\kappa^2} \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*j^*} - g^2(x) V(Z, Z^*) - \frac{1}{3\sqrt{|g|}} \epsilon^{\mu\nu\rho\sigma} \partial_\mu g(x) C_{\nu\rho\sigma} \right], \quad (3.27)$$

and the brane-source action

$$S_{\text{DW}} = - \int d^4x f(y) \left\{ |\mathcal{L}| \sqrt{|g_{(3)}|} \pm \frac{1}{4!} \epsilon^{mnp} C_{\underline{mnp}} \right\}, \quad (3.28)$$

where $f(y)$ is a distribution-function of domain walls with a common transverse direction parametrized by the coordinate $x^3 \equiv y$. For instance we could take $f(y) = \delta(y - y_0)$ as to treat the case of a single infinitely-thin domain wall placed at $y = y_0$. Having a source term for these domain walls we can do away with the Israel junction conditions [26]. It is not clear if more complicated, continuous distributions can be derived from the single brane effective action Eq. (3.22), but we will use them as toy models.

The equations of motion that follow from the total action $S \equiv S_{\text{bulk}} + S_{\text{DW}}$ are

$$\mathcal{E}_g^{\mu\nu} = -\frac{\kappa^2}{2} f(y) |\mathcal{L}| \frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}} g_{(3)}^{mn} \delta_m^\mu \delta_n^\nu, \quad (3.29)$$

$$\mathcal{G}^{ij^*} \mathcal{E}_{g i^*} = -\frac{\kappa^2}{8} f(y) \frac{\sqrt{|g_{(3)}|}}{\sqrt{|g|}} e^{i\alpha} \mathcal{N}^i, \quad (3.30)$$

$$\epsilon^{\mu\nu\rho\sigma} \partial_\sigma g(x) = \pm \frac{\kappa^2}{8} f(y) \epsilon^{mnp} \delta_m^\mu \delta_n^\nu \delta_p^\rho, \quad (3.31)$$

$$\epsilon^{\mu\nu\rho\sigma} \partial_\mu C_{\nu\rho\sigma} = 6 g(x) V(Z, Z^*), \quad (3.32)$$

where $\mathcal{E}_g^{\mu\nu}$ and $\mathcal{E}_{g i^*}$ are the Einstein and scalar equations of motion defined in Eqs. (2.15) and (2.16) after the introduction of $g(x)$. Observe that the Lagrange-multiplier term is topological and independent of the scalars and, furthermore, does not modify the Einstein nor the scalar equation of motion.

The third equation is that of the 3-form and is solved if g is a function of y satisfying

$$\partial_y g = \pm \frac{1}{8} \kappa^2 f(y). \quad (3.33)$$

The function $g(y)$ will have step-like discontinuities at the locations of the domain walls, in the case they are infinitely thin.

The fourth equation is that of the scalar $g(x)$ and, as required, simply states that the 3-form is the dual of the scalar potential.

It can be checked that the Einstein and scalar equations of motion are identically satisfied if the metric function $H(y)$ and the scalars $Z^i(y)$ satisfy the following modified *sourceful flow equations*

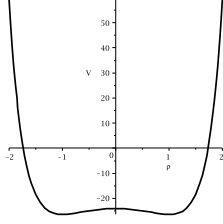


Figure 1: The potential V as a function of ρ .

$$\partial_{\underline{y}} Z^i = \pm g(y) e^{i\alpha} \mathcal{N}^i H^{1/2}, \quad (3.34)$$

$$\partial_{\underline{y}} H^{-1/2} = \pm 2g(y) |\mathcal{L}|. \quad (3.35)$$

These equations can be derived following the procedure of Section 3.1 using the modified fermion supersymmetry transformation rules

$$\delta_{\epsilon} \psi_{\mu} = \mathcal{D}_{\mu} \epsilon + ig(x) \mathcal{L} \gamma_{\mu} \epsilon^*, \quad (3.36)$$

$$\delta_{\epsilon} \chi^i = i \not{\partial} Z^i \epsilon^* + g(x) \mathcal{N}^i \epsilon. \quad (3.37)$$

It should be clear that a fully supersymmetric formulation of $N = 1, d = 4$ supergravity including all higher-rank forms is necessary in order to make sense of these modifications. Likewise, it should also be clear that these modifications have a non-trivial effect on the source-free solutions, as will be illustrated by means of a simple example.

3.4 A simple example

Let us consider model of $N = 1, d = 4$ supergravity coupled to a single chiral multiplet defined by the Kähler potential and superpotential

$$\mathcal{K} = |Z|^2, \quad W = 1, \quad (\mathcal{L} = e^{|Z|^2/2}). \quad (3.38)$$

The fermion shift \mathcal{N}^Z is given by

$$\mathcal{N}^Z = 2 Z e^{|Z|^2/2}, \quad (3.39)$$

and the scalar potential is readily seen to be

$$V = -8(3 - \rho^2) e^{\rho^2/2}, \quad (3.40)$$

where we have defined $Z \equiv \rho e^{i\beta}$, so that ρ is a radial coordinate. As one can see from the plot of this potential in figure (1), this potential has a degenerate (local) maximum at $\rho = 0$ and a degenerate (absolute) minimum at $\rho = +1$ and takes negative values at both of them. At these extrema the values of the potential are

$$V(0) = -24, \quad V(1) = -16\sqrt{e}. \quad (3.41)$$

The absolute value of these numbers is not relevant, as V is multiplied by $g^2(y)$, a factor which is determined by the sources.

The sourceful flow equation (3.34) implies that the argument of Z , β , is constant. Then, Eqs. (3.34) and (3.35) take the form

$$\partial_{\underline{y}}\rho = \pm 2g(y)\rho e^{\rho^2/2}H^{1/2}, \quad (3.42)$$

$$\partial_{\underline{y}}H^{-1/2} = \pm 2g(y)e^{\rho^2/2}. \quad (3.43)$$

The first equation implies that, in a region in which ρ (and, hence, Z) is constant, the product $g(y)\rho$ must vanish. Thus, for constant $g(y) = g$ (i.e. in absence of sources) $\rho = 0$ provides a solution with AdS_4 metric

$$H = \frac{1}{4g^2y^2}. \quad (3.44)$$

$\rho = 1$, however, can only be a solution if $g = 0$, in which case we have a Minkowski spacetime. This may look strange at first sight since we found that the value of the potential for $\rho = 1$ is $V(1) = -16\sqrt{e}$ and we might have expected an AdS_4 solution. Of course, such an AdS_4 solution exists, but it is not supersymmetric and moreover does not satisfy the sourceless flow equations.

For non-constant ρ we can combine the two sourceful flow equations to find

$$\partial_{\underline{y}}\log\rho = \partial_{\underline{y}}\log H^{-1/2}, \quad (3.45)$$

so that

$$H = c/\rho^2, \quad (3.46)$$

for some real and positive integration constant c . Substituting this expression of H into Eq. (3.42) we get

$$\rho = \sqrt{2}\operatorname{Erf}^{-1}[G(y)], \quad (3.47)$$

where Erf^{-1} is the inverse of the normalized error function⁸ and

⁸ The normalized error function is defined by

$$\operatorname{Erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du = -\operatorname{Erf}(-x), \quad (3.48)$$

$$G(y) \equiv \pm \sqrt{\frac{8c}{\pi}} \int g(y) dy + d, \quad (3.50)$$

where d is another integration constant. Observe that for ρ to be a well-defined radial coordinate, *i.e.* $\rho \geq 0$, $G(y)$ has to take values inside the interval $[0, 1)$, where we exclude the value $g(Y) = 1$ as it corresponds to $\rho = \infty$.

To solve completely our problem we must define a domain-wall source distribution function $f(y)$ to determine $g(y)$ by means of Eq. (3.33). Let us consider, first, a single, infinitely thin domain wall of positive tension $q > 0$ placed at $y = y_0$, described by the distribution function

$$f(y) = q \delta(y - y_0). \quad (3.51)$$

The corresponding local coupling constant $g(y)$ is⁹

$$g(y) = \pm \frac{\kappa^2 q}{16} [\theta(y - y_0) - \theta(y_0 - y)] = \pm \frac{\kappa^2 q}{16} \text{sgn}(y - y_0), \quad (3.52)$$

where sgn is the signum function. The above can be trivially integrated to give

$$G(y) = q\kappa^2 \sqrt{\frac{c}{32\pi}} |y - y_0| + d. \quad (3.53)$$

As it stands, the range of G is unbounded and is therefore not completely contained in the domain of Erf^{-1} , whence the solution (3.47) is not well-defined. A possible way of obtaining a G whose range is contained in the domain is by introducing a second, parallel, domain wall with opposite tension and charge at a different point ($y = -y_0$ with $y_0 > 0$ for simplicity). This means that

$$\begin{aligned} f(y) &= q\delta(y - y_0) - q\delta(y + y_0), \\ g(y) &= \pm \frac{\kappa^2 q}{16} [\theta(y - y_0) - \theta(y_0 - y) - \theta(y + y_0) + \theta(-y_0 - y)], \end{aligned} \quad (3.54)$$

$$G(y) = \sqrt{\frac{c}{32\pi}} \kappa^2 q (|y - y_0| - |y + y_0|) + d.$$

In other words: on the interval $[-y_0, y_0]$, $g(y)$ takes on the constant value $\mp \frac{\kappa^2 q}{8}$ and

and grows monotonically between $\text{Erf}(-\infty) = -1$ and $\text{Erf}(\infty) = 1$. Around the points $x = 0, 1$ it admits the expansions

$$\begin{aligned} \text{Erf}(x) &= \frac{2}{\sqrt{\pi}} \left\{ x - \frac{x^3}{3} + \dots \right\}, \\ \text{Erf}(x) &= 1 - \frac{e^{-x^2}}{\sqrt{\pi x}} \{1 + \dots\}. \end{aligned} \quad (3.49)$$

⁹ $\theta(x)$ is the Heaviside θ -function which is $\theta(x) = 1$ for $x > 0$ and zero otherwise.

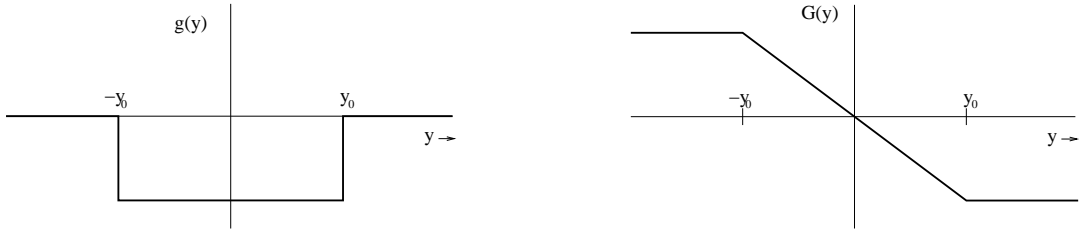


Figure 2: The profiles of the functions $g(y)$ and $G(y)$ as given in Eqs. (3.54).

vanishes identically outside said interval; this implies that outside the interval the scalars Z and Z^* vanish, whence the the spacetime for $|y| > y_0$ is Minkowski. The function G is constant outside the interval and on the interval it interpolates linearly between these two constant values which we will denominate $G(-\infty)$ and $G(\infty)$. These asymptotic values are given by

$$G(\mp\infty) = \pm q\kappa^2 \sqrt{\frac{c}{8\pi}} y_0 + d. \quad (3.55)$$

In order to create a well-behaved solution we must choose the integration constants judiciously: as $G(y)$ decreases on the interval $[-y_0, y_0]$ and Erf^{-1} is a monotonic function, whence $\rho(y)$ also decreases on the interval, it is natural to choose d so as to make $G(+\infty) = G(+y_0) = 0$. This implies that $\rho(+y_0) = 0$ and we can make ρ continuous across the domain wall located at $y = y_0$, by choosing $\rho(+\infty) = 0$. This means taking

$$d = q\kappa^2 \sqrt{\frac{c}{8\pi}} y_0. \quad (3.56)$$

It is interesting to study how the solution approaches the point $y = y_0$ from the interior of the $g(y) \neq 0$ region. The scalar field approaches zero as

$$\rho \sim \frac{q\kappa^2}{4} \sqrt{c} (y_0 - y), \quad (3.57)$$

so the metric approaches that of AdS_4

$$H \sim \frac{R^2}{(y_0 - y)^2}, \quad R = \frac{4}{\kappa^2 q}. \quad (3.58)$$

The limit $y = y_0$ is actually at an infinite proper distance in spacelike directions.

Let us now consider the other end of the $g(y) \neq 0$ region, $y = -y_0$, where $G(y)$ reaches the value $G(-y_0) = \sqrt{\frac{c}{2\pi}} \kappa^2 q y_0 = G(-\infty)$, which can be tuned by moving the domain-wall sources (y_0). Thus value has to be smaller or equal than 1 in order to have a well-defined solution.

The cases $G(-y_0) = 1$ and $G(-y_0) < 1$ are very different: if $G(-y_0) < 1$ then $\rho(-y_0)$ is finite and we can choose the constant value of ρ in the $y < -y_0$ $\rho(-\infty) = \rho(-y_0)$ to have continuity. ρ approaches $y = -y_0$ from the interior of the $g(y) \neq 0$ region as

$$\rho \sim -\sqrt{\frac{c}{2\pi}} \frac{\kappa^2 q}{\text{Erf}'[\rho(-\infty)/\sqrt{2}]} (y + y_0), \quad (3.59)$$

so the metric approaches another AdS_4 region and we can do without the exterior $y < -y_0$ region. The solution we have obtained smoothly interpolates between two AdS_4 regions one of which (the $\rho = 0$ one) corresponds to a supersymmetric vacuum of the bulk-theory.

Let us point out that the effective potential $g^2(y)V$ evaluated on this solution can become positive near $y = y_0$ when $G(-\infty) > \text{Erf}(\sqrt{3}/2)$.

The two infinitely thin domain-wall sources set-up may be understood as an approximation to the following configuration with domain-wall sources of *finite thickness* described, for instance, by

$$\begin{aligned} f(y) &= qye^{-y^2}, \\ g(y) &= \mp \frac{\kappa^2 q}{16} e^{-y^2}, \\ G(y) &= -\frac{q\kappa^2 \sqrt{2c}}{16} \text{Erf}(y) + d. \end{aligned} \quad (3.60)$$

Observe that the local coupling constant $g(y)$ vanishes only asymptotically.

This source distribution will lead to a well-defined scalar field ρ if $0 \leq d - q\kappa^2 \sqrt{2c}/16 < 1$; if we choose $d = q\kappa^2 \sqrt{2c}/16$, so that $G(y) = \frac{q\kappa^2 \sqrt{2c}}{16} [1 - \text{Erf}(y)]$, ρ will vanish asymptotically as $\rho \sim e^{-y^2}/y$ when $y \rightarrow +\infty$ and the metric will diverge as $H \sim \rho^{-2}$ in that limit. If we choose $d > q\kappa^2 \sqrt{2c}/16$, ρ will asymptote to a constant value $\rho \sim \rho(+\infty)$ and the metric will be asymptotically flat. The same happens in the $y \rightarrow -\infty$ limit if we take $d + q\kappa^2 \sqrt{2c}/16 < 1$, which can be arranged by an adequate choice of c ; ρ will, however, diverge in that limit if we choose $d + q\kappa^2 \sqrt{2c}/16 = 1$. As shown in figure (3), the fact that the asymptotically non-diverging solutions interpolate between asymptotic Minkowskian spaces is due to the fact that the effective potential as seen by the solution, *i.e.* g^2V , vanishes asymptotically.

This asymptotic behavior is qualitatively similar to that of the infinitely-thin sources case. On the other hand, if we expand the solution around any finite value of y we will find that the metric is locally AdS_4 , as happens in the infinitely-thin sources case on the interval $[-y_0, +y_0]$.

As we have seen, the introduction of sources, which forces the introduction of local coupling constants, modifies the domain-wall solutions dramatically.

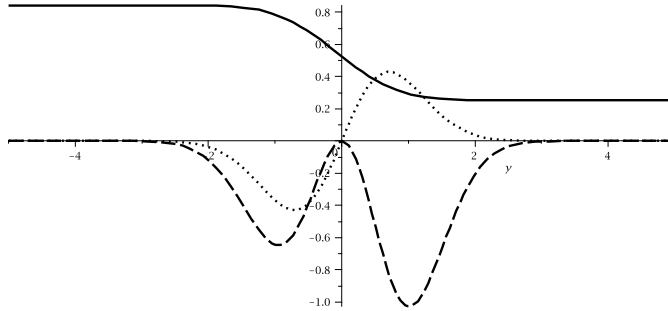


Figure 3: The various profiles of the functions occurring in the solution of the thick domain wall for the regular case: the dotted line corresponds to the chosen source, $f(y)$, the solid line to the corresponding scalar field $\rho(y)$, and the dashed line corresponds to the effective potential as seen by the solution, *i.e.* $g^2(y)V$. Observe that the degeneracy of minima of the potential is broken by the local coupling constant.

4 Supersymmetric instantons

4.1 General sourceless and sourceful instanton solutions

We are interested in (multi-) instanton solutions of $N = 1, d = 4$ supergravity generalizing the stringy D-instanton of Ref. [21], *i.e.* with flat Euclidean metric and unbroken supersymmetry.

It is convenient to start by considering the general case of a d_σ -dimensional σ -model with real coordinates ϕ^i $i = 1, \dots, d_\sigma$ and metric \mathcal{G}_{ij} coupled to gravity in d Euclidean dimensions with action (up to boundary terms)

$$S_{\text{bulk}} = \int d^d x \sqrt{|g|} \{ R + \frac{1}{2} \mathcal{G}_{ij} \partial_\mu \phi^i \partial^\mu \phi^j \}, \quad (4.1)$$

and equations of motion

$$R_{\mu\nu} + \frac{1}{2} \mathcal{G}_{ij} \partial_\mu \phi^i \partial_\nu \phi^j = 0, \quad (4.2)$$

$$\nabla^2 \phi^i + \Gamma_{jk}^i \partial_\mu \phi^j \partial^\mu \phi^k = 0. \quad (4.3)$$

The requirement that the instanton solution has flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ so $R_{\mu\nu} = 0$ in the Einstein equation implies

$$\mathcal{G}_{ij} \partial_\mu \phi^i \partial_\nu \phi^j = 0. \quad (4.4)$$

Thus, the kind of instantons we are looking for only exists when the Euclidean action is not positive-definite.

Now, following Ref. [23], we assume that the scalars in the instanton solution depend on n independent functions of the d spatial coordinates $\sigma^a(x)$, $a = 1, \dots, n$. The equations of motion (4.4) and (4.3) become, respectively,

$$\mathcal{G}_{ij} \partial_a \phi^i \partial_b \phi^j \partial_\mu \sigma^a \partial_\nu \sigma^b = 0, \quad (4.5)$$

$$\nabla^2 \sigma^a \partial_a \phi^i + \{ \partial_a \partial_b \phi^i + \Gamma_{jk}^i \partial_a \phi^j \partial_b \phi^k \} \partial_\mu \sigma^a \partial_\mu \sigma^b = 0, \quad (4.6)$$

where $\partial_a \equiv \partial / \partial \sigma^a$. If we assume that the σ^a are harmonic functions

$$\nabla^2 \sigma^a = 0, \quad (4.7)$$

(up to singular terms in the r.h.s. that will be dealt with when we consider the sources), then these solutions are satisfied if we find an n -dimensional hypersurface $\phi^i(\sigma)$ with n null and mutually orthogonal tangent vectors $\partial_a \phi^i(\sigma)$ satisfying the equations

$$\partial_a \partial_b \phi^i + \Gamma_{jk}^i \partial_a \phi^j \partial_b \phi^k = 0, \quad (4.8)$$

which imply (for each value of $a = b$) that each of the n coordinate curves obtained by taking a constant value of all but one of the σ^a s are geodesics in the scalar target space.

These equations can be solved by finding n null and mutually orthogonal vector fields $N_a^i(\phi)$ in the scalar target space satisfying the equations

$$N_{[a}{}^j \nabla_j N_{|b]}{}^i = 0, \quad \forall a, b = 1, \dots, n, \quad (4.9)$$

and then solving for $\phi(\sigma)$ the system of flow equations

$$N_a^i(\phi(\sigma)) = \partial_a \phi^i(\sigma). \quad (4.10)$$

This is possible only if the integrability conditions $\partial_{[a} \partial_{b]} \phi^i = 0$ are satisfied, i.e. if the null vector fields N_a commute over the null hypersurface $\phi^i(\sigma)$:

$$\partial_{[a} N_{b]}{}^i = \partial_{[a} \phi^j \partial_j N_{b]}{}^i = N_{[a}{}^j \partial_j N_{b]}{}^i \Big|_{\phi(\sigma)} = \frac{1}{2} [N_a, N_b]^i \Big|_{\phi(\sigma)} = 0. \quad (4.11)$$

If the vector fields N_a are Killing vectors this has further consequences that we will not explore here.

Therefore, the problem of finding D-like instanton solutions can be reduced to the problem of finding n null, mutually orthogonal vector fields (which is only possible if the dimensionality of the σ -model is greater or equal than $2n$) satisfying the parallel-transport conditions Eq. (4.9) in the space with σ -model metric \mathcal{G}_{ij} . The instanton solutions are given by the integral surfaces $\phi(\sigma)$ where the surface coordinates σ satisfy the Laplace equation in space.

It is always possible (but in no way necessary) to define coordinates $\{\phi^{a+}, \phi^M\}$ with $a = 1, \dots, n$, $M = n + 1, \dots, d_\sigma$ adapted to the null vector fields

$$N_a{}^i \partial_i \equiv \partial_{a^+}, \quad (4.12)$$

so on the integral hypersurfaces $\phi^{a^+}(\sigma) = \sigma^a$. We can also define n dual, mutually orthogonal 1-forms L^a , such that $L^a{}_i L^b{}_j \mathcal{G}^{ij} = 0$ and $L^a{}_i N_b{}^i = \delta^a_b$, whose coordinate representation reads

$$L^a{}_i d\phi^i = d\phi^{a^+} + A^{a^+}{}_M d\phi^M. \quad (4.13)$$

Let us now consider the introduction of sources for these instanton solutions: as instantons have a 0-dimensional worldvolume, their effective actions are just the value of some field at the location of the source. The most general action that we can write down, then, has the form

$$S_{\text{inst}} = \int d^d x f^a(x) \mathcal{F}_a(\phi), \quad (4.14)$$

where the $f^a(x)$ are some given distribution functions ($\sim \delta^{(d)}(x)$ for a single instanton at the origin) and the $\mathcal{F}_a(\phi)$ s are linearly-independent functions of the scalar fields whose properties will be determined by consistency (and, in due time, by requiring supersymmetry).

The coupling of this action to the bulk action Eq. (4.1) modifies the equations of motion of the scalars Eq. (4.3) to

$$\nabla^2 \phi^i + \Gamma_{jk}{}^i \partial_\mu \phi^j \partial^\mu \phi^k - \frac{f^a(x)}{\sqrt{g}} \mathcal{G}^{ij} \partial_j \mathcal{F}_a = 0. \quad (4.15)$$

Introducing the functions σ^a this equation can be solved by solving separately Eqs. (4.8) and

$$\nabla^2 \sigma^a = \frac{f^a(x)}{\sqrt{g}}, \quad (4.16)$$

if we assume that

$$\partial_a \phi^i = \mathcal{G}^{ij} \partial_j \mathcal{F}_a. \quad (4.17)$$

Observe that this condition implies that the 1-form dual to the null vectors N_a are exact: $\partial_a \phi^i \mathcal{G}_{ij} d\phi^j = d\mathcal{F}_a$. If we want to work in adapted coordinates ϕ^{a^+} then we can use the functions $\mathcal{F}_a(\phi)$ to define coordinates $\phi^{a^-} \equiv \mathcal{F}_a(\phi)$. Calling the remaining coordinates ϕ^m and using these coordinates, the σ -model metric has to be of the Walker type¹⁰

$$\mathcal{G}_{ij} d\phi^i d\phi^j = 2d\phi^{a^-} (d\phi^{a^+} + A^{a^+}{}_{b^-} d\phi^{b^-} + A^{a^+}{}_m d\phi^m) + \mathcal{C}_{mn} d\phi^m d\phi^n, \quad (4.18)$$

¹⁰In Ref. [28], A.G. Walker asked the following question: Given a spacetime of dimension n with a metric g , which admits m ($2m \leq n$) independent null-vectors N_a satisfying the propagation rule $\nabla N_a = A_a{}^b \otimes N_b$, what is the canonical form of the metric? The answer is that any such metric can be written as in Eq. (4.18).

while the instanton source will take the form

$$S_{\text{inst}} = \int d^d x f^a(x) \phi^{a-}(x). \quad (4.19)$$

In adapted coordinates the instanton solutions will have the form $\phi^{a+} = \sigma^a$, $\phi^{a-} = \text{const.}$, $\phi^m = \text{const.}$

At this point we must raise the question: given n independent null-geodesics of a metric g , under what conditions can they be embedded into simultaneously-twistfree null-congruences? It is known from the literature concerning the Penrose limit, that a null-geodesic on a Lorentzian space can always be embedded into a twistfree null-congruence and the proof, see *e.g.* Ref. [29], can in all likelihood be enhanced to the case of one null-geodesic on a pseudo-Riemannian space, but it seems unlikely that such an embedding is always possible for more geodesics. Lacking an answer to the general question, however, we shall content ourselves with the knowledge that for the metrics and null-geodesics we are interested in, namely the ones corresponding to supersymmetric instantons, this simultaneously-twistfree embedding is, as will be shown in the next section, always possible.

As we have seen so far, instanton solutions are associated only to the existence of null geodesics of the σ -model metric. The relation between instanton solutions and isometries allows us to define and compute an instanton charge for the solution. Thus, we are lead to consider the cases in which the σ -model metric is invariant under global transformations that act on the adapted coordinates ϕ^{a-} as constant shifts. The $a-$ components of the Killing vectors $k_{(a)}^i$ generating each of these transformations will, thus, have the form $k_{(a)}^{b-} = \delta_a^b$. The Noether currents associated to these invariances are, in the adapted metric Eq. (4.18)

$$j_{(a)}^\mu = k_{(a)}^i \mathcal{G}_{ij} \partial^\mu \phi^j, \quad (4.20)$$

and we can define the associated instanton charges Q_a enclosed by a 3-dimensional hypersurface Σ^3 by the integrals

$$Q_a \equiv \int_{\Sigma^3} d\Sigma^\mu j_{(a)\mu}. \quad (4.21)$$

These definitions can be rewritten as integrals over the 4-volume V^4 enclosed by Σ^3 $\partial V^{(4)} = \Sigma^3$:

$$Q_a = \int_{V^{(4)}} d^4 x \sqrt{g} \nabla_\mu j_{(a)}^\mu. \quad (4.22)$$

These expressions do not vanish because the conservation of the Noether currents is violated at the sources, which allows us to compute the charges easily: since the Noether currents are conserved on shell, i.e.

$$\nabla_\mu j_{(a)}^\mu = \nabla_\mu (k_{(a)i} \partial^\mu \phi^i) = -\frac{k_{(a)}^i}{\sqrt{g}} \frac{\delta S_{\text{bulk}}}{\delta \phi^i}. \quad (4.23)$$

In presence of instanton sources the equations of motion of the scalars are

$$\frac{\delta S_{\text{bulk}}}{\delta \phi^i} + \frac{\delta S_{\text{inst}}}{\delta \phi^i} = 0, \quad (4.24)$$

and, using the instanton source in adapted coordinates Eq. (4.19)

$$\frac{\delta S_{\text{bulk}}}{\delta \phi^i} = -\delta_i^{a-} \frac{f^a}{\sqrt{g}}, \quad (4.25)$$

so

$$\nabla_\mu j_{(a)}^\mu = \frac{f^a}{\sqrt{g}}, \quad (4.26)$$

and

$$Q_a = \int_{V^{(4)}} d^4x f^a. \quad (4.27)$$

Thus, the source for Q_a instantons of the species a placed at $x^\mu = x_0^\mu$ is just

$$f^a(x) = Q_a \delta^{(4)}(x - x_0). \quad (4.28)$$

We can try to evaluate naively the Euclidean action $S = S_{\text{bulk}} + S_{\text{inst}}$ for these instanton solutions. In principle one should add to this action the Gibbons-Hawking boundary term found in Ref. [30], but its contribution is zero for flat Euclidean space [21]. The bulk part of the action always vanishes on shell for gravity plus scalars systems (irrespective of their positive-definiteness) [31] and we are left with

$$S = S_{\text{inst}} = Q_a \phi^{a-} = Q_a \phi_\infty^{a-}. \quad (4.29)$$

(ϕ^{a-} , being constant, is also the value of ϕ^{a-} at infinity).

It has been argued in the literature (see, e.g. [32, 22]) that, in order to restore the invariance of the total action $S = S_{\text{bulk}} + S_{\text{inst}}$ under the shifts of ϕ^{a-} , an appropriate boundary term ought to be introduced. In the general case under consideration the only such term that can be introduced is

$$- \int d^4x \sqrt{g} \nabla_\mu (\phi^{a-} j_{(a)}^\mu) = - \int d^3\Sigma^\mu \phi^{a-} j_{(a)\mu}. \quad (4.30)$$

Its contribution to the action, however, cancels identically that of the source term: this was to be expected as the only way to recover the shift invariance is to eliminate all explicit occurrences of ϕ^{a-} from the result.

On the other hand, in the cases of interest, the isometries that shift the ϕ^{a-} s also shift other coordinates, so the coordinates adapted to the isometries do not coincide with the ϕ^{a-} . Let $\chi^a(\phi)$ stand for those adapted coordinates. The boundary term may then be

$$- \int d^3\Sigma^\mu \chi^a j_{(a)\mu}, \quad (4.31)$$

and then the Euclidean action would be given by the non-vanishing shift-invariant result

$$S = Q_a(\phi_\infty^{a-} - \chi_\infty^a). \quad (4.32)$$

4.2 Supersymmetric sourceless and sourceful instanton solutions

Let us now consider the case of interest for us: Wick-rotated $N = 1, d = 4$ supergravity coupled to chiral multiplets. The complex scalars Z^i of the Lorentzian theory consist of a real scalar and a pseudoscalar which, by convention, we always take to be the real part of Z^i , something which is always possible to achieve via field redefinitions. To Wick-rotate the Z^i s we are going to use the rule of thumb/prescription that says that pseudoscalars get an extra factor of i in the Wick rotation so

$$Z^i \longrightarrow iZ^{i+}, \quad Z^{*i*} \longrightarrow -iZ^{i-}, \quad (4.33)$$

where Z^{i+} and Z^{i-} are two independent real scalars related to the components of the complex scalars Z^i by

$$Z^{i\pm} \equiv \Im(Z^i) \pm \Re(Z^i)_E, \quad (4.34)$$

where the subscript E indicates that we are dealing with the Wick-rotated pseudoscalar.

In many cases (in particular in the examples considered) this prescription leads to Wick-rotated Kähler metrics \mathcal{G}_{i+j-} which are real¹¹ and to the Euclidean action

$$S = \int d^4x \sqrt{g} [R + 2\mathcal{G}_{i+j-} \partial_\mu Z^{i+} \partial^\mu Z^{j-} + 2f^a(x) \mathcal{F}_a(Z^+, Z^-)], \quad (4.35)$$

where we have included a set of sources inspired in the general case with a normalization adequate to our conventions for $N = 1, d = 4$ supergravity. From this action we get the equations of motion

$$R_{\mu\nu} + 2\mathcal{G}_{i+j-} \partial_\mu Z^{i+} \partial_\nu Z^{j-} = 0, \quad (4.36)$$

$$\nabla^2 Z^{i+} + \Gamma_{j+k+}^{i+} \partial_\mu Z^{j+} \partial^\mu Z^{k+} - f^a(x) \mathcal{G}^{i+j-} \partial_{j-} \mathcal{F}_a = 0, \quad (4.37)$$

$$\nabla^2 Z^{i-} + \Gamma_{j-k-}^{i-} \partial_\mu Z^{j-} \partial^\mu Z^{k-} - f^a(x) \mathcal{G}^{i-j+} \partial_{j+} \mathcal{F}_a = 0. \quad (4.38)$$

In order to have a flat spatial metric we must have

¹¹Actually [22] these ‘‘Wick-rotated Kähler metrics’’ must be *para-Kähler metrics*, which are real, split (signature n, n) metrics satisfying essentially the same properties as the Kähler metrics do but in terms of hyperbolic numbers which are generated over the reals by 1 and e , where $e^2 = +1$ and $e^* = -e$. Hyperbolic numbers take the general form $w = a + eb$ ($a, b \in \mathbb{R}$) and their conjugate is $w^* = a - eb$ so that $ww^* = a^2 - b^2$.

$$\mathcal{G}_{i+j-}\partial_\mu Z^{i+}\partial_\nu Z^{j-} = 0. \quad (4.39)$$

Inspired by the results of the general case, to solve this constraint we assume that in the instanton solutions only the Z^{i+} s are nontrivial, depending on up to n functions $\sigma^a(x)$, while the Z^{i-} s are constant¹²:

$$\partial_\mu Z^{i+} = \partial_\mu \sigma^a \partial_a Z^{i+}, \quad \partial_\mu Z^{i-} = 0. \quad (4.40)$$

This Ansatz (called the ‘‘extremal instanton Ansatz’’ in Ref. [22]) automatically solves the third equation of motion if

$$\partial_{j+}\mathcal{F}_a = 0, \quad (4.41)$$

a result which will be shown to be consistent with the supersymmetry of the source. After contraction with \mathcal{G}_{l-i+} , the second reduces to

$$\nabla^2 \sigma^a \mathcal{G}_{l-i+} \partial_a Z^{i+} + \partial_{(a}(\mathcal{G}_{l-i+} \partial_{b)} Z^{i+}) \partial_\mu \sigma^a \partial_\mu \sigma^b - f^a(x) \mathcal{G}^{i+j-} \partial_{j-} \mathcal{F}_a = 0, \quad (4.42)$$

which are solved by imposing, separately,

$$\nabla^2 \sigma^a = f^a(x), \quad (4.43)$$

$$\partial_{(a}(\mathcal{G}_{l-i+} \partial_{b)} Z^{i+}) = 0, \quad (4.44)$$

and the consistency constraint

$$\mathcal{G}_{i-j+} \partial_a Z^{i+} = \partial_i \mathcal{F}_a. \quad (4.45)$$

Now, given the Kähler origin of the metric, $\mathcal{G}_{l-i+} = \partial_{i+} \partial_{l-} \mathcal{K}_E$, where \mathcal{K}_E is the Wick-rotated Kähler potential, and the last equation reduces to

$$\partial_a \partial_b \partial_{i-} \mathcal{K}_E = 0, \quad (4.46)$$

which can be integrated immediately

$$\partial_{i-} \mathcal{K}_E = c_i + d_{ia} \sigma^a, \quad (4.47)$$

for some integration constants c_i, d_{ia} . These are n algebraic equations involving the $\phi^{i+}(\sigma)$ and the constants ϕ^{j-} and which, in principle, one should be able to solve for the n $\phi^{i+}(\sigma)$.

The constraint Eq. (4.45) for the sources can also be solved immediately:

$$\mathcal{F}_a = d_{ai} Z^{i-}. \quad (4.48)$$

¹²Interchanging everywhere indices $+$ and $-$ we go from instantons to anti-instantons.

At this point it is straightforward to go to adapted/Walker coordinates by effecting the coordinate-transformation¹³

$$\left. \begin{aligned} \phi^{i-} &\equiv Z^{i-} \\ \phi_i^+ &\equiv \partial_{i-}\mathcal{K}_E \end{aligned} \right\} \longrightarrow \mathcal{G}_{i+j-}dZ^{i+}dZ^{j-} = d\phi^{i-} (d\phi_i^+ + \mathbf{H}_{ij} d\phi^{j-}) , \quad (4.49)$$

whence all the information of the metric resides in

$$\mathbf{H}_{ij} \equiv \partial_{i-}\partial_{j-}\mathcal{K}_E|_{Z^\pm=Z^\pm(\phi)} , \quad (4.50)$$

and generically depends on ϕ^- and ϕ^+ .

Let us then consider the issue of the unbroken supersymmetry of these solutions: this is actually a worrisome point, one discussed at length in the literature, as it concerns the Wick rotation of spinors. The trouble is easily recognised by seeing that in Lorentzian signature there exists in 4-dimensions a spinor with four real supercharges, in fact the one used to build $N = 1$ $d = 4$ sugra, whereas in Euclidean signature the minimal spinor has eight supercharges. A consistent scheme for doing the Wick rotation was developed in [33], which does not solve the doubling problem, but leaves the form of the supersymmetry transformations invariant, which we hold to be a desirable property. So accepting the doubling of fermions, the gravitino, the chiralini and the supersymmetry parameter $\psi_\mu, \chi^i, \epsilon$ will be rotated into $\psi_\mu^+, \chi^{i+}, \epsilon^+$ whereas their complex conjugates $\psi_\mu^*, \chi^{i*}, \epsilon^*$ will be rotated into *independent* spinors $\psi_\mu^-, \chi^{i-}, \epsilon^-$. Then, the chiralini supersymmetry rules give rise to the Killing spinor equations

$$\begin{aligned} \not{\partial}Z^{i+}\epsilon^- &= 0 , \\ \not{\partial}Z^{i-}\epsilon^+ &= 0 , \end{aligned} \quad (4.51)$$

which for the Ansatz Eq. (4.40), can be solved by setting

$$\epsilon^- = 0 . \quad (4.52)$$

The gravitino supersymmetry transformation rule gives rise to another two Killing spinor equations:

$$\begin{aligned} [\nabla_\mu + \frac{1}{4}(\partial_\mu Z^{i+}\partial_{i+}\mathcal{K}_E - \partial_\mu Z^{i-}\partial_{i-}\mathcal{K}_E)]\epsilon^+ &= 0 , \\ [\nabla_\mu - \frac{1}{4}(\partial_\mu Z^{i+}\partial_{i+}\mathcal{K}_E - \partial_\mu Z^{i-}\partial_{i-}\mathcal{K}_E)]\epsilon^- &= 0 . \end{aligned} \quad (4.53)$$

The second equation is solved by the condition Eq. (4.52) and the first can be rewritten in the form

$$e^{-\mathcal{K}_E/4}\partial_\mu(e^{\mathcal{K}_E/4}\epsilon^+) = 0 , \quad (4.54)$$

¹³ Observe that this coordinate transformation is in general not para-holomorphic but that it is always invertible.

and is solved by

$$\epsilon^+ = e^{-\mathcal{K}_E/4} \epsilon_0^+, \quad (4.55)$$

for an arbitrary constant spinor ϵ_0^+ .

Since we have only used the Ansatz Eq. (4.40) and not the particular form of any solution, all the D-like instanton solutions preserve 1/2 of the supersymmetries.

Let us now consider the supersymmetry of the source. The supersymmetry transformation rules of the scalars are

$$\delta_\epsilon Z^{i\pm} = \frac{1}{4} \bar{\chi}^{i\pm} \epsilon^\pm, \quad (4.56)$$

and, thus, the supersymmetry variation of the source is

$$f^a(x) \{ \partial_{i-} \mathcal{F}_a \delta_\epsilon Z^{i-} + \partial_{i+} \mathcal{F}_a \delta_\epsilon Z^{i+} \} = \frac{1}{4} \{ \partial_{i+} \mathcal{F}_a \bar{\chi}^{i+} \epsilon^+ + \partial_{i-} \mathcal{F}_a \bar{\chi}^{i-} \epsilon^- \}, \quad (4.57)$$

and vanishes for either $\partial_{i+} \mathcal{F}_a = \epsilon^- = 0$ (our choice) or $\partial_{i-} \mathcal{F}_a = \epsilon^+ = 0$. We recover, then, the condition Eq. (4.41).

4.2.1 Example 1: $\text{Sl}(2, \mathbb{R})/\text{SO}(2)$

In this section we are going to consider the $\text{Sl}(2, \mathbb{R})/\text{U}(1)$ σ -model which is ubiquitous in supergravity theories¹⁴. Using the standard coordinate $\tau \equiv \chi + ie^{-\phi}$ which takes values in the upper half complex plane, the Kähler potential is $\mathcal{K} = -\log(\Im\tau)$ and the kinetic term takes the form

$$2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*j^*} = \frac{1}{2} \frac{\partial_\mu \tau \partial^\mu \tau^*}{(\Im\tau)^2}. \quad (4.58)$$

$\text{Sl}(2, \mathbb{R})$ acts on τ via fractional-linear transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad (4.59)$$

which leave the target-space metric invariant. These transformations are generated by 3 real Killing vectors K_T, K_D, K_K

$$K_T = \partial_\chi, \quad K_D = \partial_\phi \chi \partial_\chi, \quad K_K = \chi \partial_\phi - \frac{1}{2}(\chi^2 - e^{-2\phi}) \partial_\chi, \quad (4.60)$$

which satisfy the algebra

$$[D, T] = T, \quad [D, K] = -K, \quad [T, K] = D. \quad (4.61)$$

The corresponding components of the holomorphic Killing vectors ($K = k^\tau(\tau) \partial_\tau + \text{c.c.}$) are

¹⁴A D-like instanton solution in $N = 1, d = 4$ string compactifications was first constructed in Ref. [34] using the Kalb-Ramond 2-form instead of the dual pseudoscalar field.

$$k_T^\tau = 1, \quad k_D^\tau = -\tau, \quad k_K^\tau = -\frac{1}{2}\tau^2, \quad (4.62)$$

and the corresponding momentum maps¹⁵ are given by

$$\mathcal{P}_T = -\frac{1}{2\Im\tau}, \quad \mathcal{P}_D = -\frac{\Re\tau}{2\Im\tau}, \quad \mathcal{P}_K = -\frac{|\tau|^2}{4\Im\tau}, \quad (4.64)$$

and

$$\lambda_T = 0, \quad \lambda_D = \frac{1}{2}, \quad \lambda_K = \frac{1}{2}\tau. \quad (4.65)$$

The complex coordinate τ is adapted to the isometry T , under which it transforms by a real shift $\tau' = \tau + b$; this clearly affects only the real component χ , which can be identified as the pseudoscalar field. If we Wick-rotate τ according to the prescription we have given $\chi \longrightarrow i\chi_E$ and

$$\tau \longrightarrow i\tau^+ \equiv i(e^{-\phi} + \chi_E) \quad \tau^* \longrightarrow -i\tau^- \equiv -i(e^{-\phi} - \chi_E), \quad (4.66)$$

so¹⁶

$$\frac{1}{2} \frac{\partial_\mu \tau \partial^\mu \tau^*}{(\Im\tau)^2} \longrightarrow -\frac{2\partial_\mu \tau^+ \partial^\mu \tau^-}{(\tau^+ + \tau^-)^2}. \quad (4.67)$$

Including an instanton source associated to the coordinate τ^- the Euclidean action takes the form

$$S_T = \int d^4x \sqrt{g} \left[R + \frac{2\partial_\mu \tau^+ \partial^\mu \tau^-}{(\tau^+ + \tau^-)^2} + 2d_T f^T(x) \tau^- \right], \quad (4.68)$$

where d_T is a constant that can be set to 1.

The Noether current associated to the constant shifts $\chi_E \rightarrow \chi_E - c$, $\tau^\pm \rightarrow \tau^\pm \mp c$ is

$$j_T^\mu = 2 \frac{\partial^\mu (\tau^+ - \tau^-)}{(\tau^+ + \tau^-)^2}, \quad (4.69)$$

and

$$\partial_\mu (\sqrt{g} j_T^\mu) = -\frac{\delta S_{\text{bulk}}}{\delta \tau^+} + \frac{\delta S_{\text{bulk}}}{\delta \tau^-}, \quad (4.70)$$

which on-shell gives

¹⁵These are defined, for each isometry, by the two relations:

$$k_{\tau^*} = i\partial_{\tau^*} \mathcal{P}, \quad k^\tau \partial_\tau \mathcal{K} = i\mathcal{P} + \lambda. \quad (4.63)$$

¹⁶In the Wick rotation we go from our mostly-minus metric to a Euclidean mostly plus metric. This gives rise to a global sign in the scalar kinetic term as well as in the Ricci scalar.

$$\partial_\mu(\sqrt{g} j_T^\mu) = -\frac{\delta S_{\text{inst}}}{\delta \tau^-} = 2f^T(x). \quad (4.71)$$

For the sake of simplicity we will take $f^T(x) = \frac{1}{2}Q_T\delta^{(4)}(x)$ so that we will consider a single instanton source of charge Q_T placed at the origin.

Following the general procedure (“extremal instanton Ansatz”), the instanton solutions are given by ($d_T = 1$)

$$\begin{aligned} \tau^- &= \text{constant}, \\ \partial_{\tau^-}\mathcal{K}_E &= c_T + \sigma^T, \\ \nabla^2\sigma^T &= \frac{1}{2}Q_T\delta^{(4)}(x), \end{aligned} \quad (4.72)$$

where the para-Kähler potential is given by

$$\mathcal{K}_E = -\log(\tau^+ + \tau^-). \quad (4.73)$$

The additive constant c_T can be absorbed into the harmonic function σ^T and we define

$$H_T \equiv -(c_T + \sigma^T) = -c_T + \frac{1}{4\pi^2} \frac{Q_T/2}{r^2}. \quad (4.74)$$

The second equation can be solved for τ^+ as a function of the harmonic function H_T and the constants c_T, τ^- :

$$\tau^+ = -\tau^- + H_T^{-1}. \quad (4.75)$$

We can determine the constants c_T, τ_- in terms of $\chi_{E\infty}$ and ϕ_∞ (the asymptotic values of χ_E and ϕ) and we find that the instanton solution can be written in the final form

$$\begin{aligned} \tau^- &= e^{-\phi_\infty} - \chi_{E\infty}, \\ \tau^+ &= -\tau^- + 2e^{-\phi_\infty} \left(1 + \frac{1}{4\pi^2} \frac{e^{-\phi_\infty} Q_T}{r^2} \right)^{-1}. \end{aligned} \quad (4.76)$$

Let us now compute the Euclidean action of this instanton solution. According to the general discussion, the action Eq. (4.68) evaluated on the above instanton solution gives $\tau_\infty^- Q_T$ and adding the total-derivative term

$$\int d^4x \partial_\mu [\sqrt{g} \frac{1}{2} (\tau^+ - \tau^-) j_T^\mu] \quad (4.77)$$

needed to restore the shift-invariance, we get the standard result $\frac{1}{2}(\tau_\infty^+ - \tau_\infty^-)Q_T = e^{-\phi_\infty}Q_T$. [21]

Observe that, to obtain the instanton solution, we could have worked in the adapted coordinates ϕ^+, ϕ^- defined by

$$\tau^+ \equiv (\phi^+)^{-1} - \phi^- , \quad \tau^- \equiv \phi^- , \quad \longrightarrow \quad -\frac{d\tau^+ d\tau^-}{(\tau^+ + \tau^-)^2} = d\phi^+ d\phi^- + (\phi^+)^2 (d\phi^-)^2 . \quad (4.78)$$

Observe, however, that the reparametrization necessary to go to adapted coordinates does not respect the para-Hermitian structure (it mixes τ^+ and τ^-). According to the general discussion, it is clear that ϕ^+ is a geodesic coordinate and an instanton solution is provided by

$$\phi^+ = \sigma^T , \quad \phi^- = \text{constant} . \quad (4.79)$$

Adding a constant c_T to the σ^T we chose before, we recover exactly the same solution.

Now, however, we cannot easily recover the standard result for the Euclidean action since, in order to restore shift invariance, the term that we should add cancels automatically the contribution of the rest of the action because, in these non-para-holomorphic coordinates, only ϕ^- transforms under the shifts.

Let us now consider the instantons related to the isometry D . The complex coordinate adapted to this isometry, ξ , is implicitly defined by $k_D^\tau \partial_\tau = \partial_\xi$, so¹⁷ $\tau = ie^{-\xi}$ and the σ -model metric takes the form

$$\frac{1}{2} \frac{\partial_\mu \tau \partial^\mu \tau^*}{(\Im \tau)^2} = \frac{1}{2} \frac{\partial_\mu \xi \partial^\mu \xi^*}{\sin^2(\Im \xi)} . \quad (4.80)$$

Under the D -transformations $\xi' = \xi + c$ while $\tau' = e^{-c} \tau$.

In this case, it seems reasonable to take the imaginary part of ξ to be the pseudoscalar field, as otherwise the Wick rotation of the Kähler potential would not be real. Furthermore, we find that, thanks to the $-i$ factor in the relation between τ and ξ

$$\begin{aligned} \Re \tau &= -e^{-\Re \xi} \sin \Im \xi , \\ \Im \tau &= e^{-\Re \xi} \cos \Im \xi , \end{aligned} \quad (4.81)$$

so it will be consistent to take $\Re \tau$ and $\Im \xi$ to be pseudoscalars simultaneously.

We, then, define the complex coordinate $Z = i\xi$ and Wick-rotate it using the standard prescription

$$Z \longrightarrow iZ^+ , \quad Z^* \longrightarrow -iZ^- , \quad Z^\pm = \Im Z \pm \Re Z_E , \quad (4.82)$$

so we end up with

$$-\frac{1}{2} \frac{\partial_\mu Z^+ \partial^\mu Z^-}{\sinh^2(Z^+ - Z^-)/2} . \quad (4.83)$$

By performing the coordinate transformation

¹⁷The factor i has been chosen for consistency as will be explained.

$$Z^+ = 2 \operatorname{arccoth}(\phi^+) + 2\phi^-, \quad Z^- = 2\phi^-, \quad (4.84)$$

we end up with the metric in the Walker form

$$2d\phi^- \left(d\phi^+ - [(\phi^+)^2 - 1] d\phi^- \right). \quad (4.85)$$

In these coordinates the instanton solution is again given by Eq. (4.79) and we just have to plug these expressions into those of Z^+ and Z^- in Eq. (4.84) to find $\Re Z_E$ and $\Im Z$.

Undoing the coordinate transformations, we find complicated expressions for the original variables:

$$\begin{aligned} \chi_E &= (\Re \tau)_E = e^{-(Z^+ + Z^-)/2} \sinh(Z^+ - Z^-)/2, \\ e^{-\phi} &= \Im \tau = e^{-(Z^+ + Z^-)/2} \cosh(Z^+ - Z^-)/2, \end{aligned} \quad (4.86)$$

4.2.2 Example 2: Instantons on $\overline{\mathbb{C}\mathbb{P}^n}$

This σ -model is defined by the Kähler potential

$$\mathcal{K} = -\log(1 - |Z|^2), \quad |Z|^2 \equiv Z^i Z^{*i}, \quad (4.87)$$

which leads to the Fubini-Study metric

$$ds^2 = 2\mathcal{G}_{ij^*} dZ^i dZ^{*j^*} = 2 \frac{dZ^i dZ^{*i^*}}{1 - |Z|^2} + 2 \frac{Z^i Z^{*j^*} dZ^i dZ^{*j^*}}{(1 - |Z|^2)^2}. \quad (4.88)$$

The standard prescription for the Wick rotation

$$Z^i \longrightarrow iZ^{i+}, \quad Z^{*i^*} \longrightarrow -iZ^{i-}, \quad Z^{i\pm} = \Im Z^i \pm \Re Z_E^i, \quad (4.89)$$

and the general discussion lead us to the Euclidean action

$$S = \int d^4x \sqrt{g} \left[R + 2 \frac{\partial_\mu Z^{i+} \partial^\mu Z^{i-}}{1 - Z^{i+} Z^{i-}} + 2 \frac{Z^{i+} Z^{j-} \partial_\mu Z^{i+} \partial^\mu Z^{j-}}{(1 - Z^{i+} Z^{i-})^2} + 2d_{ia} f^a(x) Z^{i-} \right]. \quad (4.90)$$

The Wick'ed scalar manifold is that of the symmetric space $\mathrm{Sl}(n+1; \mathbb{R}) / [\mathrm{SO}(1, 1) \otimes \mathrm{Sl}(n; \mathbb{R})]$.

Putting the instantons in the Z^+ directions we can go to the adapted coordinate system by changing the coordinates as

$$Z^{i-} \equiv \phi^{i-}, \quad Z^{i+} \equiv \frac{\phi_i^+}{1 + \phi^+ \cdot \phi^-}, \quad (4.91)$$

where we have defined $\phi^+ \cdot \phi^- = \phi_i^+ \phi^{i-}$. The metric in these Walker-coordinates is given by

$$\mathcal{G}_{i+j-} dZ^{i+} dZ^{j-} = d\phi^+ \cdot d\phi^- - (\phi^+ \cdot d\phi^-)^2. \quad (4.92)$$

The above metric is adapted to the n commuting and obvious Killing vectors $k_{(i)} = \partial_{\phi^{i-}}$, and undoing the coordinate transformation in Eq. (4.91), these Killing vectors are

$$k_{(i)} = \partial_{\phi^{i-}} = \partial_{i-} - Z^{i+} Z^{j+} \partial_{j+} . \quad (4.93)$$

Observe that this Killing vector, will not lead to a holomorphic Killing vector once we undo the Wick rotation (4.89). This situation can be ameliorated by adding the n commuting, para-conjugated Killing vectors¹⁸

$$\begin{aligned} \bar{k}_{(i)} &= \partial_{i+} - Z^{i-} Z^{j-} \partial_{j-} \\ &= (1 + \phi^+ \cdot \phi^-) \partial_{\phi_i^+} + \phi^{i-} [\phi^+ \cdot \partial_+ - \phi^- \cdot \partial_-] , \end{aligned} \quad (4.94)$$

where we introduced the abbreviation $\phi^- \cdot \partial_- \equiv \phi^{j+} \partial_{\phi^{j-}}$ and similar for $\phi^+ \cdot \partial_+$.

5 Conclusions

In this paper we have constructed effective actions for supersymmetric conformally-flat domain walls and instantons of $N = 1, d = 4$ supergravity that can be used as sources for the corresponding supersymmetric solutions, which we have reviewed. In order to construct the domain-wall effective action we have clarified the situation of the two 3-forms found in Ref. [11] that transform into the gravitino, showing that there is indeed one deformation parameter associated to each of them.

In the domain-wall case we have seen how the consistent introduction of domain-wall sources modifies the scalar potential via the local, spacetime-dependent coupling constant to the superpotential. We have also seen how the introduction of this local coupling constant in the supersymmetry transformation rules leads to first order flow equations which imply the second order equations of motion including the sources. Everything is consistent with the existence of a fully supersymmetric and democratic action for $N = 1, d = 4$ in which all the higher-rank potentials found in Ref. [11] are present and all the coupling constants are local. The Killing spinor identities of such a democratic theory should imply the mentioned relation between first- and second-order equations.

This work does not exhaust the study of the effective actions of supersymmetric objects of $N = 1, d = 4$ supergravity: those of strings have not yet been constructed, although they may not be as interesting as in other dimensions. Furthermore, in these theories there are supersymmetric domain walls with AdS_3 worldvolumes whose effective actions may be different.

The construction of the effective actions of the 2- and 3-branes of $N = 2, d = 4$ supergravity (which are expected to exist) could prove very interesting since we expect non-trivial intersections with strings and supersymmetric black holes. However, it is necessary to find first the 3- and 4-form potentials of these theories. Work in this direction is in progress.

¹⁸ The fact that there are 2 sets of n mutually commuting sets of isometries is not surprising as $\mathfrak{sl}(n+1; \mathbb{R})$ admits the 3-grading $\mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1$, where $\mathfrak{L}_0 \simeq \mathfrak{so}(1, 1) \oplus \mathfrak{sl}(n; \mathbb{R})$.

Acknowledgments

T.O. would like to thank A. Achúcarro, J. Hartong and J. Urrestilla for very useful conversations. This work has been supported in part by the Spanish Ministry of Science and Education grants FPA2006-00783 and FPA2009-07692, the Comunidad de Madrid grant HEPHACOS P-ESP-00346 and by the Spanish Consolider-Ingenio 2010 program CPAN CSD2007-00042. Further, TO wishes to express his gratitude to M.M. Fernández for her permanent support.

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