Probabilities in nonorthogonal basis: Four–quark systems

J. Vijande[1] and A. Valcarce[2]

[1]Departamento de Física Atómica, Molecular y Nuclear,

Universidad de Valencia (UV) and IFIC (UV-CSIC), Valencia, Spain.

[2]Departamento de Física Fundamental,

Universidad de Salamanca, Salamanca, Spain.

(Dated: August 22, 2009)

Abstract

Four-quark states may exist as colorless meson-meson molecules or compact systems with two-body colored components. We derive an analytical procedure to expand an arbitrary four–quark wave function in terms of nonorthogonal color singlet–singlet vectors. Using this expansion we develop the necessary formalism to evaluate the probability of physical components with an arbitrary four-quark wave function. Its application to characterize bound and unbound four–quark states as meson-meson, molecular or compact systems is discussed

PACS numbers: 14.40.Nd,14.40.Lb,14.40.-n

I. INTRODUCTION

The physics of heavy quarks has become one of the best laboratories exposing the limitations and challenges of the naive quark model and also giving hints into a more mature description of hadron spectroscopy. More than thirty years after the so-called November revolution [1], heavy meson spectroscopy is being again a challenge. Its formerly comfortable world is being severely tested by new experiments reporting states that do not fit into a simple quark-antiquark configuration [2]. It seems nowadays unavoidable to resort to higher order Fock space components to tame the bewildering landscape arising with these new findings. Four–quark components, either pure or mixed with $q\bar{q}$ states, constitute a natural explanation for the proliferation of new meson states [3]. They would also account for the possible existence of exotic mesons as could be stable $cc\bar{n}\bar{n}$ states, the topic for discussion since the early 80's [4].

Four-quark systems present a richer color structure than standard baryons or mesons. While the color wave function for standard mesons and baryons leads to a single vector, working with four–quark states there are different vectors driving to a singlet color state out of colorless or colored quark-antiquark two-body components. Thus, dealing with four–quark states an important question is whether we are in front of a colorless meson-meson molecule or a compact state, i.e., a system with two-body colored components. While the first structure would be natural in the naive quark model, the second one would open a new area on the hadron spectroscopy.

In this manuscript we derive the necessary formalism to evaluate the probability of physical channels (singlet–singlet color states) in an arbitrary four–quark wave function. For this purpose one needs to expand any hidden–color vector of the four–quark state color basis, i.e., vectors with non–singlet internal color couplings, in terms of singlet–singlet color vectors. We will see that given a general four–quark state $[q_1 q_2 \bar{q}_3 \bar{q}_4]$ the above procedure requires to mix terms from two different couplings, $[(q_1\bar{q}_3)(q_2\bar{q}_4)]$ and $[(q_1\bar{q}_4)(q_2\bar{q}_3)]$. If (q_1, q_2) and $(\bar{q}_3, \bar{q}_4)$ are identical quarks and antiquarks then, a general four-quark wave function can be expanded in terms of color singlet-singlet nonorthogonal vectors and therefore the determination of the probability of physical channels becomes cumbersome. A particular case has been discussed in the literature trying to understand the light scalar mesons as $K\bar{K}$ molecules [5]. This problem has also been found in other fields, as for example in molecular

systems [6].

The manuscript is organized as follows. In Sec. II the formalism to expand the four–quark wave function in terms of singlet–singlet color vectors is derived. Without loss of generality, the evaluation of the probabilities is exemplified with the $QQ\bar{n}\bar{n}$ system. In Sec. III we discuss some examples of bound and unbound states in the charm and bottom sectors. Finally, we summarize in Sec. IV our conclusions.

II. FORMALISM

Given an arbitrary state $|\Psi\rangle$ made up of two quarks, q_1 and q_2, and two antiquarks, $\bar{q}_3$ and $\bar{q}_4$, its most general wave function will be the direct product of vectors from the color, spin, flavor and radial subspaces. We start discussing the color substructure.

A. Color substructure.

There are three different ways of coupling two quarks and two antiquarks into a colorless state:

$$[(q_1 q_2)(\bar{q}_3 \bar{q}_4)] \equiv \{|\bar{3}_{12} 3_{34}\rangle, |6_{12} \bar{6}_{34}\rangle\} \equiv \{|\bar{3}3\rangle_c^{12}, |6\bar{6}\rangle_c^{12}\} \tag{1a}$$

$$[(q_1 \bar{q}_3)(q_2 \bar{q}_4)] \equiv \{|1_{13} 1_{24}\rangle, |8_{13} 8_{24}\rangle\} \equiv \{|11\rangle_c, |88\rangle_c\} \tag{1b}$$

$$[(q_1 \bar{q}_4)(q_2 \bar{q}_3)] \equiv \{|1_{14} 1_{23}\rangle, |8_{14} 8_{23}\rangle\} \equiv \{|1'1'\rangle_c, |8'8'\rangle_c\}, \tag{1c}$$

being the three of them orthonormal basis. Each coupling scheme allows to define a color basis where the four–quark problem can be solved. The first basis, Eq. (1a), being the most suitable one to deal with the Pauli principle is made entirely of vectors containing hidden–color components. The other two, Eqs. (1b) and (1c), are hybrid basis containing singlet–singlet (physical) and octet–octet (hidden–color) vectors.

In order to express a four–quark state in terms only of physical components it is necessary to define the antiunitary transformation connecting the basis (1b) and (1c) [7]

$$\begin{aligned}
|\,11\rangle_c &= \cos\alpha \,|\,1'1'\rangle_c + \sin\alpha \,|\,8'8'\rangle_c \\
|\,88\rangle_c &= \sin\alpha \,|\,1'1'\rangle_c - \cos\alpha \,|\,8'8'\rangle_c \,,
\end{aligned} \tag{2}$$

3

and the projectors on the different vectors:

$$P = |11\rangle_c {}_c\langle 11|$$
$$Q = |88\rangle_c {}_c\langle 88|\,, \tag{3}$$

and

$$\hat{P} = |1'1'\rangle_c {}_c\langle 1'1'|$$
$$\hat{Q} = |8'8'\rangle_c {}_c\langle 8'8'|\,. \tag{4}$$

With these definitions any arbitrary state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = P|\Psi\rangle + Q|\Psi\rangle = \hat{P}|\Psi\rangle + \hat{Q}|\Psi\rangle\,. \tag{5}$$

One can extract the singlet–singlet components from the octet–octet one, $Q|\Psi\rangle$, by inserting identities $\mathbb{1} = P + Q = \hat{P} + \hat{Q}$ in the following iterative manner:

$$
\begin{aligned}
|\Psi\rangle &= P|\Psi\rangle + \hat{P}Q|\Psi\rangle + \hat{Q}Q|\Psi\rangle = \\
&= P|\Psi\rangle + \hat{P}Q|\Psi\rangle + P\hat{Q}Q|\Psi\rangle + Q\hat{Q}Q|\Psi\rangle = \\
&= P|\Psi\rangle + \hat{P}Q|\Psi\rangle + P\hat{Q}Q|\Psi\rangle + \hat{P}Q\hat{Q}Q|\Psi\rangle + \hat{Q}Q\hat{Q}Q|\Psi\rangle = \ldots = \\
&= P\left[\mathbb{1} + \hat{Q}Q + \hat{Q}Q\hat{Q}Q + \ldots\right]|\Psi\rangle + \hat{P}\left[Q + Q\hat{Q}Q + Q\hat{Q}Q\hat{Q}Q + \ldots\right]|\Psi\rangle\,. \tag{6}
\end{aligned}
$$

From the definition of the projectors in Eqs. (3) and (4) and the antiunitary transformation in Eq. (2) one can see that

$$Q\hat{Q}Q = |88\rangle_c {}_c\langle 88|8'8'\rangle_c {}_c\langle 8'8'|88\rangle_c {}_c\langle 88| = |{}_c\langle 88|8'8'\rangle_c|^2 Q = \cos^2\alpha\, Q\,. \tag{7}$$

Therefore, Eq. (6) can be rewritten as,

$$
\begin{aligned}
|\Psi\rangle &= P|\Psi\rangle + P\hat{Q}Q\left[1 + \cos^2\alpha + \cos^4\alpha + \ldots\right]|\Psi\rangle \\
&+ \hat{P}Q\left[1 + \cos^2\alpha + \cos^4\alpha + \ldots\right]|\Psi\rangle\,. \tag{8}
\end{aligned}
$$

Noting that,

$$\sum_{k=0}^{\infty}\cos\alpha^{2k} = \frac{1}{1 - \cos^2\alpha}\,, \tag{9}$$

Eq. (8) becomes,

$$|\Psi\rangle = \left[P + P\hat{Q}Q\frac{1}{1 - \cos^2\alpha}\right]|\Psi\rangle + \left[\hat{P}Q\frac{1}{1 - \cos^2\alpha}\right]|\Psi\rangle\,. \tag{10}$$

4

This equation can be simplified by considering

$$P\hat{Q}P = |\,11\rangle_c\ _c\langle 11\,|\,8'8'\rangle_c\ _c\langle 8'8'\,|\,11\rangle_c\ _c\langle 11\,| = \sin^2\alpha\ P \tag{11}$$

and thus

$$P + P\hat{Q}Q\frac{1}{1-\cos^2\alpha} = P + P\hat{Q}(1-P)\frac{1}{1-\cos^2\alpha} \tag{12}$$

$$= P\hat{Q}\frac{1}{1-\cos^2\alpha} + P - P\hat{Q}P\frac{1}{1-\cos^2\alpha} = P\hat{Q}\frac{1}{1-\cos^2\alpha}$$

Therefore Eq. (10) can be finally written in a compact form as

$$|\Psi\rangle = \frac{1}{1-\cos^2\alpha}P\hat{Q}|\Psi\rangle + \frac{1}{1-\cos^2\alpha}\hat{P}Q|\Psi\rangle = \mathcal{P}^{NH_1}_{|11\rangle_c}|\Psi\rangle + \mathcal{P}^{NH_1}_{|1'1'\rangle_c}|\Psi\rangle\,, \tag{13}$$

where $\mathcal{P}^{NH_1}_{|11\rangle_c}$ and $\mathcal{P}^{NH_1}_{|1'1'\rangle_c}$ are nonhermitian projection operators on the corresponding singlet–singlet subspaces (see Appendix A for proof of their properties). This expression demonstrates that any octet–octet color component can be expanded, in general, as an infinite sum of singlet–singlet color states [8].

To obtain hermitian operators one can repeat the same procedure using the projectors on $|\,1'1'\rangle_c$ and $|\,8'8'\rangle_c$ given in Eq. (4),

$$|\Psi\rangle = \hat{Q}|\Psi\rangle + Q\hat{P}|\Psi\rangle + P\hat{P}|\Psi\rangle =$$

$$= \hat{Q}|\Psi\rangle + Q\hat{P}|\Psi\rangle + \hat{Q}P\hat{P}|\Psi\rangle + \hat{P}P\hat{P}|\Psi\rangle =$$

$$= \hat{Q}|\Psi\rangle + Q\hat{P}|\Psi\rangle + \hat{Q}P\hat{P}|\Psi\rangle + Q\hat{P}P\hat{P}|\Psi\rangle + \hat{P}P\hat{P}|\Psi\rangle = \ldots =$$

$$= \hat{Q}\left[\mathbb{1} + P\hat{P} + P\hat{P}P\hat{P} + \ldots\right]|\Psi\rangle + Q\left[\hat{P} + \hat{P}P\hat{P} + \hat{P}P\hat{P}P\hat{P} + \ldots\right]|\Psi\rangle\,, \tag{14}$$

where

$$\hat{P}P\hat{P} = |\,1'1'\rangle_c\ _c\langle 1'1'\,|\,11\rangle_c\ _c\langle 11\,|\,1'1'\rangle_c\ _c\langle 1'1'\,| = |\ _c\langle 1'1'\,|\,11\rangle_c\,|^2\ \hat{P} = \cos^2\alpha\ \hat{P}\,, \tag{15}$$

what allows to rewrite Eq. (14)

$$|\Psi\rangle = \hat{Q}|\Psi\rangle + \hat{Q}P\hat{P}\left[1 + \cos^2\alpha + \cos^4\alpha + \ldots\right]|\Psi\rangle$$

$$+ Q\hat{P}\left[1 + \cos^2\alpha + \cos^4\alpha + \ldots\right]|\Psi\rangle\,. \tag{16}$$

Using Eq. (9) one can finally write

$$|\Psi\rangle = \left[\hat{Q} + \hat{Q}P\hat{P}\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle + \left[Q\hat{P}\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle\,. \tag{17}$$

This equation can be simplified noting that

$$\hat{Q}P\hat{Q} = |\,8'8'\rangle_c \,_c\langle 8'8'\,|\,11\rangle_c \,_c\langle 11\,|\,8'8'\rangle_c \,_c\langle 8'8'\,| = \sin^2\alpha\,\hat{Q}\,, \tag{18}$$

then

$$\hat{Q} + \hat{Q}P\hat{P}\frac{1}{1-\cos^2\alpha} = \hat{Q} + \hat{Q}P(1-\hat{Q})\frac{1}{1-\cos^2\alpha} \tag{19}$$

$$= \hat{Q}P\frac{1}{1-\cos^2\alpha} + \hat{Q} - \hat{Q}P\hat{Q}\frac{1}{1-\cos^2\alpha} = \hat{Q}P\frac{1}{1-\cos^2\alpha}\,.$$

arriving to the compact notation

$$|\Psi\rangle = \frac{1}{1-\cos^2\alpha}\hat{Q}P|\Psi\rangle + \frac{1}{1-\cos^2\alpha}Q\hat{P}|\Psi\rangle = \mathcal{P}^{NH_2}_{|11\rangle_c}|\Psi\rangle + \mathcal{P}^{NH_2}_{|1'1'\rangle_c}|\Psi\rangle\,, \tag{20}$$

where $\mathcal{P}^{NH_2}_{|11\rangle_c}$ and $\mathcal{P}^{NH_2}_{|1'1'\rangle_c}$ are nonhermitian projection operators on the corresponding singlet-singlet subspaces.

Combining Eqs. (13) and (20) one can write any arbitrary state in the following form,

$$|\Psi\rangle = \frac{1}{2}\left\{\left[P\hat{Q}\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle + \left[\hat{P}Q\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle\right\}$$

$$+ \frac{1}{2}\left\{\left[\hat{Q}P\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle + \left[Q\hat{P}\frac{1}{1-\cos^2\alpha}\right]|\Psi\rangle\right\}\,, \tag{21}$$

or equivalently

$$|\Psi\rangle = \frac{1}{2}\left(P\hat{Q} + \hat{Q}P\right)\frac{1}{1-\cos^2\alpha}|\Psi\rangle$$

$$+ \frac{1}{2}\left(\hat{P}Q + Q\hat{P}\right)\frac{1}{1-\cos^2\alpha}|\Psi\rangle\,. \tag{22}$$

Thus, one arrives to two hermitian operators that are well–defined projectors on the two physical singlet–singlet color states

$$\mathcal{P}_{|11\rangle_c} = \left(P\hat{Q} + \hat{Q}P\right)\frac{1}{2(1-\cos^2\alpha)}$$

$$\mathcal{P}_{|1'1'\rangle_c} = \left(\hat{P}Q + Q\hat{P}\right)\frac{1}{2(1-\cos^2\alpha)} \tag{23}$$

and finally,

$$|\Psi\rangle = \mathcal{P}_{|11\rangle_c}|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c}|\Psi\rangle\,. \tag{24}$$

Thus, given an arbitrary state $|\Psi\rangle$ its projection on a particular subspace E is given by $|\Psi\rangle|_E = P_E|\Psi\rangle$. Thus, the probability of finding such an state on this subspace is

$$_E|\langle\Psi|\Psi\rangle|_E = \langle\Psi|P_E^\dagger P_E|\Psi\rangle = \langle\Psi|P_E^2|\Psi\rangle = \langle\Psi|P_E|\Psi\rangle\,. \tag{25}$$

Therefore, once the projection operators have been constructed, Eq. (23), the probabilities for finding singlet–singlet components are given by,

$$P^{|\Psi\rangle}([11]) = \left\langle \Psi \mid \mathcal{P}_{|11\rangle_c} \mid \Psi \right\rangle$$

$$P^{|\Psi\rangle}([1'1']) = \left\langle \Psi \mid \mathcal{P}_{|1'1'\rangle_c} \mid \Psi \right\rangle . \tag{26}$$

Using Eq. (23) one arrives to

$$P^{|\Psi\rangle}([11]) = \frac{1}{2(1 - \cos^2\alpha)} \left[\left\langle \Psi \mid P\hat{Q} \mid \Psi \right\rangle + \left\langle \Psi \mid \hat{Q}P \mid \Psi \right\rangle \right]$$

$$P^{|\Psi\rangle}([1'1']) = \frac{1}{2(1 - \cos^2\alpha)} \left[\left\langle \Psi \mid \hat{P}Q \mid \Psi \right\rangle + \left\langle \Psi \mid Q\hat{P} \mid \Psi \right\rangle \right] \tag{27}$$

where it can be easily checked that $P^{|\Psi\rangle}([11]) + P^{|\Psi\rangle}([1'1']) = 1$.

B.   Spin substructure

For a four–quark state one has three different total spins: 0, 1 and 2. The $S_T = 2$ case is trivial, because the basis is one-dimensional. Let us discuss the other two possibilities. For $S_T = 0$ the spin basis, in analogy with Eqs. (1), are given by:

$$[(s_1 s_2)_{S_{12}} (s_3 s_4)_{S_{34}}]_0 \equiv \{|00\rangle_s^{12}, |11\rangle_s^{12}\} \tag{28a}$$

$$[(s_1 s_3)_{S_{13}} (s_2 s_4)_{S_{24}}]_0 \equiv \{|00\rangle_s, |11\rangle_s\} \tag{28b}$$

$$[(s_1 s_4)_{S_{14}} (s_2 s_3)_{S_{23}}]_0 \equiv \{|0'0'\rangle_s, |1'1'\rangle_s\} \tag{28c}$$

and the corresponding spin projectors

$$P_s \equiv |00\rangle_{s\,s}\langle 00| \tag{29}$$

$$Q_s \equiv |11\rangle_{s\,s}\langle 11|$$

$$\hat{P}_s \equiv |0'0'\rangle_{s\,s}\langle 0'0'|$$

$$\hat{Q}_s \equiv |1'1'\rangle_{s\,s}\langle 1'1'| .$$

It is important to note that the projectors used in the color space determine the coupling in the spin space. Thus, introducing the corresponding spin projectors in Eq. (24) one arrives to

$$|\Psi\rangle = \mathcal{P}_{|11\rangle_c} (P_s + Q_s) |\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c} \left(\hat{P}_s + \hat{Q}_s \right) |\Psi\rangle = \tag{30}$$

$$= \mathcal{P}_{|11\rangle_c} P_s |\Psi\rangle + \mathcal{P}_{|11\rangle_c} Q_s |\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c} \hat{P}_s |\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c} \hat{Q}_s |\Psi\rangle \equiv$$

$$\equiv \mathcal{P}_{|11\rangle_c, |00\rangle_s} |\Psi\rangle + \mathcal{P}_{|11\rangle_c, |11\rangle_s} |\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c, |0'0'\rangle_s} |\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c, |1'1'\rangle_s} |\Psi\rangle ,$$

where $\mathcal{P}_{|11\rangle_c,|00\rangle_s}$ and $\mathcal{P}_{|1'1'\rangle_c,|0'0'\rangle_s}$ stand for the projectors on the physical state made up of two $S = 0$ $q\bar{q}$ mesons, and $\mathcal{P}_{|11\rangle_c,|11\rangle_s}$ and $\mathcal{P}_{|1'1'\rangle_c,|1'1'\rangle_s}$ for the projectors on the physical state made up of two $S = 1$ $q\bar{q}$ mesons.

Following our discussion in Subsection II A the probabilities are given by

$$P[|\,11\rangle_c\,|00\rangle_s] \;=\; \langle\Psi|\mathcal{P}_{|11\rangle_c,|00\rangle_s}|\Psi\rangle \tag{31}$$

$$P[|\,11\rangle_c\,|11\rangle_s] \;=\; \langle\Psi|\mathcal{P}_{|11\rangle_c,|11\rangle_s}|\Psi\rangle$$

$$P[|\,1'1'\rangle_c\,|0'0'\rangle_s] \;=\; \langle\Psi|\mathcal{P}_{|1'1'\rangle_c,|0'0'\rangle_s}|\Psi\rangle$$

$$P[|\,1'1'\rangle_c\,|1'1'\rangle_s] \;=\; \langle\Psi|\mathcal{P}_{|1'1'\rangle_c,|1'1'\rangle_s}|\Psi\rangle,$$

and therefore, the total probabilities of finding a physical state made up of two $S = 0$ $q\bar{q}$ mesons will be given by

$$P_{MM} = P[|\,11\rangle_c\,|00\rangle_s] + P[|\,1'1'\rangle_c\,|0'0'\rangle_s] \tag{32}$$

and correspondingly the total probability of a physical state made up of two $S = 1$ $q\bar{q}$ states

$$P_{M^*M^*} = P[|\,11\rangle_c\,|11\rangle_s] + P[|\,1'1'\rangle_c\,|1'1'\rangle_s]\,. \tag{33}$$

In the $S_T = 1$ case the spin basis are

$$[(s_1 s_2)_{S_{12}}(s_3 s_4)_{S_{34}}]_1 \;\equiv\; \{|01\rangle_s^{12}, |10\rangle_s^{12}, |11\rangle_s^{12}\} \tag{34a}$$

$$[(s_1 s_3)_{S_{13}}(s_2 s_4)_{S_{24}}]_1 \;\equiv\; \{|01\rangle_s, |10\rangle_s, |11\rangle_s\} \tag{34b}$$

$$[(s_1 s_4)_{S_{14}}(s_2 s_3)_{S_{23}}]_1 \;\equiv\; \{|0'1'\rangle_s, |1'0'\rangle_s, |1'1'\rangle_s\} \tag{34c}$$

and the corresponding projectors,

$$P_s \;\equiv\; |01\rangle_{s\;s}\langle 01| \tag{35}$$

$$Q_s \;\equiv\; |10\rangle_{s\;s}\langle 10|$$

$$W_s \;\equiv\; |11\rangle_{s\;s}\langle 11|$$

$$\hat{P}_s \;\equiv\; |0'1'\rangle_{s\;s}\langle 0'1'|$$

$$\hat{Q}_s \;\equiv\; |1'0'\rangle_{s\;s}\langle 1'0'|$$

$$\hat{W}_s \;\equiv\; |1'1'\rangle_{s\;s}\langle 1'1'|\,.$$

8

Following the same procedure as in the $S_T = 0$ case one arrives to

$$
\begin{aligned}
|\Psi\rangle &= \mathcal{P}_{|11\rangle_c}(P_s + Q_s + W_s)|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c}\left(\hat{P}_s + \hat{Q}_s + \hat{W}_s\right)|\Psi\rangle = \\
&= \mathcal{P}_{|11\rangle_c}P_s|\Psi\rangle + \mathcal{P}_{|11\rangle_c}Q_s|\Psi\rangle + \mathcal{P}_{|11\rangle_c}W_s|\Psi\rangle \\
&+ \mathcal{P}_{|1'1'\rangle_c}\hat{P}_s|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c}\hat{Q}_s|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c}\hat{W}_s|\Psi\rangle \equiv \\
&\equiv \mathcal{P}_{|11\rangle_c,|01\rangle_s}|\Psi\rangle + \mathcal{P}_{|11\rangle_c,|10\rangle_s}|\Psi\rangle + \mathcal{P}_{|11\rangle_c,|11\rangle_s}|\Psi\rangle \\
&+ \mathcal{P}_{|1'1'\rangle_c,|0'1'\rangle_s}|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c,|1'0'\rangle_s}|\Psi\rangle + \mathcal{P}_{|1'1'\rangle_c,|1'1'\rangle_s}|\Psi\rangle ,
\end{aligned}
\tag{36}
$$

where $\mathcal{P}_{|11\rangle_c,|01\rangle_s}$, $\mathcal{P}_{|1'1'\rangle_c,|0'1'\rangle_s}$, $\mathcal{P}_{|11\rangle_c,|10\rangle_s}$, and $\mathcal{P}_{|1'1'\rangle_c,|1'0'\rangle_s}$ stand for the projectors on the physical state made up of one $S = 0$ and one $S = 1$ $q\bar{q}$ mesons and $\mathcal{P}_{|11\rangle_c,|11\rangle_s}$ and $\mathcal{P}_{|11\rangle_c,|1'1'\rangle_s}$ for the projectors on the physical state made up of two $S = 1$ $q\bar{q}$ mesons.

Finally, the probabilities can be expressed as

$$
\begin{aligned}
P[|\,11\rangle_c |01\rangle_s] &= \langle\Psi|\mathcal{P}_{|11\rangle_c,|01\rangle_s}|\Psi\rangle \\
P[|\,11\rangle_c |10\rangle_s] &= \langle\Psi|\mathcal{P}_{|11\rangle_c,|10\rangle_s}|\Psi\rangle \\
P[|\,11\rangle_c |11\rangle_s] &= \langle\Psi|\mathcal{P}_{|11\rangle_c,|11\rangle_s}|\Psi\rangle \\
P[|\,1'1'\rangle_c |0'1'\rangle_s] &= \langle\Psi|\mathcal{P}_{|1'1'\rangle_c,|0'1'\rangle_s}|\Psi\rangle \\
P[|\,1'1'\rangle_c |1'0'\rangle_s] &= \langle\Psi|\mathcal{P}_{|1'1'\rangle_c,|1'0'\rangle_s}|\Psi\rangle \\
P[|\,1'1'\rangle_c |1'1'\rangle_s] &= \langle\Psi|\mathcal{P}_{|1'1'\rangle_c,|1'1'\rangle_s}|\Psi\rangle ,
\end{aligned}
\tag{37}
$$

and therefore, the total probability of a physical state made up of one $S = 0$ and one $S = 1$ $q\bar{q}$ meson will be given by

$$
P_{MM^*} = P[|\,11\rangle_c |01\rangle_s] + P[|\,11\rangle_c |10\rangle_s] + P[|\,1'1'\rangle_c |0'1'\rangle_s] + P[|\,1'1'\rangle_c |1'0'\rangle_s]
\tag{38}
$$

and the total probability of a physical state made up of two $S = 1$ $q\bar{q}$ mesons by

$$
P_{M^*M^*} = P[|\,11\rangle_c |11\rangle_s] + P[|\,1'1'\rangle_c |1'1'\rangle_s] .
\tag{39}
$$

C.   Flavor substructure

The previous discussion about the color and spin substructure is general and valid for any four–quark state. For the flavor part one find several different cases depending on the number of light quarks. Although the present formalism can be applied to any four–quark

9

state, it becomes much simpler whether distinguishable quarks are present. This would be, for example, the case of the $nQ\bar{n}\bar{Q}$ system, where the Pauli principle does not apply. In this system the basis (1b) and (1c) are distinguishable due to the flavor part, they correspond to $[(i_1i_4)_{I_{14}}(i_2i_3)_{I_{23}}]_I \equiv [(q\bar{c})_{1/2}(c\bar{q})_{1/2}]_I$ and $[(i_1i_3)_{I_{13}}(i_2i_4)_{I_{24}}]_I \equiv [(q\bar{q})_I(c\bar{c})_0]_I$, and therefore they are orthogonal. This makes that the probabilities can be evaluated in the usual way for orthogonal basis as has been done in Ref. [9].

Non-orthogonal basis are necessary for the following cases: $QQ\bar{n}\bar{Q}'$, $QQ'\bar{n}\bar{n}$, $Qn\bar{n}\bar{n}$ and $nn\bar{n}\bar{n}$ (Q may be equal to Q') or their corresponding antiparticles. The isospin basis are:

- $QQ\bar{n}\bar{Q}'$

$$[(i_1i_2)_{I_{12}}(i_3i_4)_{I_{34}}]_{\frac{1}{2}} \equiv |0\tfrac{1}{2}\rangle_f^{12} \tag{40a}$$

$$[(i_1i_3)_{I_{13}}(i_2i_4)_{I_{24}}]_{\frac{1}{2}} \equiv |\tfrac{1}{2}0\rangle_f \tag{40b}$$

$$[(i_1i_4)_{I_{14}}(i_2i_3)_{I_{23}}]_{\frac{1}{2}} \equiv |0'\tfrac{1}{2}'\rangle_f . \tag{40c}$$

- $QQ'\bar{n}\bar{n}$

$$[(i_1i_2)_{I_{12}}(i_3i_4)_{I_{34}}]_I \equiv |0I\rangle_f^{12} \tag{41a}$$

$$[(i_1i_3)_{I_{13}}(i_2i_4)_{I_{24}}]_I \equiv |\tfrac{1}{2}\tfrac{1}{2}\rangle_f \tag{41b}$$

$$[(i_1i_4)_{I_{14}}(i_2i_3)_{I_{23}}]_I \equiv |\tfrac{1}{2}'\tfrac{1}{2}'\rangle_f . \tag{41c}$$

- $Qn\bar{n}\bar{n}$

  - $I = 1/2$

$$[(i_1i_2)_{I_{12}}(i_3i_4)_{I_{34}}]_{\frac{1}{2}} \equiv \{|\tfrac{1}{2}0\rangle_f^{12}, |\tfrac{1}{2}1\rangle_f^{12}\} \tag{42a}$$

$$[(i_1i_3)_{I_{13}}(i_2i_4)_{I_{24}}]_{\frac{1}{2}} \equiv \{|\tfrac{1}{2}0\rangle_f, |\tfrac{1}{2}1\rangle_f\} \tag{42b}$$

$$[(i_1i_4)_{I_{14}}(i_2i_3)_{I_{23}}]_{\frac{1}{2}} \equiv \{|\tfrac{1}{2}'0'\rangle_f, |\tfrac{1}{2}'1'\rangle_f\} . \tag{42c}$$

  - $I = 3/2$

$$[(i_1i_2)_{I_{12}}(i_3i_4)_{I_{34}}]_{\frac{3}{2}} \equiv |\tfrac{1}{2}1\rangle_f^{12} \tag{43a}$$

$$[(i_1i_3)_{I_{13}}(i_2i_4)_{I_{24}}]_{\frac{3}{2}} \equiv |\tfrac{1}{2}1\rangle_f \tag{43b}$$

$$[(i_1i_4)_{I_{14}}(i_2i_3)_{I_{23}}]_{\frac{3}{2}} \equiv |\tfrac{1}{2}'1'\rangle_f . \tag{43c}$$

- $nn\bar{n}\bar{n}$

  - $I = 0$

$$[(i_1 i_2)_{I_{12}}(i_3 i_4)_{I_{34}}]_0 \equiv \{|00\rangle_f^{12}, |11\rangle_f^{12}\} \tag{44a}$$

$$[(i_1 i_3)_{I_{13}}(i_2 i_4)_{I_{24}}]_0 \equiv \{|00\rangle_f, |11\rangle_f\} \tag{44b}$$

$$[(i_1 i_4)_{I_{14}}(i_2 i_3)_{I_{23}}]_0 \equiv \{|0'0'\rangle_f, |1'1'\rangle_f\}. \tag{44c}$$

  - $I = 1$

$$[(i_1 i_2)_{I_{12}}(i_3 i_4)_{I_{34}}]_1 \equiv \{|01\rangle_f^{12}, |10\rangle_f^{12}, |11\rangle_f^{12}\} \tag{45a}$$

$$[(i_1 i_3)_{I_{13}}(i_2 i_4)_{I_{24}}]_1 \equiv \{|01\rangle_f, |10\rangle_f, |11\rangle_f\} \tag{45b}$$

$$[(i_1 i_4)_{I_{14}}(i_2 i_3)_{I_{23}}]_1 \equiv \{|0'1'\rangle_f, |1'0'\rangle_f, |1'1'\rangle_f\}. \tag{45c}$$

  - $I = 2$

$$[(i_1 i_2)_{I_{12}}(i_3 i_4)_{I_{34}}]_2 \equiv |11\rangle_f^{12} \tag{46a}$$

$$[(i_1 i_3)_{I_{13}}(i_2 i_4)_{I_{24}}]_2 \equiv |11\rangle_f \tag{46b}$$

$$[(i_1 i_4)_{I_{14}}(i_2 i_3)_{I_{23}}]_2 \equiv |1'1'\rangle_f. \tag{46c}$$

For those cases where the basis is one-dimensional the recoupling among the three different basis introduced in Eq. (1) is straightforward. For those cases where the basis in not one-dimensional one should follow the procedure described in Sect. II B.

D. Radial substructure

In order to derive the probability of the physical channels one has finally to analyze the symmetry of the radial wave function. Such analysis will depend on the particular state chosen. Without loss of generality we will exemplify the procedure with the particular case of the $QQ\bar{n}\bar{n}$ system. Any other four–quark system discussed in Sect. II C could be analyzed in the same manner. Let us start with the $S_T = 0$ state, whose most general wave function

reads

$$|\Psi\rangle \ = \ |R_1\rangle|\bar{3}3\rangle_c^{12}|00\rangle_s^{12}|0I\rangle_f^{12} + |R_2\rangle|\bar{3}3\rangle_c^{12}|11\rangle_s^{12}|0I\rangle_f^{12} \tag{47}$$
$$+ \ |R_3\rangle|6\bar{6}\rangle_c^{12}|00\rangle_s^{12}|0I\rangle_f^{12} + |R_4\rangle|6\bar{6}\rangle_c^{12}|11\rangle_s^{12}|0I\rangle_f^{12} \,,$$

where $|R_1\rangle, |R_2\rangle, |R_3\rangle$, and $|R_4\rangle$ are radial wave functions that due to symmetry properties satisfy $\langle R_1|R_1\rangle + \langle R_2|R_2\rangle + \langle R_3|R_3\rangle + \langle R_4|R_4\rangle = 1$, $\langle R_1|R_2\rangle = \langle R_1|R_3\rangle = \langle R_2|R_4\rangle = \langle R_3|R_4\rangle = 0$, $\langle R_1|R_4\rangle \neq 0$, and $\langle R_2|R_3\rangle \neq 0$. Applying Eqs. (32) and (33) one obtains

$$P_{MM} \ = \ P[|11\rangle_c|00\rangle_s] + P[|1'1'\rangle_c|0'0'\rangle_s] \tag{48}$$
$$= \ \frac{1}{4}\left(1 + 2\langle R_2|R_2\rangle + 2\langle R_4|R_4\rangle\right) + \frac{3\sqrt{6}}{8}\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right)$$
$$P_{M^*M^*} \ = \ P[|11\rangle_c|11\rangle_s] + P[|1'1'\rangle_c|1'1'\rangle_s]$$
$$= \ \frac{1}{4}\left(1 + 2\langle R_1|R_1\rangle + 2\langle R_3|R_3\rangle\right) - \frac{3\sqrt{6}}{8}\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right) \,.$$

Finally, the $QQ\bar{n}\bar{n}$ $S_T = 1$ most general wave function reads

$$|\Psi\rangle \ = \ |R_1\rangle|\bar{3}3\rangle_c^{12}|01\rangle_s^{12}|0I\rangle_f^{12} + |R_2\rangle|\bar{3}3\rangle_c^{12}|10\rangle_s^{12}|0I\rangle_f^{12} + |R_3\rangle|\bar{3}3\rangle_c^{12}|11\rangle_s^{12}|0I\rangle_f^{12} \tag{49}$$
$$+ \ |R_4\rangle|6\bar{6}\rangle_c^{12}|01\rangle_s^{12}|0I\rangle_f^{12} + |R_5\rangle|6\bar{6}\rangle_c^{12}|10\rangle_s^{12}|0I\rangle_f^{12} + |R_6\rangle|6\bar{6}\rangle_c^{12}|11\rangle_s^{12}|0I\rangle_f^{12} \,,$$

where $|R_i\rangle$ are radial wave functions that due to symmetry properties satisfy $\sum_{i=1}^{6}\langle R_i|R_i\rangle = 1$ and all the cross products are zero except for $\langle R_1|R_5\rangle$ and $\langle R_2|R_4\rangle$. Applying Eqs. (38) and (39) one gets

$$P_{MM^*} \ = \ \frac{1}{2}\left(1 + \langle R_3|R_3\rangle + \langle R_6|R_6\rangle - \frac{3\sqrt{2}}{2}\left(\langle R_1|R_5\rangle + \langle R_2|R_4\rangle\right)\right) \tag{50}$$
$$P_{M^*M^*} \ = \ \frac{1}{2}\left(1 - \langle R_3|R_3\rangle - \langle R_6|R_6\rangle + \frac{3\sqrt{2}}{2}\left(\langle R_1|R_5\rangle + \langle R_2|R_4\rangle\right)\right) \,.$$

III. SOME ILLUSTRATIVE EXAMPLES

In the previous section we have derived the analytical expansion of an arbitrary four–quark state wave function in terms of a non-orthogonal basis containing only physical channels. The calculation of the probabilities has been exemplified with the $QQ\bar{n}\bar{n}$ system.

We now apply this formalism to discuss the four-quark nature: unbound, molecular or compact states, of some illustrative examples. The same discussion could be done with any

FIG. 1: H–type Jacobi vectors. 1,2 stand for heavy quarks and 3,4 for light antiquarks.
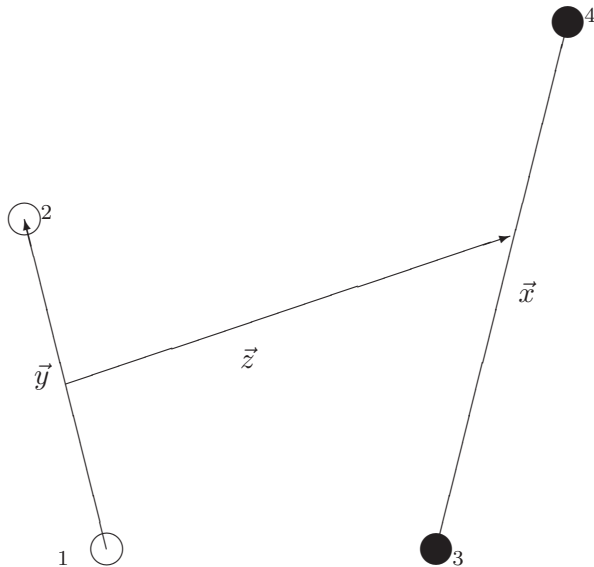
other four–quark state just changing the coupling in isospin space, see Subsect. II C. The results we are going to discuss have been obtained solving the four-body problem by means of a variational method using as trial wave function the most general linear combination of gaussians [10]. The accuracy of the variational approach has been tested by comparing with results obtained by means of the hyperspherical harmonic expansion [11]. Both approaches are in good agreement The interaction between the quarks is taken from the model of Ref. [12]. The same interacting potential used to calculate the four-quark energy is used to calculate the mass of the thresholds, i.e., the meson masses.

The stability of a four–quark state can be analyzed in terms of Δ_E, the energy difference between its mass and that of the lowest two-meson threshold,

$$\Delta_E = E_{4q} - E(M_1, M_2),$$ (51)

where E_{4q} stands for the four–quark energy and $E(M_1, M_2)$ for the energy of the two-meson threshold. Thus, $\Delta_E < 0$ indicates all fall-apart decays are forbidden, and therefore one has a proper bound state. $\Delta_E \geq 0$ will indicate that the four–quark solution corresponds to an unbound threshold (two free mesons). Thus, an energy above the threshold would simply mean that the system is unbound within our variational approximation, suggesting that the minimum of the Hamiltonian is at the two-meson threshold. Another helpful tool analyzing the structure of a four–quark state is the value of the root mean square radii: $\langle x^2 \rangle^{1/2}$, $\langle y^2 \rangle^{1/2}$, and $\langle z^2 \rangle^{1/2}$. They correspond to the Jacobi coordinates given in Fig.1. Compact four–quark

TABLE I: Probabilities for $cc\bar{n}\bar{n}$ states with quantum numbers $(S_T, I) = (1,1)$ and $(1,0)$. The notation used stands for $P[|\{\text{Color state}\}\rangle|\{\text{Spin state}\}\rangle]$ where $\{\text{Color state}\}$ corresponds to the basis vectors given in Eqs. (1) and $\{\text{Spin state}\}$ to the ones given in Eqs. (34). Flavor component $|0I\rangle_f$, $|\frac{1}{2}\frac{1}{2}\rangle_f$, and $|\frac{1'}{2}\frac{1'}{2}\rangle_f$ is understood. We list in Appendix C a summary of the expressions used.

	(1,1)	(1,0)		(1,1)	(1,0)		(1,1)	(1,0)
$P[\lvert\bar{3}3\rangle_c^{12}\lvert01\rangle_s]$	0.000	0.875	$P[\lvert 11\rangle_c\lvert01\rangle_s]$	0.277	0.094	$P[\lvert 1'1'\rangle_c\lvert0'1'\rangle_s]$	0.277	0.094
$P[\lvert\bar{3}3\rangle_c^{12}\lvert10\rangle_s]$	0.000	0.006	$P[\lvert 11\rangle_c\lvert10\rangle_s]$	0.277	0.094	$P[\lvert 1'1'\rangle_c\lvert1'0'\rangle_s]$	0.277	0.094
$P[\lvert\bar{3}3\rangle_c^{12}\lvert11\rangle_s]$	0.333	0.000	$P[\lvert 11\rangle_c\lvert11\rangle_s]$	0.002	0.186	$P[\lvert 1'1'\rangle_c\lvert1'1'\rangle_s]$	0.002	0.186
$P[\lvert6\bar{6}\rangle_c^{12}\lvert01\rangle_s]$	0.000	0.090	$P[\lvert 88\rangle_c\lvert01\rangle_s]$	0.222	0.156	$P[\lvert 8'8'\rangle_c\lvert0'1'\rangle_s]$	0.222	0.156
$P[\lvert6\bar{6}\rangle_c^{12}\lvert10\rangle_s]$	0.000	0.029	$P[\lvert 88\rangle_c\lvert10\rangle_s]$	0.222	0.156	$P[\lvert 8'8'\rangle_c\lvert1'0'\rangle_s]$	0.222	0.156
$P[\lvert6\bar{6}\rangle_c^{12}\lvert11\rangle_s]$	0.667	0.000	$P[\lvert 88\rangle_c\lvert11\rangle_s]$	0.000	0.314	$P[\lvert 8'8'\rangle_c\lvert1'1'\rangle_s]$	0.000	0.314
$P[\lvert\bar{3}3\rangle_c^{12}]$	0.333	0.881	$P[\lvert 11\rangle_c]$	0.556	0.374	$P[\lvert 1'1'\rangle_c]$	0.556	0.374
$P[\lvert6\bar{6}\rangle_c^{12}]$	0.667	0.119	$P[\lvert 88\rangle_c]$	0.444	0.626	$P[\lvert 8'8'\rangle_c]$	0.444	0.626

states can be distinguished from two free mesons by means of their root mean square radius

$$\text{RMS}_{4q(2q)} = \left(\frac{\sum_{i=1}^{4(2)} m_i \langle (r_i - R)^2 \rangle}{\sum_{i=1}^{4(2)} m_i} \right)^{1/2}, \tag{52}$$

and in particular, their corresponding ratio,

$$\Delta_R = \frac{\text{RMS}_{4q}}{\text{RMS}_{M_1} + \text{RMS}_{M_2}}. \tag{53}$$

where $\text{RMS}_{M_1} + \text{RMS}_{M_2}$ stands for the sum of the radii of the mesons corresponding to the lowest threshold.

We show in Table I the probabilities in color and spin space obtained for two $cc\bar{n}\bar{n}$ states. The first one, with quantum numbers $(S_T, I) = (1,1)$, is unbound, while the second one, $(S_T, I) = (1,0)$, is bound. We give the probabilities in the three different rearrangements in color space defined in Eqs. (1). Let us note that in the three color rearrangements of Eqs. (1), the hidden–color vectors ($|\bar{3}3\rangle$, $|6\bar{6}\rangle$, $|88\rangle$, and $|8'8'\rangle$) contain probability of physical channels as we have discussed in Sect. II. It is possible to prove from simple group theory arguments that for a system composed of two identical quarks (QQ) and two identical

14

TABLE II: Four–quark state properties for selected quantum numbers. All states have positive parity and total orbital angular momentum $L = 0$. Energies are given in MeV and distances in fm. The notation $M_1 M_2 |_\ell$ stands for mesons M_1 and M_2 with a relative orbital angular momentum ℓ. $P[|\bar{3}3\rangle_c^{12}(|6\bar{6}\rangle_c^{12})]$ stands for the probability of the $3\bar{3}(\bar{6}6)$ components given in Eq. (1a) and $P[|11\rangle_c(|88\rangle_c)]$ for the 11(88) components given in Eq. (1b). P_{MM}, P_{MM^*}, and $P_{M^*M^*}$ have been calculated as explained in text.

(S_T, I)	(0,1)	(1,1)	(1,0)	(1,0)	(0,0)
Flavor	$cc\bar{n}\bar{n}$	$cc\bar{n}\bar{n}$	$cc\bar{n}\bar{n}$	$bb\bar{n}\bar{n}$	$bb\bar{n}\bar{n}$
Energy	3877	3952	3861	10395	10948
Threshold	$DD\,|_S$	$DD^*\,|_S$	$DD^*\,|_S$	$BB^*\,|_S$	$B_1B\,|_P$
Δ_E	+5	+15	-76	-217	-153
$P[|\bar{3}3\rangle_c^{12}]$	0.333	0.333	0.881	0.974	0.981
$P[|6\bar{6}\rangle_c^{12}]$	0.667	0.667	0.119	0.026	0.019
$P[|11\rangle_c]$	0.556	0.556	0.374	0.342	0.340
$P[|88\rangle_c]$	0.444	0.444	0.626	0.658	0.660
P_{MM}	1.000	–	–	–	0.254
P_{MM^*}	–	1.000	0.505	0.531	–
$P_{M^*M^*}$	0.000	0.000	0.495	0.469	0.746
$\langle x^2 \rangle^{1/2}$	60.988	13.804	0.787	0.684	0.740
$\langle y^2 \rangle^{1/2}$	60.988	13.687	0.590	0.336	0.542
$\langle z^2 \rangle^{1/2}$	0.433	0.617	0.515	0.503	0.763
RMS_{4q}	30.492	6.856	0.363	0.217	0.330
Δ_R	69.300	11.640	0.799	0.700	0.885

antiquarks ($\bar{n}\bar{n}$), the octet–octet component probability of the wave function either in the (1b) or (1c) arrangements is restricted to the interval $[1/3, 2/3]$, see Appendix B. Do these numerical and analytical results prove the unavoidable presence of important hidden–color components in all $QQ\bar{n}\bar{n}$ states regardless of their binding energy? We shall see that the answer is no.

To respond this question we summarize in Table II the results obtained for several different four–quark states, among them those used in Table I, making use of the formal development of Sect. II. As shown in Table II, independently of their binding energy, all of them present a sizable octet-octet component when the wave function is expressed in the (1b) coupling. Let us first of all concentrate on the two unbound states, $\Delta_E > 0$, one with $S_T = 0$ and one with $S_T = 1$, given in Table II. The octet-octet component of basis (1b) can be expanded in terms of the vectors of basis (1c) as explained in the previous section. Thus, once expressions (42) and (44) are considered one finds that the probabilities are concentrated into a single physical channel, P_{MM} or P_{MM^*}. In other words, the octet-octet component of the basis (1b) or (1c) is a consequence of having identical quarks and antiquarks. Thus, four-quark unbound states are represented by two isolated mesons. This conclusion is strengthened when studying the root mean square radii, leading to a picture where the two quarks and the two antiquarks are far away, $\langle x^2 \rangle^{1/2} \gg 1$ fm and $\langle y^2 \rangle^{1/2} \gg 1$ fm, while the quark-antiquark pairs are located at a typical distance for a meson, $\langle z^2 \rangle^{1/2} \leq 1$ fm.

Let us now turn to the bound states shown in Table II, $\Delta_E < 0$, one in the charm sector and two in the bottom one. Contrary to the results obtained for unbound states, when the octet-octet component of basis (1b) is expanded in terms of the vectors of basis (1c), equations (42) and (44) drive to a picture where the probabilities in all allowed physical channels are relevant. It is clear that the bound state must be generated by an interaction that it is not present in the asymptotic channel, sequestering probability from a single singlet–singlet color vector due to the interaction between color octets. Such systems are clear examples of compact four–quark states, in other words, they cannot be expressed in terms of a single physical channel. Moreover, as can be seen in Table II, their typical sizes point to compact objects 20% smaller than a standard two–meson system.

We have studied the dependence of the probability of a physical channel on the binding energy. For this purpose we have considered the simplest system from the numerical point of view, the $(S_T, I) = (0, 1)$ $cc\bar{n}\bar{n}$ state. Unfortunately, this state is unbound for any reasonable set of parameters. Therefore, we bind it by multiplying the interaction between the light quarks by a fudge factor. Such a modification does not affect the two–meson threshold while it decreases the mass of the four–quark state. The results are illustrated in Fig. 2 (P_{MM}) and Fig. 3 (Δ_R, $\langle x^2 \rangle^{1/2}$, $\langle y^2 \rangle^{1/2}$, and $\langle z^2 \rangle^{1/2}$). In Fig. 2 it is shown how in the $\Delta_E \to 0$ limit, the four–quark wave function is almost a pure single physical channel. Close to this limit

FIG. 2: P_{MM} as a function of Δ_E.
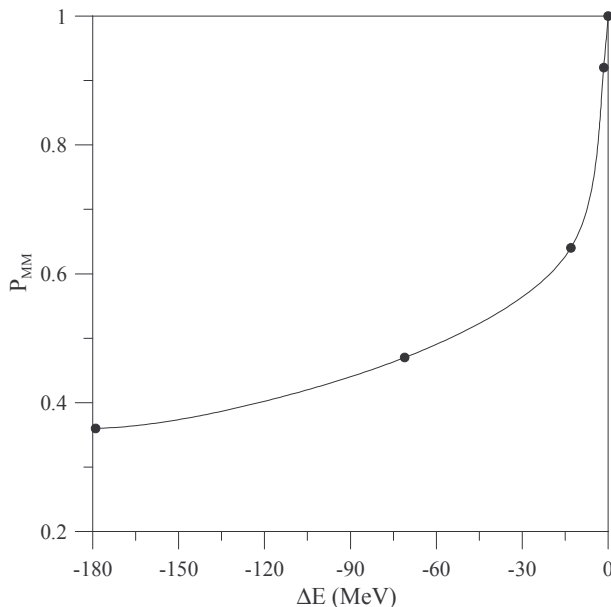
one would find what could be defined as molecular states. In Fig. 3 we see how the size of the four–quark state increases when $\Delta_E \to 0$. It can be observed that when the probability concentrates into a single physical channel ($P_{MM} \to 1$) the size of the system gets larger than the sum of two isolated mesons (Fig. 3 left panel). In Fig. 3 (right panel) we identify the subsystems responsible for increasing the size of the four–quark state. Quark-quark ($\langle x^2 \rangle^{1/2}$) and antiquark-antiquark ($\langle y^2 \rangle^{1/2}$) distances grow rapidly while the quark–antiquark distance ($\langle z^2 \rangle^{1/2}$) remains almost constant. This reinforces our previous result, pointing to the appearance of two meson like structures whenever the binding energy goes to zero. This illustrative example emphasizes the importance of performing a simultaneous analysis both of energy and wave function in order to detect bound states in the vicinity of a two-meson threshold.

To illustrate in more detail the differences observed in the calculated four–quark wave functions we depict in Fig. 4 the position distributions defined as

$$R(r_\alpha, r_\beta) = r_\alpha r_\beta \sum_i \int_V |R_i(\vec{r}_\alpha, \vec{r}_\beta, \vec{r}_\gamma)|^2 d\vec{r}_\gamma \, d\Omega_{r_\alpha} \, d\Omega_{r_\beta} \, , \qquad (54)$$

where $R_i(\vec{r}_\alpha, \vec{r}_\beta, \vec{r}_\gamma)$ are the radial wave functions introduced in Eqs. (47) and (49). We present results for an unbound, a molecular and a bound state, showing the position distribution for the different planes $(r_\alpha, r_\beta) = (z, x), (z, y)$, and (x, y). Clear differences among them can be observed. The position distribution for the unbound case spreads in the x and

FIG. 3: Δ_R (a) and $\langle x^2 \rangle^{1/2}$(solid line), $\langle y^2 \rangle^{1/2}$ (dashed line), and $\langle z^2 \rangle^{1/2}$(dashed-dotted line) (b) as a function of Δ_E.
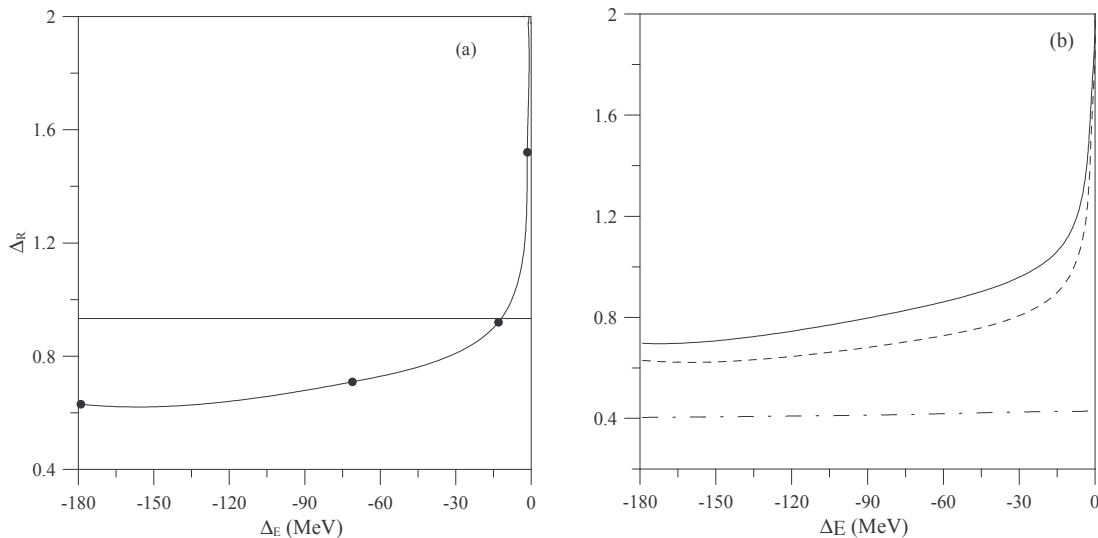
y coordinates up to 60 fm, while the bound and molecular systems are restricted to the region below 3 fm (molecular) and 1 fm (bound). In the (x, y) plane the unbound state is so widely spread that the values for the position distribution are three orders of magnitude lower than in the (z, y) and (z, x) cases, and therefore they will not appear in the picture unless artificially magnified. In the case of the molecular state a long range tail propagating in the $x = y$ region can be observed contrary to the constrained values obtained for bound systems.

The conclusions derived are independent of the quark-quark interaction used. They mainly rely on using the same hamiltonian to describe tensors of different order, two and four-quark components in the present case. Dealing with a complete basis, any four-quark bound deeply bound state has to be compact. Only slightly bound systems could be considered as molecular. Unbound states correspond to a two-meson system. A similar situation would be found in the two baryon system, the deuteron could be considered as a molecular like state with a small percentage of its wave function on the $\Delta\Delta$ channel, while the $H-$dibaryon would be a compact six–quark state. Working with central forces, the only way of getting a bound system is to have a strong interaction between the constituents that are far apart in the asymptotic limit, quarks or antiquarks in the present case. In this case the short-range interaction will capture part of the probability of a two-meson threshold to form a bound

FIG. 4: Position distribution corresponding to unbound $(S_T = 1, I = 1, cc\bar{n}\bar{n})$, molecular $(S_T = 0,$
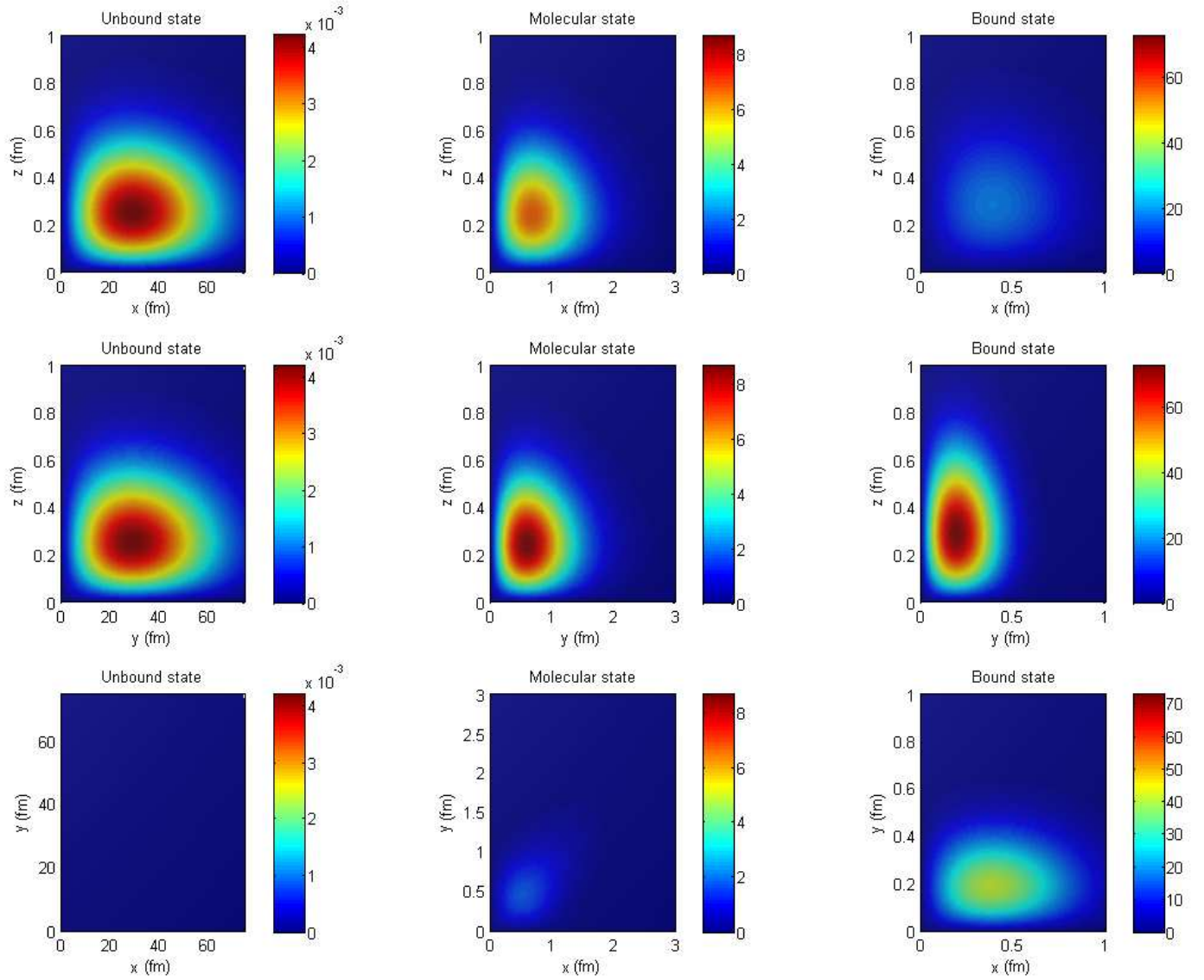
state. This can be reinterpreted as an infinite sum over physical states. This is why the analysis performed in the present manuscript is so important before concluding the existence of compact four–quark states beyond simple molecular structures.

If the prescription of using the same hamiltonian to describe all tensors in the Fock space is relaxed, new scenarios may appear. Among them, the inclusion of many–body forces is particularly relevant. In Ref. [13] the stability of $QQ\bar{n}\bar{n}$ and $Q\bar{Q}n\bar{n}$ systems was analyzed in a simple string model considering only a multiquark confining interaction given by the minimum of a flip-flop or a butterfly potential in an attempt to discern whether confining

interactions not factorizable as two–body potentials would influence the stability of four–quark states. The ground state of systems made of two quarks and two antiquarks of equal masses was found to be below the dissociation threshold. While for the cryptoexotic $Q\bar{Q}n\bar{n}$ the binding decreases when increasing the mass ratio m_Q/m_n, for the flavor exotic $QQ\bar{n}\bar{n}$ the effect of mass symmetry breaking is opposite. Others scenarios may emerge if different many–body forces, like many–body color interactions [14] or 't Hooft instanton–based three-body interactions [15], are considered.

IV. CONCLUSIONS

In this work we have developed the necessary formalism to express the wave function of a general four–quark state in terms of physical channels, i.e., those constructed by using color singlet states. Once this is done the four–quark wave function is expressed in terms of nonorthogonal vectors and hence the traditional way to compute probabilities needs to be generalized. We have obtained expressions to evaluate such probabilities for all possible nontrivial four–quark states containing two heavy antiquarks and two light quarks. We have applied these expressions to illustrative cases, where the difference among unbound, compact and molecular four–quark states has been made evident. The importance of performing a complete analysis of the system, energy and wave function, in the vicinity of a two-meson threshold has been emphasized.

APPENDIX A: PROJECTORS PROPERTIES

The following properties are proved for one particular set of projectors, $\mathcal{P}^{NH_1}_{|11\rangle_c}$ and $\mathcal{P}^{NH_1}_{|1'1'\rangle_c}$. The same procedure can be followed in all the remaining cases. By construction, see Eq. (24), they span the complete space, $\mathcal{P}^{NH_1}_{|11\rangle_c} + \mathcal{P}^{NH_1}_{|1'1'\rangle_c} = \mathbb{1}$. We demonstrate that we have

constructed idempotent operators,

$$
\left(\mathcal{P}_{|11\rangle_c}^{NH_1}\right)^2 = \left(\frac{1}{1-\cos^2\alpha}\right)^2 P\hat{Q}P\hat{Q} \tag{A1}
$$

$$
= \left(\frac{1}{1-\cos^2\alpha}\right)^2 |\,1'1'\rangle_c {}_c\langle 1'1' \,|\, 88\rangle_c {}_c\langle 88 \,|\, 1'1'\rangle_c {}_c\langle 1'1' \,|\, 88\rangle_c {}_c\langle 88 \,|
$$

$$
= \left(\frac{1}{1-\cos^2\alpha}\right)^2 |\,1'1'\rangle_c {}_c\langle 1'1' \,|\, 88\rangle_c {}_c\langle 88 \,|\,|\,{}_c\langle 1'1' \,|\, 88\rangle_c\,|^2
$$

$$
= \left(\frac{1}{1-\cos^2\alpha}\right)^2 |\,1'1'\rangle_c {}_c\langle 1'1' \,|\, 88\rangle_c {}_c\langle 88 \,|\,\sin^2\alpha
$$

$$
= \frac{1}{1-\cos^2\alpha} P\hat{Q} = \mathcal{P}_{|11\rangle_c}^{NH_1}.
$$

APPENDIX B: MINIMUM AND MAXIMUM VALUE FOR THE OCTET–OCTET COMPONENT PROBABILITY.

Without loss of generality we consider the $S_T = 0$ case. The Pauli Principle requires that the radial wave functions $|R_i\rangle$ in Eq. (41) have well-defined permutation properties under the exchange of quarks and that of antiquarks, i.e., symmetric (S) or antisymmetric (A). Hence, $|R_1\rangle$ must be antisymmetric under the exchange of the identical quarks and also under the exchange of antiquarks what we will denote by $|R_1(AA)\rangle$. Similarly for the other components: $|R_2(SS)\rangle$, $|R_3(SS)\rangle$, and $|R_4(AA)\rangle$. The transformations from (1a) to (1b) and from (28a) to (28b) are

$$
|\bar{3}3\rangle_c^{12} = \frac{1}{\sqrt{3}}\,|\,11\rangle_c - \sqrt{\frac{2}{3}}\,|\,88\rangle_c \tag{B1}
$$

$$
|6\bar{6}\rangle_c^{12} = \sqrt{\frac{2}{3}}\,|\,11\rangle_c + \frac{1}{\sqrt{3}}\,|\,88\rangle_c
$$

and

$$
|00\rangle_s^{12} = \frac{1}{2}|00\rangle_s + \frac{\sqrt{3}}{2}|11\rangle_s \tag{B2}
$$

$$
|11\rangle_s^{12} = \frac{\sqrt{3}}{2}|00\rangle_s - \frac{1}{2}|11\rangle_s
$$

21

and therefore Eq. (41) can be written as

$$
\begin{aligned}
|\Psi\rangle \;=\; & |\,11\rangle_c\,|00\rangle_s\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_f\,\frac{1}{2\sqrt{3}}\left\{|R_1\rangle + \sqrt{3}|R_2\rangle + \sqrt{2}|R_3\rangle + \sqrt{6}|R_4\rangle\right\} + \\
& |\,11\rangle_c\,|11\rangle_s\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_f\,\frac{1}{2\sqrt{3}}\left\{\sqrt{3}|R_1\rangle - |R_2\rangle + \sqrt{6}|R_3\rangle - \sqrt{2}|R_4\rangle\right\} + \\
& |\,88\rangle_c\,|00\rangle_s\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_f\,\frac{1}{2\sqrt{3}}\left\{-\sqrt{2}|R_1\rangle - \sqrt{6}|R_2\rangle + |R_3\rangle + \sqrt{3}|R_4\rangle\right\} + \\
& |\,88\rangle_c\,|11\rangle_s\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_f\,\frac{1}{2\sqrt{3}}\left\{-\sqrt{6}|R_1\rangle + \sqrt{2}|R_2\rangle + \sqrt{3}|R_3\rangle - |R_4\rangle\right\}\,.
\end{aligned}
\tag{B3}
$$

Considering the symmetry properties of the radial part of the wave function, the probabilities are calculated as

$$
\begin{aligned}
P[|\,11\rangle_c\,|00\rangle_s] \;=\; & \frac{1}{12}\left\{\langle R_1|R_1\rangle + 3\langle R_2|R_2\rangle + 2\langle R_3|R_3\rangle + 6\langle R_4|R_4\rangle \right. \\
& \left. + \; 2\sqrt{6}\,\mathrm{Re}\,\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right)\right\}
\end{aligned}
\tag{B4}
$$

$$
\begin{aligned}
P[|\,11\rangle_c\,|11\rangle_s] \;=\; & \frac{1}{12}\left\{3\langle R_1|R_1\rangle + \langle R_2|R_2\rangle + 6\langle R_3|R_3\rangle + 2\langle R_4|R_4\rangle \right. \\
& \left. - \; 2\sqrt{6}\,\mathrm{Re}\,\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
P[|\,88\rangle_c\,|00\rangle_s] \;=\; & \frac{1}{12}\left\{2\langle R_1|R_1\rangle + 6\langle R_2|R_2\rangle + \langle R_3|R_3\rangle + 3\langle R_4|R_4\rangle \right. \\
& \left. - \; 2\sqrt{6}\,\mathrm{Re}\,\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
P[|\,88\rangle_c\,|11\rangle_s] \;=\; & \frac{1}{12}\left\{6\langle R_1|R_1\rangle + 2\langle R_2|R_2\rangle + 3\langle R_3|R_3\rangle + \langle R_4|R_4\rangle \right. \\
& \left. + \; 2\sqrt{6}\,\mathrm{Re}\,\left(\langle R_1|R_4\rangle + \langle R_2|R_3\rangle\right)\right\}\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
P[|\,11\rangle_c] \;=\; & P[|\,11\rangle_c\,|00\rangle_s] + P[|\,11\rangle_c\,|11\rangle_s] \\
\;=\; & \frac{1}{12}\left\{4\langle R_1|R_1\rangle + 4\langle R_2|R_2\rangle + 8\langle R_3|R_3\rangle + 8\langle R_4|R_4\rangle\right\} \\
P[|\,88\rangle_c] \;=\; & P[|\,88\rangle_c\,|00\rangle_s] + P[|\,88\rangle_c\,|11\rangle_s] \\
\;=\; & \frac{1}{12}\left\{8\langle R_1|R_1\rangle + 8\langle R_2|R_2\rangle + 4\langle R_3|R_3\rangle + 4\langle R_4|R_4\rangle\right\}\,.
\end{aligned}
\tag{B5}
$$

By construction $P[|\bar{3}3\rangle_c^{12}] = \langle R_1|R_1\rangle + \langle R_2|R_2\rangle$ and $P[|6\bar{6}\rangle_c^{12}] = \langle R_3|R_3\rangle + \langle R_4|R_4\rangle$ with $P[|\bar{3}3\rangle_c^{12}] + P[|6\bar{6}\rangle_c^{12}] = 1$. Therefore Eqs. (B5) can be expressed as

$$
\begin{aligned}
P[|\,11\rangle_c] \;=\; & \frac{1}{3}\left\{1 + P[|6\bar{6}\rangle_c^{12}]\right\} \\
P[|\,88\rangle_c] \;=\; & \frac{1}{3}\left\{2 - P[|6\bar{6}\rangle_c^{12}]\right\}\,.
\end{aligned}
\tag{B6}
$$

and since $P[|6\bar{6}\rangle_c^{12}] \in [0,1]$ is normalized, a minimum $(1/3)$ and a maximum $(2/3)$ value for $P[|\,11\rangle_c]$ and $P[|\,88\rangle_c]$ do exist.

APPENDIX C: PROBABILITIES FOR DIFFERENT CHOICES OF BASIS.

The $S_T = 0$ case is given in Eqs. (B5), for the sake of completeness note that
$P[|\,11\rangle_c\,|00\rangle_s] = P[|\,1'1'\rangle_c\,|0'0'\rangle_s]$, $P[|\,11\rangle_c\,|11\rangle_s] = P[|\,1'1'\rangle_c\,|1'1'\rangle_s]$, $P[|\,88\rangle_c\,|00\rangle_s] = P[|\,8'8'\rangle_c\,|0'0'\rangle_s]$, $P[|\,88\rangle_c\,|11\rangle_s] = P[|\,1'1'\rangle_c\,|1'1'\rangle_s]$. For $S_T = 1$ one has

$$P[|\,11\rangle_c\,|01\rangle_s] = P[|\,1'1'\rangle_c\,|0'1'\rangle_s] = \tag{C1}$$
$$\frac{1}{6}\left(1 - \frac{1}{2}\langle R_1|R_1\rangle - \frac{1}{2}\langle R_2|R_2\rangle + \langle R_6|R_6\rangle - \sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$
$$P[|\,11\rangle_c\,|10\rangle_s] = P[|\,1'1'\rangle_c\,|1'0'\rangle_s] =$$
$$\frac{1}{6}\left(1 - \frac{1}{2}\langle R_1|R_1\rangle - \frac{1}{2}\langle R_2|R_2\rangle + \langle R_6|R_6\rangle - \sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$
$$P[|\,11\rangle_c\,|11\rangle_s] = P[|\,1'1'\rangle_c\,|1'1'\rangle_s] =$$
$$\frac{1}{6}\left(\langle R_1|R_1\rangle + \langle R_2|R_2\rangle + 2\langle R_4|R_4\rangle + 2\langle R_5|R_5\rangle + 2\sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$
$$P[|\,88\rangle_c\,|01\rangle_s] = P[|\,8'8'\rangle_c\,|0'1'\rangle_s] =$$
$$\frac{1}{6}\left(1 - \frac{1}{2}\langle R_4|R_4\rangle - \frac{1}{2}\langle R_5|R_5\rangle + \langle R_3|R_3\rangle + \sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$
$$P[|\,88\rangle_c\,|10\rangle_s] = P[|\,8'8'\rangle_c\,|1'0'\rangle_s] =$$
$$\frac{1}{6}\left(1 - \frac{1}{2}\langle R_4|R_4\rangle - \frac{1}{2}\langle R_5|R_5\rangle + \langle R_3|R_3\rangle + \sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$
$$P[|\,88\rangle_c\,|11\rangle_s] = P[|\,8'8'\rangle_c\,|1'1'\rangle_s] =$$
$$\frac{1}{6}\left(2\langle R_1|R_1\rangle + 2\langle R_2|R_2\rangle + \langle R_4|R_4\rangle + \langle R_5|R_5\rangle - 2\sqrt{2}[\langle R_1|R_5\rangle + \langle R_2|R_4\rangle]\right)$$

---

[1] J. D. Bjorken, The November Revolution: A Theorist Reminisces, in: A Collection of Summary Talks in High Energy Physics (ed. J.D. Bjorken), p. 229 (World Scientific, New York, 2003).

[2] J. L. Rosner, J. Phys. Conf. Ser. **69**, 012002 (2007).

[3] R. L. Jaffe Phys. Rep. **409**, 1 (2005); E. S. Swanson, Phys. Rep. **429**, 243 (2006); E. Klempt and A. Zaitsev, Phys. Rep. **454**, 1 (2007).

[4] J. P. Ader, J. M. Richard, and P. Taxil, Phys. Rev. D **25**, 2370 (1982); J. L. Ballot and J. M. Richard, Phys. Lett. B **123**, 449 (1983).

[5] J. Weinstein and N. Isgur, Phys. Rev. D **41**, 2236 (1990).

[6] R. S. Manning and N. De Leon, J. Math. Chem. **5**, 323 (1990).

[7] F. Stancu, *Group Theory in Subnuclear Physics*, Oxford Stud. Nucl. Phys. **19** 1 (1996).

[8] M. Harvey, Nucl. Phys. **352**, 301 (1981).

[9] J. Vijande, E. Weissman, N. Barnea, and A. Valcarce, Phys. Rev. D **76**, 094022 (2007).

[10] Y. Suzuki and K. Varga, Lect. Notes Phys. **M54**, 1 (1998).

[11] J. Vijande, A. Valcarce, and N. Barnea, Phys. Rev. D, **79** 074010 (2009).

[12] J. Vijande, F. Fernández, and A. Valcarce, J. Phys. G **31**, 481 (2005).

[13] J. Vijande, A. Valcarce, and J. M. Richard, Phys. Rev. D **76**, 114013 (2007); C. Ay, J. M. Richard, J. H. Rubinstein, Phys. Lett. B**674**, 227 (2009).

[14] V. Dmitrasinovic, Phys. Lett. B **499**, 135 (2001); Phys. Rev. D **67**, 114007 (2003).

[15] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).