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Eternity and the cosmological constant

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ABSTRACT: The purpose of this paper is to analyze the stability of interacting matter in the presence of a cosmological constant. Using an approach based on the heat equation, no imaginary part is found for the effective potential in the presence of a fixed background, which is the n -dimensional sphere or else an analytical continuation thereof, which is explored in some detail.

KEYWORDS: Gauge-gravity correspondence, Models of Quantum Gravity, Field Theories in Higher Dimensions

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Contents

1	Introduction	1
2	The composition law	3
3	The heat kernel	5
4	Green's functions in constant curvature spaces	9
4.1	Flat spacetime	10
4.2	Sphere	11
4.3	de Sitter space	12
4.4	Euclidean Anti de Sitter space	13
4.5	Anti de Sitter space	14
4.6	Projective spaces	14
5	The imaginary part of the effective potential	15
6	Conclusions	18
A	Taxonomy of constant curvature spaces	19
A.1	Global coordinates	22
A.2	Projective coordinates	23
A.3	Poincaré coordinates	23
A.4	Conformal Invariance	24
B	Conformal structure	27
C	What portion of Weierstrass coordinates do Poincaré coordinates cover?	28
D	Spherical harmonics	29

1 Introduction

A recurrent dream in theoretical physics is that a gravitational state with a nonvanishing cosmological constant is unstable. This idea was explicitly stated in [28] more than a quarter century ago, but it is perhaps older. This has been predicated mainly in the context of de Sitter space, but if the stable gravitational state should be Minkowski space there is a clear need of a similar statement concerning negative values of the cosmological constant. Given the fact that there is some evidence that classically the constant curvature, maximally symmetric spaces are stable with respect to linear perturbations irrespectively

of the sign of the curvature [1], the purported instabilities must have quantum origin. The work of Abbott and Deser established positive Killing energy for small fluctuations of the gravitational field. In cases such as de Sitter space, in which there is no Killing vector which is globally timelike, the fluctuations have got to be contained *inside* the corresponding horizon. In Anti de Sitter space they were able to show stability with respect to all asymptotically vanishing fluctuations whatever large.

The instability claim has been recently put on a new basis in a recent paper by Polyakov [30] (where some references to earlier work can be found; many that are not there can be found in the book [10]).

It is well-known that there is a one-parameter family of so-called vacuum states in de Sitter space, first uncovered by Chernikov and Tagirov [13]; a recent reference is [32]. Ariadna's thread in this maze is usually taken as the strength and physical location of the singularities of the propagators (cf. [2]). What is proposed in reference [30] is to consider instead a different guiding principle, namely the "composition principle", a property which seems natural from the first quantized path integral approach to the theory. This property uniquely selects a particular propagator.

This propagator is then used to claim that the presence of quantum fields interacting in de Sitter space an instability appears which manifest itself as an imaginary part of the free energy of the quantum fields. The stability is associated to the concept of *eternity* [30]. It is not fully clear to begin with, that this is the correct observable to consider, at least when there are horizons present (like in de Sitter space, in which no Killing energy can be globally defined). We shall nevertheless compute it, because it is anyway the first step towards more satisfactory calculations.

This claim is possibly related, but not identical, to the one put forward since quite a few years by Tsamis and Woodard [35] and recently criticized by Garriga and Tanaka [16]. The latter is a quantum gravitational effect; whereas the one we are considering in this paper is supposed to appear when considering quantum fields in a gravitational background, and neglecting backreaction effects.

The relationship of these different claims with the status of de Sitter space as a vacuum of quantum gravity [39] is not altogether clear. To the best of our knowledge, de Sitter space is at best a metastable solution of string theory [23]. But the reason for that seems to rely strongly on gravitational interactions.

The aim of the present paper is a quite modest one, namely to examine these assertions from a slightly different perspective, by studying the heat kernel which is nothing else than a particular solution of the heat equation, which is in turn a sort of euclidean version of Schrödinger's equation. This allows a straightforward determination of the free energy to one loop order. We follow the lead of the solution all the way down from the sphere towards its different analytical continuations. The setup of the problem is then as follows. The free energy is given by a path integral over the gravitational fluctuations around a background $\bar{g}_{\mu\nu}$ as well as around fluctuations of the matter fields around their backgrounds $\bar{\phi}_a$, which are assumed to be solutions of the classical equations of motion. If the gauge fixing is such that no mixing matter/gravity is generated, then the free energy is given to one loop order

by a set of determinants.

$$\begin{aligned}
 W[\bar{g}_{\mu\nu}, \bar{\phi}_a] &\equiv \bar{S}(\bar{g}_{\mu\nu}, \bar{\phi}_a) - \frac{1}{2} \text{tr} \log M_{\mu\nu\alpha\beta}^2(\bar{g}_{\mu\nu}, \bar{\phi}_a) \\
 &\quad + \text{tr} \log M_{gh}(\bar{g}_{\mu\nu}, \bar{\phi}_a) - \frac{1}{2} \text{tr} \log M^2(\bar{g}_{\mu\nu}, \bar{\phi}_a)
 \end{aligned}
 \tag{1.1}$$

where $M_{\mu\nu\alpha\beta}^2(\bar{g}_{\mu\nu}, \bar{\phi}_a)$ represents the quadratic operator acting on gravitational fluctuations using a background gauge fixing, $M_{gh}(\bar{g}_{\mu\nu}, \bar{\phi}_a)$ the corresponding operator for the ghosts and finally, $M^2(\bar{g}_{\mu\nu}, \bar{\phi}_a)$ stands for the quadratic operator for the matter fields.

Assuming, for simplicity, that all matter is composed by scalar fields, and neglecting the dynamics of the gravitational field, id est

$$\begin{aligned}
 S_m &\equiv \int d^n x \sqrt{|\bar{g}|} \frac{1}{2} \bar{g}^{\mu\nu} \delta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b - \sum_a \xi \bar{R}(\phi^a)^2 - V(\phi^a) \\
 &= \bar{S}_m - \frac{1}{2} \int d^n x \sqrt{|\bar{g}|} \phi^a \partial_a \partial_b V|_{\bar{\phi}} \phi^b + O(\phi^2)
 \end{aligned}
 \tag{1.2}$$

that is, the operator that interests us is, in an obvious notation,

$$M^2(\bar{g}_{\mu\nu}, \bar{\phi}_a) \equiv -\bar{\Delta} \delta_{ab} - \bar{V}_{ab}
 \tag{1.3}$$

Generically, we are only able to compute it in the approximation where the background scalar field is constant; that is, we are evaluating the *effective potential*.

2 The composition law

It is well known (cf. for example the discussion in [29]) that in flat space the Klein-Gordon propagator can be recovered from the first quantized path integral

$$G(x, y) \equiv \int \mathcal{D}X(s) e^{-mS(X)}$$

where the integral extends to all paths such that

$$\begin{aligned}
 X(0) &= x \\
 X(1) &= y
 \end{aligned}
 \tag{2.1}$$

and the action for each path is

$$S(X) \equiv \int_0^1 d\tau \sqrt{\delta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}$$

This representation makes manifest that the propagator enjoys a quantum mechanical composition law, at least in the euclidean case:

$$\int d^n z G(x, z) G(z, y) = \int d^n z \mathcal{D}X(s) \mathcal{D}Y(s) e^{-m\{S(X)+S(Y)\}}
 \tag{2.2}$$

where $X(s)$ goes from x to z and $Y(s)$ from z to y . Then

$$\int d^n z G(x, z) G(z, y) = \int \mathcal{D}X(s) e^{-mS(X)} \mathcal{F}(m^2, S(X)) \quad (2.3)$$

where now $X(s)$ goes from x to y , and the extra factor $\mathcal{F}(m^2, S(X))$ takes into account the integral over the intermediate point z along the curve and leads to

$$\int d^n z G(x, z) G(z, y) = -\frac{\partial}{\partial m^2} G(x, y) \quad (2.4)$$

(This is equivalent to assert that $\mathcal{F}(m^2, S(X)) = \frac{1}{2m} S(X)$. We are aware of no simple argument for this).

In a recent paper Polyakov [30] suggests that unitarity in quantum field theory is equivalent to this *path composition*. Asymptotically (for large separation between the points) the propagator should behave as

$$G(x, y) \sim e^{-ims(x,y)} \quad (2.5)$$

where $s(x, y)$ is the geodesic distance between the points x and y .

The flat space Klein-Gordon propagator can be easily recovered [29] through¹

$$G(x, y) = \int_0^\infty d\tau K(\tau; x, y) \quad (2.8)$$

where $K(\tau; x, y)$ is the Schrödinger functional

$$K(\tau; x, y) \equiv \int \mathcal{D}X e^{-i \int_0^\tau d\sigma \left(\frac{\dot{X}^2}{2\sigma} + \sigma \frac{m^2}{2} \right)} \quad (2.9)$$

and τ is the gauge invariant distance $\tau \equiv \int_0^1 e(\lambda) d\lambda$. Polyakov's path composition is then a simple consequence of Feynman's kernel quantum mechanical composition law

$$\int d^n z K(\tau_1; y, z) K(\tau_2; z, x) = K(\tau_1 + \tau_2; y, x) \quad (2.10)$$

Once these facts are understood, the temptation to choose them as the starting point for the study of quantum fields in a gravitational background is irresistible.

The preceding results are by no means restricted to flat space. We shall explain in a moment that given the heat kernel, that is, the solution of the heat equation in an arbitrary

¹In flat space this identity is true in any dimension for true propagators (id est, solutions of the inhomogeneous equation) because using the Fourier representation

$$G(x, y) = \int \frac{d^n p}{(2\pi)^n} \frac{e^{ip(x-y)}}{p^2 + m^2} \quad (2.6)$$

and

$$\int d^n z G(x, z) G(z, y) = \int d^n z \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{e^{ip(x-z)}}{p^2 + m^2} \frac{e^{ik(z-y)}}{k^2 + m^2} = -\frac{\partial}{\partial m^2} G(x, y) \quad (2.7)$$

Direct verification is more laborious.

spacetime $\partial_\tau K = (\Delta - m^2)K$ with the initial conditions $K(0; x) = \delta(x)$ we can obtain a Green's function for the Klein-Gordon equation through

$$G(x) = \int_0^\infty K(\tau; x) d\tau = \int \theta(\tau) K(\tau; x) d\tau \quad (2.11)$$

$$\begin{aligned} (\Delta - m^2)G(x) &= \int_0^\infty (\Delta - m^2)K(\tau; x) d\tau \\ &= \int_0^\infty \partial_\tau K(\tau; x) d\tau = K(\tau; x) \Big|_0^\infty = -\delta(x) \end{aligned} \quad (2.12)$$

Whenever the composition principle of Schrödinger (or the heat) equation holds

$$\int K(\tau; x, z)K(\sigma; z, y) d^m z = K(\tau + \sigma; x, y) \quad (2.13)$$

this propagator (and others related) enjoys automatically the composition law (2.4)

$$\begin{aligned} \int G(x, z)G(z, y) d^m z &= \int_C dt ds K(t, x, z)K(s, z, y) d^m z \\ &= \int_C dt ds K(t + s, x, y) = \frac{1}{2} \int_{C'} d\tau d\sigma K(\tau; x, y), \end{aligned} \quad (2.14)$$

where the integration domain in the t, s plane is the upper right quadrant C . We have performed the transformation $\tau = t + s$, $\sigma = t - s$, and the new domain C' can be parametrized as

$$\frac{1}{2} \int d\tau d\sigma \theta(\tau + \sigma)\theta(\tau - \sigma) K(\tau; x, y) = \int d\tau \tau \theta(\tau) K(\tau; x, y) = -\partial_{m^2} G(x, y) \quad (2.15)$$

where we take in account that the heat kernel for mass m is related to the massless one by $K_{m^2} = e^{-m^2\tau} K_{m=0}$. The conclusion of the above is that starting from the heat kernel, the “composition principle” is a simple consequence of the quantum mechanical closure relation

$$\sum_z |z\rangle\langle z| = 1 \quad (2.16)$$

3 The heat kernel

What we shall denote by *heat kernel* is what mathematicians call the *fundamental solution of the real heat equation* (FSRHE) made popular by Kac when he asked the question as to whether one could hear the shape of a drum [22] (the short answer is that one cannot in general). The mathematicians call heat equation to

$$\Delta K(x, y; \tau) - \mu^2 \frac{\partial K(x, y; \tau)}{\partial \tau} = 0$$

where $\Delta \equiv \nabla_\mu \nabla^\mu$, and we have introduced a mass scale μ to make τ dimensionless (or, what is equivalent, to consider the operator $\frac{\Delta}{\mu^2}$, whose eigenvalues are also dimensionless). The FSRHE is defined as the solution such that $\lim_{\tau \rightarrow 0^+} K(x, y; \tau) = \delta(x, y)$. The importance

of the FSRHE is that it is *unique* for compact connected C^∞ riemannian manifolds without boundary [9]. Formally, it can be predicated that

$$K(\tau) \equiv e^{\frac{\tau}{\mu^2} \Delta}$$

(the convention is that the operator in the exponent is negative definite for $\tau \in \mathbb{R}^+$.) so that a Green's function can be defined as

$$G \equiv -\Delta^{-1} \equiv \int_0^\infty K(\tau) d\tau$$

This Green's function is also unique under the same conditions than the FSRHE is.

We will deal with this equation with an additional mass term, as in the previous section. In the particular case of euclidean space \mathbb{R}^n (which is non compact, by the way)

$$K_0(x, y; \tau) = \frac{\mu^{n-2}}{(4\pi\tau)^{n/2}} e^{-\frac{\mu^2(x-y)^2}{4\tau} - \frac{m^2}{\mu^2}\tau}$$

(where μ is an arbitrary mass scale whose physical meaning is the same as the one appearing in dimensional regularization). The famous integral

$$\int_0^\infty dx x^{\nu-1} e^{-\frac{\beta}{x} - \gamma x} = \left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\beta\gamma}\right) \tag{3.1}$$

leads to the euclidean Green's function

$$G_0(x, y) \equiv \int_0^\infty d\tau K_0(x, y; \tau) = \frac{1}{2\pi} \left(\frac{m}{2\pi|x-y|}\right)^{n/2-1} K_{n/2-1}(m|x-y|)$$

where $|x|^2 \equiv \sum_1^n x_i^2$ and $K_n(x)$ is the Bessel function of imaginary argument. This is the mother of all Green's functions.

This whole procedure can in some sense be reversed. If we consider the heat kernel corresponding to the *massless* Klein-Gordon operator, $K_{m=0}(\tau) \equiv K(\tau)e^{\frac{m^2}{\mu^2}\tau}$, then the relationship between the heat kernel and the (massive) Green's function is just a Laplace transform

$$G_m(x) = \int_0^\infty K_{m=0}(\tau) e^{-\tau \frac{m^2}{\mu^2}} d\tau$$

This means that whenever the Green's function as a function of m^2 is bounded by a polynomial in the half plane $\text{Re } m^2 \geq c$, the Laplace transform can be inverted to yield

$$K_{m=0}(\tau) = \frac{1}{\mu^2} \int_{c-i\infty}^{c+i\infty} dm^2 e^{\tau \frac{m^2}{\mu^2}} G_m(x)$$

We shall extend this precise and beautiful mathematical framework in two ways. First of all, physics forces upon us the consideration of operators somewhat more general than the covariant laplacian, for example by allowing a generalized mass term (as well as nonminimal operators for higher spins [8]). Secondly, we are eventually interested in *pseudo-riemannian*, Lorentzian geometries which are moreover non-compact.

One of our main worries in the present paper will precisely be how to go back and forth from one signature to the other. What we have seen in the previous paragraph is that this particular Green's function also satisfies Polyakov's composition principle.

The class of spaces we are going to be interested at in this paper are all related to the sphere by analytic continuation. The sphere S_n itself can be defined as the compact form of the symmetric space $SO(n+1)/SO(n)$. It can also be usefully defined as the hypersurface

$$\sum_{A=0}^n X_A^2 \equiv \delta_{AB} X^A X^B = l^2 \tag{3.2}$$

on a flat \mathbb{R}_{n+1} space² with metric $ds^2 = \delta_{AB} dX^A dX^B$; or else the real projective space, $\mathbb{RP}_n = S_n/\mathbb{Z}_2$, where \mathbb{Z}_2 is the antipodal mapping

$$\mathbb{Z}_2 : X^A \rightarrow -X^A \tag{3.3}$$

The sphere is then the universal covering space of the projective plane, and $\pi_1(\mathbb{RP}_n) = \mathbb{Z}_2$. Functions on the projective plane are given by even functions on the sphere

$$f(X^A) = f(-X^A) \tag{3.4}$$

The projective plane is non-orientable for even values of n , but it is orientable for odd values of n . For example, $\mathbb{RP}_1 \sim S_1$.

In their work on the Schrödinger equation, Grosche and Steiner [17] are led towards the following integral, which gives what is essentially the Schrödinger propagator:

$$\begin{aligned} K(\Omega, \Omega'; \tau) &\equiv \int \mathcal{D}\Omega e^{i \int_0^\tau d\lambda \left(\frac{ml^2}{2} \dot{\Omega}^2 + \frac{n(n-2)}{8ml^2} \right)} = e^{i\tau \frac{n(n-2)}{8ml^2}} \int \mathcal{D}\Omega e^{i \int_0^\tau d\lambda \frac{ml^2}{2} \dot{\Omega}^2} \\ &\equiv e^{i\tau \frac{n(n-2)}{8ml^2}} Z(\Omega, \Omega'; \tau) \end{aligned} \tag{3.5}$$

where $\Omega \equiv \vec{n}$ is a unit vector, defining a point on the unit sphere $\vec{n} \in S_n$, and can be characterized in polar coordinates by a set of angles, $\theta_1 \dots \theta_n$.

The path integral will be done by means of Feynman's time slicing technique. The action reads

$$S = \frac{ml^2}{2} \sum_{i=1}^n \left(\vec{\Omega}_i - \vec{\Omega}_{i-1} \right)^2 = ml^2 \sum_{i=1}^n (1 - \cos \psi_{i-1}) \tag{3.6}$$

where we have defined

$$\cos \psi_{i-1} \equiv \vec{\Omega}_i \cdot \vec{\Omega}_{i-1} \tag{3.7}$$

The expansion discussed in the appendix conveys the fact that

$$e^{z \cos \psi} = \left(\frac{z}{2} \right)^{-\frac{n-1}{2}} \Gamma \left(\frac{n-1}{2} \right) \sum_{j=0}^{\infty} \left(j + \frac{n-1}{2} \right) I_{j+\frac{n-1}{2}}(z) C_j^{\frac{n-1}{2}}(\cos \psi) \tag{3.8}$$

$$Z(\theta, \theta'; \tau) = e^{i\tau \frac{n(n-2)}{8ml^2}} \int \mathcal{D}\Omega e^{i \int \frac{ml^2}{2} \dot{\Omega}^2} = e^{i\tau \frac{n(n-2)}{8ml^2}} \int \prod_i d\Omega_i e^{iml^2 \sum_i (1 - \cos \psi_{i-1})} \tag{3.9}$$

²These coordinates, which we are going to represent in capital letters, are usually denoted as *Weierstrass* coordinates.

the integrations to be done are, schematically,

$$\int d\Omega_1 \dots d\Omega_{n-1} \sum_{j_1 \vec{m}_1} \sum_{j_2 \vec{m}_2} \dots Y_{j_1 \vec{m}_1}(\Omega_1) Y_{j_1 \vec{m}_1}^*(\Omega_0) Y_{j_2 \vec{m}_2}(\Omega_2) Y_{j_2 \vec{m}_2}^*(\Omega_1) \dots$$

$$\dots \sum_{j_n \vec{m}_n} Y_{j_n \vec{m}_n}(\Omega_n) Y_{j_n \vec{m}_n}^*(\Omega_{n-1}) = \sum_{j \vec{m}} Y_{j \vec{m}}(\Omega_n) Y_{j \vec{m}}^*(\Omega_0) \sim \sum_j C_j^{\frac{n-1}{2}}(\cos \psi)$$

The final result of [17] is

$$K(\Omega, \Omega'; \tau) = \frac{1}{V(S_n)} \sum_{j=0}^{\infty} \frac{2j+n-1}{n-1} C_j^{\frac{n-1}{2}}(\Omega \cdot \Omega') e^{-\frac{i\tau}{2m^2} j(j+n-1)} \quad (3.10)$$

Our main tool in order to study the effective potential in constant curvature spaces will be the analogous of the preceding computation for our Klein-Gordon equation, as well as the representation of the delta function on the sphere S_{n-1} by means of Gegenbauer polynomials (cf. appendix), id est,

$$K(\tau; \Omega, \Omega') = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\Omega \cdot \Omega') e^{-\tau(m^2 l^2 + j(j+n-1))} \quad (3.11)$$

that is the solution of the heat equation such that

$$\lim_{\tau \rightarrow 0^+} K(\tau; \Omega, \Omega') = \delta(\Omega - \Omega') \quad (3.12)$$

where the delta function reads

$$\delta(\Omega - \Omega') = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos \theta_n) \quad (3.13)$$

We can see the heat kernel formally as

$$K(\tau) \equiv e^{-\tau \bar{M}^2} \quad (3.14)$$

where \bar{M}^2 is the positive definite operator acting on quadratic fluctuations around the background field, id est,

$$\bar{M}^2 \equiv -\Delta + \partial^2 V(\bar{\phi}) \quad (3.15)$$

and we include masses in the potential.

Let us mention that whenever the full eigenvalue problem for the operator \bar{M}^2 is known, there is a formal FSRHE. Using the discrete notation,

$$\bar{M}^2 u_n(x) = \lambda_n u_n(x) \quad (3.16)$$

with eigenfunctions which can be chosen to obey

$$(u_n, u_m) \equiv \int d\mu(x) u_n^*(x) u_m(x) = \delta_{nm} \quad (3.17)$$

(where the measure $d\mu(x)$ is usually $\sqrt{|g|}d^n x$) as well as a completeness relationship of the type

$$\sum_n u_n^*(x)u_n(y) = \delta(x - y) \tag{3.18}$$

then the following is the sought for FSRHE

$$K(x, y|\tau) = \sum_n e^{-\lambda_n \tau} u_n^*(x)u_n(y) \tag{3.19}$$

whose imaginary part is determined by the one of the eigenvalues themselves.

As we have already advertised, in order to study the free energy up to one loop order, it is much more convenient to study the heat kernel, than the Green's function, because it gives the desired result directly

$$W = \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \text{tr} \int d^n x \sqrt{|g|} K(\tau; x, x) \tag{3.20}$$

This definition *includes* the definition based to the zeta-function (which is the finite part) as well as the divergent counterterms.

Before that, however, let us clarify a few points on the relationship between Green's functions in constant curvature spaces. Although the defining equations of the different spaces themselves in Weierstrass coordinates are analytic continuations of the equation of the sphere, some subtleties appear with the analytic continuation of Green's functions.

4 Green's functions in constant curvature spaces

We shall mainly be concerned in this paper with fundamental solutions of the Klein-Gordon equation in the real sections of the sphere, invariant under the full group of isometries. Related analysis have been performed in [4, 12]. The homogeneous version of this equation takes always the same form in these spaces:

$$(z^2 - 1)G'' + nzG' \pm m^2 l^2 = 0 \tag{4.1}$$

where z is the corresponding geodesic distance for each space (cf. A.1).

The problem of finding the invariant Green's functions of this equation can be solved in a simple and general way. The full space of solutions is two-dimensional. All we have to do is extending the domain of definition of these functions to the appropriate region of the real axis for each surface.

We have to take care also of the singularities we obtain. We are interested in a single source (typically in the "north pole" $z = 1$), or perhaps in symmetric solutions under \mathbb{Z}_2 in order to obtain Green's functions for the projective case.

In the figure 4 we have summarized the results. Combining solutions of the generic Klein-Gordon equation (hypergeometric functions) with the appropriate singularity ($F(\frac{1+z}{2}, R)$), we can build several different propagators for each space. Here R is proportional to a Legendre Q function, finite at $z = \infty$. G_∞ means a Green's function that diverges at infinity. G_α stands for the Green's functions of the α -vacua.

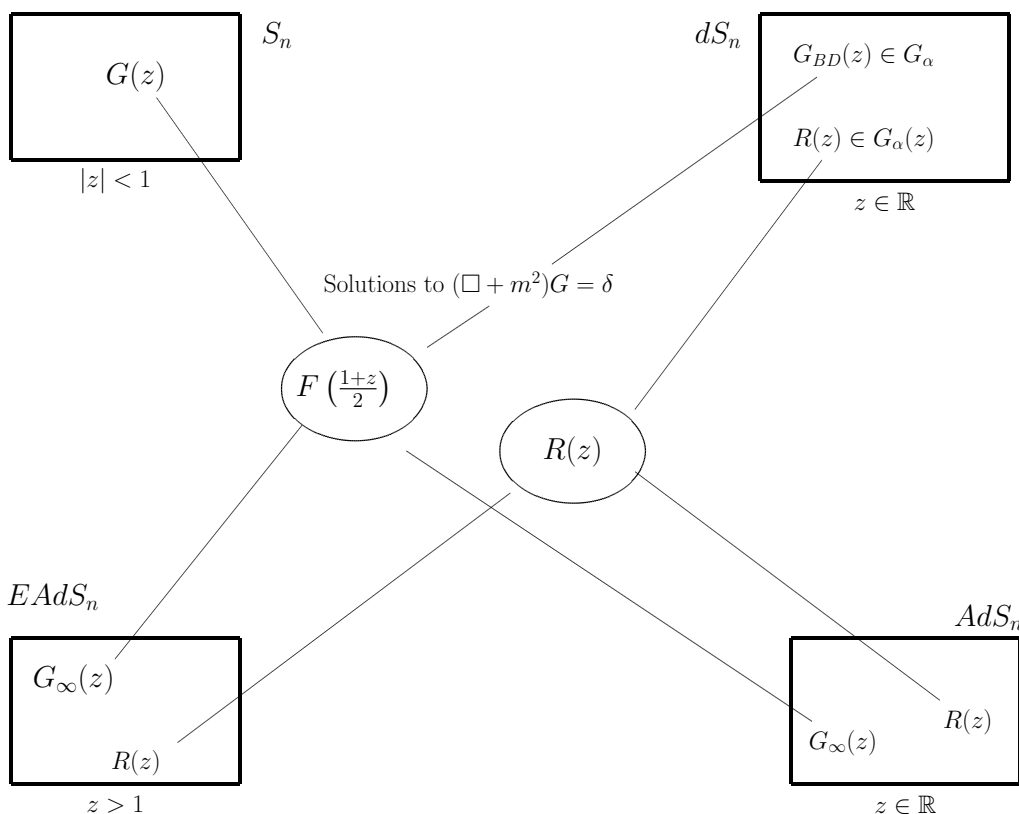


Figure 1. Route sheet of analytic continuations.

4.1 Flat spacetime

The flat spacetime case is interesting in order to know the appropriate short distance behaviour. We saw in the previous that the calculation of the n-dimensional Green's function in an euclidean flat spacetime gives

$$G(x) = \int \frac{e^{ipx}}{p^2 + m^2} \frac{d^n p}{(2\pi)^n} = \frac{1}{2\pi} \left(\frac{m}{2\pi r}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(mr) \tag{4.2}$$

When we perform the analytic continuation to the Feynman propagator in lorentzian signature, we implicitly chose the prescription such that the result is still a propagator, i.e. that keeps the appropriate singularity:

$$G_F(x) = \frac{i}{2\pi} \left(\frac{m}{2\pi\sqrt{-x^2 + i\epsilon}}\right)^{\frac{n}{2}-1} K_{\frac{n}{2}-1}(m\sqrt{-x^2 + i\epsilon})$$

That this is correct, can be checked performing the integral $\int \frac{d^n k}{(2\pi)^n} \frac{e^{ikx}}{-k^2 + m^2 - i\epsilon}$ explicitly. The branch cut of $\sqrt{-x^2}$ does not depend on the sign on time, but just on $|t|$, as was expected from a time ordering.

The singularity of this propagator is:

$$G(x) \xrightarrow{x^2 \rightarrow 0} \frac{i}{(2\pi)^{\frac{n}{2}}} 2^{\frac{n}{2}-2} \Gamma\left(\frac{n}{2} - 1\right) (-x^2 + i\epsilon)^{1-\frac{n}{2}} + [\log(-x^2 + i\epsilon)] \quad (4.3)$$

where the term in brackets appears when n is even.

This prescription precisely gives us the correct singularity to recover a delta function. Other possibilities lead to homogeneous solutions which correspond to important functions:

- Wightman function $-iW: x^2 \rightarrow -x^2 + i\epsilon t$
- Symmetric function $G^{(1)}: \text{Re } W$
- Pauli-Jordan function (commutator) $D: \text{Im } W$

4.2 Sphere

In the appendix we give some details on different metrics for constant curvature spaces with different signatures. The Klein-Gordon equation in the n -dimensional sphere reads:

$$\frac{1}{\sin \theta^{n-1}} \partial_\theta (\sin \theta^{n-1} \partial_\theta G) - m^2 l^2 G = 0 = \frac{1}{(1-z^2)^{\frac{n-2}{2}}} \partial_z ((1-z^2)^{\frac{n}{2}} \partial_z G) - m^2 l^2 G \quad (4.4)$$

where $z = \cos \theta$. This is almost an hypergeometric equation:

$$(z^2 - 1)G'' + nzG' + m^2 l^2 G = 0 \quad (4.5)$$

with the solutions:³

$$G(z) = F_\pm(z) = F\left(\frac{1 \pm z}{2}\right) \equiv F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 \pm z}{2}\right) \quad (4.6)$$

where $m^2 l^2 = \mu^2 + \frac{(n-1)^2}{4}$. Each one is singular respectively in $z = \pm 1$, and this singularity corresponds precisely to delta function in opposite points. In this way we recover the well known fact that there is a single Green's function in the sphere.

The composition law holds for this Green's function, given that is unique and therefore, proportional to the alternate expression:

$$G(\Omega \cdot \Omega') = \sum_{j\vec{k}} \frac{Y_{j\vec{k}}(\Omega) Y_{j\vec{k}}(\Omega')^*}{j(j+n-1) + m^2} \quad (4.7)$$

given in terms of eigenfunctions of Δ , i.e. spherical harmonics, and their eigenvalues. It is straightforward to check the composition law with this formula.

³The possible values of μ are real and positive, or imaginary, with $\frac{n-1}{2} > -i\mu > 0$

4.3 de Sitter space

The Klein-Gordon equation in this case reads

$$\frac{1}{\cosh \tau^{n-1}} \partial_\tau (\cosh \tau^{n-1} \partial_\tau G) - \frac{1}{\cosh \tau^2 \sin \theta^{n-2}} \partial_\theta (\sin \theta^{n-2} \partial_\theta G) + m^2 l^2 G = 0 \quad (4.8)$$

$$(z^2 - 1)G'' + nzG' + m^2 l^2 G = 0, \quad z = \cosh \tau \cos \theta \quad (4.9)$$

The solution is given by the same expression as before. In order to provide a function defined over the full de Sitter space (for all $z \in \mathbb{R}$), we must specify the values in the branch cuts. In addition, since the signature of spacetime has changed, this prescription will determine the character of the singularity, i.e. homogeneous or not.

Looking to the flat spacetime case, the solution is simple, since the short distance behaviour should match. The correct analytic continuation is:

$$G_{\text{BD}}(z) = F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1+z}{2} - i\epsilon\right) \quad (4.10)$$

and this is (proportional to) the euclidean or Bunch-Davies propagator. In addition we can continue the both solutions in such a way that they remain homogeneous, for example:

$$\text{Re}F_\pm(z) = \text{Re}F\left(i\mu + \frac{n-1}{2}, -i\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 \pm z}{2}\right) \quad (4.11)$$

where we denote by $\text{Re}, i\text{Im}f(z) = f(z + i\epsilon) \pm f(z - i\epsilon)$. This combination cancels the delta divergence.

The above expression spans the space of homogeneous invariant solutions that originates the ambiguity in the propagator:

$$G(z) = G_{\text{BD}}(z) + \alpha \text{Re}F_+(z) + \beta \text{Re}F_-(z) \quad (4.12)$$

However, if the propagator comes from a vacuum expectation value, we know [2] that just a 1-parameter family survives, the α ($\alpha > 0$) vacuum:⁴

$$G_\alpha(z) = \frac{i|\Gamma(i\mu + \frac{n-1}{2})|^2}{2(4\pi)^{\frac{n}{2}} \{-\Gamma(2 - \frac{n}{2})|\Gamma(\frac{n}{2})\}} \left\{ \cosh 2\alpha \text{Re}F\left(\frac{1+z}{2}\right) + \right. \quad (4.13)$$

$$\left. + \sinh 2\alpha \text{Re}F\left(\frac{1-z}{2}\right) - i \text{Im}F\left(\frac{1+z}{2}\right) \right\}$$

The term in the $\{\}$ corresponds to the $\{\text{odd}|\text{even}\}$ case.

⁴The most general expression, de Sitter invariant except for the discrete symmetries, is the α, β vacuum, with $\beta \in [0, 2\pi)$:

$$G_{\alpha,\beta}(x, y) = \frac{i|\Gamma(i\mu + \frac{n-1}{2})|^2}{2(4\pi)^{\frac{n}{2}} \{-\Gamma(2 - \frac{n}{2})|\Gamma(\frac{n}{2})\}} \left\{ \cosh 2\alpha \text{Re}F\left(\frac{1+z}{2}\right) + \right.$$

$$\left. + \sinh 2\alpha \left[\cos \beta \text{Re}F\left(\frac{1-z}{2}\right) - \sin \beta \sigma \text{Im}F\left(\frac{1-z}{2}\right) \right] - i \text{Im}F\left(\frac{1+z}{2}\right) \right\}$$

where σ is the sign of the time-ordering of (x^A, y) . This is defined only in the case $z < -1$, but for $z > -1$ the imaginary part of $F\left(\frac{1-z}{2}\right)$ vanishes, as in the case of the commutator function. This expression for $\beta \neq 0$ is not fully de Sitter invariant, i.e. it does not depend only on z , due precisely to the presence of this sign.

4.4 Euclidean Anti de Sitter space

Now the Klein-Gordon equation reads

$$(z^2 - 1)G'' + nzG' - m^2l^2G = 0 \tag{4.14}$$

The solutions are pretty similar to the sphere case:

$$G(z) = F\left(\mu + \frac{n-1}{2}, -\mu + \frac{n-1}{2}; \frac{n}{2}; \frac{1 \pm z}{2}\right) \tag{4.15}$$

where $\mu^2 = m^2l^2 + \left(\frac{n-1}{2}\right)^2$. This time $\mu > \frac{n-1}{2}$.

The negative sign solution is regular in $z = 1$ so it is purely homogeneous. Given that now $z \geq 1$, the positive sign solution needs a prescription in the branch cut to be meaningful. The exact behaviour near $z = 1$ depends on the parity of n , but in both cases the expressions are like:

$$F\left(\frac{1+z}{2}\right) = \dots + \dots \cdot \left(\frac{1-z}{2}\right)^{1-\frac{n}{2}} \tag{4.16}$$

where ... something regular in $z = 1$ (or a logarithm). We can see from this equation that taking the upper or lower limit in the real axis, $z \pm i\epsilon$ gives us a Green's function G_∞ .

However, this propagator G_∞ diverges in the infinity, as we can see from the expansion of the hypergeometric function near the infinity:

$$F(\alpha, \beta; \gamma; z) \xrightarrow{z \rightarrow \infty} \text{const} (-z)^{-\alpha} + \text{const} (-z)^{-\beta} \tag{4.17}$$

from which we get:

$$G_\infty(z) \xrightarrow{z \rightarrow \infty} \text{const} \left(-\frac{1+z}{2}\right)^{-\mu-\frac{n-1}{2}} + \text{const} \left(-\frac{1+z}{2}\right)^{\mu-\frac{n-1}{2}} \tag{4.18}$$

Both the imaginary and the real part of this expression diverge (this is due to the second term), so in general no prescription gives us a propagator that vanishes at infinity.⁵

An appropriate solution can be obtained combining the G_∞ with the homogeneous solutions. The exact expression can be given in terms of Legendre associated functions:

$$G(z) = (z^2 - 1)^{\frac{1-n}{4}} Q_{\mu-\frac{1}{2}}^{\frac{n-1}{2}}(z) \sim z^{-\mu-\frac{n-1}{2}} F\left(\frac{\mu}{2} + \frac{n+1}{4}, \frac{\mu}{2} + \frac{n-1}{4}; \mu+1; \frac{1}{z^2}\right) \tag{4.19}$$

This special combination, that we will abbreviate $R_{\mu-\frac{1}{2}}^{\frac{n-1}{2}}$, is a solution of (4.14). The composition principle holds for this propagator, given that this solution is the Laplace transform of the Schrödinger propagator of *EAdS* [17].

⁵In fact, some specific values of m are such that taking only the imaginary [real] part of the function, for n odd [even], this term is cancelled.

4.5 Anti de Sitter space

The Klein-Gordon equation in AdS is identical to the $EAdS$ case. The variable z can take any real value again, as in de Sitter, so the the solutions to (4.14) can be continued in the same way as in (4.10), (4.11). We have just to take in account that now $i\mu \rightarrow \mu$, where μ means the same as in the $EAdS$ case.

Since the Anti de Sitter space has a well defined spatial infinity at $z = \infty$, if we require the propagator to vanish there, we will obtain the same R expression as in the $EAdS$ case (4.19). However, in this case we have to extend the domain to the full real axis. In order to get the correct prescription, we need the relationship between the R and the hypergeometric solutions:

$$R_{\nu}^{\frac{n-2}{2}}(z) = \rho_{n,\nu} \left\{ e^{\mp i\pi\nu} F\left(\frac{1-z}{2}\right) + \varphi_{\pm} F\left(\frac{1+z}{2}\right) \right\} \quad (4.20)$$

$$\rho_{n,\nu} = \frac{2^{-\frac{n}{2}} \pi \Gamma(\frac{n}{2} + \nu)}{\Gamma(\frac{n}{2}) \Gamma(2 - \frac{n}{2} + \nu) \{i \cos \pi\nu | \sin \pi\nu\}} ; \varphi_{\pm} = \{i(-1)^{\frac{n\pm 1}{2}} | (-1)^{\frac{n}{2}}\} \quad (4.21)$$

where again we write together the {odd|even} case, and the upper (lower) sign is for positive (negative) imaginary part of z .

An expression like (4.12) is the most general Green's function. Since the delta singularities are in the imaginary part of the F solutions, and the homogeneous pieces are the real parts, we have to eliminate the imaginary part of $F_- \equiv F\left(\frac{1-z}{2}\right)$, and it is easy to see that the appropriate combination to achieve it is

$$\tilde{R}_{\nu}^{\frac{n-2}{2}}(z) = e^{i\pi\nu} R_{\nu}^{\frac{n-2}{2}}(z + i\epsilon) + e^{-i\pi\nu} R_{\nu}^{\frac{n-2}{2}}(z - i\epsilon) \quad (4.22)$$

The detailed expressions in the even and odd cases are respectively:

$$\begin{aligned} \tilde{R}_{\nu}^{\frac{n-2}{2}}(z) &\sim \text{Re}F_-(z) + (-1)^{\frac{n}{2}} \cos \pi\nu \text{Re}F_+(z) - (-1)^{\frac{n}{2}} \sin \pi\nu \text{Im}F_+(z) = \\ &= \text{Re}F_-(z) + (-1)^{\frac{n}{2}} i \sinh \pi\mu \text{Re}F_+(z) + (-1)^{\frac{n}{2}} \cosh \pi\mu \text{Im}F_+(z) \end{aligned} \quad (4.23)$$

$$\begin{aligned} \tilde{R}_{\nu}^{\frac{n-2}{2}}(z) &\sim \text{Re}F_-(z) + (-1)^{\frac{n-1}{2}} \sin \pi\nu \text{Re}F_+(z) + (-1)^{\frac{n-1}{2}} \cos \pi\nu \text{Im}F_+(z) = \\ &= \text{Re}F_-(z) - (-1)^{\frac{n-1}{2}} \cosh \pi\mu \text{Re}F_+(z) + (-1)^{\frac{n-1}{2}} i \sinh \pi\mu \text{Im}F_+(z) \end{aligned} \quad (4.24)$$

The second line in each case come from $\nu = i\mu - \frac{1}{2}$, i.e. the de Sitter case. As we can see, if and only if the dimension n is odd the R solution can be analytically continued into an alpha-beta vacuum, because of the inappropriate i factors in the even case. The parameters of that vacuum are $\sinh 2\alpha = \text{csch} \pi\mu$, and $\beta = 0$ ($\beta = \pi$) for $(-1)^{\frac{n+1}{2}}$ positive (negative).⁶

4.6 Projective spaces

A function defined over the projective version of these spaces can always be lifted to an symmetric function defined over the original space. It is very easy to obtain the most general Green's function of such an space, given the previous classification.

⁶This is valid only in the case of $m > \frac{n-1}{2}$ in de Sitter. For lower masses there is no possibility of analytic continuation, because of the i factors again.

For the projective plane $\mathbb{R}P_n = S_n/\mathbb{Z}_2$, there is a single Green function corresponding to the projection of $G(z)+G(-z)$, where $G(z)$ is the propagator in 4.6 with the positive sign.

In the projective versions of de Sitter or Anti de Sitter, dS_n/\mathbb{Z}_2 and AdS_n/\mathbb{Z}_2 , we found that the most general Green's function is:

$$G(z) = G_{\text{BD}}(z) + \alpha \text{Re } F_+(z) + \beta \text{Re } F_-(z) \tag{4.25}$$

where α and β are arbitrary constants. If we symmetrize this expression, we get the general propagator for these spacetimes:

$$G_P(z) = G_{\text{BD}}(z) + G_{\text{BD}}(-z) + \alpha (\text{Re } F_+(z) + \text{Re } F_-(z)) \tag{4.26}$$

In particular, we can symmetrize the \tilde{R} solution finite at $z = \pm\infty$.

5 The imaginary part of the effective potential

In flat space there is a systematic way of determining the ground state of a physical system, namely, to minimize the effective potential (the effective action for constant backgrounds). This is the physical principle that generalizes minimization of energy for classical systems. Things get more complicated when gravitational fields are present.

First of all there is no fully satisfactory concept of energy in general gravitational backgrounds. In de Sitter space a Killing energy with support on the space orthogonal to a given observer, u , is well-defined through

$$E(u) \equiv \int d^{n-1}x u_\mu T^{\mu\nu} k_\nu \tag{5.1}$$

where the energy-momentum tensor is defined by expanding à la Abbott-Deser around a background. The lack of global existence of the Killings means that precise statements are only possible outside the corresponding horizons. In the general situation the situation is even worse, and several definitions (such as the Hawking-Geroch, Penrose, Nester-Witten or Brown-York, [34]) of quasilocal energy exist, none of which is fully satisfactory, and besides all of them seem difficult to compute in quantum field theory.

Besides it is the case in general that

$$|0_+\rangle \neq |0_-\rangle \tag{5.2}$$

The usual Feynman path integral computes expectation values

$$\langle 0_+ | \mathcal{O} | 0_- \rangle \tag{5.3}$$

so that some modification is in order to get expectation values such as

$$\langle 0_- | \mathcal{O} | 0_- \rangle \tag{5.4}$$

One way to do it is the closed time path (CTP) formalism of Schwinger and Keldysh [31], but euclidean methods are also available [36].

The proper approach would be to study the structural stability of the Dyson-Schwinger equations for the whole system.

What we have done in this paper instead is to compute the simplest and most naive expression for the energy, namely the effective potential.

- As a matter of fact, the formula (3.11) for the sphere S_n could be directly continued to de Sitter space, given that the Gegenbauer polynomials $C_j^{\frac{n-1}{2}}$ are defined for all real z . Then, the expression:

$$K(\tau; z) = \frac{1}{V(S_n)} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(z) e^{-\tau(m^2 l^2 + V''(\bar{\phi}) + j(j+n-1))} \quad (5.5)$$

is a natural candidate for the heat kernel in de Sitter as well.⁷

Then we can evaluate the free energy given by formula (3.20):

$$\begin{aligned} W &= \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau} \int d^n x \sqrt{|g|} K(\tau; x, x) = \frac{\text{Vol}_{dS}}{2} \int_0^\infty \frac{d\tau}{\tau} K(\tau; 1) \\ &= \frac{\text{Vol}_{dS}}{2V(S_n)} \int_0^\infty \frac{d\tau}{\tau} \sum_j \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(1) e^{-\frac{\tau}{\mu^2}(m^2 + V''(\bar{\phi}) + j(j+n-1)/l^2)} \end{aligned} \quad (5.6)$$

where we have redefined the heat kernel in order to get a mass dimension 2 equation. Here $C_j^{\frac{n-1}{2}}(1) = \binom{j+n-2}{j}$. This expression, which is divergent,⁸ is purely real (the $C_l^{\frac{n-1}{2}}(1)$ are integers), so no imaginary parts appear.

- In the reference [25] the spectrum of the laplacian for de Sitter space, dS_n , anti de Sitter space AdS_n and euclidean (anti) de Sitter space $EAdS_n$ is computed and the eigenfunctions are constructed as well. The spectrum is identical⁹ for both dS_n and AdS_n and has got a discrete part (similar to the one corresponding to the sphere)

$$-L(L+n-1)/l^2$$

where

$$L = -\left[\frac{n}{2}\right] + 1, -\left[\frac{n}{2}\right] + 2, \dots -\left[\frac{n}{2}\right] + j \dots$$

and we represent by $[z]$ the integer part of z . The starting point of the spectrum is actually the only difference between the sphere and both de Sitter and anti de Sitter spaces, as long as the discrete part of the said spectrum is concerned. In terms of $j \in \mathbb{N}$, for even dimension, $n = 2m$, or else for odd dimension $n = 2m + 1$

$$L = -\frac{-j(j-1) + m(m-1)}{4l^2}$$

⁷ It seems plain that the analytic continuation, should it work at all, it not will do it term by term. The eigenvalues are not the same in the sphere as in de Sitter space, not to mention the fact that the sphere is a compact space whereas de Sitter is not. Nevertheless, there is a well-known *duality* between compact and non-compact symmetric spaces [19]. Some further caveats on the analytic continuation of the heat kernel have been made in [7]. It is true that until the whole sum is performed and then the explicit continuation is made, surprises may appear, so perhaps some wise restraint is called for.

⁸General theorems imply that the trace of the heat kernel must diverge when $\tau \rightarrow 0$ as $K \sim \mu^n \tau^{-n/2}$. This just means that the sum and the integral do not commute.

⁹Except for a sign perhaps, depending on the sign chosen for the metric for each space.

There is also a continuous piece of the spectrum, which can be written in the form

$$\frac{1}{l^2} \left(\Lambda^2 + \frac{(n-1)^2}{4} \right) \text{ where } \Lambda \in [0, \infty)$$

In the case of $EAdS_n$ only the continuous spectrum appears. So the situation is as follows: the two euclidean spaces enjoy only one type of spectrum; discrete in the case of the sphere S_n and continuum in the case of $EAdS_n$; whereas the two manifolds with lorentzian signature (AdS_n and dS_n) carry both discrete and continuous spectra. In all cases the eigenvalues are of course real.

The eigenfunctions are explicitly known and can be find in the references just quoted. It is enough for our purposes though to point out that they obey a completeness relationship,

$$\sum_L Y_L(x)^* Y_L(y) + \int d\Lambda Z_\Lambda(x)^* Z_\Lambda(y) = \delta(x, y) \tag{5.7}$$

- Let us nevertheless perform a simple approximation (in the case of the sphere; the other cases are very similar), just to get an idea of the result. We shall explore the high angular momentum region,

$$\sum_j j^{n-1} e^{-\frac{\tau}{\mu^2}(j+n-1)j/l^2} \sim \int_0^\infty dj j^{n-1} e^{-\frac{\tau}{\mu^2 l^2} j^2} = \frac{(\mu l)^n}{2\tau^{\frac{n}{2}}} \Gamma\left(\frac{n}{2}\right)$$

We then get in this approximation

$$\begin{aligned} W &\sim \mu^n l^n \int_{\frac{\mu^2}{\Lambda^2}}^\infty \frac{d\tau}{\tau^{1+\frac{n}{2}}} e^{-\frac{m^2+V''(\bar{\phi})}{\mu^2}\tau} = (m^2 l^2 + V''(\bar{\phi}) l^2)^{\frac{n}{2}} \Gamma\left(-\frac{n}{2}, \frac{m^2 + V''(\bar{\phi})}{\Lambda^2}\right) \\ &= \begin{cases} \text{odd } n : 0 \\ \text{even } n : -\frac{(-1)^{\frac{n}{2}}}{(\frac{n}{2})!} (m^2 l^2 + V''(\bar{\phi}) l^2)^{\frac{n}{2}} \log \frac{\Lambda^2}{m^2+V''(\bar{\phi})} + \frac{2\Lambda^n l^n}{n} + \dots \end{cases} \end{aligned} \tag{5.8}$$

Here, as in flat space, the only possible imaginary part comes from the logarithm, that is, when

$$\frac{m^2 + V''(\bar{\phi})}{\mu^2} \leq 0$$

This is in agreement with general theorems [18] asserting that the only way a non vanishing imaginary part can appear in a manifestly real integral is from the region in which the integral diverges.

On the other hand, his is exactly the situation when spontaneous symmetry breaking occurs in flat space and, as we shall argue in the next paragraph, it is believed to be well understood.

6 Conclusions

The effective potential of quantum fields propagating in a constant curvature space, corresponding to a cosmological constant of either sign, has been computed using the heat kernel as our main tool. Most Green's functions that appear obey Polyakov's composition principle, although other possibilities have been examined as well. The general analytic continuation of the sphere

$$S_n \sim \text{SO}(n+1)/\text{SO}(n) \tag{6.1}$$

has been considered; we believe this to be physically important, in order to determine whether the purported instability appears only for one sign of the cosmological constant, or for both, in which case it would be possible that the endpoint of the instability would have been flat Minkowski space.

No imaginary part for the effective potential has been obtained except in those cases in which the potential is such that in flat space leads to spontaneous symmetry breaking; that is, when $\partial^2 V_{\text{eff}}(\bar{\phi}) < 0$ for some range of the argument, like in the famous *mexican hat* potentials; and this particular imaginary part is in principle well understood cf. [37]. It can be shown from first principles [33] that the effective potential $V_{\text{eff}}(\bar{\phi})$ corresponds to the expectation value of the energy density in a Fock state $|\Psi\rangle$ which minimizes $\langle\Psi|H|\Psi\rangle$ subject to the constraint $\langle\Psi|\phi|\Psi\rangle = \bar{\phi}$. This implies that V_{eff} must be real and convex.

What happens for those ranges for which $\partial^2 V_{\text{eff}}(\bar{\phi}) < 0$ is that the state that minimizes the energy (let us call it $|E_0\rangle$) is a quantum superposition of two or more vacuum states, and the configurations for which the expectation value of the field is constant are unstable towards decay into $|E_0\rangle$; the imaginary part $\text{Im} V_{\text{eff}}(\bar{\phi})$ just gives half the decay rate corresponding to this process per unit volume, $\Gamma(|\bar{\phi}\rangle \rightarrow |E_0\rangle)$.

This is the only imaginary part of the effective potential within the class of models studied in this paper. Our results seem to be compatible with those in [27].

We would like to finish the paper by pointing out an argument¹⁰ clarifying when one is to expect instabilities of the background field. The fact that the functional integral of a total derivative vanishes implies

$$0 = \int \mathcal{D}g_{\mu\nu} \mathcal{D}b \mathcal{D}c \mathcal{D}\phi \frac{\delta}{\delta g^{\mu\nu}(x)} e^{i(S_{\text{grav}}(g_{\mu\nu}) + S_{gf}(g_{\mu\nu}) + S_{gh}(b,c,g_{\mu\nu}) + S_m(\phi,g_{\mu\nu}))}$$

When

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \int \sqrt{|g|} d^n x R$$

a definition of the composite operators $R_{\mu\nu}(g_{\alpha\beta})$ and $T_{\mu\nu}(g_{\alpha\beta}, \phi)$ should exist such that the Dyson-Schwinger equation holds:

$$\left\langle \chi \left| \sqrt{|g|} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \kappa^2 T_{\mu\nu} \right) \right| \psi \right\rangle = 0$$

where

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} (S_m + S_{gf} + S_{gh})$$

¹⁰ Related remarks can be found in [35]

and $|\psi\rangle$ and $|\chi\rangle$ are states that depend on the boundary conditions. Usually they are taken as $|0^\pm\rangle$.

The trace of the former equation means that

$$\bar{g}^{\mu\nu} \left\langle \chi \left| \sqrt{|g|} \left(\frac{2-n}{2} R_{\mu\nu} - \kappa^2 T_{\mu\nu} \right) \right| \psi \right\rangle = 0$$

which means in turn that when the trace of the expectation value of the energy momentum is constant, so is the trace of the expectation value of the scalar curvature. On the other hand, we insist that both the scalar curvature as well as the energy-momentum tensor are composite operators, whose definition is somewhat delicate. But this fact also tells us when a nontrivial physical effect is at least allowed. First of all, through the effect of the one-loop gravitational counterterms, namely,

$$L_{\text{count}} = \int \sqrt{|g|} d^n x (c_1 R^2 + c_2 R_{\mu\nu}^2)$$

except in the renormalization scheme when the finite parts of both c_1 and c_2 are put equal to zero. This changes the contribution of

$$\frac{\delta S_{\text{grav}}}{\delta g^{\mu\nu}}$$

Counterterms are also at the origin of the trace anomaly, i.e.

$$\bar{g}^{\mu\nu} \langle T_{\mu\nu} \rangle \neq \langle g^{\mu\nu} T_{\mu\nu} \rangle$$

which has got a piece proportional to the beta function of the theory, as well as a gravitational piece, which is non-vanishing even for conformally invariant theories (i.e., when $\beta = 0$).¹¹

The conclusion of the analysis is that we do not find any obvious reason why matter effects by themselves could not destabilize de Sitter space, causing the cosmological constant to decay. This still looks like an exciting possibility. It remains to find a self-consistent scenario implementing this general idea. Work on these lines is currently in progress.

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A Taxonomy of constant curvature spaces

The real sections of the complex sphere can be treated in an unified way. Let us choose coordinates in the embedding space in such a way that in the defining equation we have

$$X^2 = \sum_{A=0}^n \epsilon_A X_A^2 \equiv \eta_{AB} dX^A dX^B = \pm l^2 \tag{A.1}$$

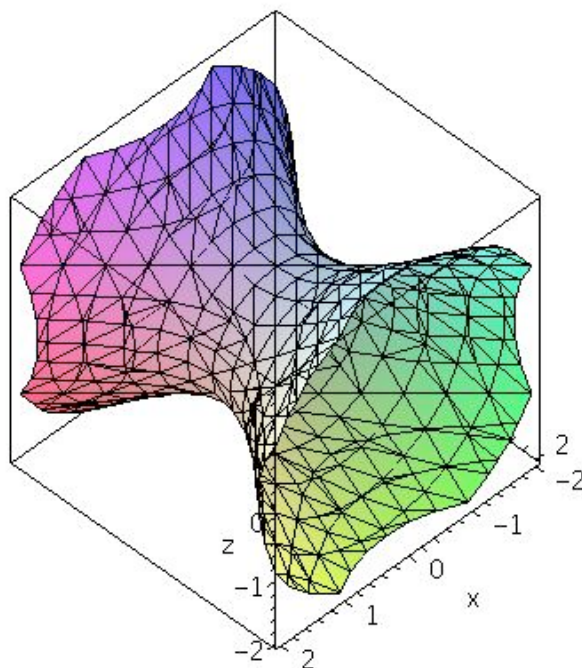


Figure 2. A pictorial representation of Anti de Sitter ($X_0^2 + X_1^2 = l^2 + \vec{X}^2$ in \mathbb{R}_{n-1}^{n+1}).

on a flat space with metric $ds^2 = \eta_{AB}dX^AdX^B$. If we change in an arbitrary manifold $g_{AB} \rightarrow -g_{AB}$, then both Christoffels and Riemann tensor remain invariant, but the scalar curvature flips sign $R \rightarrow -R$. We can furthermore group together times and spaces, in such a way that

$$\eta_{AB} = (1^t, (-1)^s) \tag{A.2}$$

If we call $n+1 \equiv t+s$, then this ambient space is Wolf's \mathbb{R}_s^{n+1} where the subindex indicates the number of *spaces*.

The standard nomenclature in Wolf's book [40] is

$$S_s^n : X \in \mathbb{R}_s^{n+1}, X^2 = l^2 \tag{A.3}$$

$$H_s^n : X \in \mathbb{R}_{s+1}^{n+1}, X^2 = -l^2 \tag{A.4}$$

The curvature scalar is given by:

$$R = \pm \frac{n(n-1)}{l^2} \tag{A.5}$$

and

$$\begin{aligned} R_{\mu\nu} &= \pm \frac{n-1}{l^2} g_{\mu\nu} \\ R_{\mu\nu\rho\sigma} &= \pm \frac{1}{l^2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \end{aligned} \tag{A.6}$$

Please note that the curvature only depends on the sign on the second member, and not on the signs ϵ_A themselves.

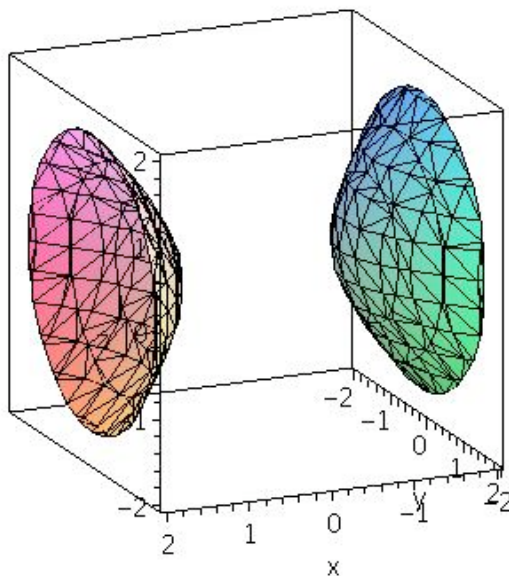


Figure 3. A pictorial representation of Euclidean Anti de Sitter (or Euclidean de Sitter) ($X_0^2 = l^2 + \vec{X}^2$ in \mathbb{R}_n^{n+1}).

It is clear, on the other hand, that the isometry group of the corresponding manifold is one of the real forms of the complex algebra $SO(n + 1)$. The Killing vector fields are explicitly given (no sum in the definition) by

$$L_{AB} \equiv \epsilon_A X^A \partial_B - \epsilon_B X^B \partial_A \equiv X_A \partial_B - X_B \partial_A \tag{A.7}$$

The square of the corresponding Killing vector is

$$L^2 = \epsilon_B X_A^2 + \epsilon_A X_B^2 \tag{A.8}$$

Our interest is concentrated on the euclidean and minkowskian cases:

- The sphere $S_n \equiv S_0^n \sim H_n^n$ is defined by $\vec{X}^2 = l^2$, with isometry group $SO(n + 1)$.
- The euclidean Anti de Sitter (or euclidean de Sitter) $EAdS_n \equiv S_n^n \sim H_0^n$ is defined by $(X^0)^2 - \vec{X}^2 = l^2$, with isometry group $SO(1, n)$.
- The de Sitter space $dS_n \equiv H_{n-1}^n \sim S_1^n$ is defined by $(X^0)^2 - \vec{X}^2 = -l^2$, with isometry group $SO(1, n)$. In our conventions de Sitter has negative curvature, but positive cosmological constant.
- The Anti de Sitter space $AdS_n \equiv S_{n-1}^n \equiv H_1^n$ is defined by $(X^0)^2 + (X^1)^2 - \vec{X}^2 = l^2$, with isometry group $SO(2, n - 1)$. For us AdS_n has positive curvature and negative cosmological constant.

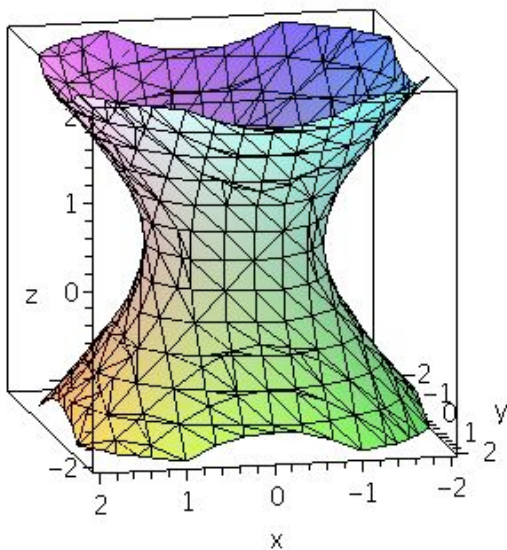


Figure 4. A pictorial representation of de Sitter ($X_0^2 = -l^2 + \vec{X}^2$) in \mathbb{R}_n^{n+1} .

A.1 Global coordinates

A very useful coordinate chart for these spaces is the one called *global* coordinates, which nevertheless do not cover the full space in any case:

$$(X^A) = l (\cosh \tau \vec{u}_t(\Omega), \sinh \tau \vec{n}_s(\Omega')) \quad (\text{A.9})$$

where \vec{u} and \vec{n} are unit vectors of both $t - 1$ and $s - 1$ dimensional spheres. This is for S_s^n spaces. For H_s^n spaces is simply:

$$(X^A) = l (\sinh \tau \vec{u}_{t-1}(\Omega), \cosh \tau \vec{n}_{s+1}(\Omega')) \quad (\text{A.10})$$

Our convention for a unit vector of a $(n - 1)$ -dimensional sphere is:

$$\vec{u}_n(\Omega) = (\cos \theta_1, \sin \theta_1 \cos \theta_2, \dots, \sin \theta_1 \dots \sin \theta_{n-1}) \quad (\text{A.11})$$

so that our convention for the “north pole” is:

$$S_s^n: N = (l, 0, \dots); H_s^n: N = (\underbrace{0, \dots, 0}_{t-1}, l, 0, \dots) \quad (\text{A.12})$$

The invariant distance, that we call z , is defined as $z(X, Y) = \pm \frac{X \cdot Y}{l^2}$, where the sign is chosen to make $z(X, X) = 1$ in every space. In our cases of interest:

- Sphere: $X = l \vec{u}_n(\Omega)$, $z = \cos \theta_1$
- Euclidean Anti de Sitter: $X = l(\cosh \tau, \sinh \tau \vec{u}_{n-1}(\Omega))$, $z = \cosh \tau$
- de Sitter: $X = l(\sinh \tau, \cosh \tau \vec{u}_{n-1}(\Omega))$, $z = \cosh \tau \cos \theta_1$
- Anti de Sitter: $X = l(\cosh \tau \cos \theta, \cosh \tau \sin \theta, \sinh \tau \vec{u}_{n-2}(\Omega'))$, $z = \cosh \tau \cos \theta$

A.2 Projective coordinates

We shall further assume that $\epsilon_k = \pm 1$, that is, the chosen coordinate has the same sign for the metric as the second member in (A.4). We then define the south pole (i.e. $X^k = -l$) stereographic projection for $\mu \neq k$, as

$$x_S^\mu \equiv \frac{2l}{X^k + l} X^\mu \equiv \frac{X^\mu}{\Omega_S} \tag{A.13}$$

The equation of the surface then leads to

$$X^k = l(2\Omega_S - 1) ; \Omega_S = \frac{1}{1 \pm \frac{x_S^2}{4l^2}} ; x_S^2 \equiv \sum_{\mu \neq k} \epsilon_\mu (x_S^\mu)^2 \tag{A.14}$$

The metric in these coordinates is conformally flat:

$$ds^2 = \Omega_S^2 \eta_{\mu\nu} dx_S^\mu dx_S^\nu \tag{A.15}$$

We could have done projection from the North pole (for that we need that $X^k \neq l$). Uniqueness of the definition of X^k needs

$$\Omega_N + \Omega_S = 1 \tag{A.16}$$

and uniqueness of the definition of X^μ

$$x_N^\mu = \frac{\Omega_S}{\Omega_N} x_S^\mu = \pm \frac{4l^2}{x_S^2} x_S^\mu \tag{A.17}$$

The antipodal \mathbb{Z}_2 map $X^A \rightarrow -X^A$ is equivalent to a change of the reference pole in stereographic coordinates

$$x_N^\mu \leftrightarrow x_S^\mu \tag{A.18}$$

A.3 Poincaré coordinates

A generalization of Poincaré's metric for the half-plane can easily be obtained by introducing the horospheric coordinates. It will always be assumed that $\epsilon_0 = +1$, that is that X^0 is a time, and also that $\epsilon_n = -1$, that is X^n is a space, in our conventions. Otherwise (like in the all-important case of the sphere S_n) it is not possible to construct these coordinates.

$$\begin{aligned} \frac{l}{z} &\equiv X^- = X^n - X^0 \\ y^i &\equiv zX^i \end{aligned} \tag{A.19}$$

The promised generalization of the Poincaré metric is:

$$ds^2 = \frac{\sum_1^{n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{A.20}$$

where the sign is the opposite to the one defined in (A.4), and the surfaces of constant z are sometimes called *horospheres*. This form of the metric is conformally flat in a manifest way.

- In *de Sitter space*, dS_n , z is a timelike coordinate, and its metric reads

$$ds_{dS_n}^2 = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j + l^2 dz^2}{z^2} \tag{A.21}$$

The square of the Killing vectors M_{0A} (candidates to be timelike) are

$$M_{0A}^2 = X_0^2 - X_A^2 = \sum_{B \neq A} X_B^2 - l^2 \tag{A.22}$$

so they are timelike only outside the *horizon* defined as

$$H_{0A} \equiv \sum_{B \neq A} X_B^2 = l^2 \tag{A.23}$$

For example, the horizon corresponding to H_{0n} is

$$\sum y_i^2 = l^2 z^2 \tag{A.24}$$

This means that de Sitter space, dS_n is not globally static.

- What one would want to call *Euclidean anti de Sitter*, $EAdS_n$, has got all its coordinates spacelike, and *positive* curvature. To be specific

$$ds_{EAdS_n}^2 = \frac{-\sum^{n-1} \delta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{A.25}$$

- Finally, when the metric is given by

$$ds_{AdS_n}^2 = \frac{\sum^{n-1} \eta_{ij} dy^i dy^j - l^2 dz^2}{z^2} \tag{A.26}$$

(where as usual, $\eta_{ij} \equiv \text{diag}(1, -1^{n-2})$) this is the *Anti de Sitter*, AdS_n . In this case there is a globally defined timelike Killing vector field, namely M_{01}

$$M_{01}^2 = X_0^2 + X_1^2 = l^2 + \sum_{A>1} X_A^2 \tag{A.27}$$

that is everywhere positive. This means that Anti de Sitter space is globally static, as opposed to de Sitter.

A.4 Conformal Invariance

Let us be very explicit with the definition of Poincaré coordinates: Let us denote

$$x^2 \equiv y^2 \mp l^2 z^2 \equiv \sum \epsilon_i y_i^2 \mp l^2 z^2 \tag{A.28}$$

Then

$$\begin{aligned} X^0 &= \frac{l^2 - x^2}{2lz} \\ X^n &= -\frac{l^2 + x^2}{2lz} \\ X^i &= \frac{y^i}{z} \quad (i = 1 \dots n-1) \end{aligned} \tag{A.29}$$

This is a legitimate change of coordinates as long as we keep the radius l itself as one of the coordinates.

Conversely,

$$\begin{aligned} y^i &= \frac{X^i}{X^0 - X^n} l \\ z &= \frac{l}{X^0 - X^n} \\ l^2 &= \mp (X_0^2 - X_n^2 + \epsilon_i X_i^2) \end{aligned} \tag{A.30}$$

Some useful formulas:

$$\begin{aligned} \frac{\partial}{\partial X_0} &= -\frac{z}{l} y^i \partial_i - \frac{z^2}{l} \partial_z \mp \frac{l^2 - x^2}{lz} \partial_{l^2} \\ \frac{\partial}{\partial X_n} &= \frac{z}{l} y^i \partial_i + \frac{z^2}{l} \partial_z \mp \frac{l^2 + x^2}{lz} \partial_{l^2} \\ \frac{\partial}{\partial X_i} &= z \partial_i \mp 2 \frac{\epsilon_i y_i}{z} \partial_{l^2} \end{aligned} \tag{A.31}$$

The full isometry group is some noncompact form of $SO(n+1)$. In Poincaré coordinates, there is a $ISO(n-1)$ manifest isometry group not involving the horographic coordinate. It will be important for us to understand all isometries in Poincaré coordinates. Let us work out the non-explicit generators:

$$L_{0n} \equiv X^0 \partial_n + X_n \partial_0 = y^i \partial_i + z \partial_z \tag{A.32}$$

$$L_{0i} = X^0 \partial_i - \epsilon_i X_i \partial_0 = \sum_j \frac{(l^2 - x^2) \delta_{ij} + 2\epsilon_i y_i y_j}{2l} \partial_j + \epsilon_i y^i \frac{z}{l} \partial_z \tag{A.33}$$

$$L_{ni} = -X^n \partial_i - \epsilon_i X_i \partial_n = \sum_j \frac{(l^2 + x^2) \delta_{ij} - 2\epsilon_i y_i y_j}{2l} \partial_j - \epsilon_i y^i \frac{z}{l} \partial_z \tag{A.34}$$

Translations of the y^i correspond to the combination:

$$k_i \equiv l \frac{\partial}{\partial y^i} = -(L_{ni} + L_{oi}) \tag{A.35}$$

All spaces we are considering in this paper, which in Poincaré coordinates enjoy the metric

$$ds^2 = \frac{\sum_{i=1}^{i=n-1} \epsilon_i dy_i^2 \mp l^2 dz^2}{z^2} \tag{A.36}$$

are obviously *scale invariant*

$$\begin{aligned} y_i &\rightarrow \lambda y_i \\ z &\rightarrow \lambda z \end{aligned} \tag{A.37}$$

This corresponds in Weierstrass coordinates to the Lorentz transformation in the plane $(X^0 X^n)$

$$\begin{aligned} (X')^0 &= \frac{(\lambda^2 + 1) X^0 + (\lambda^2 - 1) X^n}{2\lambda} \\ (X')^n &= \frac{(\lambda^2 - 1) X^0 + (\lambda^2 + 1) X^n}{2\lambda} \end{aligned} \tag{A.38}$$

id est,

$$\begin{aligned} X^- &\rightarrow \lambda X^- \\ X^+ &\rightarrow \frac{X^+}{\lambda} \end{aligned} \tag{A.39}$$

(This ought to be more or less obvious already from the previous formula for the generator L_{0n}). Not only that, but also they are invariant under *inversions*, id est,

$$\begin{aligned} y_i &\rightarrow \frac{y_i}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \\ z &\rightarrow \frac{z}{\sum \epsilon_i y_i^2 \mp l^2 z^2} \end{aligned} \tag{A.40}$$

Inversions in Weierstrass coordinates look even simpler; just exchange the two light-cone coordinates in the aforementioned plane ($X^0 X^n$):

$$X^+ \leftrightarrow X^- \tag{A.41}$$

The remaining isometries are the somewhat nasty combinations

$$L_{0i} - L_{ni} = \sum_j \frac{(-x^2) \delta_{ij} + 2\epsilon_i y_i y_j}{l} \partial_j + 2\epsilon_i y_i^i \frac{z}{l} \partial_z \tag{A.42}$$

We are now in a position to study the little group H of a given point (which can always be rotated to

$$P \equiv (\vec{y} = \vec{0}, z = 1) \tag{A.43}$$

We know that then the space will be isomorphic to $\text{SO}(n+1)/H$. The translational isometries must be generated by the n generators

$$\begin{aligned} L_{ni} + L_{0i} \\ L_{0n} \end{aligned} \tag{A.44}$$

It seems then that

$$\begin{aligned} H^+ &= \{L_{ij}, L_{ni}\} \\ H^- &= \{L_{ij}, L_{0i}\} \end{aligned} \tag{A.45}$$

The number of not compact generators is equal to the number of times in the coordinates y^i in the + case, and the number of times plus one in the minus case. This seems to imply that

$$\begin{aligned} AdS_n &= \text{SO}(2, n-1)/\text{SO}(1, n-1) \\ EAdS_n &= \text{SO}(1, n)/\text{SO}(n) \\ dS_n &= \text{SO}(1, n)/\text{SO}(1, n-1) \\ EdS_n &= \text{SO}(n, 1)/\text{SO}(n) \end{aligned} \tag{A.46}$$

Euclidean anti de Sitter $EAdS_n$ is just de Sitter dS_n with imaginary radius. Euclidean de Sitter EdS_n is Euclidean anti de Sitter dS_n with negative ambient metric.

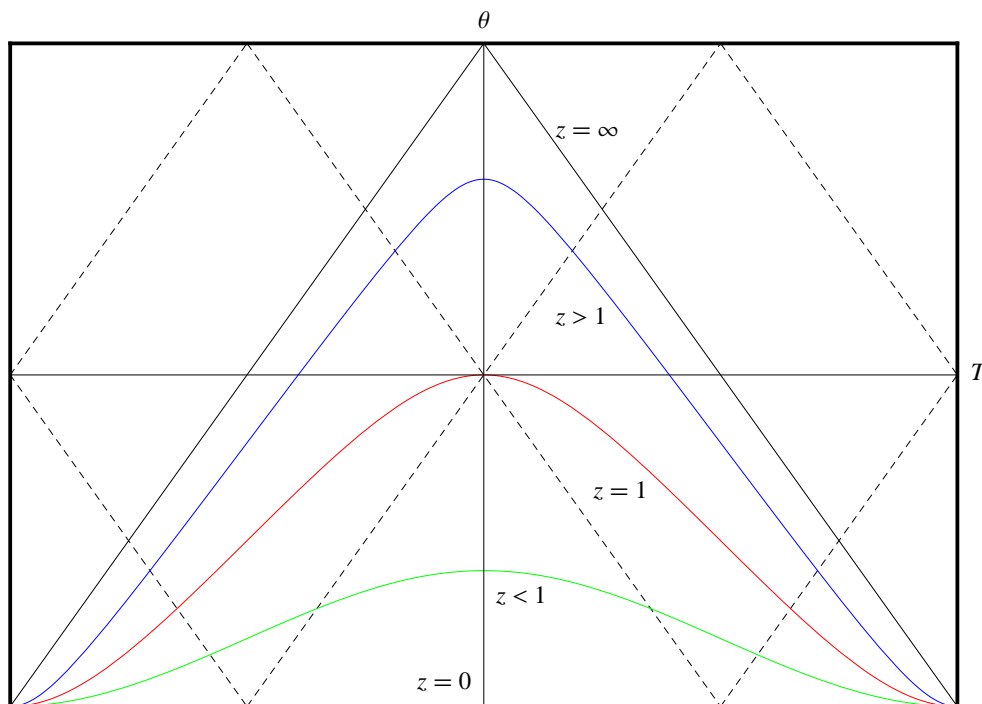


Figure 5. Conformal structure of dS_n . The coloured lines are $z = \text{const.}$ surfaces in Poincaré coordinates.

B Conformal structure

- dS_n From the global coordinates in de Sitter (cf. A.1), we can define $\cos T = \frac{1}{\cosh \tau}$ where $-\pi/2 \leq T \leq \pi/2$ so it yields

$$ds^2 = \frac{l^2}{\cos^2 T} (dT^2 - d\Omega_{n-1}^2) \tag{B.1}$$

which is conformal to a piece of $\mathbb{R} \times S_{n-1}$, which is the Einstein static universe to study conformal structure. The piece is a slab in the timelike direction, but otherwise including the full three-sphere at each time. The fact that conformal infinity is spacelike means that there are both particle and event horizons.

- AdS_n The same change of coordinates from the global chart can be used, $\cos \rho = \frac{1}{\cosh \tau}$, where $\rho \in (0, \pi/2)$. The space is again conformal to a piece of half Einstein's static universe:

$$ds^2 = \frac{l^2}{\cos^2 \rho} (d\theta^2 - d\rho^2 - \sin^2 \rho d\Omega_{n-2}^2) = \frac{l^2}{\cos^2 \rho} (d\theta^2 - d\Omega_{n-1}^2) \tag{B.2}$$

If we want to eliminate the closed timelike lines, one can consider the covering space $-\infty \leq \theta \leq \infty$. The slab of $\mathbb{R} \times S_{n-1}$ to which AdS_n is conformal to includes now the full timelike direction, but only an hemisphere at each particular time. Null and spacelike infinity can be considered as the timelike surfaces $\rho = 0$ and $\rho = \pi/2$. This implies that there are no Cauchy surfaces.

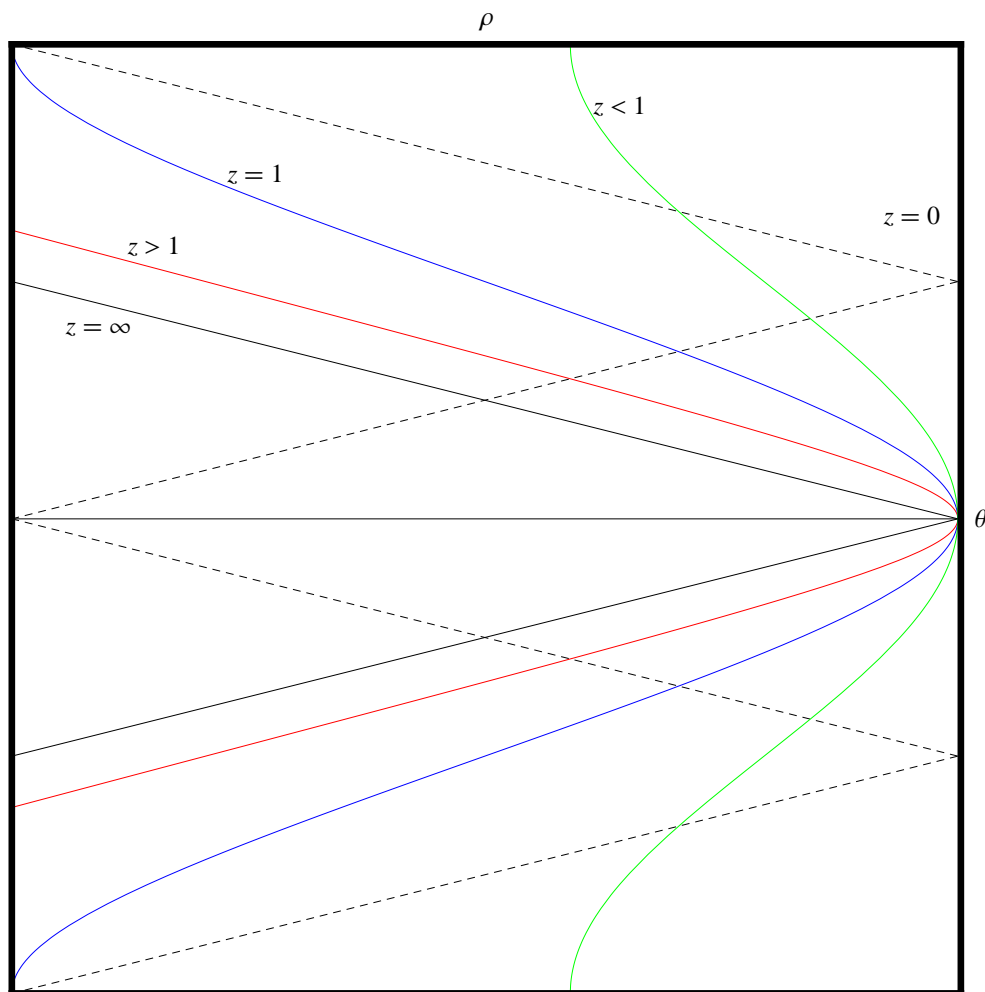


Figure 6. Conformal structure of AdS_n . The coloured lines are $z = \text{const.}$ surfaces in Poincaré coordinates.

C What portion of Weierstrass coordinates do Poincaré coordinates cover?

- dS_n

If we call u_n the n -th component of the unit vector \vec{u} , then there is a critical value of the parameter τ such that

$$\tanh \tau(u) = u_n(\Omega) \tag{C.1}$$

which is such that

$$\tau < \tau(u) \Rightarrow z > 0 \tag{C.2}$$

and

$$z \rightarrow \pm\infty \Leftrightarrow \tau \rightarrow \tau(n)^\mp \tag{C.3}$$

This means that at any given value of τ only those points on the sphere that obey

$$u_n(\Omega) \geq \tanh \tau \tag{C.4}$$

can be represented in Poincaré coordinates. For example, when $\tau = \infty$, that is $T = \pi/2$, $\tanh \tau = 1$, so that only the North pole ($n = 1$) can be covered. At the other extreme, when, $\tau = -\infty$, that is $T = -\pi/2$, $\tanh \tau = -1$, we can cover the full sphere.

On the other hand, it is clear that

$$z \rightarrow 0^\pm \Leftrightarrow \tau \rightarrow \mp\infty \tag{C.5}$$

There is a discontinuity at $\tau(n)$ which depends on the point in de Sitter space.

- AdS_n

As in the previous case, it is clear that the region $1/z = 0$ corresponds to

$$u_{n-1}(\Omega) \sin \rho = \cos \theta \tag{C.6}$$

and the region $z > 0$ to

$$u_{n-1}(\Omega) \sin \rho > \cos \theta \tag{C.7}$$

The region

$$z = 0 \tag{C.8}$$

is dubbed the *boundary* (of the Poincaré patch) of AdS and corresponds to

$$\rho = \pi/2 \tag{C.9}$$

Finally

$$z = \infty \tag{C.10}$$

is usually called the *horizon* and corresponds to (C.6)

D Spherical harmonics

- *The n-dimensional sphere.* The simplest way of getting eigenfunctions of the Laplace operator in the sphere is Helgason's (confer [19]). Consider the following harmonic polynomial in \mathbb{R}^{n+1}

$$f_{a,\lambda} \equiv (\vec{a} \cdot \vec{x})^\lambda \tag{D.1}$$

with $\vec{a} \in \mathbb{C}$, $\vec{a}^2 = 0$.

Now we know that the full laplacian in \mathbb{R}^{n+1} is

$$\Delta_{\mathbb{R}^{n+1}} = \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S_n} \tag{D.2}$$

This yields

$$\Delta_{\mathbb{R}^{n+1}} f_{a,\lambda} = 0 = \frac{\lambda^2 + (n-1)\lambda}{r^2} f_{a,\lambda} + \frac{1}{r^2} \Delta_{S_n} f_{a,\lambda} \quad (\text{D.3})$$

so that the eigenvalues of the Laplacian in the sphere S_n are

$$-\lambda(\lambda + n - 1) \quad (\text{D.4})$$

It is more or less equivalent to start from traceless homogeneous polynomials

$$P \equiv \sum P_{(i_1 \dots i_k)} x^{i_1} \dots x^{i_k} \quad (\text{D.5})$$

The number of such animals is the number of symmetric polynomials in n variables of degree λ minus the number of symmetric polynomials of degree $\lambda - 2$:

$$d(\lambda) = \binom{\lambda + n - 1}{\lambda} - \binom{\lambda + n - 3}{\lambda - 2} = \frac{(n + 2\lambda - 2)(\lambda + n - 3)!}{\lambda!(n - 2)!} \quad (\text{D.6})$$

- If we represent by μ an appropriate collection of indices, then we first build harmonic polynomials such that

$$\int_{S_n} d\Omega h_{\lambda'\mu'}^* h_{\lambda\mu} = \delta_{\lambda\lambda'} \delta_{\mu\mu'} r^{\lambda+\lambda'} \quad (\text{D.7})$$

The hyperspherical harmonics are then defined by

$$h_{\lambda\mu} \equiv r^\lambda Y_{\lambda\mu} \quad (\text{D.8})$$

and are normalized in such a way that

$$\int_{S_n} d\Omega Y_{\lambda'\mu'}^* Y_{\lambda\mu} = \delta_{\lambda\lambda'} \delta_{\mu\mu'} \quad (\text{D.9})$$

- Gegenbauer polynomials are generalizations of Legendre polynomials, in the sense that

$$\frac{1}{|\vec{x} - \vec{x}'|^{n-2}} = \frac{1}{r_{>}^{n-2} \left(1 + \left(\frac{r_{\leq}}{r_{>}}\right)^2 - 2 \left(\frac{r_{\leq}}{r_{>}}\right) \hat{x} \cdot \hat{x}' \right)^{\frac{n-2}{2}}} = \frac{1}{r_{>}^{n-2}} \sum_{\lambda=0}^{\infty} \left(\frac{r_{\leq}}{r_{>}}\right)^\lambda C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') \quad (\text{D.10})$$

Let us now prove the sum rule for hyperspherical harmonics. For concreteness, let us assume that

$$\begin{aligned} r &\equiv |\vec{x}_{<}| \\ r' &\equiv |\vec{x}_{>}| \end{aligned} \quad (\text{D.11})$$

Then it is a fact of life that

$$\Delta \frac{1}{|\vec{x} - \vec{x}'|^{n-2}} = 0 = \sum_{\lambda=0}^{\infty} \frac{1}{(r')^{\lambda+n-2}} \Delta \left(r^\lambda C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') \right) \quad (\text{D.12})$$

Imposing term by term vanishing leads to

$$\left(\frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{S^{n-1}} \right) \left(r^\lambda C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') \right) = 0 \quad (\text{D.13})$$

which conveys the fact that

$$\Delta_{S^{n-1}} C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') = -\lambda(\lambda + n - 2) C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') \quad (\text{D.14})$$

Since the hyperspherical harmonics are by assumption a complete set of eigenfunctions,

$$C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') = \sum_\mu a_{\lambda\mu}(\vec{x}') Y_{\lambda\mu}(\hat{x}) \quad (\text{D.15})$$

where

$$a_{\lambda\mu}(\vec{x}') = \int_{\hat{x}} C_\lambda^{\frac{n-2}{2}}(\hat{x} \cdot \hat{x}') Y_{\lambda\mu}^*(\hat{x}) = \frac{2(n-2)\pi^{n/2}}{\Gamma(n/2)(2\lambda + n - 2)} Y_{\lambda\mu}^*(\hat{x}') \quad (\text{D.16})$$

This is related to the degeneracy $d(\lambda)$ of hyperspherical harmonics in the following way. Choosing $\hat{x} = \hat{x}'$, the sum rule leads to

$$C_\lambda^{\frac{n-2}{2}}(1) = K_\lambda \sum_\mu Y_{\lambda\mu}^*(\vec{x}') Y_{\lambda\mu}(\hat{x}) \quad (\text{D.17})$$

Integrating now over the unit sphere

$$C_\lambda^{\frac{n-2}{2}}(1) V(S_{n-1}) = K_\lambda \sum_\mu 1 = K_\lambda d(\lambda) \quad (\text{D.18})$$

The result is

$$d(\lambda) = \frac{(n + 2\lambda - 2)(\lambda + n - 3)!}{\lambda!(n - 2)!} \quad (\text{D.19})$$

- Let us now become more specific and perform some computations in gory detail. The metric on S_n is

$$ds_n^2 = d\theta_n^2 + \sin^2 \theta_n d\theta_{n-1}^2 + \dots + \sin^2 \theta_n \sin^2 \theta_{n-1} \dots \sin^2 \theta_2 d\theta_1^2 \quad (\text{D.20})$$

and, in a recurrent form

$$\begin{aligned} ds_1^2 &= d\theta_1^2 \\ ds_n^2 &= d\theta_n^2 + \sin^2 \theta_n ds_{n-1}^2 \end{aligned} \quad (\text{D.21})$$

This corresponds to polar coordinates in \mathbb{R}^n

$$\begin{aligned} X_{n+1} &= \cos \theta_n \\ X_n &= \sin \theta_n \cos \theta_{n-1} \\ &\dots \\ X_2 &= \sin \theta_n \sin \theta_{n-1} \dots \cos \theta_1 \\ X_1 &= \sin \theta_n \sin \theta_{n-1} \dots \sin \theta_1 \end{aligned} \quad (\text{D.22})$$

Spherical harmonics have been constructed quite explicitly by Higuchi [20], are such that

$$\Delta_n Y_{j_n \dots j_1}(\theta_n \dots \theta_1) = -j_n(j_n + n - 1) Y_{j_n \dots j_1}(\theta_n \dots \theta_1) \quad (\text{D.23})$$

We shall explicitly write down the laplacian in a moment. They are orthonormal with respect to the induced riemannian measure

$$d\Omega_n \equiv \sqrt{|g|} d\theta_1 \wedge \dots \wedge d\theta_n = d\theta_1 \dots d\theta_n \sin^{n-1} \theta_n \sin^{n-2} \theta_{n-1} \dots \sin \theta_2 \quad (\text{D.24})$$

The laplacian is easily found to be

$$\begin{aligned} \Delta_{S_n} = & \left(\frac{\partial^2}{\partial \theta_n^2} + (n-1) \cot \theta_n \frac{\partial}{\partial \theta_n} \right) \\ & + \frac{1}{\sin^2 \theta_n} \left(\frac{\partial^2}{\partial \theta_{n-1}^2} + (n-2) \cot \theta_{n-1} \frac{\partial}{\partial \theta_{n-1}} \right) + \dots \\ & + \frac{1}{\sin^2 \theta_n \sin^2 \theta_{n-1} \dots \sin^2 \theta_2} \frac{\partial^2}{\partial \theta_1^2} \end{aligned} \quad (\text{D.25})$$

Another useful recurrence

$$d\Omega_n = \sin^{n-1} \theta_n d\theta_n d\Omega_{n-1} \quad (\text{D.26})$$

and

$$V(S_{n-1}) = \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \quad (\text{D.27})$$

To be specific,

$$\int d\Omega_n Y_{j_n \dots j_1}(\theta_n \dots \theta_1) Y_{j'_n \dots j'_1}^*(\theta_n \dots \theta_1) = \delta_{j_n, j'_n} \dots \delta_{j_1, j'_1} \quad (\text{D.28})$$

- It is obvious that any function on the sphere can be expanded

$$\begin{aligned} f(\Omega) &= \sum_{j_n \dots j_1} C_{j_n \dots j_1} Y_{j_n \dots j_1}(\theta_n \dots \theta_1) \\ &= \sum_{j_n \dots j_1} \int d\Omega' Y_{j_n \dots j_1}^*(\theta'_n \dots \theta'_1) f(\theta'_n \dots \theta'_1) Y_{j_n \dots j_1}(\theta_n \dots \theta_1) \end{aligned}$$

which means

$$\sum_{j_n \dots j_1} Y_{j_n \dots j_1}^*(\theta'_n \dots \theta'_1) Y_{j_n \dots j_1}(\theta_n \dots \theta_1) \equiv \delta(\Omega - \Omega') \quad (\text{D.29})$$

where by definition

$$\int d\Omega' \delta(\Omega - \Omega') f(\theta') = f(\theta) \quad (\text{D.30})$$

whence in a somewhat symbolic form,

$$\delta(\Omega - \Omega') = \delta(\theta'_1 - \theta_1) \dots \delta(\theta'_n - \theta_n) \sin^{-(n-1)} \theta'_n \sin^{-(n-2)} \theta'_{n-1} \dots \sin^{-1} \theta'_2 \quad (\text{D.31})$$

Now we can expand this function, as any other function, in series of Gegenbauer polynomials

$$\delta(\Omega - \Omega') = \sum_j d_j C_j^\nu(\cos \theta_n) \quad (\text{D.32})$$

Let us choose our reference frame in such a way that

$$\Omega \cdot \Omega' \equiv \cos \theta_n \quad (\text{D.33})$$

id est, Ω' is pointing towards the North pole.

On functions constant on S_{n-1} ,

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \sin^{n-1} \theta_n d\theta_n \quad (\text{D.34})$$

and, denoting $x \equiv \cos \theta_n$

$$d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} (1-x^2)^{\frac{n-2}{2}} dx \quad (\text{D.35})$$

as well as

$$\delta(\Omega) = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(\theta_n) \frac{1}{\sin^{n-1} \theta_n} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \delta(1-x)(1-x^2)^{\frac{2-n}{2}} \quad (\text{D.36})$$

We can now integrate the two sides of the equation (D.32) against $C_{j'}^\nu(x)(1-x)^{\nu-1/2}$. The orthogonality property

$$\int_{-1}^1 dx C_j^\nu(x) C_{j'}^\nu(x) (1-x^2)^{\nu-1/2} = \delta_{jj'} \frac{2^{1-2\nu} \pi \Gamma(j+2\nu)}{j!(\nu+j)\Gamma(\nu)^2} \quad (\text{D.37})$$

then implies

$$d_j \frac{2^{1-2\nu} \pi \Gamma(j+2\nu)}{j!(\nu+j)\Gamma(\nu)^2} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} \int_{-1}^1 dx C_j^\nu(x) (1-x^2)^{1-n/2} \delta(x-1) (1-x^2)^{\nu-1/2} \quad (\text{D.38})$$

The member of the right converges when $\nu = \frac{n-1}{2}$. Given in addition the fact that

$$C_j^\nu(1) = \frac{\Gamma(j+2\nu)}{j!\Gamma(2\nu)} \quad (\text{D.39})$$

we can write

$$d_j = \frac{\Gamma(\frac{n}{2})(j + \frac{n-1}{2})\Gamma(\frac{n-1}{2})^2}{\Gamma(n-1)\pi^{\frac{n+1}{2}}2^{3-n}} = \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} \quad (\text{D.40})$$

(using $\Gamma(2x) = 2^{1-2x}\sqrt{\pi}\Gamma(x + \frac{1}{2})/\Gamma(x)$) as well as

$$\begin{aligned} \delta(\Omega - \Omega') &= \sum_j \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos \theta_n) \\ \sum_{j_n \dots j_1} Y_{j_n \dots j_1}^*(\theta'_n = 0 \dots \theta'_1) Y_{j_n \dots j_1}(\theta_n \dots \theta_1) &= \sum_j \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos \theta_n) \end{aligned} \quad (\text{D.41})$$

If we employ the notation $j \equiv j_n$ and $\vec{m} \equiv (j_{n-1} \dots j_1)$, then the preceding formula presumably means that

$$\sum_{\vec{m}} Y_{j \dots \vec{m}}^*(\Omega_z) Y_{j \dots \vec{m}}(\Omega) = \frac{1}{V(S_n)} \frac{n-1+2j}{n-1} C_j^{\frac{n-1}{2}}(\cos \theta_n) \quad (\text{D.42})$$

- We begin by defining some eigenfunctions of the differential operator:

$$D \equiv \frac{\partial^2}{\partial \theta^2} + (N-1) \cot \theta \frac{\partial}{\partial \theta} - \frac{j(j+N-2)}{\sin^2 \theta} \quad (\text{D.43})$$

such that

$$D \bar{P}_{Nk}^j(\theta) = -k(k+N-1) \bar{P}_{Nk}^j(\theta) \quad (\text{D.44})$$

The form we are going to need is

$$\left(\frac{\partial^2}{\partial \theta^2} + (N-1) \cot \theta \frac{\partial}{\partial \theta} \right) \bar{P}_{Nk}^j(\theta) = \left(\frac{j(j+N-2)}{\sin^2 \theta} - k(k+N-1) \right) \bar{P}_{Nk}^j(\theta) \quad (\text{D.45})$$

To be specific,

$$\bar{P}_{Nk}^j(\theta) \equiv c_{Nk}^j (\sin \theta)^{-\frac{N-2}{2}} P_{k+\frac{N-2}{2}}^{-(j+\frac{N-2}{2})}(\cos \theta) \quad (\text{D.46})$$

where $P_\nu^\mu(z)$ are Legendre functions, and the normalization is given by

$$c_{Nk}^j \equiv \sqrt{\frac{2k+N-1}{2} \frac{(k+j+N-2)!}{(k-j)!}} \quad (\text{D.47})$$

The differential equation that Legendre functions $P_\nu^\mu(z)$ are solutions of is given by

$$Lw(z) \equiv (1-z^2) \frac{d^2 w}{dz^2} - 2z \frac{dw}{dz} + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2} \right) w = 0 \quad (\text{D.48})$$

Changing variables $z = \cos \theta$ this reads

$$\left(\frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{\mu^2}{\sin^2 \theta} \right) w(\cos \theta) = -\nu(\nu+1) w(\cos \theta) \quad (\text{D.49})$$

and using this it is not difficult to actually prove the basic equation (D.44).

The harmonics themselves are given by:

$$Y_{j_n \dots j_1}(\theta_n, \dots, \theta_1) \equiv \prod_{m=2}^n \bar{P}_{m j_m}^{j_{m-1}}(\theta_m) \frac{1}{\sqrt{2\pi}} e^{i j_1 \theta_1} \quad (\text{D.50})$$

It is actually easy to check. From the expression for the laplacian, the operator acting on θ_1 , just leads to

$$-\frac{j_1^2}{\sin^2 \theta_n \dots \sin^2 \theta_2} \quad (\text{D.51})$$

Next, the operator acting on θ_2 , corresponding to $N=2, k=j_2$ and $j=j_1$, yields

$$\frac{j_1^2}{\sin^2 \theta_n \dots \sin^2 \theta_2} - \frac{j_2(j_2+1)}{\sin^2 \theta_n \dots \sin^2 \theta_3} \quad (\text{D.52})$$

Next, the operator acting on θ_3 , which corresponds to $N = 3$, $k = j_3$ and $j = j_2$, gives

$$\frac{j_2(j_2 + 1)}{\sin^2 \theta_n \dots \sin^2 \theta_3} - \frac{j_3(j_3 + 2)}{\sin^2 \theta_n \dots \sin^2 \theta_4} \quad (\text{D.53})$$

After all pairwise cancellations, we are left with the last term, corresponding to $N = n$, $k = j_n$ and $j = j_{n-1}$, yielding the eigenvalue

$$- j_n(j_n + n - 1) \quad (\text{D.54})$$

- We can now employ the expansion (GR, 8.534)

$$e^{im\rho \cos \phi} = 2^\nu \Gamma(\nu) \sum_{k=0}^{\infty} (\nu + k) i^k (m\rho)^{-\nu} J_{\nu+k}(m\rho) C_k^\nu(\cos \phi) \quad (\text{D.55})$$

and using our expansion of the Gegenbauer polynomials in terms of spherical harmonics,

$$e^{iz\Omega \cdot \Omega'} = 2^{n/2-1} \Gamma(n/2 - 1) \sum_{k=0}^{\infty} (n/2 - 1 + k) i^k (z)^{-(n/2-1)} J_{n/2-1+k}(z) C_{k,n} \sum_{\vec{m}} Y_{k\dots\vec{m}}^*(\Omega) Y_{k\dots\vec{m}}(\Omega') \quad (\text{D.56})$$

where $C_{l,n}$ are appropriate constants.

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