Ghost and tachyon free gauge theories of gravity

A systematic approach

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

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Yun-Cherng Lin
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Abstract

In this thesis, we present a systematic method for determining the conditions on the parameters in the action of a parity-preserving gauge theory of gravity about a Minkowski background for it to be free of ghost or tachyon particles. The approach naturally accommodates critical cases in which the parameter values satisfying some critical conditions causing changes of particle contents and may lead to additional gauge invariances. In Chapter 1, we give an overall introduction to the field. We then introduce the systematic method in Chapter 2. The method is implemented as a computer program, and the details of its implementation are presented in Chapter 3. In Chapter 4, we apply the method to investigate the particle content of parity-conserving Poincaré gauge theory (PGT$^+$). We find 450 critical cases that are free of ghosts and tachyons and compare the no-ghost-and-tachyon conditions of some critical cases with literature. We also examine the power-counting renormalisability of some of the critical cases of PGT$^+$ and clarify the treatment of non-propagating modes in determining whether a theory is power-counting renormalisable (PCR) in Chapter 5. We identify 58 of the ghost and tachyon free PGT$^+$ critical cases that are also PCR, of which seven have 2 massless degrees of freedom (d.o.f.) in propagating modes and a massive 0^- or 2^- mode, 12 have only 2 massless d.o.f., and 39 have only massive mode(s). In chapter 6, we analyse parity-preserving Weyl gauge theory (WGT$^+$) in a similar way. Within a subset of WGT$^+$, we find 168 critical cases that are free of ghosts and tachyons. We further identify 40 of these cases that are also PCR. Of these theories, 11 have only massless tordion propagating particles, 23 have only a massive tordion propagating mode, and 6 have both. We also repeat our analysis for PGT$^+$ and WGT$^+$ with vanishing torsion or curvature, respectively. In Chapter 7, we summarise the contents in this thesis and suggest some future work.

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Notations and conventions

Units, signs, and factors

- We use natural units: $c = \hbar = 1$.
- Minkowski metric: $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$.
- Fourier transformation:

$$f(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \tilde{f}(p) e^{ip\cdot x}, \;\; \tilde{f}(p) = \int \mathrm{d}^4 x \, f(x) e^{-ip\cdot x} \tag{1}$$

  Sometimes we will drop the tilde of the fields in the momentum space if the meaning is obvious.
- Propagators: For a free Lagrangian in momentum space $\mathcal{L}(p) = \frac{1}{2}\varphi(-p)\mathcal{O}(p)\varphi(p)$, the propagator is $\mathcal{O}^{-1}(p)$.

Indices

- Greek letters $\mu, \nu \cdots$: coordinate frame
- Uppercase Latin letters $A, B \cdots$: local Lorentz frame
- Lowercase Latin letters $a, b, c \cdots$: labels for fields
- Lowercase Latin letters $i, j, k \cdots$: labels for orthogonal components in the same spin-parity sector
- Greek index with an acute accent ($\acute{\alpha}$,$\acute{\beta}$): representing the collection of the local Lorentz indices of a field

Gauge fields, field strengths, actions

- $A^{AB}{}_{\mu}$: Lorentz rotational gauge field

- $h_A{}^\mu$: translational gauge field, with inverse $b^A{}_\mu$
- $h = \det(h_A{}^\mu)$, $b = \det(b^A{}_\mu)$
- B_μ: dilational gauge field
- $\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu} A^{AB}{}_{\nu]} + A^A{}_{E[\mu} A^{EB}{}_{\nu]})$: Lorentz field strength (for PGT and WGT)
- $\mathcal{T}^A{}_{\mu\nu} = 2(\partial_{[\mu} b^A{}_{\nu]} + A^A{}_{E[\mu} b^E{}_{\nu]})$: translational field strength (for PGT)
- $\mathcal{T}^{*C}{}_{AB} = \mathcal{T}^C{}_{AB} + 2B_{[A}\delta^C_{B]}$: translational field strength (for WGT)
- $\mathcal{H}_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}$: dilational field strength (for WGT)
- $S = \int \mathrm{d}^4 x h^{-1} (\mathcal{L}_\mathrm{G} + \mathcal{L}_\mathrm{M}) = \int \mathrm{d}^4 x (L_\mathrm{G} + L_\mathrm{M})$: action for PGT/WGT, where $\mathcal{L}_\mathrm{G}$ and $\mathcal{L}_\mathrm{M}$ correspond to the free gravitational part and matter part, respectively.

Acronyms

- GR: general relativity
- SR: special relativity
- PGT: Poincaré gauge theory
- WGT: Weyl gauge theory
- eWGT: extended Weyl gauge theory
- PGT$^+$,WGT$^+$: general PGTs or WGTs with parity-preserving Lagrangians
- d.o.f.: degrees of freedom
- SPO: spin projection operator
- PC: power-counting
- PCR: power-counting renormalisable
- EC theory: Einstein–Cartan theory
- GCT: general coordinate transformation
- EP: equivalence principle

Chapter 1

Introduction

It is known that Einstein's theory of general relativity (GR) is compatible with the experimental and observational data at intermediate length scales [1]. For large scales, while the ΛCDM model seems successful, it assumes the existence of dark matter, for which there is no confirmed candidate. It also requires a non-zero positive cosmological constant. The cosmological constant we observe today is too small compared to the vacuum energy predicted by quantum field theory (the cosmological constant problem), and the energy density of the cosmological constant is surprisingly close to the matter density now (the coincidence problem).

On the other hand, the theory also has problems at small length scales from the theoretical perspective. GR is not perturbatively renormalisable. At first glance, it is not power-counting (PC) renormalisable, but a power-counting non-renormalisable theory might turn out to be renormalisable. While GR is renormalisable at the one-loop level without coupling to other particles, it is not renormalisable at the one-loop level if it couples to matter [2]. Another problem is that spacetime singularities are expected in GR [3].

An approach to solve the renormalisation problem is to modify the action of gravity. It seems that GR is an inevitable gravitational theory because of the Lovelock theorem [4, 5]. The theorem states that the only divergence-free rank-2 tensors which are constructed from the metric tensor $g_{\mu\nu}$ and its derivatives up to second differential order in four spacetime dimensions and preserving diffeomorphism invariance are the Einstein tensor and the metric tensor. However, if we relax the assumptions, we can obtain several ways to modify gravity. There are several methods that we can use to relax the assumptions of the theorem [1, 6]:

- Additional fields
- Allowing higher order derivatives
- Violations of diffeomorphism invariance
- Higher dimensions
- Violations of Weak Equivalence Principle (WEP)
- Allowing non-locality

In 1977, Stelle [7] showed that the theories with the following $\mathcal{R}+\mathcal{R}^2$ type action with appropriate parameters are renormalisable:

$$S=\int \mathrm{d}^4x\sqrt{-g}\left(\alpha\mathcal{R}+\beta\mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma}+\gamma\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu}\right). \tag{1.1}$$

However, this kind of theory is a higher-derivative theory, and it usually suffers from Ostrogradsky's instability [8, 9]. This is generally related to containing modes with negative energy (ghosts). The ghost modes make the theory unstable for small oscillations at the classical level, and violate unitarity at the quantum level [10]. The Hamiltonian is not bounded from below, so the energy of the system can be negative. This does not cause any problem for the free theory because the overall sign of the Lagrangian does not change the equation of motion in classical theories. If there are interactions with systems with positive energy, then it is possible that the system is bounded from below. However, there are infinite excited states around the ground state, and so the system can "evaporate" into these ground states and thus be unstable. This may be the underlying reason why Nature prefers theories with second-order equations of motion. At the quantum level, the ghost particles have negative norms, so the probability of finding a particle in a state can be negative or greater than one, and unitarity is therefore violated. Higher-derivative theory may also contain modes with imaginary (complex) mass (tachyons). The tachyon modes propagate faster than light and thus violate causality. They also make the theory unstable at the classical level and destroy unitarity at the quantum level. The classical field can grow exponentially

[11], and the norm of a quantum state can evolve exponentially because of the imaginary (complex) energy. There are some special types of Lagrangian with the form $\mathcal{R}+\mathcal{R}^2$ that do not contain ghosts or tachyons, such as $\mathcal{L}(\mathcal{R})$ gravity, but they do not improve the UV behaviour [12].

An approach to improve the high energy behaviour is to include infinite sets of higher derivative terms [12–16]. Biswas, Mazumdar and Siegel [13] argued that it is the only way to obtain a renormalisable and ghost-free theory if the Lagrangian is constructed from $\mathcal{R}^{\mu\nu\rho\sigma}$ and $g_{\mu\nu}$ [13]. These kinds of theories are called Infinite Derivative theories of Gravity (IDG) and can be free of ghost and asymptotic-free simultaneously. However, including infinite higher derivatives makes the theories non-local. If the field equation contains infinite derivatives, then we need the initial value of the field and its derivatives up to infinite order to solve the equation. This is equivalent to requiring the initial value everywhere to get the later value of the field at a point, not only requiring the initial values over a finite domain, and so it may violate causality.

1.1 Poincaré gauge theory

Another approach is inspired by Yang–Mills theory [17]. The Standard Model of particle physics describes the electroweak and strong interactions as gauge theories of $SU(3)_C \times SU(2)_L \times U(1)_Y$, but it does not include gravity. Hence, it is a natural assumption that gravity is a gauge theory as for the other interactions. The equivalence principle (EP) states that at each point x in an arbitrary gravitational field one can choose a locally inertial frame $S(x)$ in which the laws of physics take the same form as in special relativity (SR). Although not often pointed out, the EP is closely related to local Poincaré symmetry, since the frame $S(x)$ can be obtained from an arbitrarily fixed frame $S_0 \equiv S(x_0)$ by the combination of a translation, to bring the origin of S_0 to coincide with that of $S(x)$, and a Lorentz rotation, to bring the axes of S_0 to coincide with those of $S(x)$ [18]. These transformations are the elements of the Poincaré group, and its parameters depend on the point x at which $S(x)$ is defined. Moreover, these Poincaré transformations are symmetry transformations because the laws of physics possess the same form in all inertial frames. Thus, according to the EP, an arbitrary gravitational field is characterized by the group of local Poincaré transformations, which act on the set of all locally inertial frames. Therefore, it is reasonable to consider gravity as a gauge theory of the Poincaré group.

Following the gauging of the Lorentz group by Utiyama [19], Kibble was the first to gauge the Poincaré group [20], and the theory was also considered by Sciama [21]. This class of theories is called Poincaré gauge theory (PGT). In Kibble's model, a continuum matter field(s) φ with energy-momentum and spin-angular-momentum tensors is distributed continuously in background Minkowski spacetime. The action of the matter field $S_{\mathrm{M}} = \int \mathrm{d}^4 x \mathcal{L}_{\mathrm{M}}(\varphi, \partial_\mu \varphi)$ is invariant under global Poincaré transformation and can be made invariant under local Poincaré transformation by introducing the gauge fields ${h_A}^\mu$ and ${A^{AB}}_\mu$, which correspond to translation and Lorentz transformation respectively. Note that ${A^{AB}}_\mu$ is antisymmetric in (A,B). Greek indices denote the coordinate frame, and Latin capital indices correspond to the local Lorentz frame. We need these gauge fields to compensate for the unwanted effect of local Poincaré transformations, ensuring the theory has local symmetry. This is fulfilled by using the minimal coupling procedure, which replaces the partial derivatives $\partial_\mu \varphi$ in the original (special-relativistic) Lagrangian with their corresponding covariant derivatives $\mathcal{D}_A \varphi$, which are constructed with the gauge fields ${h_A}^\mu$ and ${A^{AB}}_\mu$. The local Poincaré invariant action is then $S_{\mathrm{M}} = \int \mathrm{d}^4 x h^{-1} \mathcal{L}_{\mathrm{M}}(\varphi, \mathcal{D}_A \varphi)$, where $h \equiv \det({h_A}^\mu)$ makes the integrand a scalar density and thus the integral is invariant.

After Kibble's initial work, several authors proposed different approaches to gauging the Poincaré group. Hehl et al. [22] performed active Poincaré transformations on the fields, rather than the passive ones on the coordinates as in Kibble's work. The active interpretation of the transformation considers the "form" variation $\delta_0 \varphi(x) \equiv \varphi'(x) - \varphi(x)$, while the passive one considers the "total" variation $\delta \varphi(x) \equiv \varphi'(x') - \varphi(x)$. The active approach makes the symmetry closer to the spirit of conventional gauge theories, where the symmetries are internal symmetries, whereas Kibble's approach is more directly related to the geometric interpretation [18, 23]. It turns out that both the active and passive interpretations give the same final structure of theories. To preserve the geometrical meaning of translation when localising the Poincaré symmetry, Hehl et al. further replaced the partial derivative in the translational generator by a covariant derivative. The variation with the replaced translational generator only differs from the original one by a Lorentz transformation, and thus invariance under $\delta_0\varphi(x)$ is equivalent to its counterpart with the replaced translational generator. One can also consider finite transformations instead of infinitesimal ones as in Kibble's work. Mukunda [24] applied passive finite local Poincaré transformations in his work. Using the powerful language of geometric algebra, Lasenby, Doran and Gull [25] constructed the gauge theory by considering active finite local Poincaré transformations and studied its astrophysical and cosmological applications. No matter how the Poincaré group is gauged, the resulting theories are equivalent. We will adopt the approach of passive infinitesimal transformations in this thesis.

Besides the matter Lagrangian, the free Lagrangian for the gravitational gauge fields is also required to make these fields dynamic. Similar to Yang-Mills theory, we can commute the covariant derivatives and get the field strengths. The commutator $[\mathcal{D}_A, \mathcal{D}_B]\varphi$ leads to the identification of the Lorentz field strength $\mathcal{R}^{AB}{}_{CD}$ and the translational field strength $\mathcal{T}^{A}{}_{BC}$. The local Poincaré invariant free gravitational action $S_{\mathrm{G}} = \int \mathrm{d}^4 x h^{-1}\mathcal{L}_{\mathrm{G}}(\mathcal{R}^{AB}{}_{CD}, \mathcal{T}^{A}{}_{BC})$ can then be constructed with the field strengths $\mathcal{R}^{AB}{}_{CD}$ and $\mathcal{T}^{A}{}_{BC}$.

Although it is natural to consider the gauge theories of gravity as a field theory in Minkowski spacetime, which will be adopted in this thesis, one can also interpret it as a geometric theory. It turns out that PGT has the geometric structure of the Riemann-Cartan spacetime (U_4). The difference between the Riemann-Cartan spacetime (U_4) and the usual Riemann spacetime (V_4) is that the torsion is zero in V_4 but in general non-zero in U_4. The U_4 manifold has a metric $g_{\mu\nu}$, and we can define local Lorentz frame with a set of orthonormal basis vectors $\mathbf{e}_A$ at each point. It can be shown that the gauge field $h_A{}^{\mu}$ is interpreted as the vierbein (or tetrad) field in the geometric interpretation, i.e. $\mathbf{e}_A = h_A{}^{\mu}\mathbf{e}_{\mu}$, where $\mathbf{e}_{\mu}$ is the basis vector for the coordinate frame. The tetrad field $h_A{}^{\mu}$ and its inverse $b^A{}_{\mu}$ can be used to convert tensors between the coordinate frame and the local Lorentz frame. Similarly, the gauge field $A^{AB}{}_{\mu}$ acts as the spin connection in the geometric construction. The rule of parallel transport for a vector u^A in the local Lorentz frame is $\delta u^A = -A^A{}_{B\mu}u^B \mathrm{d}x^{\mu}$. The field strengths $\mathcal{R}^{AB}{}_{CD}$ and $\mathcal{T}^{A}{}_{BC}$ are interpreted geometrically as the curvature and torsion, respectively.

To avoid higher derivatives, we should construct the free gravitational Lagrangian $\mathcal{L}_\mathrm{G}$ with at most quadratic field strengths, and therefore $\mathcal{L}_\mathrm{G}$ is at most quadratic in the first derivatives of the gauge fields. Because $A^{AB}{}_{\mu}$ is an independent field, quadratic field strengths do not contain higher derivatives, unlike the conventional $\mathcal{R}^2$ theories. The restriction also makes the theory satisfy the so-called pseudolinearity (or quasi-linearity) hypothesis, which suggests that the second derivatives must occur linearly in the field equations [26–28]. Unlike Yang-Mills theory, in PGT we can have an action linear in field derivatives. In Kibble's work, the gravitational action is $S_\mathrm{G} = \int \mathrm{d}^4 x h^{-1}\mathcal{R}$, where $\mathcal{R} \equiv \mathcal{R}^{AB}{}_{AB}$. This is the Einstein–Cartan–Sciama–Kibble theory (or Einstein–Cartan theory, EC theory). It is a direct generalisation of the standard Einstein–Hilbert action, but the torsion sourced by the spin-angular-momentum of matter fields is included. The propagator of the free gravitational theory has a similar structure as that of ordinary GR, and it is equivalent to GR if one does not couple it to fermionic matter [29]. Hence, it does not improve the ultraviolet behaviour when coupled to bosonic matter. However, recent research indicates that a universe with Einstein–Cartan–Sciama–Kibble theory avoids the unphysical big-bang singularity [30–32]. Terms with quadratic field strengths can also be added into $\mathcal{L}_\mathrm{G}$, so the most general $\mathcal{L}_\mathrm{G}$ is then $\mathcal{L}_\mathrm{G} \sim \mathcal{R} + \mathcal{R}^2 + \mathcal{T}^2$, where we omit the coefficients and indices, and it is natural to require parity invariance.[1] The linear term $\mathcal{R}$ typically ensures that PGT has the correct macroscopic limit as GR [33], and the quadratic terms mimic the conventional Yang-Mills theory.

In the most general free PGT, while there are 40 dynamical variables (16 in $h_A{}^{\mu}$, or equivalently $b^A{}_{\mu}$, and 24 in $A^{AB}{}_{\mu}$), 10 of the field equations represent initial data because the Lagrangian does not contain $\partial_0 b^A{}_0$ and $\partial_0 A^{AB}{}_0$ due to the antisymmetric structure in the field strengths. One can further fix the gauge of Poincaré symmetry so that ten more dynamical variables are not independent, and the number of independent variables becomes 20. One may give constraints on the Lagrangian parameters and make the properties of the theory

[1] We denote PGTs with parity-preserving Lagrangians as PGT$^+$.

different. For example, one may consider $\mathcal{L}_{\mathrm{G}} \sim \mathcal{R} + \mathcal{T}^2$, which is one of the most simple generalisations of the EC theory. Because there is no kinetic term for the A-field, it becomes a "non-propagating" field, which vanishes outside any sources. We can make the A-field propagating by adding $\mathcal{R}^2$ terms into the Lagrangian, and one may consider $\mathcal{L}_{\mathrm{G}} \sim \mathcal{R} + \mathcal{R}^2$ instead [34–36], or $\mathcal{L}_{\mathrm{G}} \sim \mathcal{R}^2 + \mathcal{T}^2$ [37]. Obukhov et al. [33] summarised the conservation laws and field equations of the $\mathcal{R} + \mathcal{R}^2 + \mathcal{T}^2$ theories and gave their correspondence with GR, exact Friedman-type solutions and weak gravitational radiation. The coupling to matter fields is studied in [38], with the gravitational Lagrangian of type $f(\mathcal{R})$. Theories with parity-odd terms in $\mathcal{L}_{\mathrm{G}}$ have also been investigated by [32, 33, 39–42], but we will only consider parity invariant $\mathcal{L}_{\mathrm{G}}$ in this thesis. It is also possible to broaden the choice of the free gravitational Lagrangian by introducing a scalar [43].

As mentioned above, we require the gravitational theory to be unitary and renormalisable. Both properties can be investigated by studying the particle spectrum. To obtain the particle spectrum, one may split the fields into irreducible tensors, find the linearised (weak field perturbation) field equations, perform variable changes so that the field equations become Klein–Gordon-like, and then read off the masses [44–46]. The particle spectrum can also be obtained by the Hamiltonian method [42, 47–50]. However, it is most convenient to obtain the particle spectrum with spin projection operators (SPOs) [51–53]. One has to linearise the gauge fields around the trivial vacuum solution and obtain the linearised Lagrangian $\mathcal{L}_{\mathrm{G}}^{(2)}$ in momentum space containing only the terms with bilinear linearised gauge fields. Then the linearised gauge fields of $h_A{}^\mu$ and $A^{AB}{}_\mu$ can be decomposed into irreducible representations of the Poincaré group, and each of the irreducible representations corresponds to a specific spin and parity J^P. If we view the 40 linearised gauge fields as a column vector, then $\mathcal{L}_{\mathrm{G}}^{(2)}$ can be viewed as a 40×40 matrix sandwiched by a row and a column vector. When projected into the SPO basis, the matrix of $\mathcal{L}_{\mathrm{G}}^{(2)}$ becomes block-diagonal, with each block corresponding to a J^P. Note that if $\mathcal{L}_{\mathrm{G}}$ violates parity invariance, then each block corresponds only to a specific spin J. By fixing the gauge and inverting the blocks of $\mathcal{L}_{\mathrm{G}}$, one obtains the propagator for each J^P sector. The particle spectrum can then be obtained by studying the poles and residues of the gauge-invariant saturated propagators, which are the propagators sandwiched by the source currents. Note that when the momentum square is zero ($k^2 = 0$), some of the SPOs have zero denominators and thus the basis is not well-defined. One needs more careful investigation for these massless particles.

To make PGT unitary, we need to require the particle spectrum of the gravitational fields to be free from ghosts and tachyons. We can obtain no-tachyon and no-ghost conditions on the parameters of $\mathcal{L}_{\mathrm{G}}$ by requiring the poles of the saturated propagator to be positive and the residues at the poles to be positive definite, respectively. The renormalisability of PGT can be examined by requiring the theory to be power-counting renormalisable (PCR) as a first step. To be PCR, the propagators of the h-field (graviton) and the A-field (tordion, or roton) [26] should behave as $\sim 1/k^4$ and $\sim 1/k^2$ in the high energy limit, respectively [29]. If we require the propagator goes as $\sim 1/k^4$, however, one may encounter some problems. If there are poles with the structure $\left(k^2 - m_1^2\right)^{-1}\left(k^2 - m_2^2\right)^{-1}$, then by the partial fractions

$$\frac{F(k)}{\left(k^2 - m_1^2\right)\left(k^2 - m_2^2\right)} = \frac{F(k)}{m_1^2 - m_2^2}\left(\frac{1}{k^2 - m_1^2} - \frac{1}{k^2 - m_2^2}\right), \tag{1.2}$$

one should find the residues at the two poles have opposite signs when F has the same sign at the two poles. If the ghost is in a lower spin-sector and massless, as in GR, it may be compensated by some degrees of freedom (d.o.f.) from massless modes in a higher spin-sector since the SPOs are not orthonormal at the massless poles. If the ghost is in the spin-2 sector, there is no mode from a higher spin-sector to compensate it. When there are $\sim 1/\left(k^2 - m_1^2\right)^4$ poles ($m_1^2 = m_2^2$), where m^2 can be zero, in the saturated propagator, there exist dipole ghosts, which again violate unitarity [11, 54–56]. It can be illustrated by setting $m_2^2 = m_1^2 + \varepsilon$ in (1.2). When $\varepsilon \to 0$, there must be a pole with positive residue and the other with a negative one, and so one of them must be a ghost. Therefore, it is widely considered that for any PCR theories, there must be ghosts in the spin-2 sector. However, the statement is not robust. As we can see in Eq. (8-10) in [57], the massless poles can be eliminated by not only compensation with higher spin-sectors but also source constraints and index symmetries[2].

[2] While there is a sign error in Eq. (8) in [57], it does not affect the discussion here.

The SPO formulation for rank-2 tensor fields was introduced by Fronsdal [51] and developed by Barnes [52] and Rivers [53]. Van Nieuwenhuizen [58] applied the SPO formulation to study the no-ghost-and-tachyon conditions for the most general quadratic rank-2 tensor field theory with at most two derivatives. Neville [36] constructed a set of SPO for the h and A-fields of PGT$^+$ and considered PGT$^+$ with $\mathcal{R}+\mathcal{R}^2$ actions. He found a ghost-and-tachyon-free action with a 0^- massive tordion in addition to the ordinary graviton. Neville also considered actions at most quadratic in the curvature tensor, the contorsion tensor $K_{\mu\lambda\nu} = -\frac{1}{2}\left(\mathcal{T}_{\mu\lambda\nu} - \mathcal{T}_{\nu\mu\lambda} + \mathcal{T}_{\lambda\nu\mu}\right)$, and covariant derivatives of $K_{\mu\lambda\nu}$ in [59]. Sezgin and van Nieuwenhuizen [29] examined the most general PGT$^+$ action with no more than two derivatives, i.e. $\mathcal{R}+\mathcal{R}^2+\mathcal{T}^2$, using a systematic method with SPOs, while they used a different basis from Neville's one. The "most general" PGT$^+$ contains a 2^+ ordinary massless graviton and massive tordions with spin-parity $J^P = 0^-, 0^+, 1^-, 1^+, 2^-, 2^+$, with $2+(1+1+3+3+5+5) = 20$ degrees of freedom in total. The result is consistent with the analysis from the aspect of field equations. However, the "most general" PGT$^+$ must contain massive ghosts or tachyons. Note that if the Lagrangian parameters meet some "critical conditions", the theory may change qualitatively. For example, massive modes may become massless or non-propagating, and the theory may gain additional gauge invariances. Therefore, ghosts and tachyons may appear or disappear in these critical theories. They found five critical theories that are free of ghosts and tachyons. Sezgin [57] further found 12 six-parameter ghost and tachyon free critical cases of PGT$^+$ by removing some massive poles from the most general theory. The paper also found a ghost and tachyon free critical case with additional gauge invariance. The theory contains a propagating massless 1^- tordion in addition to the ordinary graviton. Kuhfuss and Nitsch [27] also examined the same theory with SPO but with a different method to tackle the massless poles, but they found no propagating massless tordion. The appearance of the tordion was caused by a sign error in Sezgin's equation, which was confirmed by Blagojević and Vasilić [48], where they studied critical cases with additional gauge invariance of the most general PGT$^+$. Kuhfuss and Nitsch also examined teleparallel (setting the curvature to zero) PGT$^+$ and illustrated a ghost and tachyon free critical case, while they found the most general theory contains dipole ghosts. Battiti and Tollek [60] found three critical cases with additional propagating massless particles which are not ghosts or tachyons, using the field equation method. Karananas [41] studied the "most general" $\mathcal{R}+\mathcal{R}^2+\mathcal{T}^2$ action for PGT with parity-violating terms with SPOs but found it must contain ghosts or tachyon. Blagojević and Cvetković [42] also studied the same theory with the Hamiltonian approach and made the same conclusion, while some details are different. They also gave an outline of how to deal with the critical cases with the Hamiltonian approach. While some of the theories above contain no ghosts or tachyons, none of them is PCR simultaneously. However, it is still possible that there exist some ghost-and-tachyon-free and PCR critical cases which have not been investigated yet, and thus a complete study of the critical cases is essential.

1.2 Weyl gauge theory

Because of the difficulties encountered in PGT, one may consider gauging extra symmetries beyond the Poincaré one. Since a scale-invariant theory contains no dimensionful parameter and no absolute energy scale, it may have better ultraviolet behaviour. Thus, rather than gauging the Poincaré group, one may instead gauge the Weyl group so that the action is also invariant under local dilations. Inspired by the early work of Weyl [61] in the late 1910s, which introduced a new gauge field B_μ in an attempt to unify gravity and electromagnetism[3], the authors of [62], [63] and [64] gauged the Weyl group $W(1,3)$. The Weyl group is a subgroup of the conformal group $C(1,3)$ and extends the Poincaré group to be scale-invariant. The theory is called Weyl gauge theory (WGT). Similar to PGT, the theory can be formulated by considering active or passive, finite or infinitesimal Weyl transformation, but they are all equivalent.

As in PGT, the action of the matter field is also obtained by the minimal coupling rule $S_\mathrm{M}=\int \mathrm{d}^4x h^{-1}\mathcal{L}_\mathrm{M}(\varphi,\mathcal{D}^*_A\varphi)$, where φ is a continuum matter field(s) with energy-momentum and spin-angular-momentum tensors and dilational current distributed continuously in background Minkowski spacetime, and $\mathcal{D}^*_A\varphi$ is the covariant derivative in WGT. The gauge fields

[3]The gauge field B_μ interacts with particles and antiparticles in the same way, which is not the case for electromagnetism.

$h_A{}^\mu$ and $A^{AB}{}_\mu$ are also required to build the covariant derivatives, while $h_A{}^\mu$ has a different transformation rule in WGT. In contrast to PGT, WGT contains an additional gauge field, B_μ, which corresponds to the dilation part of the Weyl group. In WGT, each field transforms like $\varphi(x) \to \varphi'(x') = e^{w(\varphi)\rho(x)}\varphi(x)$ under the dilation transformation, where $w(\varphi)$ is called the weight of φ. Because $w(h^{-1}) = 4$, the weight of the matter Lagrangian must be $w(\mathcal{L}_\mathrm{M}) = -4$ if we require S_M to be Weyl invariant. Note that "ordinary" matter can not have mass in this case. Because the Dirac field ψ has the weight $w(\psi) = w(\bar{\psi}) = -3/2$, the mass term $m\bar{\psi}\psi$ is not allowed in the matter Lagrangian.

We can also obtain the field strengths $\mathcal{R}^{AB}{}_{CD}$ and $\mathcal{T}^{*A}{}_{BC}$ from the commutation relations of the covariant derivatives, in addition to a new field strength $\mathcal{H}_{AB}$ corresponding to the new gauge field B_μ. Note that the form of the translational field strength $\mathcal{T}^{*A}{}_{BC}$ is different from $\mathcal{T}^{A}{}_{BC}$ in PGT. The field strengths have weights $w(\mathcal{R}^{AB}{}_{CD}) = -2$, $w(\mathcal{T}^{*A}{}_{BC}) = -1$, and $w(\mathcal{H}_{AB}) = -2$, respectively. Since we require the free gravitational action $S_\mathrm{G} = \int \mathrm{d}^4x h^{-1}\mathcal{L}_\mathrm{G}(\mathcal{R}^{AB}{}_{CD}, \mathcal{T}^{*A}{}_{BC}, \mathcal{H}_{AB})$ to be Weyl invariant, the free gravitational Lagrangian must have the weight $w(\mathcal{L}_\mathrm{G}) = -4$. Hence, the most general $\mathcal{L}_\mathrm{G}$ in WGT with at most two derivatives in the gauge fields is $\mathcal{L}_\mathrm{G} \sim \mathcal{R}^2 + \mathcal{H}^2 + \mathcal{R}\mathcal{H}$, which is more restricted compared with the one in PGT. However, we may increase the choices of $\mathcal{L}_\mathrm{G}$ by introducing a scalar field (compensator [18]) ϕ with weight $w(\phi) = -1$ and allow it to couple with the field strengths non-minimally. Indeed, a scalar field can also be coupled to Einstein's gravity in a similar way so that the gravitational constant can vary over the spacetime (for example, see Brans–Dicke theory [65]), and the theory becomes free of dimensionful parameters. With the compensator, the most general $\mathcal{L}_\mathrm{G}$ with at most two derivatives in the gauge fields becomes $\mathcal{L}_\mathrm{G} \sim \phi^2\mathcal{R} + \mathcal{R}^2 + \phi^2\mathcal{T}^{*2} + \mathcal{H}^2 + \mathcal{R}\mathcal{H} + (\mathcal{D}^*\phi)^2 + \phi^4$. Dirac [66] investigated the $\phi^2\mathcal{R} + \mathcal{H}^2 + (\mathcal{D}^*\phi)^2 + \phi^4$ theory without torsion and found gauge theories of gravity with scale invariance compatible with "ordinary" matter. The matter Lagrangian also becomes more flexible if we include the compensator, and we can include "ordinary" matter in it. For the Dirac field, we may include the term $\mu\phi\bar{\psi}\psi$ with the correct weight -4, where μ is a dimensionless constant, instead of the non-scale-invariant mass term $m\bar{\psi}\psi$. After choosing the gauge $\phi = \phi_0$ (or interpreted as spontaneous symmetry breaking, see Chapter 6), the Dirac field can be interpreted as a massive field.

Similar to PGT, we also require WGT to be unitary and renormalisable. However, the unitarity and renormalisability of WGT are far less well understood, compared to PGT.

For more details of broader topics about gauge theories of gravity, one may refer to the books [18, 67–69].

1.3 Outline of thesis

In this thesis, we provide a systematic approach to investigate the no-ghost-and-tachyon conditions and the power-counting renormalisability for gauge theories of gravitation and their critical cases. We then apply it to study those properties of PGT$^+$ and WGT$^+$.

The remainder of this thesis is arranged as follows[4].

In Chapter 2, we present a systematic approach to investigate the no-ghost-and-tachyon conditions for general gauge theories of gravitation. The method can systematically deal with the no-ghost condition in the massless sector, including those with additional gauge invariances, which is not fully investigated in the literature. It also classifies all critical conditions into three categories and can find all critical cases, including those conditions preventing them from becoming other critical cases if the critical conditions contain only linear combinations of the Lagrangian parameters. In the appendix of this chapter, we explain the relation between the polarisation basis and SPO and show some details about the no-ghost condition.

In Chapter 3, we show some details of the implementation of the systematic method, which is implemented in MATHEMATICA.

In Chapter 4, we construct PGT by gauging the Poincaré symmetry and apply the method to investigate the most general parity-preserving PGT with up to two derivatives, as well as all of the critical cases. We also examine torsion-free and curvature-free PGT$^+$ in the same way. We then compare our results with those previously presented in the literature. In

[4]Chapters 2 and 4 are based on the paper [70], Chapter 5 is based on the paper [71], and Chapter 6 is based on the paper [72].

the appendix to the chapter, we list the details about the intermediate steps for finding the no-ghost-and-tachyon conditions for the most general PGT$^+$ ("root" theory).

In Chapter 5, we review the criterion of PCR and list critical cases of PGT$^+$ which are ghost and tachyon free and PCR simultaneously. We also clarify the treatment of non-propagating modes in determining whether a theory is PCR. The criterion can be implemented in addition to the method presented in Chapter 2, and the implementation is described in the appendix to the chapter. We also apply the criterion to the much simpler cases of the Proca and Stueckelberg theories in the appendix.

In Chapter 6 we apply the systematic method to study the most general parity-preserving WGT with up to two derivatives and investigate a subset of the critical cases. We found that some of them are ghost and tachyon free and PCR. We then find that some results are related to some critical cases in PGT$^+$, and the results can be extended to some critical cases of the most general WGT$^+$ outside the subset. The torsion-free and curvature-free parity-preserving WGTs are also investigated. In the appendix to the chapter, we discuss the completeness of the critical conditions and additional conditions of the critical cases.

Finally, we summarise the contents in this thesis in Chapter 7 and make some suggestions for future work.

Chapter 2

Systematic method

In this chapter, we present our systematic method to determine the conditions on the parameters in the action of a parity-preserving gauge theory of gravity for it to contain no ghost or tachyon particles. If the parameters in the action satisfy certain "critical conditions", however, the theory may possess different particle contents or additional gauge invariances or both. This increases the difficulty of obtaining the no-ghost condition of the massless sector of a gauge theory of gravity systematically. Therefore, following a brief primer on spin projection operators and notation in Section 2.1, we present in Section 2.2 a systematic approach to investigating all such critical cases and accommodating the associated additional source constraints. The technique naturally accommodates critical cases in which the parameter values lead to additional gauge invariances. The method is implemented as a computer program in MATHEMATICA, and we present the implementation in Chapter 3. The program is used in Chapters 4 and 6 to investigate the particle content of parity-conserving Poincaré gauge theory and Weyl gauge theory, respectively, which we compare with previous results in the literature.

2.1 Spin projection operators

We begin by briefly reviewing the spin projection operator (SPO) formalism [52, 53, 73] and establishing our notation. The SPOs may be used to decompose a field in momentum space into parts with definite spin J and parity P.

A field $\zeta_{\acute{\alpha}}$, where a Greek index with an acute accent ($\acute{\alpha}$, ...) represents the collection of the local Lorentz indices of the field, may be decomposed as

$$\zeta(k)_{\acute{\alpha}} = \sum_{J,P,i} \zeta_i(J^P,k)_{\acute{\alpha}}, \tag{2.1}$$

$$\zeta_i(J^P,k)_{\acute{\alpha}} \equiv P_{ii}(J^P,k)_{\acute{\alpha}}{}^{\acute{\beta}} \zeta(k)_{\acute{\beta}}, \tag{2.2}$$

where there is no sum on i in (2.2). There may be more than one component, or none, with spin-parity J^P. The index i (or, more generally, lowercase Latin letters from the middle of the alphabet) labels these components in the same spin-parity sector and also labels the SPOs. The momentum k^A is a timelike vector, but for simplicity we omit the tensor indices of the momentum k and position x when they appear as function arguments. Indeed, for brevity's sake, we will omit the dependence of fields and SPOs on k or x for the remainder of this section.

There are also off-diagonal SPOs $P_{ij}(J^P)_{\acute{\alpha}}{}^{\acute{\beta}}$, where $i \neq j$, which complete a basis for parity-conserving operators acting on $\zeta_{\acute{\alpha}}$. The SPO basis is Hermitian, complete, orthonormal, and the diagonal elements are positive (or negative) definite. Thus, they satisfy

$$P_{ij}(J^P)^{\acute{\alpha}\acute{\beta}} = P^*_{ji}(J^P)^{\acute{\beta}\acute{\alpha}}, \tag{2.3}$$

$$\sum_{i,J,P} P_{ii}(J^P)_{\acute{\alpha}\acute{\beta}} = \mathbb{I}_{\acute{\alpha}\acute{\beta}}, \tag{2.4}$$

$$P_{ik}(J^P)_{\acute{\alpha}}{}^{\acute{\mu}} P_{lj}(J'^{P'})_{\acute{\mu}\acute{\beta}} = \delta_{JJ'}\delta_{PP'}\delta_{kl}P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}, \tag{2.5}$$

$$[\varphi^*_{\acute{\alpha}} P_{ii}(J^P)^{\acute{\alpha}\acute{\beta}} \varphi_{\acute{\beta}}]P \geq 0 \quad \forall i, \varphi_{\acute{\alpha}}, \tag{2.6}$$

where $\mathbb{I}_{\acute{\alpha}\acute{\beta}}$ is the identity operator for the field ζ, and in the final condition $\varphi_{\acute{\beta}}$ is an arbitrary field in the same tensor space as ζ and P (without indices) is the parity.

Now consider the (usual) case in which the action contains multiple fields $\zeta^{(1)}_{\acute{\alpha}_1}$, $\zeta^{(2)}_{\acute{\alpha}_2}$, $\ldots$, $\zeta^{(f)}_{\acute{\alpha}_f}$, where the index $a = 1,\ldots,f$ labels the fields (generally we will use lowercase Latin letter from the start of the alphabet for this purpose). One can then generalise the SPO $P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ in the single-field case to $P^{(ab)}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$, where the latter now projects the jth part with spin-parity J^P of the field $\zeta^{(b)}_{\acute{\beta}}$ into the ith part with spin-parity J^P of the field $\zeta^{(a)}_{\acute{\alpha}}$.

It is clear from the above discussion that the description of SPOs requires the introduction of several sets of indices of different types. In an attempt to ease somewhat this notational burden, we introduce a matrix-vector formalism that removes two of these sets of indices. We begin by defining the generalised field vector

$$\hat{\zeta} \equiv \sum_{a=1}^{n} \zeta^{(a)}_{\acute{\alpha}_a} \mathbf{e}_a, \tag{2.7}$$

where $\mathbf{e}_a$ is a column vector with ath element equal to unity and the remaining elements zero. On the left-hand side (LHS) of (2.7), we have suppressed the local Lorentz indices, and it should be understood that the ath element of $\hat{\zeta}$ consists of the tensor expression $\zeta^{(a)}_{\acute{\alpha}_a}$. The contraction of two generalised field vectors $\hat{\zeta}$ and $\hat{\xi}$ is then given by

$$\hat{\zeta}^{\dagger} \cdot \hat{\xi} = \sum_{a=1}^{n} \zeta^{*(a)}_{\acute{\alpha}_a} \xi^{(a)\acute{\alpha}_a}, \tag{2.8}$$

where we have "overloaded" the dot notation on the LHS to encompass the summations both over the field index a and the collection of local Lorentz indices $\acute{\alpha}$.

Turning to the SPOs, we begin by considering the tensor quantities $P^{(ab)}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ as the elements of a block matrix $\mathsf{P}(J^P)$, for which the indices (a,b) label the $f \times f$ blocks and the indices (i,j) label the individual elements within each block. Note that since not every field has parts belonging to a given spin-parity sector J^P, some of the blocks will have zero size. We then redefine the indices (i,j) such that $P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ denotes simply the tensor expression in the ith row and jth column of $\mathsf{P}(J^P)$. Finally, for each such element, we define the $f \times f$ matrix

$$\hat{P}_{ij}(J^P) \equiv P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}} \mathbf{e}_a \mathbf{e}_b^{\dagger}, \tag{2.9}$$

where (a,b) denotes the block in $\mathsf{P}(J^P)$ to which the element belongs. By analogy with (2.7), we have again suppressed the local Lorentz indices on the LHS of (2.9) for brevity. The advantage of this notation is that these generalised quantities (denoted by a caret) satisfy relationships of an analogous form to those given in Equations (2.3)–(2.6).

The SPO block matrices $\mathsf{P}(J^P)$ used in this thesis are listed in Appendix 2.A. One can obtain the operators for other fields by the method described in [73].

2.2 Method

We determine whether a theory contains ghosts or tachyons by adapting the systematic method of spin projection operators used in [29, 41]. We apply the method to parity-preserving actions $S = \int \mathrm{d}^4 x L$ with arbitrary real tensor fields, for which the linearised Lagrangian can be written as

$$
\begin{aligned}
L &= L_{\mathrm{F}} + L_{\mathrm{I}}, \\
&= \tfrac{1}{2}\sum_{a,b} \zeta^{(a)}_{\acute{\alpha}}(x)\mathcal{O}^{(ab)}(\partial)^{\acute{\alpha}\acute{\beta}}\zeta^{(b)}_{\acute{\beta}}(x) - \sum_{a} \zeta^{(a)}_{\acute{\alpha}}(x) j^{(a)\acute{\alpha}}(x), \\
&= \tfrac{1}{2}\hat{\zeta}^{\mathrm{T}}(x)\cdot\hat{\mathcal{O}}(\partial)\cdot\hat{\zeta}(x) - \hat{\zeta}^{\mathrm{T}}(x)\cdot\hat{j}(x),
\end{aligned} \tag{2.10}
$$

where $\zeta^{(a)}_{\acute{\alpha}}(x)$ are the fields, $j^{(a)}_{\acute{\alpha}}(x)$ are the corresponding source currents, and we have defined the generalised operator $\hat{\mathcal{O}}(\partial) \equiv \mathcal{O}^{(ab)}(\partial)^{\acute{\alpha}\acute{\beta}}\mathbf{e}_a\mathbf{e}_b^\dagger$ (again suppressing local Lorentz indices on the LHS), in which $\mathcal{O}^{(ab)}(\partial)^{\acute{\alpha}\acute{\beta}}$ is a polynomial in ∂ and depends linearly on the coefficients of the terms in the free-field Lagrangian.

By Fourier transformation, the free-field part of the Lagrangian can be written

$$
L_{\mathrm{F}} = \tfrac{1}{2}\hat{\zeta}^{\mathrm{T}}(-k)\cdot\hat{\mathcal{O}}(k)\cdot\hat{\zeta}(k), \tag{2.11}
$$

where, without loss of generality, one may take $\hat{\mathcal{O}}(k)$ to be Hermitian. A theory has no tachyons if all particles have real masses, and it contains no ghost particle if the real parts of the residues of the saturated propagator at all poles are non-negative:

$$\mathrm{Re}\left[\underset{k^2=m^2}{\mathrm{Res}}(\Pi)\right] \geq 0, \tag{2.12}$$

where the saturated propagator is the propagator sandwiched between currents

$$\Pi(k) = \hat{j}^{\dagger}(k) \cdot \hat{\mathcal{O}}^{-1}(k) \cdot \hat{j}(k). \tag{2.13}$$

We will show more details about (2.12) in Appendix 2.C.

To obtain the propagator, one first decomposes $\hat{\mathcal{O}}(k)$ into sectors with definite spin and parity:

$$\hat{\mathcal{O}}(k) = \sum_{J,P} \hat{\mathcal{O}}(J^P, k) = \sum_{i,j,J,P} a_{ij}(J^P, k) \hat{P}_{ij}(J^P, k). \tag{2.14}$$

Pre- and post-multiplying (2.14) by SPOs and using the orthonormality conditions (2.5), one obtains (omitting the explicit dependence of quantities on k for brevity)

$$\begin{aligned}\hat{P}_{ii}(J^P) \cdot \hat{\mathcal{O}} \cdot \hat{P}_{jj}(J^P) &= \sum_{k,l,J',P'} a_{kl}(J'^{P'}) \hat{P}_{ii}(J^P) \cdot \hat{P}_{kl}(J'^{P'}) \cdot \hat{P}_{jj}(J^P) \\ &= a_{ij}(J^P) \hat{P}_{ij}(J^P),\end{aligned} \tag{2.15}$$

from which one can read off $a_{ij}(J^P)$ as the coefficient of $\hat{P}_{ij}(J^P)$. The quantity $a_{ij}(J^P)$ may be considered as the (i, j)th element of a $s \times s$ matrix $a(J^P)$, where s is the number of parts of spin-parity J^P across all the fields.

The next step is to invert $\hat{\mathcal{O}}(k)$ to obtain the propagator. The orthonormality property of the SPO means that inverting $\hat{\mathcal{O}}(k)$ is equivalent to inverting the matrices $a(J^P)$. One may, however, find that some of the a-matrices are singular, and so cannot be inverted.

If $a(J^P)$ is singular, then the theory possesses gauge invariances, as follows. If $a(J^P)$ has dimension $s \times s$ and rank r, then it has $(s-r)$ null right eigenvectors $v_i^{w,R}(J^P)$, where i is the vector component index and w is a label enumerating the null eigenvectors (a null eigenvector is an eigenvector that corresponds to a zero eigenvalue). Similarly, it also has $(s-r)$ null left eigenvectors $v_i^{w,L}(J^P)$. Thus, if the generalised field $\hat{\zeta}$ is subjected to a change of the form

$$\delta\hat{\zeta}^w = \sum_{k,J,P} v_k^{w,R}(J^P)\hat{P}_{kj}(J^P)\cdot\hat{\varphi}, \tag{2.16}$$

where $\hat{\varphi}$ is some arbitrary generalised field, then the equations of motion $\hat{\mathcal{O}}\cdot\hat{\zeta} = \hat{j}$ remain unchanged.

The null eigenvectors also lead to constraints on the source currents $\hat{j}$. From the equations of motion, one may show that

$$\begin{aligned}\sum_l v_l^{w,L}(J^P)\hat{P}_{kl}(J^P)\cdot\hat{j} &= \sum_{l,i,j} v_l^{w,L}(J^P)\hat{P}_{kl}(J^P)\cdot a_{ij}(J^P)\hat{P}_{ij}(J^P)\cdot\hat{\zeta}\\ &= \sum_{i,j}\left[v_i^{w,L}(J^P)a_{ij}(J^P)\right]\hat{P}_{kj}(J^P)\cdot\hat{\zeta}\\ &= 0 \qquad \forall k, J^P, w.\end{aligned} \tag{2.17}$$

Hence, one can use the $(s-r)$ field transformations in (2.16) to set the corresponding $(s-r)$ parts $\zeta_k(J^P)_{\alpha}$ of the field to zero and hence fix the gauge. This is equivalent to deleting the corresponding $(s-r)$ rows and columns in $a(J^P)$, and thereby $a(J^P)$ becomes nonsingular (this is most easily implemented by successively proposing each row/column pair for deletion, and eliminating only those for which the rank of the matrix is unchanged). We denote the $a-$matrices after deleting the rows and columns by $b(J^P)$. Note that, if the rank of $a(J^P)$ is zero, then there is no particle content in this spin-parity sector and we will ignore these spin-parity sectors in the following discussion.

The inverse of $\hat{\mathcal{O}}(J^P)$ then becomes

$$\hat{\mathcal{O}}^{-1}(J^P) = \sum_{i,j} b_{ij}^{-1}(J^P)\hat{P}_{ij}(J^P), \tag{2.18}$$

where $b_{ij}^{-1}(J^P)$ denotes the (i,j)th element of the inverse b-matrix, and the saturated propagator is thus given by

$$\Pi = \sum_{i,j,J,P} b_{ij}^{-1}(J^P)\,\hat{j}^{\dagger}\cdot\hat{P}_{ij}(J^P)\cdot\hat{j}. \tag{2.19}$$

The no-ghost condition (2.12) requires us to locate the poles of the saturated propagator. We first consider those arising from the elements of the inverse b-matrices, which can be written as

$$b^{-1}_{ij}(J^P) = \frac{1}{\det\left[b(J^P)\right]} C^{\mathrm{T}}_{ij}(J^P), \tag{2.20}$$

where $C_{ij}(J^P)$ is the cofactor of the element $b_{ij}(J^P)$. Since $C_{ij}(J^P)$ is polynomial in k, all poles of $b^{-1}_{ij}(J^P)$ are located at the zeroes of $\det\left[b(J^P)\right]$. The determinant in each spin-parity sector can be written as

$$\det\left[b(J^P)\right] = \alpha k^{2q}(k^2-m_1^2)(k^2-m_2^2)...(k^2-m_r^2), \tag{2.21}$$

where α and $m_1, m_2, \ldots, m_q$ (which we assume are nonzero) are functions of the Lagrangian parameters but independent of k, and q and r are non-negative integers. Thus, $b^{-1}_{ij}(J^P)$ has poles only at $k^2=0$ and $k^2=m_1^2$, $k^2=m_2^2, \ldots, k^2=m_r^2$.

It is worth noting that the reason why there are no odd-order k terms in the determinant is that only the off-diagonal elements of b-matrices contain odd-order k terms. Such an element must belong to a row and column corresponding to one field with odd indices and the other with even indices. The odd-order k is always accompanied by a factor i, so such elements are purely imaginary. Since the b-matrix is Hermitian, however, its determinant is real. The terms in odd powers of k must cancel because they are imaginary, and so the determinant contains only terms with even powers of k.

2.2.1 Massless sector

The no-ghost condition (2.12) in the massless sector is that the residue of the saturated propagator (2.19) at $k^2=0$ be non-negative. Besides the poles at $k^2=0$ present in $b^{-1}_{ij}(J^P)$, the SPOs $P_{ij}(J^P)$ also contain singularities of the form k^{-2n}, where n is a positive integer.

Letting $k^A = (E, \vec{p})$ and $p \equiv \sqrt{\vec{p}^2}$, the particle energy is given by $E = \sqrt{k^2+p^2}$, and the saturated propagator can be written (most conveniently in a slightly unorthodox form) as a

Laurent series in k^2 in the neighbourhood of $k^2 = 0$

$$\Pi(k^2, \vec{p}) = \sum_{n=-\infty}^{N} \frac{Q_{2n}}{k^{2n}}, \tag{2.22}$$

where N is an integer and the coefficients Q_{2n} are some functions of the on-shell momentum $\bar{k}^A \equiv (p, \vec{p})$ and the on-shell source currents $j^{(a)}_{\acute{\alpha}}(\bar{k})$. If N is zero or negative, then there is no pole at $k^2 = 0$ and there is no propagating massless particle. We will only discuss the $N > 0$ cases here. The no-ghost conditions (2.12) are that the residue of $k^2 = 0$ be non-negative, so $Q_2 \geq 0$. Furthermore, we require that the saturated propagator has a *simple* pole in k^2 at this point, since terms proportional to k^{-2n} with $n > 1$ contain ghost states [54]. For example, if the Laurent series of the saturated propagator about $k^2 = 0$ contains a term proportional to k^{-4}, one can write this as

$$\frac{1}{k^4} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\frac{1}{k^2} - \frac{1}{k^2 + \varepsilon} \right), \tag{2.23}$$

which contains a normal state and a ghost state.

To obtain the coefficients Q_{2n} in the Laurent series (2.22), one may expand the SPOs in the saturated propagator, which can then be written as a sum of terms of the form

$$\Pi(k) = \sum \frac{\mathcal{P}_1(k^2)\mathcal{C}(k^A, \eta^{AB}, j^{(a)}_{\acute{\alpha}})}{\mathcal{P}_2(k^2)}, \tag{2.24}$$

where $\mathcal{P}_1(k^2)$ and $\mathcal{P}_2(k^2)$ are polynomials of k^2, and $\mathcal{C}(\cdots)$ is a scalar that is obtained from contracting the tensors in its argument. We require that $\mathcal{C}(\cdots)$ does not contain the factor k^2 because it can be absorbed into $\mathcal{P}_1(k^2)$. Note that the coefficient Q_{2n} may not necessarily be given by

$$\sum \left\{ \operatorname*{Res}_{k^2=0} \left[k^{2(n-1)} \frac{\mathcal{P}_1(k^2)}{\mathcal{P}_2(k^2)} \right] \mathcal{C}(k^A, \eta^{AB}, j_{\phi,\acute{\alpha}})|_{k^2=0} \right\}, \tag{2.25}$$

if there exists any nonzero higher-order (larger n) terms because there may be k^A terms in $\mathcal{C}(\cdots)$. One can accommodate this situation by expanding the tensor expressions into their components before taking residues. To this end, it is convenient to choose a coordinate system such that $k^A = (E, 0, 0, p)$; this greatly simplifies the calculation without loss of generality, since the saturated propagator is Lorentz invariant.

Note that the source currents have to satisfy the source constraints (2.17). However, (2.17) is a set of tensor equations, which is difficult to use systematically in the no-ghost conditions. We thus expand the source constraints into their components, and then solve the component equation set and substitute them back to the saturated propagator. Since (2.17) is a set of homogeneous linear equations, we can write it in matrix-vector form as

$$\mathbf{C}\cdot\mathbf{j} \equiv \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1q} \\ \vdots & \dots & \dots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mq} \end{pmatrix} \begin{pmatrix} j^{(1)}_{0\cdots 0} \\ j^{(1)}_{0\cdots 1} \\ \vdots \\ j^{(f)}_{3\cdots 3} \end{pmatrix} = 0, \tag{2.26}$$

where m and q are integers, c_{ij} is the coefficient of the jth component of the source current in the ith equation, f is the total number of fields, and the subscripts of $j^{(i)}$ are Lorentz indices. The solution is

$$\mathbf{j} = \sum_i X_i \mathbf{n}_i, \tag{2.27}$$

where $\mathbf{n}_i$ are the null vectors of $\mathbf{C}$, and X_i are some free variables. Note that we have to rescale those null vectors with factors $(E-p)^n$ in the denominator to avoid introducing spurious singularities to the saturated propagator, where n is the minimum integer to make the null vector nonsingular at $E=p$. We then replace the source current components with X_i using (2.27).

Now the residue only contains the free variables X_i, and we can put them in a column matrix $\mathbf{X}$. The saturated propagator can then be written as a matrix $\mathbf{M}$ sandwiched by current vectors $\mathbf{X}$:

$$\Pi = \mathbf{X}^\dagger \cdot \mathbf{M} \cdot \mathbf{X}. \tag{2.28}$$

We can also write Q_{2n} in terms of a matrix $\mathbf{Q}_{2n}$ in a similar way:

$$Q_{2n} = \mathbf{X}^\dagger \cdot \mathbf{Q}_{2n} \cdot \mathbf{X}. \tag{2.29}$$

Since $Q_{2n} = 0$ for $n > N$, then $k^{2(N-1)}\Pi$ contains only a simple pole or no pole at $k^2 = 0$, and one obtains

$$\mathbf{Q}_{2N} = \operatorname*{Res}_{k^2=0}\left[k^{2(N-1)}\mathbf{M}\right] = \lim_{E\to p}\left[k^{2N}\mathbf{M}\right]. \tag{2.30}$$

One then calculates the remaining $\mathbf{Q}_{2n}$ by subtracting all the higher singularities:

$$\mathbf{Q}_{2n} = \lim_{E\to p}\left[k^{2N}\left(\Pi - \sum_{j=n+1}^{N}\frac{\mathbf{Q}_{2j}}{k^{2j}}\right)\right], \tag{2.31}$$

Thus, we obtain recursively all of the $\mathbf{Q}$-matrices with positive n: $\mathbf{Q}_{2N}$, $\mathbf{Q}_{2(N-1)}$, ..., $\mathbf{Q}_2$. For $\mathbf{Q}_{2n}$ with $n > 1$, one requires that each element in the matrix is zero:

$$\mathbf{Q}_{2n} = 0 \qquad \forall\, p \neq 0, n > 1. \tag{2.32}$$

For $n = 1$, corresponding to the k^{-2} pole, the no-ghost condition is equivalent to requiring that each eigenvalue of $\mathbf{Q}_2$ is non-negative:

$$\text{Eigenvalues}(\mathbf{Q}_2) \geq 0 \qquad \forall\, p \neq 0. \tag{2.33}$$

The number of nonzero eigenvalues is equal to the number of degrees of freedom of the propagating massless particles.

Solving the inequalities in (2.33) may be quite time consuming, however, in the cases where the eigenvalues contain some roots of cubic or even higher polynomials. It is therefore convenient to convert them into an alternative form. In particular, if $x_1, \cdots, x_n$ are the roots of a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ and the roots are guaranteed to be real, then

$$x_1, \cdots, x_n > 0 \Leftrightarrow (-1)^{n-i}a_i > 0 \qquad \forall\, a_i. \tag{2.34}$$

We can extend the above relation to non-negative roots using the fact that if there are exactly z zero roots, then $a_0, \cdots, a_{z-1} = 0$ and $a_z \neq 0$. We then collect the conditions with 0 to n zero roots. This gives the conditions for non-negative roots.

2.2.2 Massive sector

In the massive sector, the no-tachyon conditions are simply:

$$m_s^2 > 0 \quad \forall s \tag{2.35}$$

for every spin-parity sector. If this condition is satisfied, one must then determine if any of the massive particles is a ghost. For non-tachyonic particles, k is real around $k^2 = m_s^2$, and so the b-matrices are Hermitian. Although one can thus expand the saturated propagator and analyse its poles in a similar manner to that used in the massless sector, there is a simpler approach in the massive sector, provided all the masses in all spin sectors are distinct, which is true in PGT$^+$. We first discuss this case and discuss the other more general cases later.

From Eqs. (2.19)–(2.21), for an arbitrary current $\hat{j}$, the no-ghost condition (2.12) may be written as

$$\begin{aligned}(2.12) \Leftrightarrow &\left[\sum_{i,j,J} \frac{1}{\alpha k^{2q}} \left(\prod_{r \neq s} \frac{1}{k^2 - m_r^2}\right) C^{\mathrm{T}}_{ij}(J^P)\, \hat{j}^{\dagger} \cdot \hat{P}_{ij}(J^P) \cdot \hat{j}\right]_{k^2=m_s^2} \geq 0 \quad \forall \hat{j}, s, J, P,\\ \Leftrightarrow &\left[\sum_{i,J} \frac{1}{\alpha k^{2q}} \left(\prod_{r \neq s} \frac{1}{k^2 - m_r^2}\right) C^{\mathrm{T}}_{D,ii}(J^P)\, \hat{j}^{\dagger}_D \cdot \hat{P}_{ii}(J^P) \cdot \hat{j}_D\right]_{k^2=m_s^2} \geq 0 \quad \forall \hat{j}_D, s, J, P,\end{aligned} \tag{2.36}$$

where $C^{\mathrm{T}}_{D,ij}(J^P) = \sum_{k,l} U_{ik}(J^P) C^{\mathrm{T}}_{kl}(J^P) U^{\dagger}_{lj}(J^P)$, $\hat{j}_D = \sum_{i,j} U(J^P)_{ij} \hat{P}(J^P)_{ij} \cdot \hat{j}$ and $U(J^P)_{ij}$ are the elements of a unitary matrix of which each column is a eigenvector of the matrix with elements C^{T}_{ij} (the subscript D thus denotes a diagonal basis). We can write the last line in (2.36) safely because the matrix with elements $C^{\mathrm{T}}_{ij}(J^P, k^2 = m_s^2)$ is finite and Hermitian, so it must have finite real eigenvalues and the transform matrix with elements $U_{ij}(J^P)$ is finite even at the pole. Since the current $\hat{j}_D$ is arbitrary and $b^{-1}_{D,ii}(J^P)$ has either no singularity or a simple pole at $k^2 = m_s^2$, which we will explain later, then using (2.20) again gives

$$(2.36) \Leftrightarrow \sum_{i,J^P} \operatorname*{Res}_{k^2=m_s^2} \left[b^{-1}_{D,ii}(J^P)\right] \cdot \left[\hat{j}^{\dagger} \cdot \hat{P}_{ii}(J^P) \cdot \hat{j}\right]_{k^2=m_s^2} \geq 0 \quad \forall \hat{j}, s, J, P. \tag{2.37}$$

Since $b_{ij}(J^P,k^2)$ is Hermitian for real k^2 about $m_s^2>0$, its eigenvalue $b_{D,ii}(J^P,k^2)$ is analytic as a function of k^2 about $m_s^2>0$ [74, p. 139], and one can Taylor expand it about $k^2=m_s^2$:

$$b_{D,ii}(J^P,k^2)=b_{D,ii}(J^P,m_s^2)+b'_{D,ii}(J^P,m_s^2)\cdot(k^2-m_s^2)+\dots, \tag{2.38}$$

where the prime denotes the derivative with respect to k^2. The determinant is a polynomial in k^2, so it must also be analytic. Since it equals zero at $k^2=m_s^2$, we can write:

$$\det\left[b(J^P)\right](k^2)=\det\left[b(J^P)\right]'(m_s^2)\cdot(k^2-m_s^2)+\frac{1}{2}\det\left[b(J^P)\right]''(m_s^2)\cdot(k^2-m_s^2)^2+\dots \tag{2.39}$$

As we are assuming that all the masses are distinct, then $\det\left[b(J^P)\right]'(m_s^2)\neq 0$ and $\det\left[b(J^P)\right](k^2)\sim O(k^2-m_s^2)$ when k^2 is near m_s^2. Hence, there should be one i with $b_{D,ii}(J^P)(m_s^2)\sim O(k^2-m_s^2)$, and the other $b_{D,ii}(J^P)(m_s^2)\sim O(1)$. Thus, exactly one $\mathrm{Res}_{k^2=m_s^2}\left[b^{-1}_{D,ii}(J^P)\right]$ is nonzero. Together with the property (2.6), the massive no-ghost condition therefore becomes

$$\begin{aligned}(2.37)\Leftrightarrow &\operatorname*{Res}_{k^2=m_s^2}\left[b^{-1}_{D,ii}(J^P)\right]\cdot P\geq 0 \quad \forall s,\\ \Leftrightarrow &\operatorname*{Res}_{k^2=m_s^2}\left[\mathrm{Tr}\,b_D^{-1}(J^P)\right]\cdot P\geq 0 \quad \forall s,\\ \Leftrightarrow &\operatorname*{Res}_{k^2=m_s^2}\left[\mathrm{Tr}\,b^{-1}(J^P)\right]\cdot P\geq 0 \quad \forall s.\end{aligned} \tag{2.40}$$

Let us now examine the case where $\mathrm{Res}_{k^2=m_s^2}\left[\mathrm{Tr}\,b^{-1}(J^P)\right]=0$. This violates the conclusion that exactly one $\mathrm{Res}_{k^2=m_s^2}\left[b^{-1}_{D,ii}(J^P)\right]$ is nonzero. The violation is equivalent to that at least one of the assumptions we made fails. The assumptions are that there is no tachyon, all masses are distinct, and the Lagrangian parameters do not satisfy any of the critical conditions, which we will mention later. Therefore, if $\mathrm{Res}_{k^2=m_s^2}\left[\mathrm{Tr}\,b^{-1}(J^P)\right]=0$, there must be a tachyon or there exist identical masses.

Hence, the combined massive no-ghost-and-tachyon conditions are

$$m_s^2 > 0 \quad \forall s, \tag{2.41}$$

$$\operatorname*{Res}_{k^2=m_s^2} \left[\operatorname{Tr} b^{-1}(J^P)\right] \cdot P > 0 \quad \forall s, \tag{2.42}$$

if the masses in each spin sector are distinct. To obtain the masses, one merely has to calculate the roots of the determinants of the b-matrices. We assume that all the roots that depend on the parameters of the Lagrangian are indeed non-zero. If one sets a nonzero mass to zero, however, a massive pole becomes massless pole and one has to recalculate the massless no-ghost conditions because the additional massless pole was not included in the calculation in the previous previous step. We will discuss such "critical cases" later and assume that they do not occur here.

If any mass in a spin sector has multiplicity greater than one, Eq. (2.38) will not hold. In that case, one has to calculate $b^{-1}_{D,ii}(J^P)$ explicitly and use the condition (2.37) directly. One should also avoid higher singularities in these cases. In the PGT$^+$ case that we will consider in Chapter 4, however, there is at most one massive mode in each spin sector.

We note that the condition (2.42) is the same as Eq. (27b) in [29], but differs from Eq. (47) in [41]. The reason is that Karananas considers full PGT, with parity-violating terms, so that his spin projectors do not satisfy $P^*_{ij}(J^P)_{\acute{\alpha}\acute{\beta}} = P_{ji}(J^P)_{\acute{\beta}\acute{\alpha}}$ and the parity-even and odd parts are mixed. Hence, (2.37) is not valid in this case. It is not clear, however, how one arrives at Eq. (47) in [41] in the full PGT case.

Finally, we note that the full combination of conditions on the Lagrangian are given by (2.32), (2.33), (2.41) and (2.42).

2.2.3 Critical cases

There are a number of assumptions in the analysis outlined above, so the process is not complete. To understand this better, let us reexamine the determinants in (2.21), which can be written as

$$\det\left[b\left(J^P\right)\right] = k^{2q}\sum_{j=0}^{r}\left(A_{2j}k^{2j}\right) = k^{2q}A_{2r}\prod_{j=1}^{r}\left(k^2 - m_j^2\right), \tag{2.43}$$

where q and r are non-negative integers, and A_{2j} are some finite functions of the parameters, with $A_{2r} \neq 0$ and $A_0 \neq 0$. In the above process, we have implicitly assumed $m_j \neq 0$ and finite. We now discuss what may happen if the parameters in the Lagrangian satisfy some equalities and violate these assumptions in a given spin-parity sector J^P.

In particular, we consider the following eventualities.

1. $\det\left[b\left(J^P\right)\right] = 0$: This is equivalent to all $A_{2j} = 0$. The determinant becomes zero, and there are more gauge freedoms. Hence, we need to calculate the new source constraints and $b_{ij}^{-1}\left(J^P\right)$ matrix elements, and the massless, as well as massive poles, have different forms.

2. $\det\left[b\left(J^P\right)\right] \neq 0$, $A_{2r} \neq 0$, but $A_0 = 0$: The determinant can then be written as

$$\det\left[b\left(J^P\right)\right] = k^{2(q+l)}\sum_{j=l}^{r}\left(A_{2j}k^{2(j-l)}\right), \tag{2.44}$$

   where $A_{2l} \neq 0$, l is a positive integer and $r \geq l > 0$. Some masses becomes zero, so some massive poles of the propagator become massless. The number of massive conditions decreases, and the massless conditions change. Hence, there is no further gauge invariance, and the source constraints and the matrix elements $b_{ij}^{-1}\left(J^P\right)$ remain in the same form. One needs to calculate the new massless and massive conditions.

3. $\det\left[b\left(J^P\right)\right] \neq 0$, $A_0 \neq 0$, and $A_{2r} = 0$: The second equality of (2.43) becomes invalid since some masses become infinite. In this case, we can write the determinant as

$$\det\left[b\left(J^P\right)\right] = k^{2q}\sum_{j=0}^{r-l}\left(A_{2j}k^{2j}\right), \tag{2.45}$$

   where l is a non-negative integer. There is no new gauge freedom, but the number of the roots is decreased. The poles are "removed" in this case. Since only the k^{2q} part will affect the massless poles in the saturated propagator [see Eq. (2.19)–(2.21)], the forms of the massless poles are unchanged. Hence, one need only recalculate the massive conditions. In this case, some non-propagating modes (propagator with no pole) might appear. We do not forbid these modes in this thesis.

4. $\det\left[b\left(J^P\right)\right] \neq 0$, $A_0 = 0$, and $A_{2r} = 0$: This eventuality is a combination of 2. and 3. There is no new gauge freedom, but some masses become zero, and the number of the roots is decreased.

We can find all conditions that cause a theory to be a critical case by finding all conditions that cause $\det\left[b\left(J^P\right)\right] = 0$, $A_0 = 0$, or $A_{2r} = 0$ in any spin sectors. While some conditions may cause more than one of the above situations, we can still divide all the critical conditions into three categories.

A. Those causing $\det\left[b\left(J^P\right)\right] = 0$ in any spin-parity sector: The source constraints, $b_{ij}^{-1}\left(J^P\right)$ matrix elements, and the massless as well as massive poles have different forms.

B. Those causing $A_0 = 0$ in any spin-parity sector, and not belonging to Type A: The form of the source constraints and the $b_{ij}^{-1}\left(J^P\right)$ matrix elements are the same, but the massless and massive conditions have different forms.

C. Those conditions not belonging to Type A and Type B: These conditions cause $A_{2r} = 0$ in some spin sectors. Only the form of the massive condition is changed. We can substitute the conditions into the massless condition directly.

We can then traverse all possible critical cases. First, we find the type A, B and C conditions for the parameters in the original Lagrangian satisfying only one equality. Each type A and B condition is a child theory of the original theory. For the type C conditions, any combination of type C conditions of a theory is also a type C condition of the theory, provided they are not contradictory. Note that we are assuming that a child theory does not satisfy the other sibling critical conditions, and it does not include the critical cases of itself. Hence, some combinations of type C conditions might be contradictory, and we have to remove these cases. We first calculate the no-ghost-and-tachyon conditions for all the type C child theories. We then calculate the no-ghost-and-tachyon conditions for the first type A or B child theory and then find its critical cases.

We traverse the "tree" in a pre-ordered way: we repeat the above process until the theory we are investigating has no type A or B child theory, and then return to its parent theory and consider the next unevaluated child theory of the parent theory. Because it is possible to reach the same theory by different routes, we have to check whether the child theory has been evaluated. If it has been evaluated, we neither calculate it again nor find its child theories. The reason why we do not have to find the child theories for type C conditions is that their type A and B child conditions must be evaluated in some other branches of their sibling type A or B conditions. As for the type C child theories, they are already included in the combination of the sibling type C conditions. We can then find all possible critical cases and collect all no-ghost-and-tachyon conditions.

This process is best illustrated by examples, which we provide in Chapter 4, in the context of PGT$^+$.

Appendix 2.A List of spin projection operators

The block matrices $\mathsf{P}(J^P)$ containing the most general spin projection operators for theories used in this paper are as follows (see Section 2.1 for details). In PGT$^+$, only the rows/columns of A, $\mathfrak{s}$, and $\mathfrak{a}$-fields are used. In WGT$^+$, the rows and columns of B-field are also required. While the scalar ϕ-field is not used in this thesis, we also include it here for completeness.

$$\mathsf{P}\left(0^-\right) = \begin{array}{c} A_{ABC} \\ A^*_{IJK}\left(\frac{2}{3}\Theta_{IC}\Theta_{JA}\Theta_{KB}+\frac{1}{3}\Theta_{IA}\Theta_{JB}\Theta_{KC}\right) \end{array}, \tag{2.46}$$

$$\mathsf{P}\left(0^+\right) =$$

$$\begin{array}{c} \\ A^*_{IJK} \\ \mathfrak{s}^*_{IJ} \\ \mathfrak{s}^*_{IJ} \\ \phi^* \\ B^*_K \end{array} \begin{array}{c} \begin{array}{ccccc} A_{ABC} & \mathfrak{s}_{AB} & \mathfrak{s}_{AB} & \phi & B_C \end{array} \\ \begin{pmatrix} \frac{2}{3}\Theta_{CB}\Theta_{KJ}\Omega_{IA} & \frac{\sqrt{2}}{3}\tilde{k}_J\Theta_{AB}\Theta_{KI} & \sqrt{\frac{2}{3}}\tilde{k}_J\Theta_{KI}\Omega_{BA} & \sqrt{\frac{2}{3}}\tilde{k}_J\Theta_{KI} & -\sqrt{\frac{2}{3}}\Theta_{KJ}\Omega_{IC} \\ \frac{\sqrt{2}}{3}\tilde{k}_B\Theta_{CA}\Theta_{IJ} & \frac{1}{3}\Theta_{AB}\Theta_{IJ} & \frac{1}{\sqrt{3}}\Theta_{IJ}\Omega_{AB} & \frac{1}{\sqrt{3}}\Theta_{IJ} & \frac{1}{\sqrt{3}}\tilde{k}_C\Theta_{IJ} \\ \sqrt{\frac{2}{3}}\tilde{k}_B\Theta_{CA}\Omega_{JI} & \frac{1}{\sqrt{3}}\Theta_{AB}\Omega_{IJ} & \Omega_{AB}\Omega_{IJ} & \Omega_{IJ} & \tilde{k}_C\Omega_{IJ} \\ \sqrt{\frac{2}{3}}\tilde{k}_B\Theta_{CA} & \frac{1}{\sqrt{3}}\Theta_{AB} & \Omega_{AB} & 1 & \tilde{k}_C \\ -\sqrt{\frac{2}{3}}\Theta_{CB}\Omega_{AK} & \frac{1}{\sqrt{3}}\tilde{k}_K\Theta_{AB} & \tilde{k}_K\Omega_{AB} & \tilde{k}_K & \Omega_{KC} \end{pmatrix} \end{array}, \tag{2.47}$$

$$\mathsf{P}\left(1^-\right) =$$

$$\begin{array}{c} \\ A^*_{KIJ} \\ A^*_{KIJ} \\ \mathfrak{s}^*_{IJ} \\ a^*_{IJ} \\ B^*_K \end{array} \begin{array}{c} \begin{array}{ccccc} A_{ABC} & A_{ABC} & \mathfrak{s}_{AB} & a_{AB} & B_C \end{array} \\ \begin{pmatrix} \Theta_{CB}\Theta_{IA}\Theta_{KJ} & \sqrt{2}\Theta_{IA}\Theta_{KJ}\Omega_{CB} & \sqrt{2}\tilde{k}_B\Theta_{IA}\Theta_{KJ} & \sqrt{2}\tilde{k}_B\Theta_{IA}\Theta_{KJ} & \Theta_{IC}\Theta_{KJ} \\ \sqrt{2}\Theta_{AI}\Theta_{CB}\Omega_{KJ} & 2\Theta_{IA}\Omega_{CB}\Omega_{KJ} & 2\tilde{k}_J\Theta_{IA}\Omega_{KB} & 2\tilde{k}_J\Theta_{IA}\Omega_{KB} & \sqrt{2}\Theta_{IC}\Omega_{KJ} \\ \sqrt{2}\tilde{k}_J\Theta_{AI}\Theta_{CB} & 2\tilde{k}_B\Theta_{AI}\Omega_{CJ} & 2\Theta_{IA}\Omega_{JB} & 2\Theta_{IA}\Omega_{JB} & \sqrt{2}\tilde{k}_J\Theta_{IC} \\ \sqrt{2}\tilde{k}_J\Theta_{AI}\Theta_{CB} & 2\tilde{k}_B\Theta_{IA}\Omega_{CJ} & 2\Theta_{IA}\Omega_{JB} & 2\Theta_{IA}\Omega_{JB} & \sqrt{2}\tilde{k}_J\Theta_{IC} \\ \Theta_{AK}\Theta_{CB} & \sqrt{2}\Theta_{AK}\Omega_{CB} & \sqrt{2}\tilde{k}_B\Theta_{AK} & \sqrt{2}\tilde{k}_B\Theta_{AK} & \Theta_{KC} \end{pmatrix} \end{array}, \tag{2.48}$$

$$
\mathsf{P}\left(1^{+}\right)=\begin{array}{c} \\ A^{*}_{IJK} \\ A^{*}_{IJK} \\ a^{*}_{IJ} \end{array}\begin{array}{c} \begin{array}{ccc} A_{ABC} & A_{ABC} & a_{AB} \end{array} \\ \begin{pmatrix} \Theta_{IC}\Theta_{KB}\Omega_{JA}+\Theta_{IA}\Theta_{KC}\Omega_{JB} & -\sqrt{2}\Theta_{JA}\Theta_{KB}\Omega_{IC} & \sqrt{2}\tilde{k}_{J}\Theta_{IA}\Theta_{KB} \\ -\sqrt{2}\Theta_{BI}\Theta_{CJ}\Omega_{AK} & \Theta_{IA}\Theta_{JB}\Omega_{KC} & \tilde{k}_{K}\Theta_{IA}\Theta_{JB} \\ \sqrt{2}\tilde{k}_{B}\Theta_{AI}\Theta_{CJ} & \tilde{k}_{C}\Theta_{AI}\Theta_{BJ} & \Theta_{AI}\Theta_{BJ} \end{pmatrix} \end{array}, \tag{2.49}
$$

$$
\mathsf{P}\left(2^{-}\right)=A^{*}_{IJK}\overset{A_{ABC}}{\left(\tfrac{2}{3}\Theta_{IC}\Theta_{JB}\Theta_{KA}+\tfrac{2}{3}\Theta_{IA}\Theta_{JB}\Theta_{KC}-\Theta_{CB}\Theta_{IA}\Theta_{KJ}\right)}, \tag{2.50}
$$

$$
\mathsf{P}\left(2^{+}\right)=
$$

$$
\begin{array}{c} \\ A^{*}_{IJK} \\ \mathfrak{s}^{*}_{IJ} \end{array}\begin{array}{c} \begin{array}{cc} A_{ABC} & \mathfrak{s}_{AB} \end{array} \\ \begin{pmatrix} -\tfrac{2}{3}\Theta_{CB}\Theta_{KJ}\Omega_{IA}+\Theta_{IC}\Theta_{KA}\Omega_{JB}+\Theta_{IA}\Theta_{KC}\Omega_{JB} & \sqrt{2}\tilde{k}_{J}\left(\Theta_{IA}\Theta_{KB}-\tfrac{1}{3}\Theta_{AB}\Theta_{KI}\right) \\ \sqrt{2}\tilde{k}_{B}\left(\Theta_{CJ}\Theta_{IA}-\tfrac{1}{3}\Theta_{CA}\Theta_{IJ}\right) & -\tfrac{1}{3}\Theta_{AB}\Theta_{IJ}+\Theta_{IA}\Theta_{JB} \end{pmatrix} \end{array}. \tag{2.51}
$$

where $\tilde{k}_A = k_A/\sqrt{k^2}$, $\Omega^{AB} = k^A k^B/k^2$, and $\Theta^{AB} = \eta^{AB} - k^A k^B/k^2$. The operators are adapted from [41]. The fields have some symmetry properties: the A_{ABC} field is antisymmetric in AB, the a_{AB} field is antisymmetric in AB, and the s_{AB} field is symmetric in AB. Note that the spin projection operators satisfy the symmetry properties implicitly. For example, although $P_{33}(1^-) = P^{(ss)}_{11}(1^-)$ is notated as $2\Theta_{IA}\Omega_{JB}$ above, its correctly symmetrised form is $(\Theta_{IA}\Omega_{JB} + \Theta_{IB}\Omega_{JA} + \Theta_{JA}\Omega_{IB} + \Theta_{JB}\Omega_{IA})/2$. We have verified that the above set of spin projection operators satisfies (2.4) and (2.5).

Appendix 2.B Polarisation basis vectors [1]

If we consider $SO(3)$ space rotation, we can decompose Lorentz tensors into $SO(3)$ representation spaces. For a Lorentz vector, v^A, the timelike component v^0 is invariant under space rotation, and v^i is a 3-vector. If we perform a parity transform, v^0 is invariant, but all the components of v^i gain a different sign. Hence, we can decompose v^A into $\mathbf{0}^+ \oplus \mathbf{1}^-$.

[1] We are taking inspiration from [75, 76] in this appendix.

We can find a set of polarisation basis vectors for a vector in the momentum space[2]. We choose the momentum k^A to be $k^A = (k^0, 0, 0, k^3)$ without loss of generality[3]. The bases $\varepsilon_{(J^P,m)}$ are[4]

$$\varepsilon^A_{(1^-,1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix},\quad \varepsilon^A_{(1^-,-1)} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\i\\0\end{pmatrix},\quad \varepsilon^A_{(1^-,0)} = \frac{1}{k}\begin{pmatrix}k^3\\0\\0\\k^0\end{pmatrix},\quad \varepsilon^A_{(0^+,0)} = \frac{1}{k}\begin{pmatrix}k^0\\0\\0\\k^3\end{pmatrix}. \tag{2.52}$$

The basis vectors satisfy the orthonormal and completeness conditions,

$$\varepsilon^{*A}_{(J_1^{P_1},m_1)}\varepsilon_{(J_2^{P_2},m_2),A} = P_1\delta_{J_1,J_2}\delta_{P_1,P_2}\delta_{m_1,m_2}, \tag{2.53}$$

$$\sum_{J,P,m} P\,\varepsilon^A_{(J^P,m)}\varepsilon^*_{(J^P,m),B} = \delta^A_B. \tag{2.54}$$

Hence, the operator $\varepsilon^A_{(0^+,m)}\varepsilon^*_{B,(0^+,m)}$ projects a Lorentz vector to the spin-0^+ part, and similarly $\sum_m\left(-\varepsilon^A_{(1^-,m)}\varepsilon^*_{B,(1^-,m)}\right)$ projects a Lorentz vector to the spin-1^- part.

The numbers of the basis vectors are consistent with the fact that an off-shell spin-j particle has $2j+1$ degrees of freedom. For the massless on-shell particle, we can set $k^3 \to k^0$ so that $\varepsilon^A_{(1^-,0)} \to \varepsilon^A_{(0^+,0)}$ and the degrees of freedom decrease.

Now we define the tensors:

$$\Omega^{AB} = \frac{k^A k^B}{k^2},\quad \Theta^{AB} = \eta^{AB} - \frac{k^A k^B}{k^2}, \tag{2.55}$$

and we can find they satisfy the following properties:

$$\begin{aligned}&\Omega^A{}_C\Omega^C{}_B = \Omega^A{}_B,\ \Theta^A{}_C\Theta^C{}_B = \Theta^A{}_B,\ \Omega^A{}_C\Theta^C{}_B = \Theta^A{}_C\Omega^C{}_B = 0,\\ &\Omega^{AB} + \Theta^{AB} = \eta^{AB}.\end{aligned} \tag{2.56}$$

[2]From now on, all the calculations are in the momentum space if not specified.

[3]We are considering the off-shell case here, so in general $k^2 \neq 0$ even if the particle is massless.

[4]The $\varepsilon^\mu_{(0^+,0)}$ does not correspond to the v^0 part because now we are in a different representation from that in the last paragraph.

The operators have the following relations with the polarisation basis:

$$\Omega^{AB} = \varepsilon^{A}_{(0^+,m)}\varepsilon^{*B}_{(0^+,m)}, \quad \Theta^{AB} = \sum_m \left(\varepsilon^{A}_{(1^-,m)}\varepsilon^{*B}_{(1^-,m)}\right), \tag{2.57}$$

so they are proportional to the spin projector operators which project a Lorentz vector to spin-0^+ and spin-1^- space respectively.

For the higher rank tensors, we can apply the addition rules for angular momentum. For example, a $(2,0)$ tensor f^{AB} can be decomposed as

$$\begin{aligned} f^{AB} &\in (\mathbf{0}^+ \oplus \mathbf{1}^-)\otimes(\mathbf{0}^+ \oplus \mathbf{1}^-) \\ &= (\mathbf{0}^+ \otimes \mathbf{0}^+)\oplus(\mathbf{0}^+ \otimes \mathbf{1}^-)\oplus(\mathbf{1}^- \otimes \mathbf{0}^+)\oplus(\mathbf{1}^- \otimes \mathbf{1}^-) \\ &= \mathbf{0}^+ \oplus \mathbf{1}^- \oplus \mathbf{1}^- \oplus (\mathbf{0}^+ \oplus \mathbf{1}^+ \oplus \mathbf{2}^+). \end{aligned} \tag{2.58}$$

The polarisation basis can be obtained by the Clebsch-Gordan coefficients[5]. For example, some bases $\varepsilon^{AB}_{(J_1^{P_1},J_2^{P_2},J'^{P'},m_{J'})}$ obtained by $J_1^{P_1}\otimes J_2^{P_2}$ are

$$\begin{aligned} \varepsilon^{AB}_{(1^-,1^-,2^+,+2)} &= \varepsilon^{A}_{(1^-,1)} \otimes \varepsilon^{B}_{(1^-,1)}, \\ \varepsilon^{AB}_{(1^-,1^-,2^+,+1)} &= \frac{1}{\sqrt{2}}\left(\varepsilon^{A}_{(1^-,1)} \otimes \varepsilon^{B}_{(1^-,0)} + \varepsilon^{A}_{(1^-,0)} \otimes \varepsilon^{B}_{(1^-,1)}\right). \end{aligned} \tag{2.59}$$

We can decompose any $(0,2)$ tensor into $f^{AB} = \mathfrak{s}^{AB} + \mathfrak{a}^{AB}$, where $\mathfrak{s}$ is symmetric and $\mathfrak{a}$ is antisymmetric. We can observe from the Clebsch-Gordan coefficients table that the $\mathbf{2}^+$ and the $\mathbf{0}^+$s are symmetric, and the $\mathbf{1}^+$ is antisymmetric in A and B. We can make a linear combination of the two $\mathbf{1}^-$ sectors to get a symmetric sector and an antisymmetric sector:

$$\varepsilon^{AB}_{(\mathtt{sym},1^-,m)} \equiv \frac{1}{\sqrt{2}}\left(\varepsilon^{AB}_{(0^+,1^-,1^-,m)} + \varepsilon^{AB}_{(1^-,0^+,1^-,m)}\right) \tag{2.60}$$

$$\varepsilon^{AB}_{(\mathtt{ant},1^-,m)} \equiv \frac{1}{\sqrt{2}}\left(\varepsilon^{AB}_{(0^+,1^-,1^-,m)} - \varepsilon^{AB}_{(1^-,0^+,1^-,m)}\right). \tag{2.61}$$

[5] We are using the same notations as the Particle Data Group, which can be found at http://pdg.lbl.gov/2008/reviews/clebrpp.pdf.

Hence, we can conclude that the symmetric part of Equation (2.58) is $\mathbf{2}^+ \oplus \mathbf{1}^- \oplus \mathbf{0}^+ \oplus \mathbf{0}^+$, which has $5+3+1+1 = 10$ degrees of freedom, and the antisymmetric part is $\mathbf{1}^+ \oplus \mathbf{1}^-$, which has $3+3 = 6$ degrees of freedom, all as expected.

One can similarly decompose the A^{ABC} fields, which are antisymmetric on A and B, into

$$\begin{aligned} A^{ABC} &\in (\mathbf{1}^+ \oplus \mathbf{1}^-) \otimes (\mathbf{0}^+ \oplus \mathbf{1}^-) \\ &= \mathbf{1}^+ \oplus (\mathbf{0}^- \oplus \mathbf{1}^- \oplus \mathbf{2}^-) \oplus \mathbf{1}^- \oplus (\mathbf{0}^+ \oplus \mathbf{1}^+ \oplus \mathbf{2}^+). \\ &= \mathbf{0}^- \oplus \mathbf{0}^+ \oplus 2(\mathbf{1}^-) \oplus 2(\mathbf{1}^+) \oplus \mathbf{2}^- \oplus \mathbf{2}^+, \end{aligned}$$

for which the basis is straightforwardly constructed following an analogous approach to that illustrated above.

The bases for higher rank tensors satisfy similar orthonormality and completeness conditions to (2.53) and (2.54),

$$\varepsilon^{*\acute{\alpha}}_{(i_1,J_1^{P_1},m_1)} \varepsilon_{\acute{\alpha},(i_2,J_2^{P_2},m_2)} = P_1 \delta_{i_1,i_2} \delta_{J_1,J_2} \delta_{P_1,P_2} \delta_{m_1,m_2} \tag{2.62}$$

$$\sum_{i,j,P,m} \left(P\, \varepsilon^{\acute{\alpha}}_{(i,J^P,m)} \varepsilon^{*}_{\acute{\beta},(i,J^P,m)} \right) = \mathbb{I}^{\acute{\alpha}}_{\acute{\beta}}, \tag{2.63}$$

where i is the label of the basis in the spin sector J^P, as there might be more than one basis in a sector. The $\acute{\alpha}$ and $\acute{\beta}$ indices are shorthand for some generic indices, such as $\acute{\alpha} = A_1 A_2 ... A_n$.

We can write the basis vectors together with its corresponding column vector $\mathbf{e}_a$ indicating the field (see (2.7)) in bra-ket notation $|i, J^P, m\rangle$,

$$|i, J^P, m\rangle \equiv \varepsilon^{\acute{\alpha}}_{(i,J^P,m)} \mathbf{e}_a, \tag{2.64}$$

$$\langle i, J^P, m| \equiv \varepsilon^{*\acute{\alpha}}_{(i,J^P,m)} \mathbf{e}^{\dagger}_a. \tag{2.65}$$

The SPOs in Appendix 2.A are related with those polarisation basis vectors by

$$\hat{P}_{ij}(J^P) = \sum_m P\, |i, J^P, m\rangle \langle j, J^P, m| \,. \tag{2.66}$$

Note that the bras and kets here do not denote a quantum state, but are used merely to denote the field decomposition in a straightforward manner.

These operators forms a complete and orthonormal basis for the $\mathcal{O}$ operator, as shown in Equations (2.4), (2.5) and (2.14).

We can view $\mathcal{O}$ as a block diagonal matrix with diagonal sub-blocks numbered by J^P, and in each sub-block the matrix indices are i and j. Note that each matrix element possesses the same Lorentz indices as the corresponding $\hat{P}_{ij}(J)$. Note that for any matrices of the form (2.14), they obey,

$$\begin{aligned}
\hat{M}_{ij}(J) &= m_{ij}(J)\hat{P}_{ij}(J), \\
\hat{N}_{ij}(J) &= n_{ij}(J)\hat{P}_{ij}(J), \\
\sum_k \hat{M}_{ik}(J)\hat{N}_{kj}(J) &= \sum_k m_{ik}(J)n_{kj}(J)\hat{P}_{ik}(J)\hat{P}_{kj}(J), \\
&= \sum_k m_{ik}(J)n_{kj}(J)\hat{P}_{ij}(J).
\end{aligned} \tag{2.67}$$

Hence for each J, we can treat $\hat{M}_{ij}(J)$ as a matrix with elements $m_{ij}(J)$ (with out $\hat{P}_{ij}(J)$), and it obeys matrix addition and multiplication rules (and the rules derived from them). Note that the inverse of $\hat{M}_{ij}(J)$ share the same form of $m^{-1}_{ij}(J)$, since the $\sum_k \hat{M}_{ik}(J)\left[m^{-1}_{kj}(J)\hat{P}_{kj}(J)\right] = \delta_{ij}\hat{P}_{ij}(J) = \hat{\mathbb{I}}_{ij}(J)$, and the same for $\sum_k \left[m^{-1}_{ik}(J)\hat{P}_{ik}(J)\right]\hat{M}_{kj}(J)$.

Note that the spin projection operators are Hermitian by construction. From Equations (2.64)–(2.66), we obtain

$$\begin{aligned}
\left[\hat{P}_{ji}(J)\right]^* &= \left(\sum_m |j,J^P,m\rangle\langle i,J^P,m|\right)^* \\
&= \sum_m P\varepsilon^{*\acute{\alpha}_a}_{(j,J^P,m)}\varepsilon^{\acute{\alpha}_b}_{(i,J^P,m)}\mathbf{e}_a\mathbf{e}_b^\dagger \\
&= \sum_m P_{ij}(J^P)^{\acute{\alpha}_b\alpha_a}\mathbf{e}_a\mathbf{e}_b^\dagger,
\end{aligned} \tag{2.68}$$

and therefore

$$P_{ij}(J^P)^{\acute{\alpha}\acute{\beta}} = P^*_{ji}(J^P)^{\acute{\beta}\acute{\alpha}}. \tag{2.69}$$

Appendix 2.C The no-ghost condition [6]

We consider here a Lagrangian quadratic in the real fields $\hat{\varphi}$ with integer spins. Starting from the path integral, we can obtain the vacuum-vacuum amplitude:

$$L[\hat{\varphi}] = \frac{1}{2}\hat{\varphi}^T\hat{\mathcal{O}}\hat{\varphi} \tag{2.70}$$

$$Z_0(J) = \langle 0,\infty|0,-\infty\rangle = \int \mathcal{D}\hat{\varphi}e^{i\int \mathrm{d}^4x\left(L[\hat{\varphi}]-\hat{J}^T\hat{\varphi}\right)}, \tag{2.71}$$

where $\hat{\varphi}$ is a boson field, and $\hat{J}$ is the source current. Fourier transforming the integrand, we obtain,

$$\int \mathrm{d}^4x\left(L[\hat{\varphi}]-\hat{J}^T\hat{\varphi}\right) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\left(\hat{\varphi}^T(-k)\hat{\mathcal{O}}(k)\hat{\varphi}(k) - \hat{J}^T(-k)\hat{\varphi}(k) - \hat{J}^T(k)\hat{\varphi}(-k)\right). \tag{2.72}$$

Now we define $\hat{\psi} \equiv \hat{\varphi} - \hat{\theta}$ and set $\hat{\theta}(k) = \hat{\mathcal{O}}^{-1}(k)\hat{J}(k)$, where $\hat{\theta}$ is independent of $\hat{\varphi}$. Since the fields are real and $\hat{\mathcal{O}}(k) = \hat{\mathcal{O}}^\dagger(k)$ [7], we obtain

$$\int \mathrm{d}^4x\left(L[\hat{\varphi}]-\hat{J}^T\hat{\varphi}\right) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}(\hat{\psi}^\dagger(k)\hat{\mathcal{O}}(k)\hat{\psi}(k) - \hat{J}^\dagger(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(k)). \tag{2.73}$$

Since $\mathcal{D}\hat{\varphi} = \mathcal{D}\hat{\psi}$, the path integral becomes,

$$\begin{aligned} Z_0\left(\hat{J}\right) &= \langle 0,\infty|0,-\infty\rangle = \int \mathcal{D}\hat{\psi}e^{i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}(\hat{\psi}^\dagger(k)\hat{\mathcal{O}}(k)\hat{\psi}(k)-\hat{J}^\dagger(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(k))} \\ &= Z_0(0)\exp\left(-i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\hat{J}^\dagger(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(k)\right). \end{aligned} \tag{2.74}$$

[6] This appendix is adapted from Buoninfante [76] with more clarification.

[7] Let φ and ξ be two different fields. Since there are even derivatives in $\mathcal{O}_{\varphi\varphi}(\partial)$, we obtain $\mathcal{O}_{\varphi\varphi}(k) = \mathcal{O}_{\varphi\varphi}(-k)$. Odd order of derivatives can only exist in the terms with two different fields. The corresponding action has contribution from two $\mathcal{O}$ components:

$$\begin{aligned} S_{\varphi\xi} &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\left[\varphi(-k)\mathcal{O}_{\varphi\xi}(k)\xi(k) + \xi(-k)\mathcal{O}_{\xi\varphi}(k)\varphi(k)\right] \\ &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{2}\varphi(-k)\left[\mathcal{O}_{\varphi\xi}(k) + \mathcal{O}_{\xi\varphi}(-k)\right]\xi(k). \end{aligned}$$

The only relevant value is the sum $\mathcal{O}_{\varphi\xi}(k) + \mathcal{O}_{\xi\varphi}(-k)$, so we have the freedom to set $\mathcal{O}(k)_{\xi\varphi} = \mathcal{O}(-k)_{\varphi\xi}$.

The integrand is proportional to the saturated propagator, so only the saturated propagator is physical. The terms which appear in the propagator but which disappear after sandwiching the currents do not affect the physics. From Equations (2.20) and (2.21), we can write the integrand as

$$i\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k) = \sum_{n_1,n_2\cdots} F_{n_1 n_2\cdots}(\hat{J}(k),k^A)\prod_j \frac{1}{\left(k^2-m_i^2\right)^{n_j}} \tag{2.75}$$

where $F_{n_1 n_2\cdots}(\hat{J}(k),k^A)$ is a polynomial of $\hat{J}(k)$ and k^A (and η_{AB}). Now we let $k^A=(E,\vec{p})$ and $p\equiv\sqrt{\vec{p}^2}$. Restoring the small ε term in the path integral and doing the E integral first, we obtain

$$\begin{aligned} i\int\frac{\mathrm{d}^4k}{(2\pi)^4}\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k) &= i\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^4}\int \mathrm{d}E\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k)\\ &= i\int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^4}\int\mathrm{d}E\sum_{n_1,n_2\cdots}\prod_j\frac{F_{n_1n_2\cdots}(\hat{J}(k),k^A)}{(k^0-(\sqrt{p^2+m_j^2}-i\varepsilon))^{n_j}(k^0+(\sqrt{p^2+m_j^2}-i\varepsilon))^{n_j}}. \end{aligned} \tag{2.76}$$

We choose the contour from $E=-\infty$ to $E=\infty$ on the real axis and go back to $-\infty$ by the lower half circle. The integral on the lower half circle is assumed to be zero, and the poles on the lower half plane is at $E=\sqrt{p^2+m_j^2}-i\varepsilon$. By the residue theorem, we obtain

$$i\int\frac{\mathrm{d}^4k}{(2\pi)^4}\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k) = \int\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\sum_j \mathrm{Res}\left[\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k)\right]_{E=\sqrt{p^2+m_j^2}-i\varepsilon}. \tag{2.77}$$

With the relations $\int_\gamma f(z)dz=\int_{g^{-1}(\gamma)} f\left[(g(s))\right]g'(s)\mathrm{d}s$, $E(k^2)=\sqrt{k^2+p^2}$ around $E=\sqrt{p^2+m_j^2}$, and $\mathrm{Res}\left[f(z)\right]_{z=c}=\frac{1}{2\pi i}\int_{|z-c|=\varepsilon} f(z)\mathrm{d}z$, we obtain

$$\mathrm{Res}\left[\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k)\right]_{E=\sqrt{p^2+m_j^2}} = \frac{1}{2\sqrt{m_j^2+p^2}}\mathrm{Res}\left[\hat{J}^\dagger(k)\hat{O}(k)^{-1}\hat{J}(k)\right]_{k^2=m_j^2}. \tag{2.78}$$

Hence, with the normalisation choice $Z_0(0) = 1$ and making $\varepsilon \to 0$, (2.74) becomes

$$\langle 0,\infty|0,-\infty\rangle = \exp \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_j \frac{-\mathrm{Res}\left\{\hat{J}(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(-k)\right\}_{k^2=m_j^2}}{2\sqrt{p^2+m_j^2}}. \tag{2.79}$$

If $\mathrm{Re}\left\{\mathrm{Res}\left[\hat{J}(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(-k)\right]_{k^2=m^2}\right\} \geq 0$ for all j and $\hat{J}(k)$, then the absolute value of (2.79) is always no more than one. However, if $\mathrm{Re}\left\{\mathrm{Res}\left[\hat{J}(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(-k)\right]_{k^2=m^2}\right\} < 0$ for any j or $\hat{J}(k)$, then the absolute value of (2.79) can be more than one, and the probability for a ground state at $t = -\infty$ remaining in ground state at $t = +\infty$ is greater than one. Since the probability is always positive and sums up to one, the case violates unitarity. Hence the condition for not violating unitarity is

$$\mathrm{Re}\left\{\mathrm{Res}\left[\hat{J}(k)\hat{\mathcal{O}}(k)^{-1}\hat{J}(-k)\right]_{k^2=m_j^2}\right\} \geq 0 \qquad \forall j, \hat{J}(k). \tag{2.80}$$

Note that Equation (2.80) (i.e. requiring no ghost) is only a necessary condition for unitarity. It may still be violated for other reasons.

Chapter 3

Details of the implementation

In this chapter, we will introduce the details of the implementation of our systematic method in Chapter 2. The discussion of these implementation details is inevitably rather lengthy and intricate. Although this material is important for an understanding of how our systematic method has been automated, the reader more interested in its application to PGT$^+$ and WGT$^+$ can safely omit this chapter, since it is not relied upon in the remainder of the thesis.

We use MATHEMATICA 11.1 with the MATHGR package [77] to deal with tensor calculations. We write the program as a package, so we do not have to copy the same codes whenever we are analysing different theories. The package consists of the main part with the functions to calculate the no-ghost-and-tachyon conditions and find the critical cases, a part extending the MATHGR package, a part with tool functions, and a part converting the expressions to LaTeX codes. There is also a part analysing the critical cases further, such as counting the number of no-ghost-and-tachyon and PCR cases and exporting the results to the form used in the webpages. We will introduce only the details of the main part in this chapter and mention only some contents in the other parts, and we leave the part about power-counting renormalisability to Appendix 5.B. The codes shown here are simplified to keep only those relevant to the core process, and we sometimes put sentences in `(*...*)` to replace some trivial steps. We also omit all the codes printing the results in the main part, as well as the codes letting us control the program better, such as time and memory limits. Note that some of the notations in the code are different from those in the other parts of this thesis, such as the order of the tensor indices and some signs of the fields. We will always convert results to the standard notation when they are outputted to LaTeX or webpages.

3.1 The MATHGR package

We are using the MATHGR package to deal with tensor calculations. The tensors are notated in the code, for example, ${X^A}_{BC}$ as `X[UP["A"],DN["B"],DN["C"]]`. It also has the function to display the tensors in the sub/superscript format and turn them to LaTeX codes. We can assign symmetry properties to the tensors and use "`Simp`" to simplify the tensor expressions. For example, we declare the symmetry for the gauge fields at the beginning,

```
DeclareSym[A,{DN,DN,DN},Antisymmetric[{2,3}]];
DeclareSym[a,{DN,DN},Antisymmetric[{1,2}]];
DeclareSym[s,{DN,DN},Symmetric[{1,2}]];
```

where we declare the A-field to be antisymmetric in the second and third indices when the indices are all lowered, $\mathfrak{a}$-field to be antisymmetric, and $\mathfrak{s}$-field to be symmetric, respectively. Note that the order of the indices in the code is different from those in the other parts in this thesis. The Greek indices are at the left of the Latin indices in the code for historical reasons, for example, $A_{\mu AB}$, ${b_\mu}^A$, and ${h^\mu}_A$. We then convert the notations to the correct order at the output stage.

We use the default metric `g` in `gr.m`, and set it to the Minkowski metric by setting its partial derivative to zero,

```
PdT[g[__],_]:=0;
```

and set the spacetime dimension to 4,

```
DefaultDim:=4;
```

The `Simp` function does not raise and lower the indices when contracting tensors with metrics by default, but we can assign the replacement rules to the variable `SimpHook`. Note that the indices in the tensors with partial derivatives can also be raised or lowered by the Minkowski metric. We assign this rule to the variable `SimpHookDefault` so that we can switch the `SimpHook` between this value and the empty rule `{}`. We also add a rule `k[DN[x_]]k[UP[x_]]:>kk` at the end of the `SimpHookDefault` so that the momentum contraction $k_A k^A$ yields k^2 (the symbol `kk`).

Before starting the calculation, we have to set the list of constant parameters in this calculation and set the SPOs. The derivatives of the parameters are then set to zeros, and we will discuss the SPOs in the next section.

3.2 Setting the spin projection operators

We write a set of default SPO in the package so that one can choose one of the default SPO sets. The default SPO contains the A-, $\mathfrak{s}$-, $\mathfrak{a}$-, ϕ- (scalar), and the vector B-field. For the theories not containing all of them, such as PGT$^+$, the package will select their corresponding submatrices of the most general default SPO.

The most general SPO is a list of length 6, which corresponds to $(0^-, 0^+, 1^-, 1^+, 2^-, 2^+)$. The SPOs of the theory we want to calculate, the a- and b-matrices, and the source currents are all arranged in the same way. We also create a list to label the fields corresponding to the rows/columns of the SPOs. The numbers corresponding to the fields are

$$1 : A, 2 : \mathfrak{s}, 3 : \mathfrak{a}, 4 : \phi, 5 : B, \tag{3.1}$$

and for example, PGT$^+$ will have

```
fieldListOld={{1},{1,2,2},{1,1,2,3},{1,1,3},{1,2}},
```

where “Old” means the variable corresponds to the a-matrices, where we have not deleted the rows and columns to fix the gauge.

We then write down the most general default SPO in Θ^{AB}, Ω^{AB} and $\tilde{k}^A = k^A/\sqrt{k^2}$ as in Appendix 2.A and the corresponding fields.

```
POnR[c_,a_,b_,k_,i_,j_]:={{1/3*Θ[c,k]Θ[a,i]Θ[b,j]+2/3*Θ[a,k]Θ↩
↪[b,i]Θ[c,j]}};
fieldOn={1};
```

```
...
tmpP={P0nR,P0pR,P1nR,P1pR,P2nR,P2pR};
tmpF={field0n,field0p,field1n,field1p,field2n,field2p};
```

We then select the fields we will be using for the theory we want to calculate to build the SPO and the list of corresponding fields. For example, for PGT we have

```
containField={1,2,3};
validIdx=Map[Join@@Position[#,Alternatives@@containField]&
  ,tmpF];
PListRawOrigin=MapThread[Function[{c,a,b,k,i,j},
  If[#2==={},{{}},#1[c,a,b,k,i,j][[#2,#2]]]
]&,{tmpP,validIdx}];
fieldListOld=MapThread[#1[[#2]]&,{tmpF,validIdx}];
```

Note that we still have to apply the corresponding symmetries to the indices in the SPO. We make those indices corresponding to A_{CAB} or $\mathfrak{a}_{AB}$ antisymmetric in AB, those corresponding to $\mathfrak{s}_{AB}$ symmetric in AB, and the remaining unchanged. The `AntiSym` and `Sym` can antisymmetrise or symmetrise a tensor with the given indices without multiplying the normalise factor $1/n!$ to it, so we have to put it back.

```
fieldSym={AntiSym,Sym,AntiSym,NoSym,NoSym};
fieldSymListOld=Map[fieldSym[[#]]&,fieldListOld,{2}];
ApplySym2P[PP_,ffx_,ffy_,c_,a_,b_,k_,i_,j_]:=MapThread[
  Function[{PPe,ffxe,ffye},ffye[ffxe[PPe,{i,j}],{a,b}]/4]
,{PP[c,a,b,k,i,j],ffx,ffy},2];
meshgrid[x_List,y_List]:={ConstantArray[x,Length[x]],↩
↪Transpose@ConstantArray[y,Length[y]]};
PListOrigin=MapThread[Function[{PP,ff},Function[{c,a,b,k,i,j},
  If[PP[c,a,b,k,i,j]==={{}},{{}},ApplySym2P[PP,meshgrid[ff,ff↩
  ↪][[1]],meshgrid[ff,ff][[2]],c,a,b,k,i,j]]
]],{PListRawOrigin,fieldSymListOld}];
```

Now we have the SPO elements $P_{ij}(J^P)^{CABKIJ}$ that we will use to obtain a-matrices. We then define Ω_{AB} and Θ_{AB} in terms of k_A and η_{AB} so that the SPOs are now expressed in k_A and η_{AB}.

```
Θ[x_,y_]:= g[x,y]-ks[x]*ks[y];
Ω[x_,y_]:=ks[x]ks[y];
ks[x_]:=k[x]/Sqrt[kk];
```

We also create the lists of the source current for later use. The "L"s, such as the one in σL, are to denote the current is on the left (i.e. the momentum is $-k$).

```
PListOld=Map[#[UP["C"],UP["A"],UP["B"],UP["K"],UP["I"],UP["J"]↩
↪]]&,PListOrigin];
jLeftListOld=Map[#[DN["C"],DN["A"],DN["B"]]&,jListOld,{2}]/.{σ↩
↪->σL,τ-> τL,ϕ-> ϕL,χ-> χL,ζ-> ζL};
jRightListOld=Map[#[DN["K"],DN["I"],DN["J"]]&,jListOld,{2}];
```

3.3 Linearising the Lagrangian

We define the field strengths $\mathcal{R}$, $\mathcal{T}$, $\mathcal{T}^*$, $\mathcal{H}$, and the covariant derivative $\mathcal{D}\phi$ in the package as

```
RR[m_,n_,a_,b_]:=Pd[A1[n,a,b],m]-Pd[A1[m,a,b],n]+↩
↪MetricContract[A1[m,a,DG["RR1"]]A1[n,b,DG["RR1"]]]-↩
↪MetricContract[A1[n,a,DG["RR1"]]A1[m,b,DG["RR1"]]];
T[m_,n_,r_]:=Pd[bb[n,r],m]-Pd[bb[m,r],n]-MetricContract[A1[m,r↩
↪,DG["T1"]]bb[n,DG["T1"]]]+MetricContract[A1[n,r,DG["T1"]]bb[↩
↪m,DG["T1"]]];
Ts[m_,n_,r_]:=T[m,n,r]+g[n,r]B1[m]-g[m,r]B1[n];
HH[x_,y_]:=Pd[B1[y],x]-Pd[B1[x],y];
SetAttributes[𝒟,HoldFirst];
𝒟[ϕ0,x_]:=-ϕ0*B1[x];
𝒟[ϕ1,x_]:=Pd[ϕ1,x]-ϕ1*B1[x];
```

We also define the contractions of the $\mathcal{R}$ tensor

```
RR2[m_,n_]:=(RR[DN[#1],m,DN[#2],n]g[UP[#1],UP[#2]]&@Uq[2]);
RR4:=hh[DN[#1], DN[#3]]*hh[DN[#2], DN[#4]]*
RR[DN[#5], DN[#6], DN[#7], DN[#8]]*g[UP[#1], UP[#5]]*
g[UP[#2], UP[#6]]*g[UP[#3], UP[#7]]*g[UP[#4], UP[#8]] &@Uq[8];
```

Note that we can use η^{AB} to contract $\mathcal{R}^{AB}{}_{\mu\nu}$ because $\mathcal{R}_{AB}$ only appears in the $\sim \mathcal{R}^2$ terms, and only the terms $\sim O(f)$ in $\mathcal{R}^{AB}{}_{\mu\nu}$ can affect the $O(f^2)$ terms in the Lagrangian. Since $\mathcal{R}^{AB}{}_{\mu\nu}$ is already at order $O(f)$, the $f_A{}^{\mu}$ in $h_A{}^{\mu}$ does not make any difference, and we can use $\delta_A{}^{\mu}$ instead of $h_A{}^{\mu}$. However, the scalar $\mathcal{R}$ can appear as $\sim \mathcal{R}^1$ in the Lagrangian as $\sim \det(b) h_A{}^{\mu} h_B{}^{\nu} \mathcal{R}^{AB}{}_{\mu\nu}$. Because $\det(b) h_A{}^{\mu} h_B{}^{\nu} \sim O(1)$ and $\mathcal{R}^{AB}{}_{\mu\nu} \sim O(f)$, the terms at order $O(f)$ in $\det(b) h_A{}^{\mu} h_B{}^{\nu}$ are important after linearisation, and we have to use the full expression. One may wonder whether choosing to linearise $A_{AB\mu}$ or A_{ABC} can give different linearisation in the $\mathcal{R}$ term. The difference is at $O(f^2)$ in the $\partial(A)$ part of $\mathcal{R}^{AB}{}_{\mu\nu}$, and to affect the $O(f^2)$ order of the Lagrangian, the corresponding part of $\det(b) h_A{}^{\mu} h_B{}^{\nu}$ must be $O(1)$, which is $\delta_A{}^{\mu} \delta_B{}^{\nu}$ and can be moved into the partial derivative. Hence, the difference is just a total derivative and does not change the result at order $O(f^2)$ of the Lagrangian.

For each theory, we can then build the Lagrangian in the notebook file with the elements we defined above, for example,

```
LagTTABCABC:=
1/12*(4t1+t2+3λ)*(T[DN[#1], DN[#2], DN[#3]]*T[DN[#4], DN[#5], ↩
↪DN[#6]]*g[UP[#1], UP[#4]]*g[UP[#2], UP[#5]]*g[UP[#3], UP↩
↪[#6]] &@Uq[6]);
LagRRABAB:=(r4+r5)*(RR2[DN[#1], DN[#2]] RR2[DN[#3], DN[#4]]*g[↩
↪UP[#1], UP[#3]]*g[UP[#2], UP[#4]] &@Uq[4]);
(*... *)
LagRaw=LagTTABCABC+LagRRABAB+(*... *);
Lag=bbb*LagRaw;
```

We then declare the parameters in the Lagrangian and which type of theory it is (e.g. PGT$^+$, WGT$^+$, ...), and we can start the main process in the package.

We then linearise the fields. The tetrads are linearised around the Minkowski metric as

$$h_A{}^{\mu} = \delta_A{}^{\mu} + f_A{}^{\mu}, \tag{3.2}$$

while A and B-fields are linearised around zero. The inverse of h becomes

$$b^A{}_{\mu} = \delta^A{}_{\mu} - f_{\mu}{}^A + O\left(f^2\right), \tag{3.3}$$

and its determinant is

$$\det(b) = 1 - f + \frac{1}{2}\left(f^2 - f_{AB}f^{BA}\right) + O(f^3), \tag{3.4}$$

where $f = f^A{}_A$. We expand $\det(b)$ to order $O(f^2)$ in case it is needed, while theories in this thesis only need it up to $O(f)$ order.

```
hh[x_,y_]:=g[x,y]+f[x,y];
bb[x_,y_]:=g[x,y]-f[y,x];
bbb:=(1-ff+1/2*(ff*ff-(f[DN[#1],DN[#2]]f[DN[#3],DN[#4]]g[UP↩
↪[#1],UP[#4]]g[UP[#2],UP[#3]]&@Uq[4])));
ff:=(f[DN[#1],DN[#2]]g[UP[#1],UP[#2]]&@Uq[2]);
f[x_,y_]:=a1[x,y]+s1[x,y];
```

The Lagrangian `Lag` is now expressed by the small fields `A1`, `s1`, `a1`, `ϕ1`, `B1`. We then set the small fields proportional to a parameter t. The terms with t^2 are then quadratic in the small fields.[1]

```
a1[x_,y_]:=t*a[x,y];
s1[x_,y_]:=t*s[x,y];
A1[x_,y_,z_]:=-t*A[x,y,z];
B1[x_]:=t*B[x];
ϕ1:=t*ϕ;
```

[1]There is a minus sign in the linearised A field because the definitions of the field strengths $\mathcal{R}$ and $\mathcal{T}$ are different in the code and the main text. The `A1` in the code is indeed $-A$ in the main text.

Note that we have to declare the Lagrangian parameters and t as constant with respect to partial derivatives.

```
PdT[t|Alternatives@@parameters,_]:=0;
```

3.4 Obtaining a-matrices

We then collect the part with t^2 and do the Fourier transformation. Note that the rule of transformation is $\int \mathrm{d}xX(x)Y(x) = \int \mathrm{d}kX(-k)Y(k)$, so the terms with odd partial derivatives will have sign problems. The terms with odd derivatives are those terms with a field with odd indices (A, B) and one with even indices ($\mathfrak{s}$, $\mathfrak{a}$, ϕ). For these terms, we have to know which field is on the left and which is on the right. We will always follow the "standard field order" (3.1) so that A is always on the left and B is always on the right. Because a field will only carry up to two derivatives in the theories we will study in this thesis, the patterns of the terms with derivatives can only be $\partial^2 X \partial^2 Y$, $\partial^2 XY$, or $\partial X \partial Y$. Both the first two cases have a minus sign, but the third case does not. For the terms with odd derivatives, we have to replace ∂AY with $-ikAY$, $A\partial Y$ with $ikAY$, ∂BY with $ikBY$, and $B\partial Y$ with $-ikBY$. Now we have the terms with two different fields all in the order (3.1), and it will be easier to assign one field to the left in those terms with two same fields. We do this by replacing one of the fields with a different symbol (e.g. `A` $\rightarrow$ `AA`) to mark that it is on the right.

```
SimpHook={Pd[Pd[x_,y_],z_]:> -k[y] k[z]x,Pd[x_,y_]Pd[u_,v_]:> ↩
↪k[y]k[v] x u,Pd[A[x__],y_]z_[u__]:> -I k[y] A[x]z[u],A[x__]↩
↪Pd[z_,y_]:> I k[y] A[x]z,Pd[B[x__],y_]z_[u__]:> I k[y] B[x]z↩
↪[u],B[x__]Pd[z_,y_]:> -I k[y] B[x]z,k[DN[x_]]g[UP[x_],UP[y_↩
↪]]:> k[UP[y]],k[DN[x_]]k[UP[x_]]:> kk};
Lag2=Simp[Coefficient[ReleaseHold[Lag]/.OptionValue[↩
↪LagReplaceRule],t^2]];
SimpHook={A[DN["d"],x__]A[y__]:> A[DN["d"],x]AA[y],s[DN["a"],↩
↪x__]s[y__]:> s[DN["a"],x]ss[y],a[DN["a"],x__]a[y__]:> a[DN["↩
↪a"],x]aa[y],ϕ*ϕ:> ϕ*ϕϕ,B[DN["a"]]B[y__]:> B[DN["a"]]BB[y]};
```

```
Lag2New=Simp[Lag2];
```

The next step is to obtain the operator $\hat{\mathcal{O}}$ in $L = (1/2)\,\hat{\varphi}^{\mathrm{T}} \cdot \hat{\mathcal{O}} \cdot \hat{\xi}$ where $\hat{\varphi}$ and $\hat{\xi}$ are fields, and then decompose it into $\hat{\mathcal{O}} = \sum_{i,j,J,P} a_{ij}(P)\hat{P}_{ij}(J^P)$. To obtain $\hat{\mathcal{O}}$, we first expand the Lagrangian to a summation of operators sandwiched by two fields. We then write a tool function `fixIndices` to fix the dummy indices of the fields to $A(-k)_{CAB}$, $\mathfrak{s}(-k)_{AB}$, $\mathfrak{a}(-k)_{AB}$, $B(-k)_C$, $A(k)_{KIJ}$, $\mathfrak{s}(k)_{IJ}$, $\mathfrak{a}(k)_{IJ}$, and $B(k)_K$. Therefore, the operators sandwiched by the same pair of fields can be summed up directly.

```
Lag2New=toTermList@Lag2New;
LagAANew=fixIndices[Total[Cases[Lag2New,A[__]*AA[___]*_]],{{A↩
↪,{"C","A","B"}},{AA,{"K","I","J"}}},DN];
LagAsNew=fixIndices[Total[Cases[Lag2New,A[__]*s[_,_]*_]],{{A,{↩
↪"C","A","B"}},{s,{"I","J"}}},DN];
(*... *)
OAA=Replace[LagAANew, A[DN["C"],DN["A"],DN["B"]] AA[DN["K"],DN↩
↪["I"],DN["J"]]*z_:> z,All];
(*... *)
```

We can obtain $\tilde{\mathcal{O}}^{\acute{\alpha}\acute{\beta}}_{\varphi\xi}$ in the Lagrangian in the form of $(1/2)\sum_{\varphi\xi}\varphi(-k)_{\acute{\alpha}}\tilde{\mathcal{O}}_{\varphi\xi}(k)^{\acute{\alpha}\acute{\beta}}\xi(k)_{\acute{\beta}}$, where $\tilde{\mathcal{O}}^{\acute{\alpha}\acute{\beta}}_{\varphi\xi}$ is the summation of the "operators" from all the terms with the φ on the left and the ξ-field on the right. We then create a matrix for each sector with the same size as the default most general SPO, with each element in the row corresponding to field φ and the column corresponding to field ξ being $\tilde{\mathcal{O}}^{\acute{\alpha}\acute{\beta}}_{\varphi\xi}$. Note that the value of an element with the index of φ greater than that of ξ is zero by construction. Before we decompose each element with the SPOs, we should note that we can make the *a*-matrices Hermitian. The *a*-matrices are Hermitian if and only if the operator in the Fourier space satisfies $\mathcal{O}_{\xi\varphi}(k)^{\acute{\alpha}\acute{\beta}} = \mathcal{O}_{\varphi\xi}(-k)^{\acute{\beta}\acute{\alpha}}$. If $\varphi = \xi$, then there are only even-derivative terms, and we obtain $\mathcal{O}_{\varphi\varphi}(k)^{\acute{\alpha}\acute{\beta}} = \mathcal{O}_{\varphi\varphi}(-k)^{\acute{\alpha}\acute{\beta}}$. Hence, the operators should satisfy $\mathcal{O}_{\varphi\varphi}(k)^{\acute{\alpha}\acute{\beta}} = \mathcal{O}_{\varphi\varphi}(k)^{\acute{\beta}\acute{\alpha}}$, and we have to symmetrise the raw operators $\tilde{\mathcal{O}}$ to obtain $\mathcal{O}$: $\mathcal{O}^{\acute{\alpha}\acute{\beta}}_{\varphi\varphi} = \left(\tilde{\mathcal{O}}^{\acute{\alpha}\acute{\beta}}_{\varphi\varphi} + \tilde{\mathcal{O}}^{\acute{\beta}\acute{\alpha}}_{\varphi\varphi}\right)/2$. We can do this because symmetrising the raw operator is equivalent to exchanging $k \rightleftarrows -k$ of the fields in a part of the operator, and it does not change the Lagrangian. For those

$\varphi \neq \xi$, the raw operators with the fields obeying (3.1), contain two parts from the operator $\mathcal{O}$, $\tilde{\mathcal{O}}_{\xi\varphi}(k)^{\acute{\alpha}\beta} = \mathcal{O}_{\xi\varphi}(k)^{\acute{\alpha}\beta} + \mathcal{O}_{\varphi\xi}(-k)^{\beta\acute{\alpha}}$, while those with the fields not obeying (3.1) are zeros. Because the Lagrangian is real, we know $\mathcal{O}_{\xi\varphi}(k)^{\acute{\alpha}\beta} = \mathcal{O}_{\varphi\xi}(-k)^{\beta\acute{\alpha}} = \mathcal{O}_{\varphi\xi}(k)^{*\acute{\beta}\acute{\alpha}}$. Therefore, if $\phi\xi$ satisfies (3.1), the operators $\mathcal{O}$ can be obtained from the raw operators by $\mathcal{O}_{\varphi\xi} = \frac{1}{2}\tilde{\mathcal{O}}_{\varphi\xi}$ and $\mathcal{O}_{\xi\varphi} = \frac{1}{2}\tilde{\mathcal{O}}^*_{\varphi\xi}$. Note that because the momentum k and the Lagrangian parameters are real, so we have to simplify the expressions with the real conditions (`realCondition`), or the conjugate will also be applied to them. We also choose only the rows and columns corresponding to the fields we will be using from the most general $\mathcal{O}$-matrix.

```
iRule[c_,a_,b_,k_,i_,j_]:={UP["C"]:> c,UP["A"]:> a,UP["B"]:> b↩
↪,UP["K"]:> k,UP["I"]:> i,UP["J"]:> j,gg[x__]:> g[x]};
OOAA[c_,a_,b_,k_,i_,j_]:= (Replace[OAA,iRule[c,a,b,k,i,j],All↩
↪]+Replace[OAA,iRule[c,a,b,k,i,j],All])/2;
OOAs[c_,a_,b_,k_,i_,j_]:= Replace[OAs,iRule[c,a,b,k,i,j],All];
(*···*)
OS2p[c_,a_,b_,kt_,i_,j_]:=Simplify[{
  {OOAA[c,a,b,kt,i,j],OOAs[c,a,b,kt,i,j]/2}
  ,{OOAs[kt,i,j,c,a,b]\[Conjugate]/2,OOss[c,a,b,kt,i,j]}
},realCondition];
OSListOld={OS0n,OS0p,OS1n,OS1p,OS2n,OS2p};
OSList=MapThread[(Function[{c,a,b,k,i,j},
  If[#2==={},{{}},#1[c,a,b,k,i,j][[#2,#2]]]
])&,{OSListOld,validIdx}];
```

The next step is to put spin projectors at the front and back of $\mathcal{O}$ to obtain $a_{ij}(J^P)$ as in (2.15)[2]

$$P^{[\varphi\varphi]}_{ii}(J^P)_{\acute{\alpha}\acute{\mu}}\mathcal{O}^{\mu\nu}_{\varphi\xi}P^{[\xi\xi]}_{jj}(J^P)_{\nu\acute{\beta}} = \sum_{k,l,J',P'} a_{kl}(J'^{P'})P_{ii}(J^P)_{\acute{\alpha}\acute{\mu}}P_{kl}(J'^{P'})^{\acute{\mu}\acute{\nu}}P_{jj}(J^P)_{\acute{\nu}\acute{\beta}}$$
$$= a_{ij}(J^P)P_{ij}(J^P)_{\acute{\nu}\acute{\beta}}. \tag{3.5}$$

[2]The square brackets in $P^{[\varphi\xi]}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ denote the fields to which the SPO corresponds, which is similar to the parentheses in $P^{(\varphi\xi)}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$. However, the numbering of ij of the former is the same as $P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$, which is different from that in $P^{(\varphi\xi)}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$.

We then divide the left hand side by $P_{ij}(J^P)_{\acute{\nu}\acute{\beta}}$ to obtain $a_{ij}(J^P)$. Note that we have treated $\hat{\mathcal{O}}$ as if $L = \sum \hat{\varphi} \cdot \hat{\mathcal{O}} \cdot \hat{\xi}$, so we recover the factor 2 in the a-amtrices here to obtain $L = \frac{1}{2}\sum \hat{\varphi} \cdot \hat{\mathcal{O}} \cdot \hat{\xi}$.

```
PDiaCol[P_]:=meshgrid[Diagonal[P],Diagonal[P]];
GetFrontBackProject[P_,o_,c_,a_,b_,k_,i_,j_]:=(
  PDiaCol[P[c,a,b,DN[#1],DN[#2],DN[#3]]][[2]]*
  o[UP[#1],UP[#2],UP[#3],UP[#4],UP[#5],UP[#6]]*
  PDiaCol[P[DN[#4],DN[#5],DN[#6],k,i,j]][[1]]&@Uq[6]);
SimpHook=SimpHookDefault;
getaa[OS_,PP_]:=Map[Simp[#,"Method"->"Fast"]&,
  GetFrontBackProject[PP,OS,DN["C"],DN["A"],DN["B"]
    ,DN["K"],DN["I"],DN["J"]],{2}]/
  PP[DN["C"],DN["A"],DN["B"],DN["K"],DN["I"],DN["J"]]//↩
  ↪Simplify;
aaListOrigin=2*MapThread[If[#1[1,2,3,4,5,6]==={{}},{{}}
  ,getaa[#1,#2]]&,{OSList,PListOrigin}];
```

We then obtain the b-matrices by deleting rows/columns. We accomplish this by finding all possible submatrices with the dimensions equal to the rank of the undeleted a-matrices and filter out those submatrices with the correct ranks. We then use the first valid submatrices in all sectors to analyse the unitarity but still save all the possibilities for the PCR analysis. Note that when we are doing a critical case derived from a type A condition, we should replace the a-matrices with the critical condition before we find the nonsingular matrices.

```
getAllNonsingularMatrix[mat_]:=Module[{rank,dim,possibleIdx,↩
↪validIdx},
  If[mat==={{}},Return[{{{{}},{}}}]];
  dim=Length@mat;
  rank=MatrixRank[mat];
  If[rank===0,Return[{{{{}},{}}}]];
  possibleIdx=Subsets[Table[i,{i,1,dim}],{rank,rank}];
  validIdx=Select[possibleIdx,MatrixRank[mat[[#,#]]]===rank&];
```

```
  Map[{mat[[#,#]],#}&,validIdx]
];
aaList=aaListOrigin/.OptionValue[LagReplaceRule];
nonsingularAA=Map[getAllNonsingularMatrix,aaList];
bbList=Map[#[[1,1]]&,nonsingularAA];
bbInvIdxListAll=Map[Function[nonSin,Map[
  {If[#[[1]]==={{}},{{}},Inverse[#[[1]]]],#[[2]]}&
,nonSin,{1}]],nonsingularAA];
```

We can also obtain the inverse b^{-1} and the determinant $\det(b)$.

```
bbInvList=Map[If[#==={{}},{{}},Inverse[#]]&,bbList,{1}];
bbDet=Map[If[#==={{}},Null,Expand@Det[#]]&, bbList,{1}];
```

We can also obtain the saturated propagator expressed in tensors by deleting rows/columns of SPO and source currents to match the b-matrices and sandwiching b^{-1} by the source currents to obtain Π.

```
PList=MapThread[If[#2==={},{{}},Part[#1,#2,#2]]&
  ,{PListOld,bbIdxList}];
jLeftList=MapThread[Part[#1,#2]&,{jLeftListOld,bbIdxList}];
jRightList=MapThread[Part[#1,#2]&,{jRightListOld,bbIdxList}];
SimpHook={};
Π=Total[MapThread[If[#1==={},0,Simp[(#1.(#2*#3).#4)]]&
  ,{jLeftList,bbInvList,PList,jRightList}]];
```

With the determinants, we can obtain the critical parameters as described in Section 2.2.3. We first factorise the coefficients of k^2 in the determinants of all spin-parity sectors, collect the distinct factors, and use `Solve` to obtain the replacement rules for the factors. We can use the tool function `isRealCondsEquiv` to check whether two conditions are equivalent if all the parameters and coefficients are real.

```
isRealCondsEquiv[e1_,e2_]:=Reduce[!Xor[e1,!e2],Reals]===False;
```

The idea is that two conditions are equivalent if and only if one of the conditions together with the complement of the other condition cover the whole domain, and there is no overlap. Note that theoretically, in some extreme situations, the function may return `False` for two equivalent conditions. However, we do not find this kind of "false-negative" in our cases, and the consequence of the false-negative does not affect the physical results.

```
getAllPossibleParas[det_,masses_]:=Module[{aP,paras,ans,coeffs↩
↪,typeOfParas},
  det=det/.{Null-> 0};
  aP=Map[(Select[FactorList[#][[All,1]],!NumericQ[#]&])&
    ,Cases[CoefficientList[#,kk],Except[0]]&/@det,{2}];
  paras=DeleteDuplicates[Sort@Flatten@allPossible
    ,isRealCondsEquiv[#1==0,#2==0]&];
  coeffs=(trimHeadZeroes/@(CoefficientList[#,kk]&/@dets))↩
  ↪/.{{}->{0}};
  typeOfParas=(*Get the type of each parameter*)
  ans=<||>;
  (ans[#]=Sort@Cases[typeOfParas,{_,#}][[All,1]])&
    /@{"A","B","C","D"};
  ans["All"]=Sort@Join[ans["A"],ans["B"],ans["C"],ans["D"]];
  ans
];
```

To obtain the type of each parameter, we use the replacement rule of each factor to substitute the coefficients. We first check whether the factor is linear by checking whether the factor is a polynomial of the Lagrangian parameters, and the total power of the parameters in each term is one. If not, the nonlinear critical parameters exist, and our algorithm will not hold. We mark these parameters as "type D". If there is no nonlinear factor and all of the coefficients in any sector become zero after the substitution, then the factor is type A. If it is not type A and the lowest coefficient in any sector becomes zero, then it is type B. If the factor is not type A or B and the highest coefficient in any sector becomes zero, then the factor is type C.

```
typeOfParas=Map[Function[p1,Module[{turnToZero,isA,isB,isC,↩
↪type},
  turnToZero=Map[MapThread[#1=!=0&&FullSimplify[#2]===0&,#]&
    ,Transpose@{coeffs,coeffs/.(Solve[p1==0][[1]])}];
  isD=!ruleListLinearHomoQ[{p1-> 0}];
  isA=Or@@Map[And@@#&,turnToZero];
  isB=Or@@Map[#[[1]]&,turnToZero];
  isC=Or@@Map[#[[-1]]&,turnToZero];
  type=If[isD,"D",If[isA,"A",If[isB,"B",If[isC,"C",False]]]];
  {p1,type}
]],paras];
```

We can then obtain the child additional condition for the theory and put together the child and sibling additional condition to obtain the total additional condition. We will explain how the sibling additional conditions are obtained later.

```
parasAll=getAllPossibleParas[bbDet,bbDetRootNonZero];
addConds=Simplify[(And@@Map[#!=0&,parasAll["All"]])&&↩
↪addCondsSibling];
```

3.5 Source constraints

To obtain the no-ghost condition for the massless sector, we have to obtain the source constraint first. We obtain the source constraints (2.17) in the form of tensor equations and then turn it into component equations. We first obtain the null left eigenvectors of the a-matrices, and this is equivalent to getting the columns of the nullspace matrices of the transposes of the a-matrices. We then obtain the left hand side of (2.17) in tensor expressions.

```
nullLeftVec=Map[If[#==={{}},{},NullSpace[Transpose[#]]]&
  ,aaList]//Simplify;
jTemp=Map[#[DN["c"],DN["a"],DN["b"]]&,jListOld,{2}];
```

```
scTen=Simp/@Flatten@MapThread[Function[{P,j,nv},Map[
  Function[v,P[DN["k"],DN["i"],DN["j"],UP["c"],UP["a"],UP["b"]↩
  ↪]].(v*j)]
,nv,{1}]],{PListOrigin,jTemp,nullLeftVec},1];
```

We then turn the tensor expression of the left hand side of (2.17) into component equations. The tool function getTensorValue turns a tensor expression into component expressions, and we appoint that the momentum is pointing to $\hat{z}$ so that $k^A = (E,0,0,p)$. We also write all k^2 as $E^2 - p^2$ here, and E and p are denoted as k0 and kr in the code respectively. The tool function solveHomoLinearSet then converts the set of linear homogeneous equations, the component equations of the source constraints, into matrix expression (2.26) and find the solution (2.27) in terms of the free variables X_i. We are requiring here that the coefficients of X_i should not contain any factor $\sim 1/(E-p))$ so that we do not bring in spurious poles.

```
scVal=Map[getTensorValue[#,Massless-> False,Dir-> 3]&
  ,scTen]/.{kk-> (k0^2-kr^2)};
sceqns=(#==0&)/@DeleteCases[Flatten@Map[#["Mat"]&
  ,(Flatten@scVal)],0];
sourceConstraintsRule=solveHomoLinearSet[sceqns,cComps,↩
↪NullVectorNoFactors-> {k0-kr}]["Rules"];
```

We then save the result in a map (Association) with the key as the null vector. When we are evaluating the critical cases later, we can check whether the source constraints of the same null vector has been already solved and just read the saved result.

3.6 Massless sector

We then turn the saturated propagator into tensor components with the same setting as the source constraints and replace the source current components with the free variables X_i by (2.27). After substituting, we have to expand the complex conjugates with the fact that the momentum is real. The tool function getBilinearTensorMatrix then extracts the **M**-matrix in (2.28) from the saturated propagator by direct calculation.

```
PiValue=getTensorValue[Π/.{σL[x__]-> Conjugate[σ[x]], (*Similar ↩
↪ for the other currents *)},Massless->False,Dir-> 3]["Mat"][[1]];
PiSimp=FunctionExpand[PiValue/.sourceConstraintsRule,{k0∈Reals↩
↪,kr∈Reals,k0^2>kr^2}];
PiMat=getBilinearTensorMatrix[Expand[(PiSimp/.{kk-> k0^2-kr↩
↪^2})],(*The pattern of Xi*)];
```

The next step is to obtain the Q_{2n}-matrices in (2.29). In the theories we are studying, n can be at most 3, so we only have to find Q_6, Q_4, and Q_2. We first apply (2.30) and (2.31) to obtain those $n > 1$.

```
PiMat6=Map[Limit[(k0-kr)^3(2kr)^3 #,k0-> kr]&,PiMat,{2}];
PiMat4=Map[Limit[(k0-kr)^2(2kr)^2 #,k0-> kr]&,(PiMat-PiMat6/((↩
↪k0-kr)(2kr))^3),{2}];
```

We then apply (2.32), and requiring the additional condition at the same time because sometimes the results of (2.32) may conflict the additional conditions. We are using `Solve` here because we can then use the result to simplify Q_2.

```
PiMat46Eles=DeleteDuplicates@Flatten[Join[PiMat4,PiMat6]];
k4k6CondSolve=Quiet[Solve[ForAll[kr,kr!=0,addConds&&And@@Map↩
↪[#==0&,PiMat46Eles]],parameters,MaxExtraConditions->All,↩
↪Method->Reduce],Solve::svars];
```

We then extract the replacement rules to `k4k6Rule`. Note that the equation set may have multiple solutions, so `k4k6Rule` is a list of rules, and we make it like `{{{(*rules1*)}},{{(*rules2*)}},...}`. If there is no solution, then `k4k6Rule` will be `{{}}`. Each solution may only be valid when some extra conditions hold, and we save them in `k4k6AddCond`. Note that it contains both additional conditions and the extra conditions of the solution, but this does not matter because we will put everything together eventually.

Now we obtain Q_2 and use `k4k6Rule` to simplify it. If there is no solution for `k4k6Rule`, we will keep Q_2 unchanged if we want to do the remaining calculation in any case. After the replacement by `k4k6Rule`, there are sometimes infinities in Q_2. This is because the expressions are generally very complicated, and for example, MATHEMATICA sometimes cannot find some structures like x/x. If we set $x \to 0$ then the result will be `Indeterminate`. If any problem about infinity happens, we will undo the replacement. We then find the eigenvalues for each of Q_2, and the number of nonzero eigenvalues is the d.o.f. of massless propagating mode. We store all the d.o.f. gotten by using different solutions of `k4k6Rule` in a list, but we expect that all the d.o.f. should be the same.

```
PiMat2=Map[Limit[(k0-kr)(2kr)#,k0-> kr]&,(PiMat-PiMat6/((k0-kr↩
↪)(2kr))^3-PiMat4/((k0-kr)(2kr))^2),{2}];
k4k6RuleUsing=If[k4k6Rule==={{}},{{{}}},k4k6Rule];
Pikm2MatWithk4k6Cond=Quiet[Map[
  Simplify[PiMat2/.#[[1]]]&,k4k6RuleUsing]
,{Power::infy,Infinity::indet,Simplify::infd}];
Pikm2MatWithk4k6Cond=Map[If[Length@Cases[Flatten@#,Infinity|-↩
↪Infinity|ComplexInfinity|DirectedInfinity[___]|Indeterminate↩
↪|__*Infinity]>0,PiMat2,#]&,Pikm2MatWithk4k6Cond];
Pikm2Eigenvalues=Map[Eigenvalues[#]&,Pikm2MatWithk4k6Cond];
masslessDim=Map[Length@Cases[#,x_/;x=!=0]&,Pikm2Eigenvalues];
```

Before we solve (2.33), we have to note that some eigenvalues are roots of some high degree polynomials (≥ 2) multiplied by a rational function of the Lagrangian parameters. Solving inequalities with these complicated expressions may block the process for a very long time ($\gtrsim$ a few weeks until we gave up). However, we can transform the inequality into a different form to speed it up. For the non-rational roots of a polynomial with degree ≥ 3, MATHEMATICA does not calculate them explicitly by default. Instead, they are shown as `Root[ploy,# of the roots]`. We write a tool function `getAllNonNegRootsCond` to convert the condition in the form $c \cdot$`Root[...]`≥ 0 to the form as in (2.34) with some modification to include the "$= 0$" part. The function first extract the coefficient of the polynomial equation and call the function `allRootNonPosCond` which implements (2.34) to obtain the simplified condition. Note that (2.34) is easier to be implemented if we set

$x \to -x$ so that the condition becomes requiring non-positive roots, and we do not have to deal with the $(-1)^{n-i}$ factors in (2.34).

```
allRootNonPosCond[coeff_List]:=Module[{},
  (And@@Map[#>0&,coeff] )|| Or@@Table[
    (And@@Map[#==0&,coeff[[i+1;;-1]]]&&
    And@@Map[#>0&,coeff[[1;;i]]])
  ,{i,1,Length[coeff]-1}]];
getAllNonNegRootsCond[c_ Root[f_,n_]]:=Module[{var,cl,alt},
  var=Unique[X];
  cl=Reverse@CoefficientList[f[var],var];
  alt=Table[(-1)^i,{i,0,Length[cl]-1}];
  (c==0)||(c>0&&allRootNonPosCond[cl*alt])||
  (c<0&&allRootNonPosCond[cl])];
getAllNonNegRootsCond[Root[f_,n_]]:=getAllNonNegRootsCond[2 ↩
↪Root[f,n]];
getAllNonNegRootsCond[x_]:=x>=0;
```

There are also expressions with square roots in the non-rational roots of quadratic polynomials, and they may slow down the calculation, too. Similarly, we use the tool function `getQuadNonNegCond` to convert these roots into the form of (2.34). Note that by construction the roots are eigenvalues of Hermitian matrices, so they are automatically real, and we do not have to require that again.

```
getQuadNonNegCond[x__+y_*Sqrt[z__]]:=Module[{X,Z},
  X=Total@{x};Z=Total@{z};
  Simplify[X^2-y^2*Z>=0&&X>=0]
]
getQuadNonNegCond[y_*Sqrt[z__]]:=(y^2*z<= 0);
getQuadNonNegCond[Sqrt[z__]]:=(z==0);
getQuadNonNegCond[x_]:=x>=0;
```

Now we can apply the functions above and obtain the condition (2.33) requiring the eigenvalues of $\mathbf{Q}_2$ to be non-negative, with each of the solutions of `k4k6Rule`, respectively.

```
Pikm2EvWithRoots=Map[Select[#,Length@Cases[#,Root[__]]>0&]&
  ,Pikm2Eigenvalues];
Pikm2EvNoRoots=(*Same but >0 → ===0*)
k2CondList=MapThread[{
  Map[ForAll[kr,kr!=0,#]&,getQuadNonNegCond/@Expand[#1]],
  Map[ForAll[kr,kr!=0,getAllNonNegRootsCond@#]&,#2]
}&,{Pikm2EvNoRoots,Pikm2EvWithRoots}];
```

Theoretically, we can solve all the Q_2 conditions with the same `k4k6Cond` solution together and then combine the results with `Or`. However, in many cases this method gets stuck. Instead, we write a tool function to speed up the calculation. The main idea is that if any condition in the conditions connected by `And` is `False` (no solution), then the connected condition will be `False` as well. We do this in two rounds of calculations. In the first round, we evaluate each condition one by one and set a reasonable time limit for calcualtion. If a calculation finishes before the time limit, we will check whether it is `False` or not. If it is `False`, then we stop the proccess and return the whole condition as `False`. If the calculation does not finish in time, we will stop it and mark the condition as "unevaluated". If none of the condition is `False` after the first round, we then collect the finished parts of the first round, evaluate them together, and check whether it is `False`. If not, we then evaluate the unevaluated part and solve everthing together. We also make the function able to let us know where the `False` comes from. It will also check whether the conditions conflict with additional condition and return `False` earlier if so. We do not put the codes here because of their length.

Now we can put everything together and obtain the massless no-ghost condition and check whether it conflicts with the additional condition again.

```
MasslessCond=Simplify[Or@@MapThread[#1&&#2&
  ,{k2CondList,k4k6CondList}]];
MasslessCondAddCond=Reduce[MasslessCond&&addConds,Reals];
```

3.7 Massive sector

To obtain the no-ghost-and-tachyon conditions for the massive sector, we have to obtain the masses in each spin-parity sector first. They are simply the nonzero roots of the determinants of b-matrices.

```
bbDetRoot=Map[If[#===Null,{}
  ,{ToRules@Roots[Expand@#==0,kk]}]&,det];
bbDetRootAll=Map[(kk/.#)&,bbDetRoot];
bbDetRootNonZero=Map[Select[#,(!(# ===0)&)]&,bbDetRootAll];
massiveDim=Length/@bbDetRootNonZero;
```

We then evaluate the no-tachyon, no-ghost, and the combined conditions in each sector straightforwardly. We can mark the massive condition as `False` and stop the evaluation if any of the no-ghost or no-tachyon condition in any sector is `False`. The codes for each sector are in outline,

```
ResAtMasses[mat_,m2s_]:=Map[Residue[Tr[mat],{kk,#}]&,m2s];
parityList={-1,1,-1,1,-1,1};
multParity[listEle_,parity_]:=Map[#*parity>0&,listEle];
(*For each sector *)
(*no-tachyon*)=Map[#>0&,(*masses*)];
bbInvTrRes=ResAtMasses[(*b^-1*),(*masses*)];
(*no-ghost*)=multParity[bbInvTrRes,(*parity *)];
(*combined*)=Reduce[(And@@ans["ghost"])&&(And@@ans["tachyon"]),↩
↪Reals];
```

However, for the unsimplified root theory of full WGT$^+$, which is the most complicated case we will study in this thesis, MATHEMATICA may be stuck in these steps. We can still use another way to obtain some relatively simple expressions of the massive conditions when we need them. The 1^- sector causing problems has a 3×3 b-matrix with two nonzero masses.

Its determinant is of the form

$$\begin{aligned}\det(b) &= k^{2n}\left(Ak^4+Bk^2+C\right)\\ &= Ak^{2n}\left(k^2-m_+^2\right)\left(k^2-m_-^2\right),\end{aligned} \tag{3.6}$$

where $m_\pm^2 = \left(-B\pm\sqrt{B^2-4AC}\right)/(2A)$ are the roots of Ak^4+Bk^2+C. Note that $A\neq 0$ and $C\neq 0$, or it will violate the additional condition and becomes a critical case. The no-tachyon conditions are

$$-AB>0, AC>0, B^2-4AC>0, \tag{3.7}$$

where we also require the roots to be real and distinct.

From Cayley–Hamilton theorem we can write b^{-1} as

$$b^{-1} = \frac{1}{\det(b)}\left\{\frac{1}{2}\left[(\operatorname{tr}b)^2-\operatorname{tr}\left(b^2\right)\right]\mathbb{I}-b\operatorname{tr}b+b^2\right\}, \tag{3.8}$$

and so

$$\begin{aligned}\operatorname{tr}\left(b^{-1}\right) &= \frac{1}{2\det(b)}\left[(\operatorname{tr}b)^2-\operatorname{tr}\left(b^2\right)\right]\\ &= \frac{1}{2Ak^{2n}\left(k^2-m_+^2\right)\left(k^2-m_-^2\right)}\left[(\operatorname{tr}b)^2-\operatorname{tr}\left(b^2\right)\right].\end{aligned} \tag{3.9}$$

Because each element of b is a polynomial of ik and b is Hermitian, $\left[(\operatorname{tr}b)^2-\operatorname{tr}\left(b^2\right)\right]$ can be written as

$$\left[(\operatorname{tr}b)^2-\operatorname{tr}\left(b^2\right)\right]\equiv Dk^4+Ek^2+F. \tag{3.10}$$

The residues at the poles are then

$$\operatorname*{Res}_{k^2=m_\pm^2}\operatorname{tr}\left(b^{-1}\right) = \frac{Dm_\pm^4+Em_\pm^2+F}{2Am_\pm^{2n}\left(m_\pm^2-m_\mp^2\right)}. \tag{3.11}$$

Because $m_\pm$ are roots of Ak^4+Bk^2+C, we can use $Dm_\pm^4+D\frac{B}{A}m_\pm^2+D\frac{C}{A}=0$ to simplify the numerator and obtain

$$\operatorname*{Res}_{k^2=m_\pm^2} \operatorname{tr}\left(b^{-1}\right) = \frac{\left(E-D\frac{B}{A}\right)m_\pm^2+\left(F-D\frac{C}{A}\right)}{2Am_\pm^{2n}\left(m_\pm^2-m_\mp^2\right)}. \tag{3.12}$$

The no-ghost conditions (2.42) are then

$$P\left[(AE-BD)m_\pm^2+(AF-CD)\right]\gtrless 0, \tag{3.13}$$

where m_+^2 corresponds to $>$ and m_-^2 correspond to $<$. We can use the relation $X>0, Y<0 \Leftrightarrow X-Y>0, XY<0$ to get rid of the square roots in $m_\pm^2$. The no-ghost condition then becomes

$$P(AE-BD)>0 \tag{3.14}$$

$$(AF-CD)^2-\frac{B}{A}(AF-CD)(AE-BD)+\frac{C}{A}(AE-BD)^2<0. \tag{3.15}$$

We can then substitute the actual value of A, B, C, D, E, and F to obtain the no-ghost condition in terms of the Lagrangian parameters.

Now we can combine the massless and massive conditions to obtain the total no-ghost-and-tachyon conditions.

3.8 Critical cases and parallelisation

We then try to find all the descendant critical cases of the root theory. The tree-like structure can be easily implemented in object-oriented programming (OOP), but MATHEMATICA is not an OOP language. However, we can still emulate some essential functions of object and class in MATHEMATICA[3]. A critical case (including the root theory) is stored in a "node". To create a node, we just have to call `new`.

```
Module[{parent,children,value,flag,bTree},
```

[3]This idea is from https://stackoverflow.com/questions/6097071/

```
  children[_]:={};
  value[_]:=Null;
  parent[_]:={};
  bTree[_]:=Null;
  node/:new[node[]]:=node[Unique[]];
  node/:new[node[v_]]:=Module[{obj},
    obj=node[Unique[]];obj.setValue[v];obj];
  node/:node[tag_].(*function *)
(*......  *) ];
```

Each node can have multiple parents and children. We can also define member functions to build the relations between the nodes and access the value of the node. The values contain information about the critical condition, no-ghost-and-tachyon condition, additional condition, and some results of the intermediate steps. We also store the information about the critical parameter, its type, and the sibling additional condition between the node and each of its parents.

```
(*In ......  in the previous block*)
(*Only showing parts after ‘‘ node/: node[tag_].’’ for each function *)
addParent[pt_,path_] := (
  parent[tag]=Append[parent[tag],<|"node"->pt,"path"->path|>];
  pt.getTree[].addNode[node[tag]];);
getParents[] := parent[tag];
getChildren[] := children[tag];
addChild[child_node,path_] := (
  child.addParent[node[tag],path];
  children[tag] = Append[children[tag], child];);
getValue[] := value[tag];
setValue[val_] := value[tag] = val;
getTree[] := bTree[tag];
setTree[val_] := bTree[tag] = val;
```

All the nodes derived from the same root theory belong to a "tree". Besides storing its root node in a tree, we also save all nodes in a list in it so that we do not need to traverse all the nodes when we need them. It is accomplished by adding a new node to the list when it is added as a child to a node in the tree.

```
Module[{root,nodes},
  root[_] := Null;
  nodes[_] := {};
  tree/:new[tree[]]:=tree[Unique[]];
  tree/:new[tree[n_]]:=Module[{obj},
    obj=tree[Unique[]];obj.setRoot[n];obj];
  tree/:tree[tag_].setRoot[nd_node]:=(
    root[tag] = n;tree[tag].addNode[n]);
  tree/:tree[tag_].getRoot[]:=root[tag];
  tree/:tree[tag_].getNodes[]:=nodes[tag];
  tree/:tree[tag_].addNode[n_node] := (
    nodes[tag]=If[!MemberQ[nodes[tag],n],n.setTree[tree[tag]];
    Append[nodes[tag], n],nodes[tag]]; );
];
```

Now we can find the critical cases and evaluate their no-ghost-and-tachyon conditions recursively. Start from the root node, we divide its critical parameters into several groups, and the critical parameters in each group has the same type. We also require that all type C parameters are in the same group. We will call these groups "missions", and results of the parent node "model". The function below can evaluate the no-ghost-and-tachyon condition and repeat the process recursively for each mission.

```
doMission[mission_,model_,(*options*)]:=Module[{(*... *)},
  (*Deal with critical parameters, additional conditions *)
  (*Calculate unitary conditions for the child nodes,
  and do the critical cases of the child nodes*)];
```

After the evaluation of the root theory, we can call `doMission` in series for each mission, and all critical cases will be found. We can also divide the missions evenly into some sets and use `ParallelEvaluate` to evaluate these sets in series in the parallel kernels at the same time. This is the simplest way to parallelise the calculation of critical cases. While the evaluation of some sets will finish earlier than the others causing wastes of the computation power of these kernels, it is enough for our purpose to use this simple method rather than making the code more complicated. Before starting the evaluation, We also set some functions as shared functions by `SetSharedFunction` so that they are executed on the master kernel synchronously. These functions include `requestDoRule` and the functions accessing the map of the source constraints. Furthermore, because `node` and `tree` are not working properly in the parallel kernels, we write some shared functions and will always access them through these shared functions. Because the parallel kernels in MATHEMATICA use their own memory, we have to distribute the definitions of the functions and variables except those related to `node` and `tree` to the parallel kernels by `DistributeDefinitions`.

```
sets={(*Evenly gourped missions*)};
rootModel=<|(*results  of the root*)|>;
ParallelEvaluate[If[Length@mission>=$KernelID,
  doMission[#,rootModel,opt]&/@sets[[$KernelID]]]
];
```

In `doMission`, we first “combine” the critical parameters is the mission is type C. This is done by finding all possible combinations of the parameters and eliminate the empty set. If the mission is type A or B, then each critical parameter is itself a set. The sibling additional condition corresponding to each set is then the complement of the set with respect to all critical parameters of the parent node. We then use `Solve` to obtain the replacement rule of each set of the critical condition with its sibling additional condition. If there is no solution for a set, then it means that the critical condition contradicts its additional condition, and we remove the set. This will only happen when the mission is type C because each set in type A or B mission contains only one parameter, and the contradiction can only happen when the parameter is proportional to one of the sibling additional condition, which cannot happen.

```
(*Deal with critical parameters, additional conditions *)
allParaSets=If[mission["type"]==="C",
  DeleteCases[Subsets[mission["paras"]],{}],
  Map[{#}&,mission["paras"]]];
allAddConds=Map[Function[paraSet,And@@Map[#!=0&,
  Complement[model["degenParas"]["All"],paraSet]
]],allParaSets];
allRuleObjs=MapThread[Module[{svRule},
  svRule=Solve[mission["oldParas"]==0&&#1==0&&#2];
  <|"rule"-> svRule,"paras"-> #1,"addConds"-> #2|>
]&,{allParaSets,allAddConds}];
allRuleObjs=DeleteCases[allRuleObjs,<|"rule"-> {},__|>];
```

We then deal with each critical parameter set. We first reset the values such as a and b-matrices to the values of the parent node. A new node is created, as well as a path with the information between the new node and the parent node, such as the critical parameter(s) and the type, all the critical parameters accumulated from the root except the current one(s), sibling additional condition, all the critical parameters of the parent node. Before evaluating the unitary conditions for each set, we check whether the theory with the same critical conditions is already evaluated. The function `requestDoRule` converts all equivalent critical conditions to a unique form and returns whether the evaluation of the critical condition has been started or not. If so, we add the existing node as a child of the parent node, skip the evaluation and go to the next set; otherwise, we add the new node instead and evaluate its unitary condition. If the mission is type A or B, we then find the critical cases of the new node and repeat the process. Hence, the process goes recursively until all the critical cases are found.

```
Do[
  rule=allRuleObjs[[i]]["rule"]; (*Same for paras, addConds*)
  (*setting the a, b−matrices, etc to the values in ''model''*)
  nodePath=<|(*info of the relation between the node and parent *)|>;
```

```
  If[!requestDoRule[rule,condNodeK=newNode[]],
    condNodeK=getNodeFromRule[rule];
    nodeAddChild[model["cNode"],condNodeK,nodePath];
  ,
    nodeAddChild[model["cNode"],condNodeK,nodePath];
    doCondsWithRule[rule,paras,mission["type"],model,mission["↩
    ↪oldParas"],AdditionalCondition-> addConds,opt];
    If[mission["type"]=!="C",doSubCritical[{rule,paras},↩
    ↪mission,i===Length@allRuleObjs,opt]];
  ];
,{i,Length@allRuleObjs}];
```

In the function `doCondsWithRule`, we run the codes in the previous sections again to obtain the no-ghost-and-tachyon conditions for the node. If the mission is type A, then we have to replace the a-matrices with the `rule`, obtain the b-matrices and source constraints, and obtain the massless as well as massive no-ghost-and-tachyon conditions. For the type B missions, we can reuse the a, b-matrices and the saturated propagator of its parent node and replace them with the `rule`, but we still have to run the codes of the massless and massive parts again. As for the type C missions, we can replace everything except the masses and massive condition with the `rule` and re-evaluate the massive condition. However, if the massless condition is already `False`, then the total condition should be `False` as well, and we do not need to process further. We then set the result to the value of the node. Note that we are indeed setting time and memory limit in the steps evaluating the conditions, which we do not show in the codes. If some step runs over the time or memory limit, then the results are incomplete, and we mark the massless and massive conditions as skipped. For the type C missions of a skipped node, we will also skip them because the massless condition is invalid.

```
doCondsWithRule[rules_,newParas_,type_,model_,oldParas_,(*... *)↩
↪]:=Module[{conds},
  Switch[type,
  "A",
```

```
    (*Get b–matrices with ‘‘ rule ’’ *)
    (*Get Source constraints , massless, massive conditions with ‘‘ rule ’’ *),
  "B",
    (*Replace a, b–matrices related variables and Π with ‘‘rule’’ *)
    (*Get massless, massive conditions with ‘‘ rule ’’ *),
  "C",
    (*Replace a, b–matrices related variables and Π with ‘‘rule’’ *)
    (*Replace massless conditions with ‘‘ rule ’’ *)
    If[MasslessCond===False,
      conds=(*Total and massless conditions are False*),
      (*Get massive conditions with ‘‘ rule ’’ *)];];
  nodeSetValue[condNodeK,(*Results, note that the accumulated critical ↩
  ↪ parameters are Join[oldParas, newParas]*)];
];
```

In the function `doSubCritical`, we first set the a, b-matrices, masses, and the saturated propagator to the correct value if the evaluation of the condition for the node run out of time or memory. We then find the new critical parameters, the new accumulated critical parameters, and save the information for the current node to `model`. We can then call `doMission` to evaluate the child nodes of the current node. Note that we should avoid accessing the nodes because it will occupy the main kernel to do this.

```
doSubCritical[rule,paras_,mission_,(*options*)]:=Module[{oldParas↩
↪,model},
  (*If the evaluation run out of time/memory,set variables to correct values*)
  paras=getAllPossibleParas[bbDet,bbDetRootNonZero];
  nodeSetValue[condNodeK,(*add ‘‘paras’’ into the value*)];
  oldParas=Flatten[{paras,mission["oldParas"]}];
  model=(*Save the current results to ‘‘model’’*);
  If[Length@paras[#]>0,doMission[Mission[oldParas,paras[#],#],↩
  ↪model,opt]]&/@{"C","A","B"};
];
```

Chapter 4

Ghost and tachyon free PGT

In this chapter, we will first construct and introduce PGT in more detail in Section 4.1 and then apply the method in Chapter 2 to it and compare our results with those previously presented in the literature.

4.1 Poincaré Symmetry and Poincaré gauge theory

In this section, we will gauge the Poincaré symmetry mainly following the method in [18] and construct the local Poincaré invariant matter and gravitational actions.

In a Minkowski spacetime M_4, we can choose a global coordinate x^μ which has the metric $\eta_{\mu\nu}$. If we perform an infinitesimal coordinate transformation,

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{4.1}$$

then the metric transforms as

$$\eta_{\mu\nu} \to g'_{\mu\nu}(x) = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\eta_{\rho\sigma} \approx \eta_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu}) \tag{4.2}$$

Now we require the new metric to be flat, $g'_{\mu,\nu}(x) = \eta_{\mu\nu}$. The most general solution $\zeta^\mu(x)$ to (4.2) has the form,

$$\xi^\mu(x) = \varepsilon^\mu + \omega^\mu{}_\nu x^\nu, \tag{4.3}$$

where $\omega^{\mu\nu}$ is antisymmetric. Here ε^{μ} generates translation, and $\omega^{\mu\nu}$ generates Lorentz transformation. We then write the infinitesimal transformation rule for an arbitrary field $\varphi(x)$ which may carry spatial or spin indices (which we suppress) as

$$\varphi(x) \to \varphi'(x') = \left(1 + \frac{1}{2}\omega^{\mu\nu}\Sigma^{\varphi}_{\mu\nu}\right)\varphi(x), \tag{4.4}$$

where $\Sigma^{\varphi}_{\mu\nu}$ is the spin part of the Lie algebra generator and carries indices correspond to $\varphi(x)$. For example, $\Sigma_{\mu\nu}$ for a Dirac spinor one has $\Sigma^{D}_{\mu\nu} = \frac{1}{4}[\gamma_{\mu}, \gamma_{\nu}]$, for a vector field V^{ρ} one has $(\Sigma^{1}_{\mu\nu})^{\lambda}{}_{\rho} = \delta^{\lambda}_{\mu}\eta_{\nu\rho} - \delta^{\lambda}_{\nu}\eta_{\mu\rho}$, and for a rank-2 tensor $T^{\mu\nu}$ one has $(\Sigma^{2}_{\mu\nu})^{\lambda\rho}{}_{\sigma\delta} = (\Sigma^{1}_{\mu\nu})^{\lambda}{}_{\sigma}\delta^{\rho}_{\delta} + (\Sigma^{1}_{\mu\nu})^{\rho}{}_{\delta}\delta^{\lambda}_{\sigma}$. We then compare the transformed and untransformed fields at the same point and get

$$\delta_0\varphi(x) \equiv \varphi'(x) - \varphi(x) = \left[\frac{1}{2}\omega^{\mu\nu}(\Sigma^{\varphi}_{\mu\nu} + L_{\mu\nu}) + \varepsilon^{\mu}P_{\mu}\right]\varphi(x) \tag{4.5}$$

$$= \left[\frac{1}{2}\omega^{\mu\nu}\Sigma^{\varphi}_{\mu\nu} + \xi^{\mu}P_{\mu}\right]\varphi(x) \equiv U^{\varphi}\varphi(x) \tag{4.6}$$

where

$$L_{\mu\nu} = x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}, \quad P_{\mu} = -\partial_{\mu} \tag{4.7}$$

are the generators of Lorentz transformation and translation respectively.

It is convenient to define a local Lorentz frame at each point, with the orthonormal basis $\mathbf{e}_A(x) \equiv h_A{}^{\mu}(x)\mathbf{e}_{\mu}(x)$ satisfying $\mathbf{e}_A \cdot \mathbf{e}_B = \eta_{AB}$, where $\mathbf{e}_{\mu}(x)$ satisfying $\mathbf{e}_{\mu} \cdot \mathbf{e}_{\nu} = g_{\mu\nu}$ is the coordinate basis vector. Note that $g_{\mu\nu} = \eta_{\mu\nu}$ here because the spacetime is Minkowski, and so we can choose $h_A{}^{\mu} = \delta^{\mu}_{A}$. The capital Latin indices refer to local Lorentz frames, and the Greek indices refer to coordinate frames. We can also find the inverse $b^{A}{}_{\mu}$ of $h_A{}^{\mu}$ satisfying $b^{A}{}_{\mu}h_B{}^{\mu} = \delta^{A}_{B}$ and $b^{A}{}_{\mu}h_A{}^{\nu} = \delta^{\nu}_{\mu}$. The tetrad vectors $b^{A}{}_{\mu}$ and $h_A{}^{\mu}$ can then be used to convert Latin indices of tensors into Greek indices and vice versa. We then express φ in the matter Lagrangian with only Latin indices. We then let the Σ operators act only on Lorentz indices, which does not affect the result.

The matter action $S_{\mathrm{M}} = \int \mathrm{d}^4x \mathcal{L}_{\mathrm{M}}(\varphi, \partial_A \varphi)$ is invariant under the global Poincaré transformation. An action is transformed as

$$\delta S = \int_{\Omega'} \mathrm{d}^4x' \mathcal{L}'(x') - \int_{\Omega} \mathrm{d}^4x \mathcal{L}(x). \tag{4.8}$$

Since the Jacobian is $|\partial(x')/\partial(x)| \approx 1 + \partial_\mu \xi^\mu$, the action is invariant if [20]

$$\Delta\mathcal{L} \equiv \delta\mathcal{L} + (\partial_\mu \xi^\mu)\mathcal{L} = \delta_0\mathcal{L} + \xi^\mu \partial_\mu \mathcal{L} + (\partial_\mu \xi^\mu)\mathcal{L}, \tag{4.9}$$

where $\delta\mathcal{L} = \mathcal{L}'(x') - \mathcal{L}(x)$. Because $\partial_\mu \xi^\mu = 0$ for global Poincaré transformation, we get $\delta\mathcal{L}_{\mathrm{M}} = 0$.

We can now make the transformation local by allowing the constants $\omega^{\mu\nu}$ and ε^μ to vary over the spacetime. Note that in the local transformation, ξ^μ and $\omega^{\mu\nu}$ are independent parameters since we can always choose proper ε^μ so that $\xi^\mu = 0$ even if $\omega^{\mu\nu} \neq 0$. We will use them as the independent parameters for the local transformation from now on, and the Poincaré group is split into the general coordinate transformation (GCT) corresponding to ξ^μ and the local Lorentz rotation (SO(3,1) or SL(2,C) to accommodate spinors) corresponding to ω^{AB}. While the transformation rule for φ remains the same ($\delta_0\varphi = U^\varphi \varphi$), the transformation rule for $\partial_A \varphi(x)$ is different from its global one. Under the global transformation, $\partial_A \varphi(x)$ transforms as

$$\delta_0 \partial_A \varphi(x) = \frac{\partial}{\partial x^A}\varphi'(x) - \frac{\partial}{\partial x^A}\varphi(x) = \partial_A \delta_0 \varphi(x) = U^\varphi \partial_A \varphi - \omega^B{}_A \partial_B \varphi, \tag{4.10}$$

while under local transformation, it transforms as

$$\delta_0 \partial_A \varphi(x) = U^\varphi \partial_A \varphi + \frac{1}{2}\partial_A \omega^{CD} \Sigma^\varphi_{CD} \varphi - \partial_A \xi^\nu \partial_\nu \varphi. \tag{4.11}$$

Therefore, $\delta\mathcal{L}_{\mathrm{M}}$ is not zero any more, and S_{M} becomes not invariant. To make S_{M} invariant, we can first make $\delta\mathcal{L}_{\mathrm{M}} = 0$ in a minimal coupling way by replacing the derivative $\partial_A \varphi$ in $\mathcal{L}_{\mathrm{M}}(\varphi, \partial_A \varphi)$ with the covariant derivative $\mathcal{D}_A \varphi$, whose transformation rule is the same as the global one

$$\delta_0 \mathcal{D}_A \varphi = U^{\varphi} \mathcal{D}_A \varphi - \omega^B{}_A \mathcal{D}_B \varphi. \tag{4.12}$$

The new Lagrangian $\mathcal{L}_{\mathrm{M}}(\varphi, \mathcal{D}_A \varphi)$ then satisfies $\delta \mathcal{L}_{\mathrm{M}} = 0$ because the matter field and its covariant derivative satisfies the "old" transformation rule. In contrast to the Yang-Mills theory, we accomplish this in two steps. The first step is to eliminate the term $\frac{1}{2}\omega^{AB}\Sigma^{\varphi}_{AB}\varphi$ in (4.11) by introducing the "rotational" gauge field $A^{AB}{}_{\mu}$. Similar to the replacement $\partial_\mu \to \mathcal{D}_\mu = \partial_\mu + A^a_\mu T^a$ (T^a: generators) in the Yang-Mills theory, we define

$$\mathcal{D}_\mu \varphi = \partial_\mu \varphi + \frac{1}{2} A^{AB}{}_{\mu} \Sigma^{\varphi}_{AB} \varphi \tag{4.13}$$

and require it to transform as

$$\delta_0 \mathcal{D}_\mu \varphi = U^{\varphi} \mathcal{D}_\mu \varphi - \partial_A \xi^\nu \mathcal{D}_\nu \varphi. \tag{4.14}$$

We find the rotational gauge field has to transform as

$$\begin{aligned} \delta_0 A^{AB}{}_{\mu} &= \omega^A{}_S A^{SB}{}_{\mu} + \omega^B{}_S A^{AS}{}_{\mu} - \xi^\nu \partial_\nu A^{AB}{}_{\mu} - \partial_\mu \xi^\nu A^{AB}{}_{\nu} - \partial_\mu \omega^{AB} \\ &= \frac{1}{2}\omega^{EF} (\Sigma^2_{EF})^{AB}{}_{CD} A^{CD}{}_{\mu} - \xi^\nu \partial_\nu A^{AB}{}_{\mu} - \partial_\mu \xi^\nu A^{AB}{}_{\nu} - \partial_\mu \omega^{AB}. \end{aligned} \tag{4.15}$$

We now define $\mathcal{D}_A \varphi$ as

$$\mathcal{D}_A \varphi = h_A{}^{\mu} \mathcal{D}_\mu \varphi \tag{4.16}$$

and require it to obey the "global transformation" (4.12). The $h_A{}^{\mu}$ acts as the "translational" gauge field here to compensate for the effect of the last term in (4.14) so that the transformation rule can become (4.12). Then the translation rule for $h_A{}^{\mu}$ must be

$$\begin{aligned} \delta_0 h_A{}^{\mu} &= -\omega^S{}_A h_S{}^{\mu} - \xi^\nu \partial_\nu h_A{}^{\mu} + h_A{}^{\nu} \partial_\nu \xi^\mu \\ &= \frac{1}{2}\omega^{CD} (\Sigma^1_{CD})_A{}^B h_B{}^{\mu} - \xi^\nu \partial_\nu h_A{}^{\mu} + h_A{}^{\nu} \partial_\nu \xi^\mu. \end{aligned} \tag{4.17}$$

Note that we have let the Σ operator only act on the Latin indices, so we can extend the covariant derivatives for the fields carrying Greek indices by simply ignoring them, or equivalently, neglecting the effect of ξ^μ. For example,

$$\mathcal{D}_\mu h_A{}^\nu = \partial_\mu h_A{}^\nu + \frac{1}{2} A^{CD}{}_\mu (\Sigma^1_{CD})_A{}^B h_A{}^\nu = \partial_\mu h_A{}^\nu - A^S{}_{A\mu} h_S{}^\nu. \tag{4.18}$$

If we do not neglect GCT in $\delta_0\varphi$, and φ carries some Greek indices, then since

$$\varphi'^{\mu_1'\cdots}{}_{\nu_1'\cdots}(x') = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \varphi^{\mu_1\cdots}{}_{\nu_1\cdots}(x) \tag{4.19}$$

and $\partial x^{\mu_1'}/\partial x^{\mu_1} = \delta^{\mu_1'}_{\mu_1} + \partial_{\mu_1}\xi^{\mu_1'}$ we get

$$\begin{aligned}
\delta_0(\omega+\Gamma)\varphi^{\mu_1\cdots}{}_{\nu_1\cdots}(x) = U^\varphi \varphi^{\mu_1\cdots}{}_{\nu_1\cdots}(x) + \big[\partial_\rho \xi^{\mu_1} \varphi^{\rho\mu_2\cdots}{}_{\nu_1\cdots}(x) + \ldots \\
-\partial_{\nu_1}\xi^\rho \varphi^{\mu_1\cdots}{}_{\rho\nu_2\cdots}(x) - \ldots\big] \\
\equiv \left[U^\varphi + G^\varphi(\xi)\right] \varphi^{\mu_1\cdots}{}_{\nu_1\cdots}(x),
\end{aligned} \tag{4.20}$$

where $G^\varphi(\xi) = \partial_\rho \xi^\sigma \mathrm{X}^\rho{}_\sigma$, and $\mathrm{X}^\rho{}_\sigma$ is the $\mathrm{GL}(4,R)$ generator matrices appropriate to the GCT tensor character of the field to which the operator is applied. Note that the Σ in U^φ operators still only act on Latin indices here.[1] Hence, from the transformation rules Equations (4.12), (4.14) and (4.17), we find $h_A{}^\mu$, $\mathcal{D}_\mu\varphi$ and $\mathcal{D}_A\varphi$ all transform covariantly under both local Lorentz rotation and GCT. As a consequence, we can use the $h_A{}^\mu$ and its inverse to convert the Latin/Greek indices in a covariantly transformed tensor, and the resulting tensor also transforms covariantly.

The matter action S_M is still not invariant because $\partial_\mu\xi^\mu \neq 0$ under local Poincaré transformation. One can show that S_M is invariant if we multiply the Lagrangian by $b \equiv \det(b)$

$$S_\mathrm{M} = \int \mathrm{d}^4 x b \mathcal{L}_M(\varphi, \mathcal{D}_A\varphi). \tag{4.21}$$

[1] If we go back to the global transformation, then we get $\partial_\mu \xi^\nu = \omega^\nu{}_\mu$ and $G^\varphi(\xi) = \frac{1}{2}\omega \cdot \Sigma^\varphi_{\text{Greek}}$, where $\Sigma^\varphi_{\text{Greek}}$ only acts on Greek indices. One can then recover (4.6), where the $\Sigma^\varphi = \Sigma^\varphi_{\text{Greek}} + \Sigma^\varphi_{\text{Latin}}$.

As mentioned in Chapter 1, we should also include the free action for the gravitational gauge fields $h_A{}^\mu$ and $A^{AB}{}_\mu$. Now we can obtain them explicitly from the commutators of the covariant derivatives. The commutators are

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi = \frac{1}{2}R^{AB}{}_{\mu\nu}\Sigma^\varphi\varphi \tag{4.22}$$

$$[\mathcal{D}_C, \mathcal{D}_D]\varphi = \frac{1}{2}R^{AB}{}_{CD}\Sigma^\varphi\varphi - T^S{}_{CD}\,\mathcal{D}_S\varphi, \tag{4.23}$$

where $\mathcal{R}^{AB}{}_{CD} = h_C{}^\mu h_D{}^\nu \mathcal{R}^{AB}{}_{\mu\nu}$ is the Lorentz field strength, $\mathcal{T}^A{}_{BC} = h_B{}^\mu h_C{}^\nu \mathcal{T}^A{}_{\mu\nu}$ is the translational field strength, and

$$\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu}A^{AB}{}_{\nu]} + A^A{}_{E[\mu}A^{EB}{}_{\nu]}), \tag{4.24}$$

$$\mathcal{T}^A{}_{\mu\nu} = 2(\partial_{[\mu}b^A{}_{\nu]} + A^A{}_{E[\mu}b^E{}_{\nu]}). \tag{4.25}$$

The action in PGT has the general form

$$S = \int \mathrm{d}^4x\, b\left[\mathcal{L}_\mathrm{G}\left(\mathcal{R}^{AB}{}_{CD}, \mathcal{T}^A{}_{BC}\right) + \mathcal{L}_\mathrm{M}(\varphi, \mathcal{D}_A\varphi)\right], \tag{4.26}$$

where $\mathcal{L}_\mathrm{G}$ is the free gravitational Lagrangian.

PGT is most naturally interpreted as a field theory in Minkowski spacetime[23, 25, 28]. It is more common, however, to reinterpret it geometrically in terms of a Riemann–Cartan spacetime (U_4). Riemann–Cartan spacetime is a manifold with linear connection (Γ) and metric ($g_{\mu\nu}$) which satisfy the metricity condition

$$\mathcal{D}_\rho(\Gamma)g_{\mu\nu} = 0, \tag{4.27}$$

where the covariant derivative is defined by

$$\mathcal{D}_\mu(\Gamma) = \partial_\mu + \Gamma^\sigma{}_{\rho\mu}\mathrm{X}^\rho{}_\sigma. \tag{4.28}$$

One may solve for the connection Γ, which is given by

$$\Gamma^{\mu}{}_{\nu\rho} = \left\{ {\mu \atop \nu\rho} \right\} + K^{\mu}{}_{\nu\rho}\,, \tag{4.29}$$

where $\left\{ {\mu \atop \nu\rho} \right\}$ is the ordinary Christoffel symbol and $K^{\mu}{}_{\nu\rho}$ is the contorsion tensor (discussed further below).

As mentioned above, a local Lorentz frame at each point on the manifold describes the tangent space and is determined by the tetrad basis $h_A{}^{\mu}$ with its inverse $b^A{}_{\mu}$; these quantities may be used to convert between coordinate and local Lorentz indices. The local frame has a connection $A^{AB}{}_{\mu}$, and the covariant derivative $\mathcal{D}_A(A)$ has properties similar to (4.28), where

$$\mathcal{D}_\rho(A)\eta_{AB} = 0. \tag{4.30}$$

One may also define the "total covariant derivative" $\mathcal{D}_\rho(\Gamma+A)$ to act on quantities with both coordinate and local Lorentz indices

$$\mathcal{D}_\rho(\Gamma+A)\varphi = \left(\mathcal{D}_\rho(\Gamma) + \mathcal{D}_\rho(A) - \partial_\rho\right)\varphi. \tag{4.31}$$

Since the total covariant derivative $\mathcal{D}_\rho(\Gamma+A)V^A$ of the local Lorentz components of a vector is a coordinate tensor in U_4 spacetime, the relation $\mathcal{D}_\rho(\Gamma+A)V^A = b^A{}_{\mu}\mathcal{D}_\rho(\Gamma+A)V^{\mu}$ should hold, from which one obtains the so-called "tetrad postulate"

$$D_\mu(\Gamma+A)b^A{}_{\nu} \equiv \partial_\mu b^A{}_{\nu} + A^A{}_{B\mu}b^B{}_{\nu} - \Gamma^{\sigma}{}_{\nu\mu}b^A{}_{\sigma} = 0. \tag{4.32}$$

One can therefore express the affine connection in the quantities corresponding to gauge fields as

$$\Gamma^{\lambda}{}_{\nu\mu} = h_A{}^{\lambda}(\partial_\mu b^A{}_{\nu} + A^A{}_{B\mu}b^B{}_{\nu}), \tag{4.33}$$

and hence show that the translational gauge field strength is equivalent to (minus) the geometric torsion tensor

$$\mathcal{T}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu}, \tag{4.34}$$

in terms of which the contorsion is given by

$$K_{\mu\lambda\nu} = -\tfrac{1}{2}\left(\mathcal{T}_{\mu\lambda\nu} - \mathcal{T}_{\nu\mu\lambda} + \mathcal{T}_{\lambda\nu\mu}\right). \tag{4.35}$$

From (4.33), (4.34), and (4.35), one also obtains

$$A_{AB\mu} = \Delta_{AB\mu} + K_{AB\mu}, \tag{4.36}$$

where we define the quantities

$$\Delta_{AB\mu} \equiv \frac{1}{2}\left(c_{ABC} - c_{CAB} + c_{BCA}\right) b^{C}{}_{\mu}, \tag{4.37}$$

$$c^{A}{}_{\mu\nu} \equiv \partial_{\mu} b^{A}{}_{\nu} - \partial_{\nu} b^{A}{}_{\mu}, \tag{4.38}$$

where $\Delta_{AB\mu}$ are the Ricci rotation coefficients or “reduced" A-field [28]. One then finds that, the geometric (Riemann) curvature tensor is equivalent to the rotational gauge field strength $\mathcal{R}^{\rho}{}_{\sigma\mu\nu}$,

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu}. \tag{4.39}$$

4.2 Application to PGT^{+}

The most general free-field PGT^{+} Lagrangian that is at most quadratic in the gravitational gauge fields may be written as:

$$\begin{aligned}
\mathcal{L} = &-\lambda\mathcal{R} + (r_4 + r_5)\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_4 - r_5)\mathcal{R}^{AB}\mathcal{R}_{BA} + \left(\frac{r_1}{3} + \frac{r_2}{6}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} \\
&+ \left(\frac{2r_1}{3} - \frac{2r_2}{3}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD} + \left(\frac{r_1}{3} + \frac{r_2}{6} - r_3\right)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB} \\
&+ \left(\frac{\lambda}{4} + \frac{t_1}{3} + \frac{t_2}{12}\right)\mathcal{T}^{ABC}\mathcal{T}_{ABC} + \left(-\frac{\lambda}{2} - \frac{t_1}{3} + \frac{t_2}{6}\right)\mathcal{T}^{ABC}\mathcal{T}_{BCA} \\
&+ \left(-\lambda - \frac{t_1}{3} + \frac{2t_3}{3}\right)\mathcal{T}_{B}{}^{AB}\mathcal{T}_{CA}{}^{C},
\end{aligned} \tag{4.40}$$

where $\mathcal{R}^A{}_B = \mathcal{R}^{AC}{}_{BC}$, $\mathcal{R} = \mathcal{R}^A{}_A$, and we have adopted the conventions in [29] for the parameters, which simplifies calculations and enables a straightforward comparison with the literature. Note that we have applied the Gauss-Bonnet identity [78] to remove the term proportional to $\mathcal{R}^2$ in (4.40),

$$\int \mathrm{d}^4x\, b\left(\mathcal{R}^2 - 4\mathcal{R}_{AB}\mathcal{R}^{BA} + \mathcal{R}_{ABCD}\mathcal{R}^{CDAB}\right) = 0. \tag{4.41}$$

To determine the particle spectrum, one must first linearise the Lagrangian. We expand it around a Minkowski background with

$$h_A{}^\mu = \delta_A{}^\mu + f_A{}^\mu, \tag{4.42}$$

and we set the A-field to be $O(f)$. The inverse of h becomes

$$b^A{}_\mu = \delta^A{}_\mu - f_\mu{}^A + O\left(f^2\right). \tag{4.43}$$

Since the effect of transforming Greek indices to Latin indices is only $O\left(f^2\right)$, we can ignore the difference between them and only use Latin indices in the linearised theory. We can decompose f into symmetric and antisymmetric parts:[2]

$$f_{AB} = s_{AB} - a_{AB}. \tag{4.44}$$

Note that one may add a constant term c_0 to the right-hand side of (4.40), but after the weak field expansion the Lagrangian becomes

$$b\mathcal{L} = c_0 + t\left(2\lambda\partial_A A^{BA}{}_B - c_0 s\right) + \mathcal{O}\left(t^2\right). \tag{4.45}$$

The constant term in (4.45) does not affect the equation of motion, so we can neglect it, and the ∂A term can be eliminated by partial integration regardless of whether c_0 is zero. If $c_0 \neq 0$,

[2] The minus sign before a_{AB} is due to the difference of the indices between the main text and the code. In the code, the h-field is $h^\mu{}_A$, while it is $h_A{}^\mu$ in the main text.

however, the $c_0 s$ term in the $\mathcal{O}(t)$ part of the Lagrangian results in the equation of motion $c_0 = 0$ at order t, which contradicts $c_0 \neq 0$. Furthermore, we consider only the Minkowski background here, and adding a cosmological constant term will cause the background to be de Sitter. Hence, c_0 must always be zero, and so we do not add the constant term to (4.40).

Before considering the general case of PGT$^+$, however, we begin by first studying the simpler cases of PGT$^+$ with vanishing torsion and curvature, respectively, which one should note are *not* merely special cases of (4.40), because additional constraints are placed not only the coefficients, but also on the fields.

4.2.1 Zero-torsion PGT$^+$

The translational field strength $\mathcal{T}^{\mathcal{A}}{}_{\mu\nu}$ is interpreted as torsion in the geometric interpretation. If we set torsion to zero, it will impose a relation between the A- and h-fields, and so they are not independent fields any more. Since the A-field can be written as $A_{AB\mu} = \Delta_{AB\mu} + K_{AB\mu}$ as shown in (4.36), setting the torsion to zero is equivalent to replacing $A_{AB\mu}$ with $\Delta_{AB\mu}$. The Lagrangian of torsionless PGT is thus

$$\mathcal{L} = -\lambda\mathcal{R} + 2r_4\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_1 - r_3)\,\mathcal{R}^{ABCD}\mathcal{R}_{ABCD}. \tag{4.46}$$

We employ the general method described in Sec. 2.2 to this case, and present our results in Fig. 4.1, which also illustrates our methodology in diagrammatic form. The top "node" in the figure (entitled "root") represents the full theory described by (4.46), without imposing any relationship between the parameters in the Lagrangian. The line "l" in each node lists the number of degrees of freedom in the massless sector and the condition for that sector to be ghost-free; alternatively it is marked with "G" to denote that the sector must contain a ghost, or "dip.G" to denote that it must contain a dipole ghost. The line "v" in each node lists the massive particles and the conditions that must be satisfied for them to be neither ghosts nor tachyons; alternatively, it is marked with a "G" if one of them must be a ghost or tachyon. If there is no massive particle, then $\times$ is written.

The arrows between nodes point from parent theories to their child theories. The first line of the label on each arrow indicates the type of the critical case, and the second line denotes that one is setting the expressions in [...] to zero in the parent theory to obtain the child theory. The first line in each node (except the "root" node) contains the full set of critical conditions for that theory. Note that for each theory, the conditions that make it critical (the expressions in the arrows from that node) are required not to hold. For example, for the theory with $\lambda = 0$ in the second row of Fig. 4.1, one requires $2r_1 - 2r_3 + r_4 \neq 0$ and $r_1 - r_3 + 2r_4 \neq 0$. The bottom node corresponds to the Lagrangian vanishing identically.

For the subset of cases considered previously by other authors, we compare our results with those in the literature in Sec. 4.2.4.

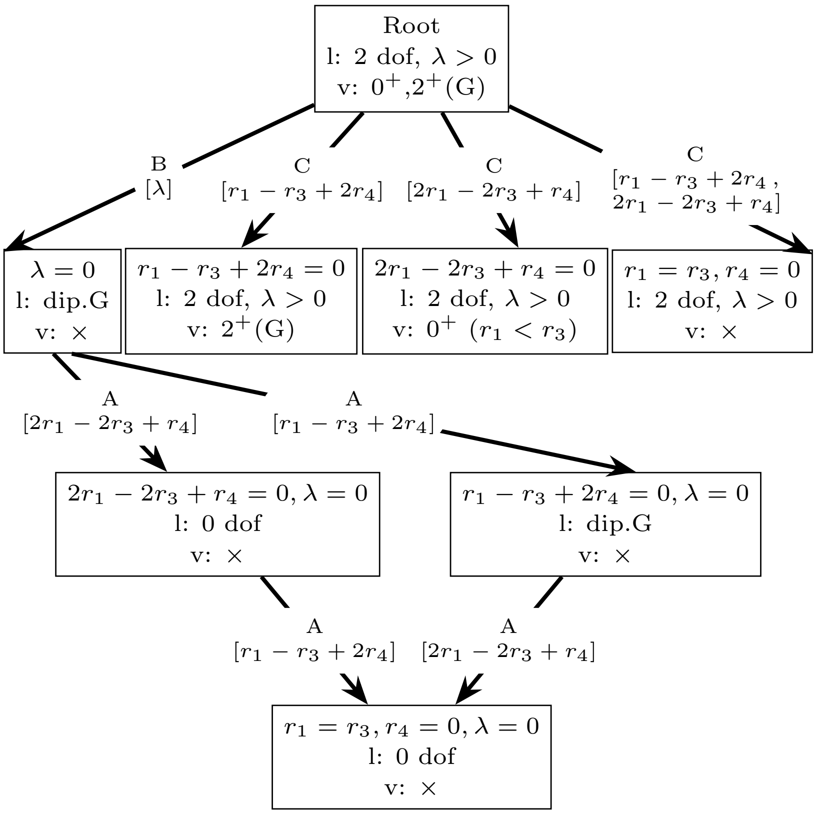


Fig. 4.1 The critical cases of zero-torsion PGT$^+$, for which the Lagrangian has the form (4.46). See the text for details.

4.2.2 Zero-curvature PGT$^+$

One may impose zero curvature in PGT$^+$ (to obtain teleparallel PGT$^+$) by setting $A_{AB\mu} = 0$ [18], and the corresponding Lagrangian is given by

$$\mathcal{L} = \left(\frac{t_1}{3} + \frac{t_2}{12}\right) \mathcal{T}^{ABC}\mathcal{T}_{ABC} + \left(-\frac{t_1}{3} + \frac{t_2}{6}\right) \mathcal{T}^{ABC}\mathcal{T}_{BCA} + \left(-\frac{t_1}{3} + \frac{2t_3}{3}\right) \mathcal{T}_{BA}{}^{B}\mathcal{T}_{C}{}^{AC}. \quad (4.47)$$

Applying the method described in Sec. 2.2 to this case yields the results presented in Fig. 4.2, which uses the same conventions as in Fig. 4.1. We again compare our results with the literature in Sec. 4.2.4.

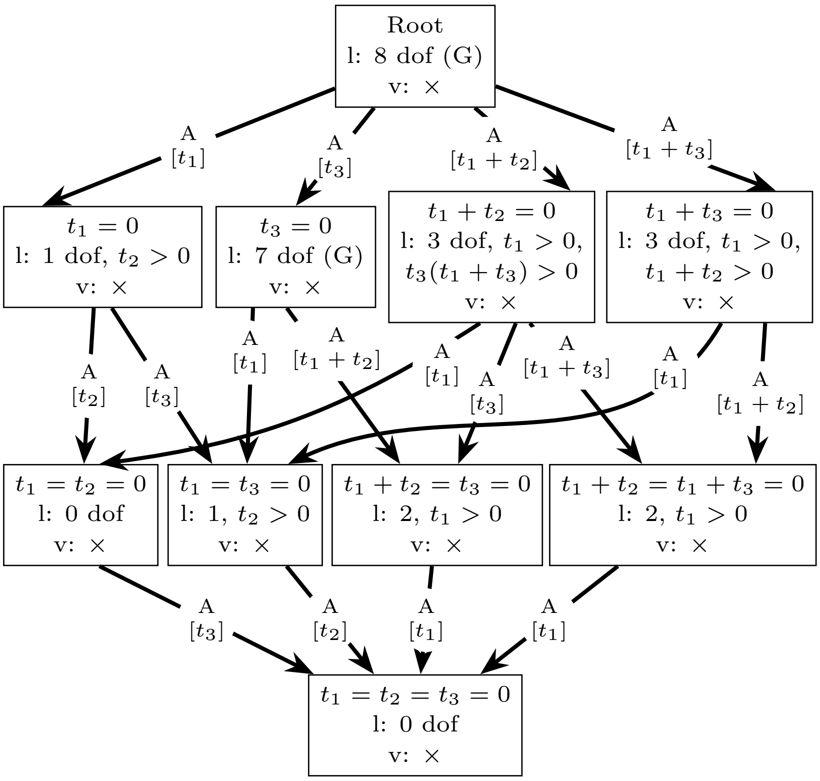


Fig. 4.2 The critical cases of zero-curvature (teleparallel) PGT$^+$, for which the Lagrangian has the form (4.47). See text for details.

4.2.3 Full PGT$^+$

We now turn our attention back to the general case of full PGT$^+$, for which the Lagrangian is given by (4.40). Starting from the "root" theory, for which no relationship is imposed on the parameters in the Lagrangian, our method outlined in Sec. 2.2 systematically identifies 1918 critical cases (excluding the "vanishing" Lagrangian case for which all parameters are zero), which thus cannot be displayed in diagrammatic form such as in Figs 4.1 and 4.2. Of these critical cases, we find that 450 can be free of ghosts and tachyons, provided the parameters in each case satisfy some conditions without generating another critical case. The full set of results displayed in an interactive form can be found at: http://www.mrao.cam.ac.uk/projects/gtg/pgt/. We show some screenshots in Figure 4.3.

4.2.4 Comparison with previous results

We content ourselves here with presenting in Table 4.1 our results for the root PGT$^+$ theory and the small subset of critical cases that have been studied previously in the literature. We also list those critical cases of the torsionless and teleparallel PGT$^+$ theories (see Figs. 4.1 and 4.2) that have been considered previously in the literature. Overall, we find that our results are indeed consistent with those reported by other authors, apart from a few minor differences that are most likely the result of typographical errors in earlier papers.

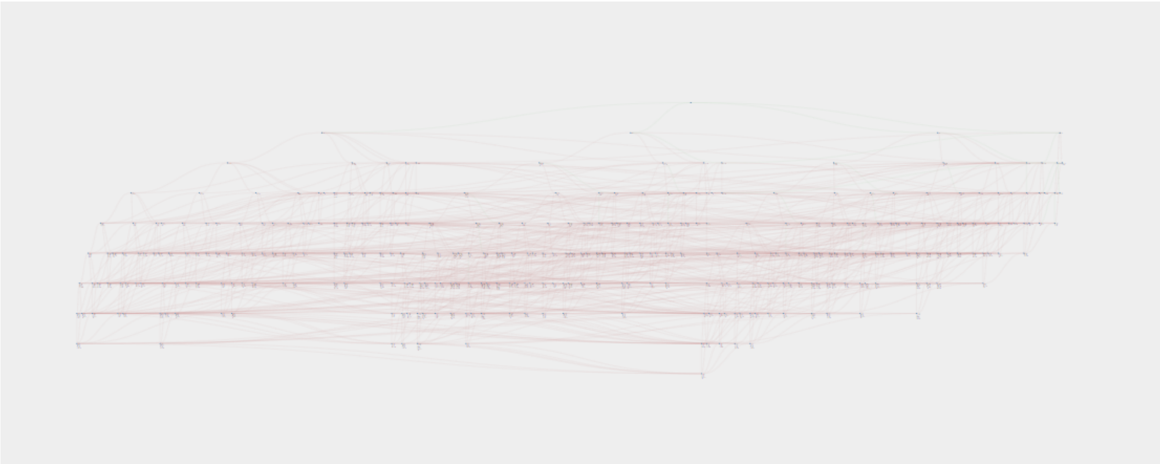
Some of the cases listed in Table 4.1 are worthy of further discussion, as follows:

- Case 1: This is the "root" PGT$^+$ theory, in which no critical condition holds. We find the massless no-ghost condition $\lambda > 0$, which agrees with [29]. In the massive case, we find the no-tachyon condition in each spin-parity sector to be:

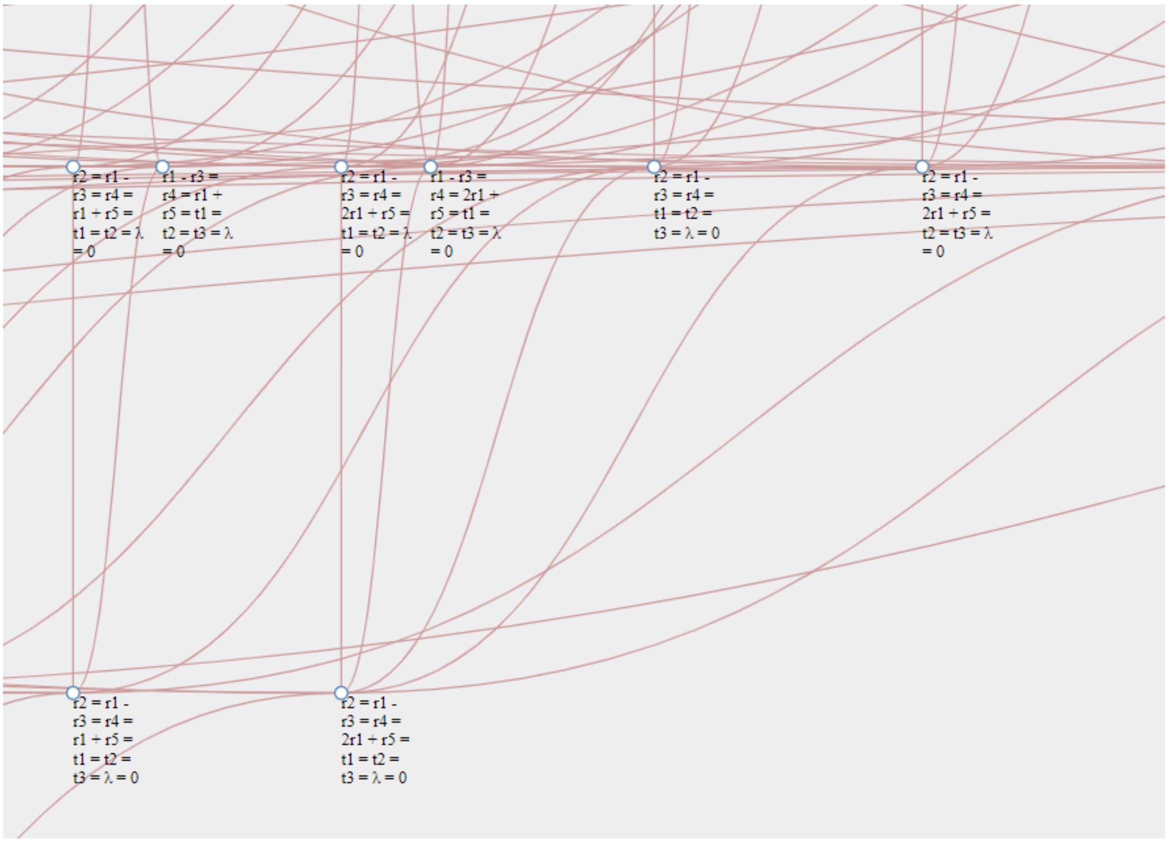
$$
\begin{aligned}
&0^- : -\frac{t_2}{r_2} > 0\\
&0^+ : \frac{t_3\lambda}{2\left(r_1 - r_3 + 2r_4\right)\left(t_3 - \lambda\right)} > 0\\
&1^- : -\frac{3t_1t_3}{2\left(r_1 + r_4 + r_5\right)\left(t_1 + t_3\right)} > 0\\
&1^+ : -\frac{3t_1t_2}{2\left(2r_3 + r_5\right)\left(t_1 + t_2\right)} > 0\\
&2^- : -\frac{t_1}{2r_1} > 0\\
&2^+ : -\frac{t_1\lambda}{2\left(2r_1 - 2r_3 + r_4\right)\left(t_1 + \lambda\right)} > 0,
\end{aligned}
\tag{4.48}
$$

Table 4.1 Conditions for no ghosts or tachyons for the PGT^{+} root theory and a subset of critical cases analyzed previously in the literature. "Massless/massive" denotes the particle content found in the literature, and the parentheses contain the number of degrees of freedom of particles in the massless sector. "Dip. G" means the massless sector contains a dipole ghost. Where our results differ from those in the literature, ours are put in squared brackets. Cells marked with "*" are discussed further in the text, and "-" means the particle content is not mentioned in the cited paper.

#	Paper	Critical Conditions	No-ghost-and-tachyon Conditions	Massless	Massive
1	[29]	$\times$	Ghost (massive)	2^+ (2)	0^-,0^+,1^-, 1^+,2^-,2^+
2	[29]	$t_1=t_2=t_3=r_1=r_2=r_3$ $=r_4=r_5=0$	$\lambda>0$	2^+ (2)	$\times$
3	[29]	$t_1=-t_2=-t_3=-\lambda$, $r_1=r_2=r_3=r_4=r_5=0$	$\lambda>0$	2^+ (2)	$\times$
4	[29]	$t_1=-t_2=-t_3=-\lambda$, $r_1=r_3=r_4=r_5=0$	$\lambda>0,r_2<0$	2^+ (2)	0^-
5	[29]	$t_1=-t_2,r_1=r_3$, $r_4=r_2=0$	$\lambda>0,r_1>0[<0]$, $r_1+r_5>0[<0]$, $t_1>0,t_3(t_1+t_3)>0$	2^+ (2)	1^-, 2^-
6	[29]	$t_1=-t_2,r_1=r_3$, $r_4=r_2=0$, torsionless	$\lambda>0$	2^+ (2)	$\times$
7	[29]	$t_1=-t_3=-\lambda,r_1=0$, $r_4=-r_5$	$t_2>\lambda>0,r_2<0$, $2r_3+r_5>0$	2^+ (2)	0^-, 1^+
8	[29]	$t_1=-t_3=-\lambda,r_1=0$, $r_4=-r_5,r_2=0$	$2r_3+r_5>0,\lambda>0$, $t_2(t_2-\lambda)>0$	2^+ (2)	1^+
9	[29]	$t_1=-t_3=-\lambda,r_1=0$, $r_4=-r_5,r_2=0$, torsionless	Ghost	-	-
10	[29]	$r_1=0,2r_3=r_4=-r_5$	$\lambda>0,r_2<0,r_3>0,t_2>0$, $t_3(\lambda-t_3)<0$	2^+ (2)	0^-, 0^+
11	[29, 79]	$r_1=0,2r_3=r_4=-r_5$, torsionless	$\lambda>0,r_3>0$	- [(2)]	- [0^+]
12	[57]	(1)-(12)*	*	2^+ (2)	*
13	[57]	$t_1=t_2=t_3=0,r_1=r_3$, $r_4=0,2r_3+r_5=0$	$\lambda>0,r_1>0$ [$\lambda>0$]*	2^+,1^-* (4) [(2)]	$\times$
14	[27]	$t_1=t_2=t_3=0,r_1=r_3$, $r_4=0,2r_3+r_5=0$	$\lambda>0$*	2^+ (2)	$\times$
15	[27]	$t_1=-t_3$, teleparallel	$t_1+t_2>0,t_1+\lambda>0$*	2^+,0^+ (3)	$\times$
16	[80]	$t_1=-t_3$, teleparallel	$t_1+\lambda>0$	2^+ (2) [(3)]	$\times$
17	[60]	$r_4=-(r_1/2)+r_3/2,t_3=0$	$r_1+r_3+2r_5<0$, $\lambda>0$ (massless)	2^+,1 (4)	-
18	[60]	$r_2=0,t_2=0$	$2r_3+r_5>0,\lambda>0$ (massless)	2^+,1 (4)	-
19	[60]	$t_2=t_3=r_1-r_3+2r_4=r_2=0$	$2r_3+r_5>0$, $r_1+r_3+2r_5<0$, $\lambda>0$ (massless)	2^+,1,1 (6)	-
20	[79]	torsionless	Ghost (massive 2^+)	2^+ (2)	0^+,2^+
21	[12, 79]	$r_1-r_3+2r_4=0$, torsionless	Ghost (massive 2^+)	2^+ (2)	2^+
22	[81]	$r_1-r_3+2r_4=\lambda=0$, torsionless	Ghost (massless)	2^+,1,2^+ (dip. G)*	$\times$

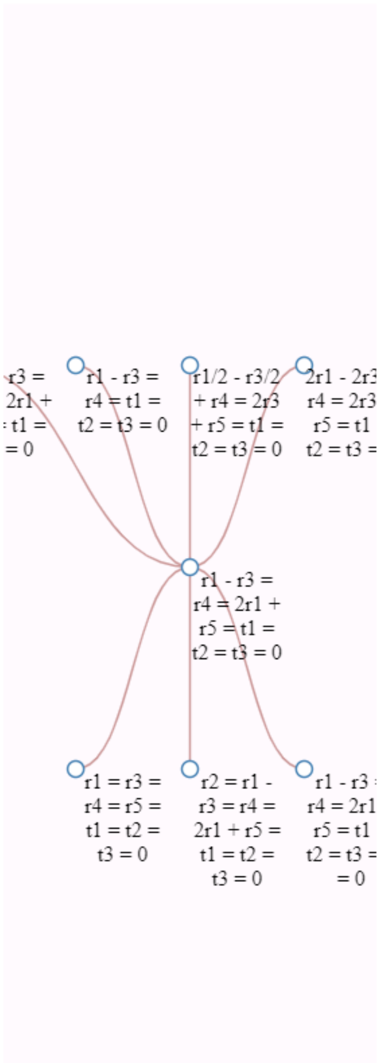


(a) The 1918+2 critical cases with those derived from type C condition hidden. It is impossible to show them clearly in an A4 paper (even in full page).



(b) A zoomed-in picture of a small area of Figure 4.3a.

Fig. 4.3 Screenshots of the website http://www.mrao.cam.ac.uk/projects/gtg/pgt/ showing the 1918+2 ("root" theory and zero Lagrangian) critical cases with those derived from type C condition hidden.



Critical Cond.	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_2 = t_3 = 0$
G/T Cond.	$\lambda > 0$
Add. Cond.	$r_1 \neq 0 \wedge r_2 \neq 0 \wedge \lambda \neq 0$
PC renormalizable	False
Critical Paras. (type A)	(r_1, r_2, λ)
Critical Paras. (type B)	×
Critical Paras. (type C)	×
Massless no-ghost cond.	$\lambda > 0$
Massless DOF	2
Massive modes	×
$a_{ij}(J^P)$	$a(0^-) = (2k^2 r_2)$ $a(0^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4k^2\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $a(1^-) = \begin{pmatrix} -2k^2 r_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $a(1^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $a(2^-) = (2k^2 r_1)$ $a(2^+) = \begin{pmatrix} 0 & 0 \\ 0 & 2k^2\lambda \end{pmatrix}$
	$b^{-1}(0^-) = \left(\frac{1}{2k^2 r_2}\right)$ rows $= (1)$

Fig. 4.4 A screenshot of the website http://www.mrao.cam.ac.uk/projects/gtg/pgt/ showing details for a critical case.

and the no-ghost condition in each sector is:

$$
\begin{aligned}
0^- &: -\frac{1}{r_2} > 0 \\
0^+ &: \frac{-r_1 t_3 + r_3 t_3 - 2r_4 t_3 - t_3\lambda + \lambda^2}{2\left(r_1 - r_3 + 2r_4\right)\lambda\left(-t_3 + \lambda\right)} > 0 \\
1^- &: -\frac{3\left(t_1^2 + 2t_3^2\right)}{2\left(r_1 + r_4 + r_5\right)\left(t_1 + t_3\right)^2} > 0 \\
1^+ &: \frac{3\left(t_1^2 + 2t_2^2\right)}{2\left(2r_3 + r_5\right)\left(t_1 + t_2\right)^2} > 0
\end{aligned}
$$

$$
\begin{aligned}
&2^- : -\frac{1}{r_1} > 0 \\
&2^+ : \frac{-2r_1t_1 + 2r_3t_1 - r_4t_1 + t_1\lambda + \lambda^2}{(2r_1 - 2r_3 + r_4)\lambda(t_1 + \lambda)} > 0.
\end{aligned} \tag{4.49}
$$

These conditions are again equivalent to those in [29], as expected, and cannot be satisfied simultaneously. Hence, the theory contains a massive ghost, as is well known. We show details of obtaining Equations (4.48) and (4.49) in Appendix 4.A.

- Case 3: This is Einstein–Cartan theory, and our results are consistent with the literature.

- Case 5: Our conditions $\lambda > 0, r_1 < 0, r_1 + r_5 < 0, t_1 > 0, t_3(t_1 + t_3) > 0$ differ from the conditions $\lambda > 0, r_1 > 0, r_1 + r_5 > 0, t_1 > 0, t_3(t_1 + t_3) > 0$ found in [29] in that two of the inequalities have the opposite sign. We believe these are typos in [29].

- Case 6: This torsionless theory corresponds to that in node 4 of row 2 in Fig. 4.1. We obtain the condition $\lambda > 0$, with only 2 massless degrees of freedom, but [29] also set $2t_3 - t_1 = 3\lambda, r_5 = 0$. These additional conditions neither cause the theory to become a critical case nor contradict the other conditions, so adding them has no effect on the particle content. [29] finds that the action reduces to the Einstein action, which is consistent with our result.

- Case 12: We find that the critical cases that contain three coefficient equations and only type C critical conditions are precisely the 12 cases listed in Table I of [57], and we obtained the same particle content for each theory.

- Case 13: Our no-ghost conditions and massless particle content are different from those found in [57]. However, [27] studied the same theory and obtained the same conditions and particle content as ours. Moreover, our result that there is no massless propagating tordion in this theory is also found in [48]. We notice that, compared to our analysis, some terms in Eq. (8) in [57] have different signs, which we believe to be typos.

- Case 14 and 15: We find that there is an overall sign difference between our linearised Lagrangian and that in [27], so the conditions also have an overall sign difference. We assume that this is a minor error either in their calculation or our conversion of it to our notation. We have thus added an overall minus sign to their conditions.

- Case 15: This theory was also studied in [80] (along with Case 16), who found only a spin-2 massless mode with the condition $t_1 + \lambda > 0$. However, they studied only the spin-2 particles, so our results are consistent.

- Case 19: We believe that the condition $\alpha - \gamma_3 = 0$ quoted in [60] contains a typo and should instead read $\alpha - \gamma_3 \neq 0$, which is equivalent to $t_1 = 0 \to t_1 \neq 0$ in our notation, thus yielding our result.

- Case 22: This is conformal gravity. [81] showed it has a normal spin-2, a normal spin-1, and a ghost spin-2 mode, all massless. We find there is no massive mode, and there must be dipole ghost(s) in the massless sector. Our method can determine the existence of ghosts, but not the degrees of freedom in the massless sector if there are dipole ghost(s). Nonetheless, the results are consistent.

4.2.5 Source constraints

As mentioned previously, if the parameters in the PGT$^+$ Lagrangian (4.40) satisfy some specific conditions (type A critical cases), then the resulting theory may possess extra gauge invariances beyond the Poincaré symmetry assumed in its construction. For example, for Case 13 in Table 4.1, it is noted in [27, 57] that the theory is additionally invariant under the gauge transformation

$$\delta A_{ABC} = \partial_A \Lambda_{BC} - \partial_B \Lambda_{AC} + \partial_C \theta_{AB}, \tag{4.50}$$

where $\partial^B \Lambda_{AB} = 0$, $\theta_{AB} = \partial_A V_B - \partial_B V_A$, $\partial^A V_A = 0$ and Λ and V are arbitrary (see also [48]), and has the additional source constraints

$$\partial^B \tau_{ABC} = 0, \qquad \partial^C \tau_{ABC} = 0, \tag{4.51}$$

beyond the standard ones $\partial^B \sigma_{AB} = 0$ and $\sigma_{[AB]} - \partial^C \tau_{ABC} = 0$ arising from the Poincaré symmetry. Here, σ_{AB} and τ_{ABC} are the source currents of the f_{AB} (graviton) and A_{ABC} (tordion) gravitational fields, respectively.

Our approach also found the same source constraints for this theory, although not directly as tensor equations, but instead in component form for k aligned with the z-direction. Indeed, we found there are 310 different sets of source constraints among the root PGT^{+} theory and its 1918 critical cases. We are not able to convert all of them automatically into their corresponding tensor equations, but it is possible to make such a conversion in some cases. This is performed by first suggesting possible tensor equations from the patterns present in the component equations, then converting the possible tensor equations into component forms, and finally comparing whether they are equivalent. In table 4.2, we present the results for all the sets of sources constraints that we were able to convert into tensor form. We find that the same set of source constraints may hold for more than one critical case, so in the table we list only the case having the simplest critical conditions. It is worth noting that the first case listed is the root PGT^{+} theory, for which we recover the two well-known source constraints arising from the Poincaré symmetry alone. We also note that, aside from the root theory, the numbering of cases in the table is not related to that used in table 4.1.

Table 4.2 Source constraints for the root PGT^{+} theory and those critical cases for which the constraints could be found in tensor form. Note that there may be more than one critical case sharing the same source constraints, so we list only the case having the simplest critical conditions. The numbering of cases is not related to that used in Table 4.1.

#	Critical Conditions	Source constraints
1	$\times$	$k^B \sigma_{AB} = \sigma_{AB} - \sigma_{BA} - 2ik^C \tau_{ABC} = 0$
2	$r_1 - r_3 = r_4 = \lambda = 0$	$i\sigma_{AB} + i\sigma_{BA} + 2k^C \tau_{CAB} + 2k^C \tau_{CBA} = i\sigma_{AB} - i\sigma_{BA} + 2k^C \tau_{ABC} = 0$

Table 4.2 (continued): Source constraints for some PGT$^+$ critical cases.

#	Critical Conditions	Source constraints
3	$r_1/2-r_3/2+r_4=r_1/2+r_3/2+r_5=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}-2ik^C\tau_{ABC}=g^{BC}\tau_{ACB}=0$
4	$r_1/2-r_3/2+r_4=r_1/2+r_3/2+r_5=t_1=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}-2ik^C\tau_{ABC}=k^B\sigma_{BA}=g^{BC}\tau_{ACB}=0$
5	$r_1=r_3=r_4=r_5=t_1=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}+2ik^C\tau_{CBA}=\tau_{ACB}+\tau_{BCA}=0$
6	$r_1=r_3=r_4=r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=k^C\tau_{CBA}=\tau_{ACB}+\tau_{BCA}=0$
7	$r_1=r_3=r_4=r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=k^C\tau_{CBA}=\tau_{ACB}+\tau_{BCA}=0$
8	$r_1=r_2=r_3=r_4=r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=\tau_{BCA}=0$
9	$r_1=r_3=r_4=r_5=t_1=t_3=\lambda=0$	$\sigma_{AB}+ik^C\tau_{CBA}=\tau_{ACB}+\tau_{BCA}=0$
10	$r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}-ik^C\tau_{CAB}+ik^C\tau_{CBA}=g^{BC}\tau_{ACB}=2k^C\tau_{ABC}-k^C\tau_{CAB}+k^C\tau_{CBA}=0$
11	$r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}-k^C\tau_{CBA}=k^C\tau_{ABC}=0$
12	$r_2=r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}-k^C\tau_{CBA}=\tau_{ABC}-\tau_{ACB}+\tau_{BCA}=0$
13	$r_2=r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}-k^C\tau_{CBA}=\tau_{ABC}-\tau_{ACB}+\tau_{BCA}=0$
14	$r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}-k^C\tau_{CBA}=k^C\tau_{ABC}=0$
15	$r_1/3-r_3=r_1/3+r_4=2r_1/3+r_5=t_1=t_3=\lambda=0$	$\sigma_{AB}-ik^C\tau_{ABC}=g^{BC}\tau_{ACB}=2i\sigma_{AB}+k^C\tau_{CAB}-k^C\tau_{CBA}=0$
16	$r_1-r_3=r_4=r_1+r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}+k^C\tau_{CBA}=k^C\tau_{ABC}=0$
17	$r_1-r_3=r_4=r_1+r_5=t_1=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}-2ik^C\tau_{ABC}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}+k^C\tau_{CBA}=0$
18	$r_1-r_3=r_4=r_1+r_5=t_1=t_3=\lambda=0$	$\sigma_{AB}-ik^C\tau_{ABC}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}+k^C\tau_{CBA}=0$
19	$r_1-r_3=r_4=r_1+r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=g^{BC}\tau_{ACB}=k^C\tau_{CAB}+k^C\tau_{CBA}=k^C\tau_{ABC}=0$
20	$r_1/2-r_3/2+r_4=r_1/2+r_3/2+r_5=t_1=t_2=t_3=0$	$k^B\sigma_{AB}=\sigma_{AB}-\sigma_{BA}=g^{BC}\tau_{ACB}=k^C\tau_{ABC}=0$
21	$r_2=r_1-r_3=r_4=2r_1+r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=k^C\tau_{CBA}=\tau_{ABC}-\tau_{ACB}+\tau_{BCA}=0$
22	$r_1/2-r_3/2+r_4=r_1/2+r_3/2+r_5=t_1=t_2=t_3=\lambda=0$	$\sigma_{AB}=g^{BC}\tau_{ACB}=k^C\tau_{ABC}=0$

Table 4.2 (continued): Source constraints for some PGT$^+$ critical cases.

#	Critical Conditions	Source constraints
23	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} = k^C\tau_{CBA} = k^C\tau_{ABC} = 0$
24	$r_1/2 - r_3/2 + r_4 = r_1/2 + r_3/2 + r_5 = t_1 = t_3 = \lambda = 0$	$k^B\sigma_{AB} = g^{BC}\tau_{ACB} = i\sigma_{AB} + k^C\tau_{ABC} = 0$
25	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_3 = \lambda = 0$	$\sigma_{AB} + ik^C\tau_{CBA} = 2\sigma_{AB} - ik^C\tau_{ABC} - ik^C\tau_{CAB} = 0$
26	$r_1 - r_3 = r_4 = r_1 + r_5 = t_3 = \lambda = 0$	$g^{BC}\tau_{ACB} = i\sigma_{AB} + i\sigma_{BA} + 2k^C\tau_{CAB} + 2k^C\tau_{CBA} = i\sigma_{AB} - i\sigma_{BA} + 2k^C\tau_{ABC} = 0$
27	$r_1 - r_3 = r_4 = t_1 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} = k^C\tau_{CAB} + k^C\tau_{CBA} = k^C\tau_{ABC} = 0$
28	$r_1 = r_3 = r_4 = r_5 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} - 2ik^C\tau_{CBA} = \sigma_{AB} - 2ik^C\tau_{ABC} - 2ik^C\tau_{CAB} = g^{BC}\tau_{ACB} = 0$
29	$r_1 - r_3 = r_4 = t_1 = t_3 = \lambda = 0$	$k^C\tau_{CAB} + k^C\tau_{CBA} = i\sigma_{AB} + k^C\tau_{ABC} = 0$
30	$r_1 = r_2 = r_3 = r_4 = r_5 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} - 2ik^C\tau_{CBA} = g^{BC}\tau_{ACB} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = 0$
31	$r_2 = r_1 - r_3 = r_4 = 2r_1 + r_5 = t_2 = \lambda = 0$	$\sigma_{AB} - 2ik^C\tau_{CBA} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = 0$
32	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_2 = \lambda = 0$	$\sigma_{AB} - 2ik^C\tau_{CBA} = \sigma_{AB} - 2ik^C\tau_{ABC} - 2ik^C\tau_{CAB} = 0$
33	$r_2 = r_1/3 - r_3 = r_1/3 + r_4 = 2r_1/3 + r_5 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} + 2ik^C\tau_{CAB} - 2ik^C\tau_{CBA} = g^{BC}\tau_{ACB} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = 0$
34	$r_1/3 - r_3 = r_1/3 + r_4 = 2r_1/3 + r_5 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} + 2ik^C\tau_{CAB} - 2ik^C\tau_{CBA} = \sigma_{AB} - \sigma_{BA} - 2ik^C\tau_{ABC} = g^{BC}\tau_{ACB} = 0$
35	$r_1 - r_3 = r_4 = t_1 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} - 2ik^C\tau_{ABC} = k^C\tau_{CAB} + k^C\tau_{CBA} = 0$
36	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} + 2ik^C\tau_{CBA} = \sigma_{AB} - \sigma_{BA} - ik^C\tau_{ABC} - ik^C\tau_{CAB} = 0$
37	$r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = k^C\tau_{CBA} = k^C\tau_{ABC} = 0$
38	$r_2 = r_1 - r_3 = r_4 = 2r_1 + r_5 = t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = k^C\tau_{CBA} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = 0$
39	$r_1 - r_3 = r_4 = t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = k^C\tau_{CAB} + k^C\tau_{CBA} = k^C\tau_{ABC} = 0$
40	$r_2 = 2r_3 + r_5 = t_1 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = k^C\tau_{CAB} - k^C\tau_{CBA} = 0$
41	$t_1 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} - 2ik^C\tau_{ABC} = k^B\sigma_{BA} = 0$
42	$2r_3 + r_5 = t_1 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} = k^C\tau_{CAB} - k^C\tau_{CBA} = k^C\tau_{ABC} = 0$
43	$r_2 = 2r_3 + r_5 = t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = k^C\tau_{CAB} - k^C\tau_{CBA} = 0$
44	$2r_3 + r_5 = t_1 = t_3 = \lambda = 0$	$3i\sigma_{AB} + k^C\tau_{ABC} + k^C\tau_{CAB} - k^C\tau_{CBA} = 2i\sigma_{AB} + k^C\tau_{CAB} - k^C\tau_{CBA} = 0$
45	$2r_3 + r_5 = t_1 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} - ik^C\tau_{CAB} + ik^C\tau_{CBA} = \sigma_{AB} - \sigma_{BA} - 2ik^C\tau_{ABC} = 0$
46	$2r_3 + r_5 = t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = k^C\tau_{CAB} - k^C\tau_{CBA} = k^C\tau_{ABC} = 0$

Table 4.2 (continued): Source constraints for some PGT$^+$ critical cases.

#	Critical Conditions	Source constraints
47	$t_1 = t_2 = t_3 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} = k^C\tau_{ABC} = 0$
48	$t_1 = t_2 = t_3 = \lambda = 0$	$\sigma_{AB} = k^C\tau_{ABC} = 0$
49	$t_1 = t_3 = \lambda = 0$	$k^B\sigma_{AB} = i\sigma_{AB} + k^C\tau_{ABC} = 0$
50	$r_2 = 2r_3 + r_5 = t_2 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} + 2ik^C\tau_{CAB} - 2ik^C\tau_{CBA} = \tau_{ABC} - \tau_{ACB} + \tau_{BCA} = 0$
51	$2r_3 + r_5 = t_2 = 0$	$k^B\sigma_{AB} = \sigma_{AB} - \sigma_{BA} + 2ik^C\tau_{CAB} - 2ik^C\tau_{CBA} = i\sigma_{AB} - i\sigma_{BA} + 2k^C\tau_{ABC} = 0$

4.3 Discussion and Conclusions

We have presented a systematic method for obtaining the no-ghost-and-tachyon conditions for all critical cases of a parity-preserving gauge theory of gravity. We have implemented the method as a computer program and examined the critical cases of PGT$^+$, as well as of torsionless PGT$^+$ and teleparallel PGT$^+$. In comparing our results with the literature for the (small) subset of critical cases that have been analysed previously, we find that they are consistent, apart from a few minor differences that most probably arise from typographical errors in previous works.

Our method does, however, have the shortcoming that it does not yield the spins or parities of the massless particles, but only their total number of degrees of freedom (when there is no dipole ghost). Moreover, in the presence of a dipole ghost, our method can determine only that the dipole ghost exists, but does not yield the number of degrees of freedom.

Although not a shortcoming of our method *per se*, it is also difficult to classify the results obtained. In particular, care must be taken since, for a given ghost and tachyon free critical case, it is not guaranteed that all of its child critical cases do not contain ghosts or tachyons. Furthermore, in general, a theory has multiple child critical theories, and it also has multiple parent theories, so it is difficult to divide the theories into some categories without cutting lots of relations between parent and child theories. Our interactive interface available at http://www.mrao.cam.ac.uk/projects/gtg/pgt/ is intended to assist in navigating this space of theories.

An alternative method to that presented here is the Hamiltonian approach, which has recently been used to study the particle spectrum of parity-violating PGT by Blagojević and Cvetković [42]. Their results can be straightforwardly reduced to PGT$^+$ by setting all the $\bar{a}$ and $\bar{b}$ to zero in their paper. This will not cause any new "critical parameters" to vanish. By comparing their "critical parameters" with our "critical conditions," we find that our type C critical conditions are identical to their critical parameters. These critical parameters are second class constraints [47, 82], so they do not lead to additional gauge invariance, which is consistent with our definition of type C critical cases. As for the type A critical conditions, we believe that they correspond to first class if-constraints because first class constraints represent additional gauge invariance. In Blagojevic's book [18], the critical parameters for the most general teleparallel PGT$^+$ are listed, and found to be first class. Our method found 4 type A conditions from the theory, which is the same as Blagojevic. This is consistent with our supposition. As for the type B critical cases, however, [42] does not mention its consequences (massive particle becomes massless), but only requires the mass squares to be positive. Blagojević and Vasilić [48] studied what happens when massive modes become massless. In particular, they claim that if any massive tordion becomes massless, there will be extra gauge invariance. However, in their analysis they always include other critical condition(s) in addition to setting the mass to zero to make the theory healthy, so they are not purely applying type B conditions. It is possible that we combine some type B conditions with some other conditions to get a type A condition and extra gauge invariance appears, so their conclusion does not conflict with ours. The Hamiltonian approach also gives more information. Indeed, it is shown in [49, 50, 83, 84] that linearizing a theory can change its structure qualitatively, so that the degrees of freedom and gauge invariances may differ. One must therefore perform a full non-linear analysis to determine whether this is the case for the theories considered here.

Finally, although we demonstrated our method only for PGT$^+$ in this chapter, it may be applied to more complex theories such as Weyl gauge theory (WGT) [62–64], which we will present in Chapter 6, or extended Weyl gauge theory (eWGT) [28]. It is also applicable to conventional metric theories such as $\mathcal{R}^2$ theories.

Appendix 4.A The “root” Poincaré gauge theory

In this section, we will show more details about the process of obtaining the no-ghost-and-condition for the root PGT. The same topic has already been discussed discussed in [29] using similar methods (and also [41] which also included parity-odd terms), and one can compare our process to theirs.

We first linearise the root PGT free gravitational Lagrangian (4.40). The first order Lagrangian is

$$L_1 = 2\lambda \partial_B A^A{}_A{}^B, \tag{4.52}$$

and it is a total derivative. The quadratic Lagrangian is

$$\begin{aligned}
L_2 = {} & \left(\frac{t_1}{3} - \frac{2t_2}{3}\right) A_A{}^{CB} A_{BC}{}^A + \left(\frac{t_1}{3} - \frac{2t_3}{3}\right) A_{AB}{}^A A^B{}_C{}^C + \left(\frac{t_1}{3} + \frac{t_2}{3}\right) A_{BC}{}^A A^{BC}{}_A \\
& + 2\lambda \mathfrak{a}^{AB} \partial_B A_{AC}{}^C - 2\lambda \mathfrak{s}^{AB} \partial_B A_{AC}{}^C + \left(\frac{2r_1}{3} - \frac{2r_2}{3}\right) \partial^d A_A{}^{CB} \partial_d A_{BC}{}^A + 2\lambda \mathfrak{a}^{AB} \partial_C A_B{}^C{}_A \\
& + 2\lambda \mathfrak{s}^{AB} \partial_C A_B{}^C{}_A + (r_4 - r_5) \partial_d A_A{}^{dB} \partial_C A_B{}^{CA} + \left(\frac{4r_1}{3} + \frac{2r_2}{3} - 4r_3\right) \partial_C A_A{}^{dB} \partial_d A_B{}^{CA} \\
& - 2\lambda \mathfrak{s}_A{}^A \partial_C A_B{}^{CB} + (2r_4 + 2r_5) \partial_C A_A{}^{BA} \partial_d A_B{}^{dC} + (-r_4 - r_5) \partial_C A_{BA}{}^A \partial^C A_d{}^{Bd} \\
& + (r_4 + r_5) \partial_d A_B{}^{dA} \partial_C A^{BC}{}_A + \left(\frac{2r_1}{3} - \frac{2r_2}{3}\right) \partial_C A_B{}^{dA} \partial_d A^{BC}{}_A \\
& + \left(\frac{2r_1}{3} + \frac{r_2}{3}\right) \partial_d A_{BC}{}^A \partial^d A^{BC}{}_A + \left(-\frac{2r_1}{3} - \frac{r_2}{3}\right) \partial_d A_{BC}{}^A \partial_A A^{BCd} \\
& + (-r_4 + r_5) \partial_C A_A{}^{BA} \partial_B A^C{}_d{}^d + (2r_4 - 2r_5) \partial_C A_{AB}{}^A \partial_d A^{CdB} \\
& + \left(-\frac{4r_1}{3} + \frac{4r_2}{3}\right) \partial_d A_B{}^{CA} \partial_A A^d{}_C{}^B + \left(\frac{2t_1}{3} - \frac{4t_2}{3} + 2\lambda\right) A^{BCA} \partial_C \mathfrak{a}_{AB} \\
& + \left(-\frac{2t_1}{3} - \frac{2t_2}{3}\right) A^{BCA} \partial_A \mathfrak{a}_{BC} + \left(-\frac{2t_1}{3} + \frac{4t_3}{3} - 2\lambda\right) A_A{}^{BA} \partial_C \mathfrak{a}_B{}^C \\
& + \left(\frac{t_1}{3} + \frac{t_2}{3}\right) \partial^C \mathfrak{a}_A{}^B \partial_C \mathfrak{a}^A{}_B + \left(\frac{t_1}{3} - \frac{2t_2}{3} + \lambda\right) \partial_C \mathfrak{a}_A{}^B \partial_B \mathfrak{a}^{AC} \\
& + \left(-\frac{t_1}{3} + \frac{2t_3}{3} - \lambda\right) \partial_B \mathfrak{a}_A{}^B \partial_C \mathfrak{a}^{AC} + (2t_1 + 2\lambda) A^{BCA} \partial_C \mathfrak{s}_{AB}
\end{aligned}$$

$$
\begin{aligned}
&+\left(-2t_1-2\lambda\right)\partial_C\mathfrak{a}^{AB}\partial_B\mathfrak{s}_A{}^C+\left(\frac{2t_1}{3}-\frac{4t_3}{3}+2\lambda\right)\partial_B\mathfrak{a}^{AB}\partial_C\mathfrak{s}_A{}^C\\
&+\left(\frac{2t_1}{3}-\frac{4t_3}{3}+2\lambda\right)A_A{}^{BA}\partial_C\mathfrak{s}_B{}^C+\left(-\frac{2t_1}{3}+\frac{4t_3}{3}-2\lambda\right)A_A{}^{BA}\partial_B\mathfrak{s}_C{}^C\\
&+\left(\frac{2t_1}{3}-\frac{4t_3}{3}+2\lambda\right)\partial_A\mathfrak{a}^{AB}\partial_B\mathfrak{s}_C{}^C+\left(-\frac{t_1}{3}+\frac{2t_3}{3}-\lambda\right)\partial_B\mathfrak{s}_A{}^A\partial^B\mathfrak{s}_C{}^C\\
&+\left(t_1+\lambda\right)\partial^C\mathfrak{s}_{AB}\partial_C\mathfrak{s}^{AB}+\left(-t_1-\lambda\right)\partial_C\mathfrak{s}_A{}^B\partial_B\mathfrak{s}^{AC}+\left(-\frac{t_1}{3}+\frac{2t_3}{3}-\lambda\right)\partial_B\mathfrak{s}_A{}^B\partial_C\mathfrak{s}^{AC}\\
&+\left(\frac{2t_1}{3}-\frac{4t_3}{3}+2\lambda\right)\partial_B\mathfrak{s}_A{}^A\partial_C\mathfrak{s}^{BC}
\end{aligned}
\tag{4.53}
$$

After Fourier transformation, (4.53) is in the form of (2.11). By applying (2.14) which decomposes (2.11) by the SPOs, one obtains the a-matrices

$$
a\left(0^-\right)=\begin{array}{cc} & A \\ A & \left(2\left(k^2r_2+t_2\right)\right)\end{array}
\tag{4.54}
$$

$$
a\left(0^+\right)=\begin{array}{cc} & \begin{array}{ccc} A & \mathfrak{s} & \mathfrak{s}\end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \\ \mathfrak{s}\end{array} & \begin{pmatrix} 2\left(2k^2\left(r_1-r_3+2r_4\right)+t_3\right) & 2i\sqrt{2}kt_3 & 0 \\ -2i\sqrt{2}kt_3 & 4k^2\left(t_3-\lambda\right) & 0 \\ 0 & 0 & 0\end{pmatrix}\end{array}
\tag{4.55}
$$

$$
a\left(1^-\right)=
\tag{4.56}
$$

$$
\frac{1}{3}\begin{array}{cc} & \begin{array}{cccc} A & A & \mathfrak{s} & \mathfrak{a}\end{array} \\ \begin{array}{c} A \\ A \\ \mathfrak{s} \\ \mathfrak{a}\end{array} & \begin{pmatrix} 6k^2\left(r_1+r_4+r_5\right)+t_1+4t_3 & -\sqrt{2}\left(t_1-2t_3\right) & -i\sqrt{2}k\left(t_1-2t_3\right) & i\sqrt{2}k\left(t_1-2t_3\right) \\ -\sqrt{2}\left(t_1-2t_3\right) & 2\left(t_1+t_3\right) & 2ik\left(t_1+t_3\right) & -2ik\left(t_1+t_3\right) \\ i\sqrt{2}k\left(t_1-2t_3\right) & -2ik\left(t_1+t_3\right) & 2k^2\left(t_1+t_3\right) & -2k^2\left(t_1+t_3\right) \\ -i\sqrt{2}k\left(t_1-2t_3\right) & 2ik\left(t_1+t_3\right) & -2k^2\left(t_1+t_3\right) & 2k^2\left(t_1+t_3\right)\end{pmatrix}\end{array}
\tag{4.57}
$$

$$a\left(1^{+}\right)=\frac{1}{3}\begin{array}{cc} & \begin{array}{ccc} A & A & \mathfrak{a} \end{array} \\ \begin{array}{c} A \\ A \\ \mathfrak{a} \end{array} & \begin{pmatrix} 6k^{2}\left(2r_{3}+r_{5}\right)+t_{1}+4t_{2} & \sqrt{2}\left(t_{1}-2t_{2}\right) & -i\sqrt{2}k\left(t_{1}-2t_{2}\right) \\ \sqrt{2}\left(t_{1}-2t_{2}\right) & 2\left(t_{1}+t_{2}\right) & -2ik\left(t_{1}+t_{2}\right) \\ i\sqrt{2}k\left(t_{1}-2t_{2}\right) & 2ik\left(t_{1}+t_{2}\right) & 2k^{2}\left(t_{1}+t_{2}\right) \end{pmatrix} \end{array} \tag{4.58}$$

$$a\left(2^{-}\right)=\begin{array}{cc} & A \\ A & \begin{pmatrix} 2\left(k^{2}r_{1}+\frac{t_{1}}{2}\right) \end{pmatrix} \end{array} \tag{4.59}$$

$$a\left(2^{+}\right)=\begin{array}{cc} & \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \end{array} & \begin{pmatrix} 2k^{2}\left(2r_{1}-2r_{3}+r_{4}\right)+t_{1} & i\sqrt{2}kt_{1} \\ -i\sqrt{2}kt_{1} & 2k^{2}\left(t_{1}+\lambda\right) \end{pmatrix} \end{array} \tag{4.60}$$

We then fix the gauge by removing rows and columns in the singular a-matrices. After deleting the third row/column in $a(0^{+})$, the third and fourth row/column in $a(1^{-})$, and the third row/column in $a(1^{+})$, we obtain the b-matrices

$$b\left(0^{-}\right)=\begin{array}{cc} & A \\ A & \begin{pmatrix} 2\left(k^{2}r_{2}+t_{2}\right) \end{pmatrix} \end{array} \tag{4.61}$$

$$b\left(0^{+}\right)=\begin{array}{cc} & \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \end{array} & \begin{pmatrix} 2\left(2k^{2}\left(r_{1}-r_{3}+2r_{4}\right)+t_{3}\right) & 2i\sqrt{2}kt_{3} \\ -2i\sqrt{2}kt_{3} & 4k^{2}\left(t_{3}-\lambda\right) \end{pmatrix} \end{array} \tag{4.62}$$

$$b\left(1^{-}\right)=\frac{1}{3}\begin{array}{cc} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A \\ A \end{array} & \begin{pmatrix} 6k^{2}\left(r_{1}+r_{4}+r_{5}\right)+t_{1}+4t_{3} & -\sqrt{2}\left(t_{1}-2t_{3}\right) \\ -\sqrt{2}\left(t_{1}-2t_{3}\right) & 2\left(t_{1}+t_{3}\right) \end{pmatrix} \end{array} \tag{4.63}$$

$$b\left(1^{+}\right)=\frac{1}{3}\begin{array}{cc} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A \\ A \end{array} & \begin{pmatrix} 6k^{2}\left(2r_{3}+r_{5}\right)+t_{1}+4t_{2} & \sqrt{2}\left(t_{1}-2t_{2}\right) \\ \sqrt{2}\left(t_{1}-2t_{2}\right) & 2\left(t_{1}+t_{2}\right) \end{pmatrix} \end{array} \tag{4.64}$$

$$b\left(2^{-}\right)=\begin{array}{cc} & A \\ A & \begin{pmatrix} 2\left(k^{2} r_{1}+\frac{t_{1}}{2}\right) \end{pmatrix} \end{array} \tag{4.65}$$

$$b\left(2^{+}\right)=\begin{array}{cc} & \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \end{array} & \begin{pmatrix} 2k^{2}\left(2r_{1}-2r_{3}+r_{4}\right)+t_{1} & i\sqrt{2}kt_{1} \\ -i\sqrt{2}kt_{1} & 2k^{2}\left(t_{1}+\lambda\right) \end{pmatrix} \end{array}. \tag{4.66}$$

The inverses of the b-matrices are

$$b^{-1}\left(0^{-}\right)=\begin{array}{cc} & A \\ A & \begin{pmatrix} \frac{1}{2\left(k^{2} r_{2}+t_{2}\right)} \end{pmatrix} \end{array} \tag{4.67}$$

$$b^{-1}\left(0^{+}\right)=\frac{1}{2k^{2}\left(r_{1}-r_{3}+2r_{4}\right)\left(t_{3}-\lambda\right)-t_{3}\lambda} \begin{array}{cc} & \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \end{array} & \begin{pmatrix} \frac{1}{2}\left(t_{3}-\lambda\right) & -\frac{it_{3}}{2\sqrt{2}k} \\ \frac{it_{3}}{2\sqrt{2}k} & \frac{2k^{2}\left(r_{1}-r_{3}+2r_{4}\right)+t_{3}}{4k^{2}} \end{pmatrix} \end{array} \tag{4.68}$$

$$b^{-1}\left(1^{-}\right)=$$

$$\frac{1}{3t_{1}t_{3}+2k^{2}\left(r_{1}+r_{4}+r_{5}\right)\left(t_{1}+t_{3}\right)} \begin{array}{cc} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A \\ A \end{array} & \begin{pmatrix} t_{1}+t_{3} & \frac{t_{1}-2t_{3}}{\sqrt{2}} \\ \frac{t_{1}-2t_{3}}{\sqrt{2}} & \frac{1}{2}\left(6k^{2}\left(r_{1}+r_{4}+r_{5}\right)+t_{1}+4t_{3}\right) \end{pmatrix} \end{array} \tag{4.69}$$

$$b^{-1}\left(1^{+}\right)=\frac{1}{3t_{1}t_{2}+2k^{2}\left(2r_{3}+r_{5}\right)\left(t_{1}+t_{2}\right)} \begin{array}{cc} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A \\ A \end{array} & \begin{pmatrix} t_{1}+t_{2} & -\frac{t_{1}-2t_{2}}{\sqrt{2}} \\ -\frac{t_{1}-2t_{2}}{\sqrt{2}} & 6k^{2}r_{3}+3k^{2}r_{5}+\frac{t_{1}}{2}+2t_{2} \end{pmatrix} \end{array} \tag{4.70}$$

$$b^{-1}\left(2^{-}\right)=\begin{array}{cc} & A \\ A & \begin{pmatrix} \frac{1}{2\left(k^{2} r_{1}+\frac{t_{1}}{2}\right)} \end{pmatrix} \end{array} \tag{4.71}$$

$$b^{-1}\left(2^{+}\right)=\frac{1}{t_1\lambda+2k^2\left(2r_1-2r_3+r_4\right)\left(t_1+\lambda\right)}\begin{array}{c|c} & \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \end{array} & \begin{pmatrix} t_1+\lambda & -\frac{it_1}{\sqrt{2}k} \\ \frac{it_1}{\sqrt{2}k} & 2r_1-2r_3+r_4+\frac{t_1}{2k^2} \end{pmatrix} \end{array}. \tag{4.72}$$

The saturated propagator is obtained by sandwiching the b^{-1}-matrices by source currents.

To obtain the no-ghost condition in the massless sector, we have to obtain the source constraints first. The null left eigenvectors $v_i^{w,L}(J^P)$ in (2.17) are

$$\begin{aligned} 0^{+} &:(0,0,1) \\ 1^{-} &:(0,-ik,0,1),(0,ik,1,0) \\ 1^{+} &:(0,-ik,1) \end{aligned} \tag{4.73}$$

We then expand the SPOs in (2.17), solve the tensor component equations by Equations (2.26) and (2.27). By expanding the saturated propagator into tensor components and applying (2.27) to express it in the form of (2.28), we obtain the residue matrices $\mathbf{Q}_{2N}$. Note that the form of (2.27) is not unique, and therefore the $\mathbf{Q}_{2N}$-matrices are not unique, either. We found the higher poles $\mathbf{Q}_4$ and $\mathbf{Q}_6$ vanish identically. The non-zero eigenvalues of $\mathbf{Q}_2$ are

$$\frac{1+6|\vec{k}|^2}{\lambda},\frac{1+8|\vec{k}|^2}{2\lambda}, \tag{4.74}$$

and therefore the massless no-ghost condition is

$$\lambda>0. \tag{4.75}$$

For the massive conditions, we first obtain the determinants of the b-matrices

$$\begin{aligned} \det(b(0^{-})) &= 2t_2+2r_2k^2 \\ \det(b(0^{+})) &= -8t_3\lambda k^2+16\left(r_1-r_3+2r_4\right)\left(t_3-\lambda\right)k^4 \end{aligned}$$

$$\begin{aligned}
\det(b(1^-)) &= 2t_1t_3 + \frac{4}{3}(r_1+r_4+r_5)(t_1+t_3)k^2 \\
\det(b(1^+)) &= 2t_1t_2 + \frac{4}{3}(2r_3+r_5)(t_1+t_2)k^2 \\
\det(b(2^-)) &= t_1 + 2r_1k^2 \\
\det(b(2^+)) &= 2t_1\lambda k^2 + 4(2r_1-2r_3+r_4)(t_1+\lambda)k^4.
\end{aligned} \tag{4.76}$$

The non-zero masses are the non-zero zeros of k^2 in the determinants

$$\begin{aligned}
m^2\left(0^-\right) &= -\frac{t_2}{r_2} \\
m^2\left(0^+\right) &= \frac{t_3\lambda}{2(r_1-r_3+2r_4)(t_3-\lambda)} \\
m^2\left(1^-\right) &= -\frac{3t_1t_3}{2(r_1+r_4+r_5)(t_1+t_3)} \\
m^2\left(1^+\right) &= -\frac{3t_1t_2}{2(2r_3+r_5)(t_1+t_2)} \\
m^2\left(2^-\right) &= -\frac{t_1}{2r_1} \\
m^2\left(2^+\right) &= -\frac{t_1\lambda}{2(2r_1-2r_3+r_4)(t_1+\lambda)}
\end{aligned} \tag{4.77}$$

The massive no-tachyon conditions simply require the squares of the masses to be positive, as shown in (4.48).

The no-ghost conditions (2.40) require the residue of the traces of the b^{-1}-matrices at the mass to be positive (parity even) or negative (parity odd). The results are shown in (4.49).

Combining the conditions in each sector, we obtain the no-ghost-and-tachyon conditions for each massive J^P sector

$$\begin{aligned}
&0^- : t_2 > 0, r_2 < 0 \\
&0^+ : r_1 + 2r_4 > r_3, t_3(t_3-\lambda)\lambda > 0 \\
&1^- : r_1+r_4+r_5 < 0, t_1t_3(t_1+t_3) > 0 \\
&1^+ : 2r_3+r_5 > 0, t_1t_2(t_1+t_2) < 0 \\
&2^- : t_1 > 0, r_1 < 0
\end{aligned}$$

$$2^+ : 2r_1 + r_4 > 2r_3, t_1\lambda\left(t_1 + \lambda\right) < 0$$

However, it is impossible to satisfy all of the conditions at the same time, and so the root PGT must contain ghosts and/or tachyons.

Chapter 5

Power-counting renormalisable, ghost and tachyon free PGT

In the Chapter 4, we have discussed the no-ghosts-and-tachyons conditions for PGT. In addition to possessing no ghosts or tachyons, a healthy physical theory should also be renormalisable. The first step in assessing whether this is possible is to determine whether the theory is power-counting (PC) renormalisable. We will first introduce the criterion of power-counting renormalisability in Section 5.1, and then discuss power-counting renormalisable PGT critical cases in Section 5.2.

5.1 Power counting renormalisablity

In this section, we will introduce the PCR criterion requiring the dimensions of the coupling constants to be non-negative, following the arguments in textbooks, for example, [85–87], see also [88]. We will then clarify the relation between the asymptotic power of the propagators and the PCR criterion and apply it to PGT.

We consider a quantum field theory in d dimensional spacetime with some fields labelled by i, and we assume for each field the propagator $\to p^{-l_i}$ as $p \to \infty$. The interactions are labelled by a, and for each interaction it has $N_{a,i}$ legs of field i, with a coupling constant λ_a and δ_a derivatives. Let us consider a Feynman diagram with I_i internal legs and E_i external legs of field i, and with v_a vertices a and L loops. The numbers have the following relations:

$$\sum_i E_i + \sum_i I_i = \sum_{a,i} (N_{a,i} v_a) - \sum_i I_i \tag{5.1}$$

$$\sum_i E_i + \sum_i I_i = \sum_{a,i} (N_{a,i} v_a) - \left(\sum_a v_a - 1\right) - L, \tag{5.2}$$

so we have

$$L = \sum_i I_i - \sum_a v_a + 1. \tag{5.3}$$

The amplitude is proportional to $|\mathcal{M}| \sim \int \mathrm{d}^{dL} p \, \prod_{a,i} \left(p^{v_a \delta_a} / p^{l_i I_i} \right)$. The superficial degree of divergence is

$$\begin{aligned} D &= dL - \sum_i (l_i I_i) + \sum_a (\delta_a v_a) \\ &= d + \sum_i (d - l_i) I_i + \sum_a (\delta_a - d) v_a \\ &= d - \sum_i (d - l_i) \frac{E_i}{2} + \frac{1}{2} \sum_{i,a} (d - l_i) N_{a,i} v_a + \sum_a (\delta_a - d) v_a \end{aligned} \tag{5.4}$$

for each Feynman diagram. For a renormalisable theory, only finite types of divergent D are allowed, so D has a upper bound. Since $E_i \geq 0$ and we assume $l_i \leq d$, the conditions are[1]

$$\delta_a - d + \frac{1}{2} \sum (d - l_i) N_{a,i} \leq 0 \quad \forall a. \tag{5.5}$$

We now define the canonical, or engineering dimension [85] of the field φ_i as

$$[\varphi_i] \equiv (d - l_i)/2. \tag{5.6}$$

The canonical dimension of a field sometimes coincides with the mass dimension of the field in natural units. The mass dimension of a field can be inferred from the fact that each term in the Lagrangian density has mass dimension d. For example, for the Klein-

[1] Note that if any i satisfies $d - l_i = 0$, then infinitely many values of E_i can give the same value of D. Even if the theory satisfies (5.4) and the superficial divergence is bounded, we may still need infinitely many parameters. We will not discuss this issue further for simplicity. Readers may refer to [85] for more details.

Gordon field ($\mathcal{L}_F = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2$), the Dirac field ($\mathcal{L}_F = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi$), and the electromagnetic field ($\mathcal{L}_F = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu)$) the mass dimension can be read off directly from the corresponding Lagrangian, and matches the canonical dimension in each case. However, for example, the dimensions do not match for a massive vector field ($\mathcal{L}_F = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu$). In this case, the propagator is $D(p)_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)/\left(p^2 - m^2\right)$, and it tends to a constant when $p^2 \to \infty$. We can make a field redefinition by multiplying by a constant and make the two dimensions coincide. For example, if we set $A'_\mu = mA_\mu$, then the mass dimension of A'_μ becomes 2, which is the same as $[A_\mu]$. We will assume the fields are redefined and the two dimensions coincide in the following steps.

For the interaction terms, we can obtain the dimension of the coefficient $[\lambda_a] = d - \delta_a - \sum_i N_{a,i}[\varphi_i] = d - \delta_a - 1/2\sum_i N_{a,i}(d - l_i)$, so the PC renormalisable condition is

$$[\lambda_a] \geq 0 \quad \forall a. \tag{5.7}$$

There should not be any coupling constant with negative dimension, or the theory is not PC renormalisable. However, PC is not the ultimate criterion for renormalisability. Some PCR theories may be non-renormalisable because of some deeper problems such as anomalies, and non-PCR theories may turn out to be renormalisable (for example, see [89]).

Now we can check whether GR itself is PC renormalisable as an example. The Lagrangian of GR is $L = \frac{M_{\rm pl}^2}{2}\sqrt{-g}\mathcal{R}$, and if we make the perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ we obtain:

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\sqrt{-g}\mathcal{R} \sim M_{\rm pl}^2\left[(\partial h)^2 + (\partial h)^2 h + \ldots\right] \tag{5.8}$$

The propagator is

$$D_{\mu\nu\rho\sigma}(p) = \frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{2p^2}, \tag{5.9}$$

which goes as p^{-2} at high energy, so the canonical dimension of the h-field is $(4-2)/2 = 1$. The mass dimension of the h-field is 0, so we can make the field redefinition $\tilde{h}_{\mu\nu} = M_{\rm pl}h_{\mu\nu}$

so that the two dimensions of $\tilde{h}_{\mu\nu}$ coincide, and the Lagrangian becomes

$$\mathcal{L} \sim (\partial\tilde{h})^2 + \frac{1}{M_{\rm pl}}(\partial\tilde{h})^2\tilde{h} + \dots \tag{5.10}$$

Since it contains negative dimension coefficients, it is not PC-renormalisable. Let us assume that the propagator (5.9) has a different asymptotic behaviour $\sim p^{-l}$. Then $[h_{\mu\nu}]$ becomes $2-l/2$, and we have to redefine the field as $\tilde{h}_{\mu\nu} = M_{\rm pl}^{2-l/2}h_{\mu\nu}$. If $2-l/2>0$, there will be negative dimension for the coefficient of the interaction terms. Hence, for a theory with the form $\mathcal{L} \sim M_{\rm pl}^2\left[(\partial h)^2 + (\partial h)^2 h + \dots\right]$, we need $l \geq 4$ to make it PCR.

In PGT, the Lagrangian is

$$\begin{aligned} b\mathcal{L}_G &\sim b\left(\lambda\mathcal{R} + r\mathcal{R}^2 + t\mathcal{T}^2\right) \\ &\sim \left(1+f+f^2+\dots\right)\left\{\lambda\left(1+f\right)^2\left(\partial A + A^2\right) + r\left(1+f\right)^4\left(\partial A + A^2\right)^2 \right. \\ &\quad \left. + t\left(1+f\right)^2\left[\partial(f+f^2+\dots) + (1+f+f^2+\dots)A\right]^2\right\}, \end{aligned} \tag{5.11}$$

where we do not show the detailed structures of the indices and coefficients, and the mass dimension of the parameters and fields are $[\lambda]_{\rm M}=2$, $[r]_{\rm M}=0$, $[t]_{\rm M}=2$, $[A]_{\rm M}=1$, and $[f]_{\rm M}=0$. Assuming the propagators of A and h behave as p^{-l_A} and p^{-l_h} respectively, we need to redefine the fields as $\tilde{A} = M_A^{1-l_A/2}A$ and $\tilde{h} = M_h^{2-l_h/2}h$. Therefore we require $l_A \geq 2$ and $l_h \geq 4$ for the theory to be PCR. [2]

5.2 Power-counting renormalisability of PGT

In Chapters 2 and 4, we presented a systematic method for identifying the ghost-and-tachyon-free critical cases of parity-preserving gauge theories of gravity, and applied it to parity-preserving Poincaré gauge theory (PGT$^+$). We can now use the criterion in Section 5.1 to identify PCR ghost-and-tachyon-free critical cases. However, even this condition can

[2] If $r=0$, then the interaction terms with the highest degree of A are $\sim A^2$ with coefficients of dimension 2. Hence, in this case, we may have a looser condition $l_A \geq 0$. However, there is no dynamical term for A if $r=0$, so we consider A not propagating. We will discuss this later.

be quite difficult to establish in the general case in which the propagator for the theory contains terms that mix different fields, which is the case for PGT$^+$. Nonetheless, in the decomposition of the propagator using SPOs, there are some critical cases for which the mixing terms in the b-matrices vanish. In these cases, the physical meaning is much clearer. We therefore focus only on the PGT$^+$ critical cases that satisfy this property.

The key quantity for determining whether a theory is PCR is the propagator

$$\hat{D} = \sum_{J,P,i,j} b_{ij}^{-1} \hat{P}(J^P)_{ij}. \tag{5.12}$$

In particular, if the b-matrices contain no elements linking any of the A-, $\mathfrak{s}$- and $\mathfrak{a}$-fields , then it is straightforward to obtain the propagators for these fields separately from $\hat{D}$.[3] In Section 5.1 we pointed out that the PCR criterion on the A- and f-fields of PGT requires the propagator of the A-field to decay at least as quickly as k^{-2} at high energy, and those of the $\mathfrak{s}$- and $\mathfrak{a}$-fields to fall off at least as k^{-4}. This criterion is also used in [29], and we will call it the "original PCR criterion". By contrast, here we propose an alternative PCR criterion, which also permits the presence of non-propagating fields (for which the propagator decays no faster than $\sim k^0$)[4], since these should completely decouple from the rest of the theory; we will compare these two criteria further below.

In the remainder of this chapter, we will study only those cases for which the b-matrices are block diagonal, with each block containing only one field, ensuring that there are no mixing terms in the gauge-fixed Lagrangian.[5] It is worth noting that, the behaviour of the propagators at high energy goes as the highest power of the corresponding elements in the b^{-1}-matrices. Moreover, in the PGT$^+$ cases we consider, any non-diagonal block of a b-matrix that does not mix fields is always the only block in the matrix, contains only the A-field, and has size 2×2. Moreover, these blocks occur only in the 1^- or 1^+ sector. The

[3] Because the b-matrices are generally not unique, we should find out all possible b-matrices and pick only those "non-mixing".

[4] We will see later in Table 5.1 that for the PGT cases satisfying the restriction which will be mentioned later in this chapter, the propagator decaying no faster than $\sim k^0$ is equivalent to there being fields in some spin sectors without dynamical terms in the linearised Lagrangian

[5] We note that this extension therefore does not include Einstein-Cartan theory.

most general forms of the b-matrices in these sectors are

$$b(1^-) = \frac{1}{3} \begin{array}{c c} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A^* \\ A^* \end{array} & \begin{pmatrix} 6(r_1+r_4+r_5)k^2+(t_1+4t_3) & -\sqrt{2}(t_1-2t_3) \\ -\sqrt{2}(t_1-2t_3) & 2(t_1+t_3) \end{pmatrix} \end{array}, \tag{5.13}$$

$$b(1^+) = \frac{1}{3} \begin{array}{c c} & \begin{array}{cc} A & A \end{array} \\ \begin{array}{c} A^* \\ A^* \end{array} & \begin{pmatrix} 6(2r_3+r_5)k^2+(t_1+4t_2) & -\sqrt{2}(t_1-2t_2) \\ -\sqrt{2}(t_1-2t_2) & 2(t_1+t_2) \end{pmatrix} \end{array}, \tag{5.14}$$

and they are both of the form

$$b = \begin{pmatrix} rk^2+(x+4y) & -\sqrt{2}(x-2y) \\ -\sqrt{2}(x-2y) & 2(x+y) \end{pmatrix}, \tag{5.15}$$

where x, y, and r are real linear combinations of the parameters in the Lagrangian. The determinant is

$$\det(b) = 2r(x+y)k^2 + 18xy, \tag{5.16}$$

and the inverse is

$$b^{-1} = \frac{1}{\det(b)} \begin{pmatrix} 2(x+y) & \sqrt{2}(x-2y) \\ \sqrt{2}(x-2y) & rk^2+(x+4y) \end{pmatrix}. \tag{5.17}$$

Hence, the element with the highest power of k in b^{-1} is always a diagonal element. Note that when $x+y=0$ and $r,x,y \neq 0$, the element with the highest power in b^{-1} goes as k^2, not k^{-2}, even though the highest power in b is also k^2. This is a similar case to that summarised in Eqs. (1.2)–(1.4) of [59]. Since there is no pole in the determinant $\det(b) = -18x^2$ in this case, there is no propagating mode in this sector. We list all the possible forms of (5.15) in all critical cases of PGT$^+$ and some of their properties in Table 5.1. Note that we also include here the situation that the matrix becomes diagonal, but the condition that makes it diagonal does not make it critical in the sense defined in Chapter 2. All the possible "extended critical

Table 5.1 Summary of the properties of all the forms of b-matrix. The column "b-sector" denotes the properties of the diagonal elements in b^{-1}-matrix. it is notated as φ_v^n or φ_l^n, where φ is the field, $-n$ is the power of k in the element in the b^{-1}-matrix when k goes to infinity, v means massive pole, and l means massless pole. If $n = \infty$, it represents that the diagonal element is zero. The "&" connects the diagonal elements in the same b^{-1}-matrix. The superscript "N" represents that there is non-zero off-diagonal term in the b^{-1}-matrix. The column "k power" is the highest power of k in all non-zero elements of b^{-1}. The column "consistent" indicate whether the particle content is consistent with $\det(b)$ after we integrate out the non-propagating field(s).

#	b-matrix	b-sector	$\det(b)$	k power	Consis-tent
A	$\begin{pmatrix} 4y+k^2r+x & -\sqrt{2}(-2y+x) \\ -\sqrt{2}(-2y+x) & 2(y+x) \end{pmatrix}$	$\left\{\left(A_{\mathrm{v}}^2 \& A_{\mathrm{v}}^0\right)^{\mathrm{N}}\right\}$	$2k^2r(x+y)$ $+18yx$	0	○
B	$\begin{pmatrix} 4y+x & -\sqrt{2}(-2y+x) \\ -\sqrt{2}(-2y+x) & 2(y+x) \end{pmatrix}$	$\left\{\left(A^0 \& A^0\right)^{\mathrm{N}}\right\}$	$18yx$	0	○
C	$\begin{pmatrix} 4y+k^2r & 2\sqrt{2}y \\ 2\sqrt{2}y & 2y \end{pmatrix}$	$\left\{\left(A_{\mathrm{l}}^2 \& A_{\mathrm{l}}^0\right)^{\mathrm{N}}\right\}$	$2k^2yr$	0	○
D	$\begin{pmatrix} k^2r+x & -\sqrt{2}x \\ -\sqrt{2}x & 2x \end{pmatrix}$	$\left\{\left(A_{\mathrm{l}}^2 \& A_{\mathrm{l}}^0\right)^{\mathrm{N}}\right\}$	$2k^2rx$	0	○
E	$\begin{pmatrix} k^2r-3x & -3\sqrt{2}x \\ -3\sqrt{2}x & 0 \end{pmatrix}$	$\left\{\left(A^\infty \& A^{-2}\right)^{\mathrm{N}}\right\}$	$-18x^2$	2	○
F	$\begin{pmatrix} k^2r+3x & 0 \\ 0 & 3x \end{pmatrix}$	$\left\{A_{\mathrm{v}}^2 \& A^0\right\}$	$3k^2rx+9x^2$	0	○
G	$\begin{pmatrix} 4y \end{pmatrix}$	$\left\{A^0\right\}$	$4y$	0	○
H	$\begin{pmatrix} x \end{pmatrix}$	$\left\{A^0\right\}$	x	0	○
I	$\begin{pmatrix} -3x & -3\sqrt{2}x \\ -3\sqrt{2}x & 0 \end{pmatrix}$	$\left\{\left(A^\infty \& A^0\right)^{\mathrm{N}}\right\}$	$-18x^2$	0	○
J	$\begin{pmatrix} 3x & 0 \\ 0 & 3x \end{pmatrix}$	$\left\{A^0 \& A^0\right\}$	$9x^2$	0	○
K	$\begin{pmatrix} k^2r \end{pmatrix}$	$\left\{A_{\mathrm{l}}^2\right\}$	k^2r	-2	N/A

parameters" in the root case are x, y, r, $x+y$, and $x-2y$. All the child theories do not have "extended critical parameters" other than those listed above.[6]

We find 58 cases that are PCR and free from ghosts and tachyons, and we list them in Tables 5.2–5.7, in which the old cases are indicated with an asterisk followed by the old number of the case as given in the previous paper [70]. Tables 5.2 and 5.3 summarise the 7 cases with both massless and massive modes, all of which have two massless degrees of freedom (d.o.f.) in propagating modes and a massive 0^- or 2^- mode. Tables 5.4 and 5.5

[6] We are only using the "extended critical parameters" here. In the remaining parts of this thesis, all "critical parameter" should refer to the meaning defined in Chapter 2 unless otherwise specified.

summarise the 12 cases with only massless modes, of which all contain only 2 massless d.o.f. Finally, Tables 5.6 and 5.7 summarise the 39 cases with only massive modes. For each set of tables, the first lists the various conditions for each critical case, and the second lists the "particle content" in terms of the diagonal elements in the b^{-1}-matrix of each spin-parity sector in the sequence $\{0^-,0^+,1^-,1^+,2^-,2^+\}$. Note that within the 58 cases, Case 9, 10, 11, and 13, each of which contains only 2 d.o.f. massless mode, satisfy both the original PCR criterion and the alternative PCR criterion. All the others satisfy only the alternative one.

Since we are using the alternative PCR criterion, which differs from the original criterion used in [29] by allowing the presence of non-propagating fields, it is worth discussing further the status of such fields in the determination of whether a theory is PCR. We begin by noting that an important consequence of allowing the existence of non-propagating fields is that whether some critical cases obey our PCR criterion may depend on the choice of gauge fixing. For example, in the spin-parity sector 0^+ in Case 8, the a-matrix is

$$a(0^+) = \begin{array}{c} \begin{array}{ccc} A & \mathfrak{s} & \mathfrak{s} \end{array} \\ \begin{pmatrix} 2t_3 & 2i\sqrt{2}kt_3 & 0 \\ -2i\sqrt{2}kt_3 & 4k^2t_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}, \tag{5.18}$$

which is singular, indicating the presence of gauge invariances. One may render this matrix non-singular by deleting rows and columns in two different ways, corresponding to two different gauge fixings, which in this case correspond simply to keeping either the first or the second column and row. If one chooses to keep only the second row and column, then this sector contains only an $\mathfrak{s}$-field, with a propagator that goes as $\sim k^{-2}$ at high energy, which thus violates both our alternative PCR criterion and the original one. Conversely, if one chooses to retain only the first column and row, then the 0^+ spin-parity sector contains only a non-propagating A-field, which we contend is harmless and thus satisfies our alternative PCR criterion, while violating the original one. The conclusions regarding PCR are therefore gauge dependent.

Overall, we take the view that a theory is PCR if one can find a gauge in which it satisfies our PCR criterion, irrespective of the existence of other gauge choices in which the PCR criterion is violated. The rationale for this view is that a theory should describe the same physics independently of which gauge one adopts. Thus, if one uses a particular gauge to make a physical prediction, then one should, in principle, be able to draw the same physical conclusion in any other gauge, although most often not in such a transparent manner.

We therefore consider the 0^+ sector of Case 8 to satisfy our PCR criterion, whereas it violates the original one in [29]. Moreover, although the total propagator for a field is the sum of the propagators across all sectors, it cannot satisfy either PCR condition if that same condition is violated by the propagator in any sector individually. This occurs since the high-energy asymptotic behaviour is determined by the term(s) with the highest power, unless they cancel out, but the SPO decomposition guarantees that such cancellations cannot happen if $k^2 \neq 0$, which is the case we are considering here. Thus, Case 8 as a whole violates the original PCR criterion in [29] because of the nature of the 0^+ sector, whereas one finds that it satisfies our alternative PCR criterion, and is hence listed in Tables 5.4 and 5.5.

We now explain why this does not, in fact, lead to a contradiction. If one chooses to keep only the first column and row in (5.18), the resulting b^{-1}-matrix is clearly

$$b^{-1}(0^+) = \left(\frac{1}{2t_3}\right), \tag{5.19}$$

so the field in this sector is not propagating, and the corresponding propagator is $\sim k^0$ at high energy. The key point, however, is that there is no dynamical term in the Lagrangian for the field corresponding to (5.19). Thus, one can integrate out this non-propagating field in the path integral, which is equivalent to substituting for it in the Lagrangian using its classical equation of motion obtained by varying the non-propagating field. This is most transparently achieved by first introducing polarisation basis vectors to decompose the fields and the SPOs, as discussed in Appendix 2.B. One then expands the fields in terms of these basis vectors,

$$|A\rangle = \sum_{J,P,i,m} \bar{A}_{i,J^P,m} \, |i, J^P, m\rangle \,, \tag{5.20}$$

from which one obtains the relation

$$\hat{P}_{ji}(J^P)\,|A\rangle = \bar{A}_{i,J^P,m}\,|j,J^P,m\rangle\,. \tag{5.21}$$

The Lagrangian corresponding to the 0^+ sector then becomes

$$L(0^+) = t_3\bar{A}^2_{1,0^+,0}, \tag{5.22}$$

and the equation of motion is simply $\bar{A}_{1,0^+,0} = 0$, so one can simply ignore this sector. One might alternatively use the Lagrangian containing the source current here, so that the equation of motion becomes $2t_3\bar{A}_{1,0^+,0} = \bar{j}_{1,0^+,0}$, where $\bar{j}_{1,0^+,0}$ is appropriate expansion of the source current in the polarisation. Since we are considering only free-field theories, however, the source currents can themselves be due only to the gauge fields and thus at least quadratic. Hence, these source currents can only affect the fields to the next order, so we can neglect them in the linearised Lagrangian.

The 1^- sector of Case 8 can also contain non-propagating fields. The a-matrix for this sector is

$$a(1^-) = 2\;\begin{array}{c} \begin{array}{cccc} \quad A \quad\quad & \quad A \quad & \mathfrak{s} & \quad\mathfrak{a} \end{array} \\ \begin{pmatrix} 3k^2(r_1+r_5)+2t_3 & \sqrt{2}t_3 & i\sqrt{2}kt_3 & -i\sqrt{2}kt_3 \\ \sqrt{2}t_3 & t_3 & ikt_3 & -ikt_3 \\ -i\sqrt{2}kt_3 & -ikt_3 & k^2t_3 & -k^2t_3 \\ i\sqrt{2}kt_3 & ikt_3 & -k^2t_3 & k^2t_3 \end{pmatrix} \end{array}, \tag{5.23}$$

which is singular as a result of gauge invariances. One may render the matrix non-singular and thereby fix the gauge by, for example, choosing the first two rows and columns to form the corresponding b-matrix, in which case the sector contains a propagating A-particle and a non-propagating A-particle with some mixing term. The resulting determinant is

$$\det[b(1^-)] = \frac{4}{3}(r_1+r_5)\,t_3k^2, \tag{5.24}$$

so there can only be massless modes in this sector. Using the expansion (5.21) to reconstruct the Lagrangian corresponding to the 1^- sector, one obtains

$$L(1^-) = -\sum_{m=-1}^{1}\left\{\bar{A}_{1,1^-,m}[-3(r_1+r_5)\partial^2+2t_3]\bar{A}_{1,1^-,m}+2\sqrt{2}t_3\bar{A}_{1,1^-,m}\bar{A}_{2,1^-,m}+t_3\bar{A}^2_{2,1^-,m}\right\}. \tag{5.25}$$

Hence, it is clear that there is a propagating $\bar{A}_{1,1^-,m}$ field that is mixed with a $\bar{A}_{2,1^-,m}$ field without a dynamical term. One can thus integrate out the latter field using its classical equation of motion,

$$\bar{A}_{2,1^-,m} = -\sqrt{2}\bar{A}_{1,1^-,m}, \tag{5.26}$$

and the Lagrangian becomes

$$L(1^-) = -\sum_{m=-1}^{1}\{\bar{A}_{1,1^-,m}[-3(r_1+r_5)\partial^2]\bar{A}_{1,1^-,m}\}. \tag{5.27}$$

This is consistent with there being no massive mode in this sector. Furthermore, one finds that the effect of integrating out the non-propagating fields in the 0^+ and 1^- sectors in Case 8 is the same as setting t_3 to zero, and all the b-matrices become exactly the same as those of Case 9. Hence, at least in the free-field case we are considering, in which the gauge fields do not couple to external matter fields, Case 8 and 9 are actually describing the same theory. Moreover, since Case 9 may be shown to satisfy Sezgin's original PCR criterion in [29], there is thus no contradiction in Case 8 satisfying our alternative PCR criterion. Indeed, the alternative criterion allows us to identify Case 8 as PCR, which would be missed using the original PCR criterion.

For all Cases 1–58, one may similarly check whether, after integrating out the non-propagating fields, the remaining fields are consistent with the particle contents that their determinants of b-matrices indicate. Because all the b-matrices containing non-propagating terms in these cases are in the form of (5.15), one can perform this check by examining only all the "special cases" of the form (5.15) (including the critical cases and those with the parameters making any of the elements zero). We find that all of them are consistent.

Note that after integrating out non-propagating fields in a critical case, its Lagrangian can be written in the form of (5.27), but it may not match any other critical case.

Moreover, as one might expect, one may show that similar equivalences as Case 8 and 9 exist between other cases. For example, one may further demonstrate in the manner outlined above that: Case 2 is equivalent to Case 1; Cases 12, 14, and 15 are equivalent to Case 10; Case 16 is equivalent to Case 11; Case 25 is equivalent to Case 26; Case 29 is equivalent to Case 30; Case 37 is equivalent to Case 35; and Case 41 is equivalent to Case 27. Unfortunately, it is not so straightforward to establish the equivalences amongst the other cases. For the critical cases we do not list in this section, we anticipate that there will similarly be some groups of equivalent cases in the above sense, provided they do not couple to external matter fields, so that one may simplify the "tree" of critical cases. We leave this analysis for future work. Nonetheless, we do find that after integrating out all the non-propagating fields in Cases 1-58, all the resulting theories satisfy the original PCR condition. Hence, allowing for non-propagating fields does not violate this criterion in practice.

We also investigated the PGT$^+$ theories with either zero torsion or zero curvature, discussed in Secs. 4.2.1 and 4.2.2 respectively, but found that no cases are both unitary and PC renormalisable.

5.3 Conclusions

In conclusion, we have found 58 critical cases of PGT$^+$ that are both PCR and free of ghosts and tachyons. Note that while a theory may pass our PCR criterion, this is no guarantee that the theory is renormalisable, and this would take independent investigation and the inclusion of interactions. We have also clarified the role played by non-propagating modes in determining whether a theory is PCR. We illustrate this issue further in Appendix 5.A, where we demonstrate the methods used in this section in the more familiar and much simpler cases of the Proca and Stueckelberg theories for a massive spin-1 particle.

In future work, we plan to investigate all these theories further, but especially those that possess massless propagating particles, by considering their phenomenology in the context both of cosmological and compact object solutions. Although these theories contain no graviton, only tordions, they may provide some insights into the construction of a self-consistent quantum theory of long-range gravitational interactions. In particular, cases 10 and 11 might be of interest, since they may possess particles in the 2^+ sector. Indeed, it is worth noting that in the absence of torsion the action for both of these cases reduces to that of conformal gravity, which is PC renormalisable but not unitary, as discussed in Case 22 in Section 4.2.4.

Table 5.2 Parameter conditions for the PC renormalisable critical cases that are ghost and tachyon free and have both massless and massive propagating modes. The parameters listed in "Additional conditions" must be non-zero to prevent the theory becoming a different critical case.

#	Critical condition	Additional conditions	No-ghost-and-tachyon condition
1	$r_1, \frac{r_3}{2} - r_4, t_1, t_3, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_2$	$t_2 > 0, r_2 < 0, r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
2	$r_1, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_2, r_1 - r_3, 2r_3 + r_5, r_1 + r_3 + 2r_5, t_2, t_3$	$t_2 > 0, r_2 < 0, r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
3	$r_1, r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$r_2 < 0, r_5 < 0, t_1 < 0$
4	$r_2, r_1 - r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$t_1 > 0, r_1 + r_5 < 0, r_1 < 0$
5	$r_2, r_1 - r_3, r_4, t_2, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_3$	$r_5 > 0, 2r_1 + r_5 > 0, t_1 > 0, r_1 < 0$
6	$r_1, 2r_3 - r_4, t_1 + t_2, t_3, \lambda = 0$	$r_2, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$r_2 < 0, 2r_3 + r_5 < 0, t_1 < 0$
7	$r_2, 2r_1 - 2r_3 + r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$t_1 > 0, r_1 < 0, 2r_3 + r_5 < r_1$

Table 5.3 Particle content of the PC renormalisable critical cases that are ghost and tachyon free and have both massless and massive propagating modes. All of these cases have 2 massless d.o.f. in propagating modes, and also a massive mode. The column "b-sectors" describes the diagonal elements in the b^{-1}-matrix of each spin-parity sector in the sequence $\{0^-, 0^+, 1^-, 1^+, 2^-, 2^+\}$. Here and in Tables 5.5 and 5.7 it is notated as φ_v^n or φ_l^n, where φ is the field, $-n$ is the power of k in the element in the b^{-1}-matrix when k goes to infinity, v means massive pole, and l means massless pole. If $n = \infty$, it represents that the diagonal element is zero. If $n \leq 0$, the field is not propagating. The "$|$" notation denotes the different form of the elements of the b^{-1}-matrices in different choices of gauge fixing, and the "&" connects the diagonal elements in the same b^{-1}-matrix. The superscript "N" represents that there is non-zero off-diagonal term in the b^{-1}-matrix.

#	Massless mode d.o.f.	Massive mode	b-sectors
1	2	0^-	$\left\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
2	2	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
3	2	0^-	$\left\{A^2_{\mathrm{v}}, \times, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty \& A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}\right\}$
4	2	2^-	$\left\{A^0, \times, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty \& A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}\right\}$
5	2	2^-	$\left\{\times, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}, \left(A^\infty \& A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{s}^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}\right\}$
6	2	0^-	$\left\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty \& A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}\right\}$
7	2	2^-	$\left\{A^0, A^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}} \& A^0_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}} \vert \left(A^2_{\mathrm{l}} \& \mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty \& A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty \& \mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0 \vert \mathfrak{s}^2_{\mathrm{l}}\right\}$

Table 5.4 Parameter conditions for the PC renormalisable critical cases that are ghost and tachyon free and have only massless propagating modes. The cases found previously in [70] are indicated with an asterisk followed by its original numbering.

#	Critical condition	Additional condition	No-ghost-and-tachyon condition
8	$r_2, r_1 - r_3, r_4, t_1, t_2, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_3$	$r_1 (r_1 + r_5)(2r_1 + r_5) < 0$
^{*1}9	$r_2, r_1 - r_3, r_4, t_1, t_2, t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5$	$r_1 (r_1 + r_5)(2r_1 + r_5) < 0$
^{*3}10	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, t_3, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
^{*4}11	$r_1, \frac{r_3}{2} - r_4, t_1, t_2, t_3, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
12	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_3, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_2$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
^{*2}13	$r_2, 2r_1 - 2r_3 + r_4, t_1, t_2, t_3, \lambda = 0$	$r_1, r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5$	$r_1 (r_1 - 2r_3 - r_5)(2r_3 + r_5) > 0$
14	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$2r_3 - r_4, 2r_3 + r_5, r_4 + r_5, t_3$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
15	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
16	$r_1, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$	$r_3 (2r_3 + r_5)(r_3 + 2r_5) < 0$
17	$r_1, r_2, r_3, r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_2$	$r_5 < 0, t_1 \neq 0$
18	$r_1, r_2, r_3, r_4, t_2, t_1 + t_3, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_3$	$r_5 > 0, t_1 \neq 0$
19	$r_1, r_2, 2r_3 - r_4, t_1 + t_2, t_3, \lambda = 0$	$r_1 - r_3, r_1 - 2r_3 - r_5, 2r_3 + r_5, t_1, t_2$	$r_3 < -\frac{r_5}{2}, t_1 \neq 0$

Table 5.5 Particle content of the PC renormalisable critical cases that are ghost and tachyon free and have only massless propagating modes. All of these cases have 2 massless d.o.f. of propagating mode. The cases found previously in [70] are indicated with an asterisk followed by its original numbering.

#	Massless mode d.o.f.	b-sectors
8	2	$\left\{\times, A^0 \vert \mathfrak{s}^2_\mathrm{I}, \left(A^2_\mathrm{I} \& A^0_\mathrm{I}\right)^\mathrm{N} \vert \left(A^2_\mathrm{I} \& \mathfrak{s}^2_\mathrm{I}\right)^\mathrm{N} \vert \left(A^2_\mathrm{I} \& \mathfrak{a}^2_\mathrm{I}\right)^\mathrm{N}, A^2_\mathrm{I}, A^2_\mathrm{I}, \times\right\}$
^{*1}9	2	$\{\times, \times, A^2_\mathrm{I}, A^2_\mathrm{I}, A^2_\mathrm{I}, \times\}$
^{*3}10	2	$\{\times, \times, A^2_\mathrm{I}, A^2_\mathrm{I}, \times, A^2_\mathrm{I}\}$
^{*4}11	2	$\{A^2_\mathrm{I}, \times, A^2_\mathrm{I}, A^2_\mathrm{I}, \times, A^2_\mathrm{I}\}$
12	2	$\left\{A^0, \times, A^2_\mathrm{I}, \left(A^2_\mathrm{I} \& A^0_\mathrm{I}\right)^\mathrm{N} \vert \left(A^2_\mathrm{I} \& \mathfrak{a}^2_\mathrm{I}\right)^\mathrm{N}, \times, A^2_\mathrm{I}\right\}$
^{*2}13	2	$\{\times, A^2_\mathrm{I}, A^2_\mathrm{I}, A^2_\mathrm{I}, A^2_\mathrm{I}, \times\}$
14	2	$\left\{\times, A^0 \vert \mathfrak{s}^2_\mathrm{I}, \left(A^2_\mathrm{I} \& A^0_\mathrm{I}\right)^\mathrm{N} \vert \left(A^2_\mathrm{I} \& \mathfrak{s}^2_\mathrm{I}\right)^\mathrm{N} \vert \left(A^2_\mathrm{I} \& \mathfrak{a}^2_\mathrm{I}\right)^\mathrm{N}, A^2_\mathrm{I}, \times, A^2_\mathrm{I}\right\}$

Table 5.5 (continued)

#	Massless mode d.o.f.	b-sectors
15	2	$\left\{A^0, A^0\vert\mathfrak{s}_1^2, \left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{s}_1^2\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}},\right.$ $\left.\left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}}, \times, A_1^2\right\}$
16	2	$\left\{A_1^2, A^0\vert\mathfrak{s}_1^2, \left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{s}_1^2\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}}, A_1^2, \times, A_1^2\right\}$
17	2	$\left\{A^0, \times, \left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{s}_1^2\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}}, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty\&\mathfrak{a}_1^0\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}_1^2\right\}$
18	2	$\left\{\times, A^0\vert\mathfrak{s}_1^2, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty\&\mathfrak{s}_1^0\right)^{\mathrm{N}} \vert \left(A^\infty\&\mathfrak{a}_1^0\right)^{\mathrm{N}}, \left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}_1^2\right\}$
19	2	$\left\{A^0, A_1^2, \left(A_1^2\&A_1^0\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{s}_1^2\right)^{\mathrm{N}} \vert \left(A_1^2\&\mathfrak{a}_1^2\right)^{\mathrm{N}}, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}} \vert \left(A^\infty\&\mathfrak{a}_1^0\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}_1^2\right\}$

Table 5.6 Parameter conditions for the PC renormalisable critical cases that are ghost and tachyon free and have only massive propagating modes. The cases found previously in [70] are indicated with an asterisk followed by its original numbering.

#	Critical condition	Additional conditions	No-ghost-and-tachyon condition
20	$r_1, r_3, r_4, r_5, \lambda = 0$	$r_2, t_1, t_2, t_1 + t_2, t_3, t_1 + t_3$	$t_2 > 0, r_2 < 0$
21	$r_1, r_3, r_4, r_5, t_1 + t_2, \lambda = 0$	$r_2, t_1, t_2, t_3, t_1 + t_3$	$r_2 < 0, t_1 < 0$
22	$r_1, r_3, r_4, r_5, t_1 + t_3, \lambda = 0$	$r_2, t_1, t_2, t_1 + t_2, t_3$	$t_2 > 0, r_2 < 0$
23	$r_1, r_3, r_4, r_5, t_1 + t_2, t_1 + t_3, \lambda = 0$	r_2, t_1, t_2, t_3	$r_2 < 0, t_1 < 0$
24	$r_1, r_3, r_4, t_1, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_2, t_3$	$t_2 > 0, r_2 < 0$
^{*5}25	$r_1, r_3, r_4, r_5, t_1, \lambda = 0$	r_2, t_2, t_3	$t_2 > 0, r_2 < 0$
^{*6}26	$r_1, r_3, r_4, r_5, t_1, t_3, \lambda = 0$	r_2, t_2	$t_2 > 0, r_2 < 0$
27	$r_1, \frac{r_3}{2} - r_4, \frac{r_3}{2} + r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
28	$r_1, r_3, r_4, t_1, t_3, \lambda = 0$	r_2, r_5, t_2	$t_2 > 0, r_2 < 0$
29	$r_1 - r_3, r_4, 2r_1 + r_5, t_1, \lambda = 0$	$r_1, r_2, r_1 + r_5, t_2, t_3$	$t_2 > 0, r_2 < 0$
^{*7}30	$r_1 - r_3, r_4, 2r_1 + r_5, t_1, t_3, \lambda = 0$	r_1, r_2, t_2	$t_2 > 0, r_2 < 0$
^{*8}31	$r_1, 2r_3 - r_4, 2r_3 + r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
32	$r_1, r_3, r_4, r_5, t_3, \lambda = 0$	$r_2, t_1, t_2, t_1 + t_2$	$t_2 > 0, r_2 < 0$
33	$r_1, r_3, r_4, r_5, t_1 + t_2, t_3, \lambda = 0$	r_2, t_1, t_2	$r_2 < 0, t_1 < 0$
34	$r_1, 2r_3 - r_4, t_1, t_3, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, t_2$	$t_2 > 0, r_2 < 0$
^{*9}35	$r_1, \frac{r_3}{2} - r_4, 2r_3 + r_5, t_1, t_3, \lambda = 0$	r_2, r_3, t_2	$t_2 > 0, r_2 < 0$
^{*10}36	$2r_1 - 2r_3 + r_4, 2r_3 + r_5, t_1, t_3, \lambda = 0$	$r_1, r_2, r_1 - r_3, t_2$	$t_2 > 0, r_2 < 0$

Table 5.6 (continued)

#	Critical condition	Additional conditions	No-ghost-and-tachyon condition
37	$r_1, \frac{r_3}{2} - r_4, 2r_3 + r_5, t_1, \lambda = 0$	$r_2, 2r_3 - r_4, t_2, t_3$	$t_2 > 0, r_2 < 0$
38	$r_1, 2r_3 - r_4, 2r_3 + r_5, t_3, \lambda = 0$	$r_2, r_1 - r_3, t_1, t_2, t_1 + t_2$	$t_2 > 0, r_2 < 0$
39	$r_1, 2r_3 - r_4, 2r_3 + r_5, t_1 + t_2, t_3, \lambda = 0$	$r_2, r_1 - r_3, t_1, t_2$	$r_2 < 0, t_1 < 0$
40	$r_1, r_4 + r_5, t_1, t_3, \lambda = 0$	$r_2, r_3 - 2r_4, 2r_3 - r_4, t_2$	$t_2 > 0, r_2 < 0$
41	$r_1, \frac{r_3}{2} - r_4, \frac{r_3}{2} + r_5, t_1, \lambda = 0$	$r_2, 2r_3 - r_4, t_2, t_3$	$t_2 > 0, r_2 < 0$
42	$r_1, r_3, r_4, t_1 + t_2, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_3,$ $t_1 + t_3$	$t_3 > 0, r_2 < 0, r_5 < 0,$ $t_1 < 0, t_1 + t_3 < 0$
43	$r_1, r_3, r_4, t_1 + t_3, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_1, t_2,$ $t_1 + t_2, t_3$	$r_5 > 0, t_2 > 0, t_1 + t_2 > 0,$ $r_2 < 0, t_1 < 0$
44	$r_2, r_1 - r_3, r_4, t_1 + t_2, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_3,$ $t_1 + t_3$	$t_1 > 0, r_1 < 0, r_1 + r_5 < 0,$ $t_3 (t_1 + t_3) > 0$
45	$r_2, r_1 - r_3, r_4, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_2,$ $t_1 + t_2, t_3$	$r_5 > 0, 2r_1 + r_5 > 0, t_1 > 0,$ $t_1 + t_2 > 0, r_1 < 0, t_2 < 0$
46	$r_1 - r_3, r_4, 2r_1 + r_5, t_1 + t_3, \lambda = 0$	$r_1, r_2, r_1 + r_5, t_1, t_2, t_1 + t_2, t_3$	$t_1 > 0, t_2 > 0, r_1 < 0, r_2 < 0$
47	$r_1, r_2, r_3, r_4, t_1 + t_2, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_3, t_1 + t_3$	$r_5 < 0, t_1 t_3 (t_1 + t_3) > 0$
48	$r_1, r_2, r_3, r_4, t_1 + t_3, \lambda = 0$	$r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_1 + t_2, t_3$	$r_5 > 0, t_1 t_2 (t_1 + t_2) < 0$
49	$r_1, r_3, r_4, t_1 + t_2, t_1 + t_3, \lambda = 0$	$r_2, r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_3$	$r_2 < 0, t_1 < 0$
50	$r_2, r_1 - r_3, r_4, r_1 + r_5, t_1 + t_2, \lambda = 0$	$r_1, 2r_1 + r_5, t_1, t_2, t_3, t_1 + t_3$	$t_1 > 0, r_1 < 0$
51	$r_2, r_1 - r_3, r_4, 2r_1 + r_5, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, t_1, t_2, t_1 + t_2, t_3$	$t_1 > 0, r_1 < 0$
52	$r_2, r_1 - r_3, r_4, t_1 + t_2, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
53	$r_2, r_1 - r_3, r_4, r_1 + r_5, t_1 + t_2, t_1 + t_3, \lambda = 0$	$r_1, 2r_1 + r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
54	$r_2, r_1 - r_3, r_4, 2r_1 + r_5, t_1 + t_2, t_1 + t_3, \lambda = 0$	$r_1, r_1 + r_5, t_1, t_2, t_3$	$t_1 > 0, r_1 < 0$
55	$r_2, r_1 - r_3, r_4, r_1 + r_5, t_1 + t_2, t_3, \lambda = 0$	r_1, t_1, t_2	$t_1 > 0, r_1 < 0$
56	$r_2, r_1 - r_3, r_4, 2r_1 + r_5, t_2, t_1 + t_3, \lambda = 0$	r_1, t_1, t_3	$t_1 > 0, r_1 < 0$
57	$r_1 - r_3, r_4, 2r_1 + r_5, t_2, t_1 + t_3, \lambda = 0$	r_1, r_2, t_1, t_3	$t_1 > 0, r_1 < 0$
58	$r_2, 2r_1 - 2r_3 + r_4, r_1 - 2r_3 - r_5,$ $t_1 + t_2, t_3, \lambda = 0$	$r_1, r_1 - r_3, t_1, t_2$	$t_1 > 0, r_1 < 0$

Table 5.7 Particle content of the PC renormalisable critical cases that are ghost and tachyon free and have only massive propagating modes. The cases found previously in [70] are indicated with an asterisk followed by its original numbering. Note that there are typos of the *b*-sectors of Cases 30 and 31 (old numbers 7 and 8) in [70].

#	Massive mode	*b*-sectors
20	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
21	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
22	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
23	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
24	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, \times\right\}$
^{*5}25	0^-	$\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
^{*6}26	0^-	$\{A^2_{\mathrm{v}}, \times, \times, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
27	0^-	$\left\{A^2_{\mathrm{v}}, \times, \times, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
28	0^-	$\left\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, \times\right\}$
29	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\right\}$
^{*7}30	0^-	$\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\}$
^{*8}31	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, \times, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
32	0^-	$\left\{A^2_{\mathrm{v}}, \times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
33	0^-	$\left\{A^2_{\mathrm{v}}, \times, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
34	0^-	$\left\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, \times\right\}$
^{*9}35	0^-	$\{A^2_{\mathrm{v}}, \times, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\}$
^{*10}36	0^-	$\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\}$
37	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0\vert\mathfrak{a}^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\right\}$
38	0^-	$\left\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, \left(A^0\&A^0\right)^{\mathrm{N}}\vert\left(A^0\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
39	0^-	$\left\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, \left(A^\infty\&A^0\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
40	0^-	$\left\{A^2_{\mathrm{v}}, A^2_{\mathrm{l}}, \times, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
41	0^-	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\vert\mathfrak{a}^2_{\mathrm{l}}, \left(A^2_{\mathrm{l}}\&A^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}}\right)^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
42	$0^-, 1^-$	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{v}}\&A^0_{\mathrm{v}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{v}}\&\mathfrak{s}^2_{\mathrm{vl}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{v}}\&\mathfrak{a}^2_{\mathrm{vl}}\right)^{\mathrm{N}}, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
43	$0^-, 1^+$	$\left\{A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{s}^0_{\mathrm{l}}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, \left(A^2_{\mathrm{v}}\&A^0_{\mathrm{v}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{v}}\&\mathfrak{a}^2_{\mathrm{vl}}\right)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$
44	$1^-, 2^-$	$\left\{A^0, A^0\vert\mathfrak{s}^2_{\mathrm{l}}, \left(A^2_{\mathrm{v}}\&A^0_{\mathrm{v}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{v}}\&\mathfrak{s}^2_{\mathrm{vl}}\right)^{\mathrm{N}}\vert\left(A^2_{\mathrm{v}}\&\mathfrak{a}^2_{\mathrm{vl}}\right)^{\mathrm{N}}, \left(A^\infty\&A^{-2}\right)^{\mathrm{N}}\vert\left(A^\infty\&\mathfrak{a}^0_{\mathrm{l}}\right)^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0\vert\mathfrak{s}^2_{\mathrm{l}}\right\}$

Table 5.7 (continued)

#	Massive mode	b-sectors
45	$1^{+},2^{-}$	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A_{\mathrm{v}}^{2}\&A_{\mathrm{v}}^{0})^{\mathrm{N}}\mid(A_{\mathrm{v}}^{2}\&\mathfrak{a}_{\mathrm{vl}}^{2})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
46	$0^{-},2^{-}$	$\left\{A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A^{0}\&A^{0})^{\mathrm{N}}\mid(A^{0}\&\mathfrak{a}_{\mathrm{l}}^{2})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
47	1^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A_{\mathrm{v}}^{2}\&A_{\mathrm{v}}^{0})^{\mathrm{N}}\mid(A_{\mathrm{v}}^{2}\&\mathfrak{s}_{\mathrm{vl}}^{2})^{\mathrm{N}}\mid(A_{\mathrm{v}}^{2}\&\mathfrak{a}_{\mathrm{vl}}^{2})^{\mathrm{N}},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
48	1^{+}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A_{\mathrm{v}}^{2}\&A_{\mathrm{v}}^{0})^{\mathrm{N}}\mid(A_{\mathrm{v}}^{2}\&\mathfrak{a}_{\mathrm{vl}}^{2})^{\mathrm{N}},A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
49	0^{-}	$\left\{A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
50	2^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{0}\&A^{0})^{\mathrm{N}}\mid(A^{0}\&\mathfrak{s}_{\mathrm{l}}^{2})^{\mathrm{N}}\mid(A^{0}\&\mathfrak{a}_{\mathrm{l}}^{2})^{\mathrm{N}},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
51	2^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A^{0}\&A^{0})^{\mathrm{N}}\mid(A^{0}\&\mathfrak{a}_{\mathrm{l}}^{2})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
52	2^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
53	2^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{2})^{\mathrm{N}},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
54	2^{-}	$\left\{A^{0},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},(A^{\infty}\&A^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{2})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
55	2^{-}	$\left\{A^{0},\times,A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\mid\mathfrak{a}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
56	2^{-}	$\left\{\times,A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A^{0}\mid\mathfrak{a}_{\mathrm{l}}^{2},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
57	2^{-}	$\left\{A_{\mathrm{l}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{s}_{\mathrm{l}}^{0})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A^{0}\mid\mathfrak{a}_{\mathrm{l}}^{2},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$
58	2^{-}	$\left\{A^{0},A_{\mathrm{l}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\mid\mathfrak{a}_{\mathrm{l}}^{2},(A^{\infty}\&A^{-2})^{\mathrm{N}}\mid(A^{\infty}\&\mathfrak{a}_{\mathrm{l}}^{0})^{\mathrm{N}},A_{\mathrm{v}}^{2},A^{0}\mid\mathfrak{s}_{\mathrm{l}}^{2}\right\}$

Appendix 5.A Proca and Stueckelberg theories

In this appendix, we illustrate the methods used in Section 5.2 in the context of the more familiar and much simpler Proca and Stueckelberg theories.

Proca theory contains a massive vector field B_μ and has the free-field Lagrangian

$$\mathcal{L}_{\text{Pr}} = -\tfrac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)\left(\partial^\mu B^\nu - \partial^\nu B^\mu\right) + \tfrac{1}{2} m^2 B_\mu B^\mu, \tag{5.28}$$

with $m > 0$, which has no gauge freedoms. The corresponding SPOs are

$$\mathsf{P}(0^+) = \begin{array}{cc} & B_\mu \\ B^*_\rho & \left(\Omega_{\mu\rho}\right) \end{array}, \qquad \mathsf{P}(1^-) = \begin{array}{cc} & B_\mu \\ B^*_\rho & \left(\Theta_{\mu\rho}\right) \end{array}, \tag{5.29}$$

where $\Omega^{\mu\rho} = k^\mu k^\rho / k^2$, and $\Theta^{\mu\rho} = \eta^{\mu\rho} - k^\mu k^\rho / k^2$. The a-matrices of the theory are

$$a(0^+) = \begin{array}{cc} & B_\mu \\ B^*_\mu & \left(m^2\right) \end{array}, \quad a(1^-) = \begin{array}{cc} & B_\mu \\ B^*_\mu & \left(-k^2 + m^2\right) \end{array}, \tag{5.30}$$

which are identical to the b-matrices because there are no gauge invariances and source constraints. Therefore, the 0^+ sector is non-propagating and the 1^- sector corresponds to a k^{-2} propagator. Thus, Proca theory satisfies the alternative PCR condition in [70], and hence we classify it as PCR.

Conversely, Proca theory clearly violates the original PCR condition[7]. Indeed, Proca theory is generally considered to be non-PCR in the literature, because the propagator is

$$D(k)_{\mu\nu} = \frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}}{k^2 - m^2}, \tag{5.31}$$

[7]The original PCR condition can be obtained by considering a Dirac field minimally coupled to the vector field $i\gamma^\mu \bar{\psi}\left(\partial_\mu - ieB_\mu\right)\psi - m\bar{\psi}\psi$. The interaction term is $e\gamma^\mu \bar{\psi}\psi B_\mu$, with $[e]_{\text{M}} = 0$, $[\psi]_{\text{M}} = [\bar{\psi}]_{\text{M}} = 3/2$, and $[B_\mu]_{\text{M}} = 1$. We have to redefine $\tilde{B}_\mu = m^{1-l_B/2} B_\mu$ so that the mass dimension and canonical dimension of $\tilde{B}_\mu$ coincide, where the propagator of B_μ behaves as $\sim k^{-l_B}$ as $k^2 \to \infty$. The coupling constant then becomes $\sim e m^{l_B/2-1}$, and so we require $l_B \geq 2$ to prevent the coupling constant from having negative dimension.

so some components of it become $\sim k^0$ when $k^2 \to \infty$ and the offending term $k_\mu k_\nu$ cannot be eliminated by the renormalisation procedure [90]. Using the polarisation basis method mentioned in the main text, however, we can integrate out the non-propagating 0^+ part. The free Lagrangian then becomes $\mathcal{L}_{\text{Pr}}$ with the condition $\partial^\mu B_\mu = 0$, and the resulting propagator goes as k^{-2}, so the theory is PCR.

One may gain some insight into this apparent contradiction by noting that Proca theory may be considered as a gauge-fixed version of a gauge theory, namely the Stueckelberg theory, for which the Lagrangian is [91–93]

$$\mathcal{L}_{\text{St}} = -\tfrac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)\left(\partial^\mu B^\nu - \partial^\nu B^\mu\right) + \tfrac{1}{2}m^2 B_\mu B^\mu + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + m\phi\partial_\mu B^\mu \tag{5.32}$$

and which possesses the gauge invariance,

$$B'_\mu = B_\mu + \partial_\mu\Lambda, \qquad \phi' = \phi + m\Lambda. \tag{5.33}$$

The nonzero a-matrices are

$$a(0^+) = \begin{array}{c c} & \begin{array}{cc} \phi & B_\mu \end{array} \\ \begin{array}{c} \phi^* \\ B^*_\mu \end{array} & \begin{pmatrix} k^2 & -ikm \\ ikm & m^2 \end{pmatrix} \end{array}, \tag{5.34}$$

$$a(1^-) = \begin{array}{c c} & B_\mu \\ B^*_\mu & \begin{pmatrix} -k^2 + m^2 \end{pmatrix} \end{array}, \tag{5.35}$$

and the corresponding SPOs are

$$\mathsf{P}(0^+) = \begin{array}{c c} & \begin{array}{cc} \phi & B_\mu \end{array} \\ \begin{array}{c} \phi^* \\ B^*_\rho \end{array} & \begin{pmatrix} 1 & \tilde{k}_\mu \\ \tilde{k}_\rho & \Omega_{\mu\rho} \end{pmatrix} \end{array}, \qquad \mathsf{P}(1^-) = \begin{array}{c c} & B_\mu \\ B^*_\rho & \begin{pmatrix} \Theta_{\mu\rho} \end{pmatrix} \end{array}, \tag{5.36}$$

where $\tilde{k}_\mu = k_\mu/\sqrt{k^2}$. As might be expected, the matrix $a(0^+)$ is singular, with rank one, and so we can choose to keep either the ϕ-column/row or the B-column/row. If we choose to keep B, then one recovers Proca's theory. If we instead choose to keep ϕ, then the b^{-1}-matrices all go as $\sim k^{-2}$ in the high-energy limit and the theory thus satisfies the original PCR condition. Hence, Stueckelberg theory is PCR, and so Proca theory must also be PCR, since the two theories are physically equivalent. Thus, our alternative PCR criterion succeeds in identifying Proca theory as being PCR, whereas the theory violates the original PCR criterion.

Appendix 5.B Implementation of PCR criterion

In this appendix, we will continue the discussion of the implementation of the systematic method in Chapter 3. After the evaluation of the whole "tree", we can perform some further analysis of these nodes, for example, to check whether a node is PCR. Because we are only checking PCR for those theories with no mixing, we have to check whether there is any choice of b-matrices without mixing terms. We first write a function to check whether a b-matrix in a sector is non-mixing. It checks whether there are any nonzero element whose row and column correspond to different fields. We are viewing the $\mathfrak{s}$- and $\mathfrak{a}$-field as the same field here because both of them come from the f-field.

```
isNoMixedSector[bM_,fIdx_]:=Module[{idxMesh,mixMat},
  idxMesh=If[fIdx==={},{{{}},{{}}},meshgrid[fIdx,fIdx]];
  mixMat=MapThread[Function[{x,y},
    !(x===y||(x===2&&y===3)||(x===3&&y===2))
  ],idxMesh,2];
  And@@Flatten@(Function[{m,b},MapThread[
    If[#1===True,FullSimplify[#2==0]===True,True]&
  ,{m,b},2]][mixMat,bM])
];
```

We use `isNoMixedSector` to check whether those b-matrices are non-mixing. The criterion is about the asymptotic behaviour of the diagonal elements of the b^{-1}-matrices as k^2 goes to infinity, so we also need to check it. The tool function `mainTerm[exp_,kk]` returns the power of k^2 of an expression when $k^2 \to \infty$, and it returns $-\infty$ when the expression is zero. We thus use `mainTerm` to obtain the powers of all diagonal elements in all possible b-matrices in each spin-parity sector. We then use `Tuple` to obtain the above properties of all possible combinations of b-matrices from each sector. We also create the list `GIIdx` which denotes the fields in the rows/columns in the b-matrices of all combinations.

```
isAllPC[no_]:=Module[{(*... *)},
  nV=no.getValue[];
  kPower=Map[Function[bG,
    Map[mainTerm[#,kk]["expon"]&,bG[[1]],{2}]
  ],nV["mats"]["bAll"],{2}];
  isNoMix=MapThread[Map[Function[bG,
    isNoMixedSector[bG[[1]],#2[[bG[[2]]]]]
  ],#1]&,{nV["mats"]["bAll"],fieldListOld}];
  isNoMixTpl=Map[And@@#&,Tuples[isNoMix]];
  allKPD=Tuples@Map[Diagonal[#]&,kPower,{2}];
  GIIdx=Tuples@MapThread[Function[{ba,fIdx},
    Map[fIdx[[#[[2]]]]&,ba]
  ],{nV["mats"]["bAll"],fieldListOld}];
  (*Filter out those PCR combinations*)
  <|"PC"-> Length@PCIdx>0,"PCIdx"-> PCIdx|>
];
```

With the variables mentioned above, we can now test whether each b-matrix in all combinations satisfies the PCR criterion. For each b-matrix without mixing fields, we first put the asymptotic behaviour of the diagonal elements with the same field into the same element in a list. We then check whether the powers of k^2 satisfy the PCR criterion correspondingly. The power n should satisfy $n \geq 0$, $n \leq -n_F$, or $n = -\infty$, where $n_F = 2$ for the $\mathfrak{s}$ and $\mathfrak{a}$-fields, and $n_F = 1$ for the remaining fields. We then collect the combinations with all its b-matrices passing the PCR criterion. If there are more than one combinations collected, then the node is PCR.

```
PCIdx={};
MapThread[Function[{kpd,idx,noMix},Module[{},If[noMix,
  pPowers={{},{},{},{},{}};
  MapThread[AppendTo[pPowers[[#2]],#1]&
    ,{Flatten@kpd,Flatten@idx}];
  APC=And@@Map[#<=-1||#>=0||#===-Infinity&,pPowers[[1]]];
  sPC=And@@Map[#<=-2||#>=0||#===-Infinity&,pPowers[[2]]];
  aPC=And@@Map[#<=-2||#>=0||#===-Infinity&,pPowers[[3]]];
  ϕPC=And@@Map[#<=-1||#>=0||#===-Infinity&,pPowers[[3]]];
  BPC=And@@Map[#<= -1||#>=0||#===-Infinity&,pPowers[[5]]];
  If[APC&&sPC&&aPC&&[\Phi]PC&&BPC,AppendTo[PCIdx,idx]];
]]],{allKPD,GIIdx,isNoMixTpl}];
```

Chapter 6

Application to Weyl gauge theory

In this chapter, we apply our systematic method for identifying ghost-and-tachyon-free critical cases to parity-preserving Weyl gauge theory (WGT$^+$), the ground-state particle spectrum of which has rarely been discussed in the literature before.

This chapter is arranged as follows. In Section 6.1, we give a brief introduction to WGT$^+$, and in Section 6.2 we consider the unitarity of the "root" theory, where none of the critical conditions are satisfied. In Section 6.3 we apply our systematic approach to investigating its critical cases and accommodating the associated additional source constraints, as well as identifying some unitary critical cases that are also power-counting renormalisable. We repeat our analysis for WGT$^+$ with vanishing torsion in Section 6.4 and for WGT$^+$ with vanishing curvature in Section 6.5. We conclude in Section 6.6.

6.1 Weyl gauge theories

The action of an infinitesimal element of the Weyl group $W(1,3)$ on Cartesian coordinates in Minkowski spacetime has the form

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu + \omega^\mu{}_\nu x^\nu + \rho x^\mu, \tag{6.1}$$

where ε^μ denotes a translation, $\omega^\mu{}_\nu$ denotes a Lorentz rotation, and ρ denotes a dilation. The corresponding form variation $\delta_0\varphi(x) \equiv \varphi'(x) - \varphi(x)$ of a field φ (belonging to an irreducible representation of the Lorentz group) is $\delta_0\varphi = \delta_0^{\mathrm{P}}\varphi + w\rho\varphi$, where δ_0^{P} means the variation under a Poincaré transformation and w is a dimensionless constant known as the (Weyl) weight of the field.

Similar to the procedure in Section 4.1, one gauges the Weyl group $W(1,3)$ by demanding that the action be invariant with respect to (infinitesimal, passively interpreted) general coordinate transformations (GCTs) and the local action of the subgroup $H(1,3)$ (the homogeneous Weyl group), obtained by setting the translation parameters ε^μ of $W(1,3)$ to zero (which leaves the origin $x^\mu = 0$ invariant), and allowing the remaining group parameters to become independent arbitrary functions of position. In this way, one is led to the introduction of the gravitational gauge fields $h_A{}^\mu$, $A^{AB}{}_\mu$ and B_μ, corresponding to the translational, rotational and dilational parts of the Weyl group, respectively, which transform under the gauged Weyl group as $\delta_0 h_A{}^\mu = \delta_0^{\mathrm{P}} h_A{}^\mu - \rho h_A{}^\mu$, $\delta_0 A^{AB}{}_\mu = \delta_0^{\mathrm{P}} A^{AB}{}_\mu$ and $\delta_0 B_\mu = -\partial_\mu\rho$.

The gauge fields are used to assemble the WGT covariant derivative [18, 28]

$$\mathcal{D}^*_A\varphi = h_A{}^\mu\mathcal{D}^*_\mu\varphi = h_A{}^\mu\left(\partial_\mu + \tfrac{1}{2}A^{AB}{}_\mu\Sigma_{AB} + wB_\mu\right)\varphi, \tag{6.2}$$

where w is the weight of φ and $\Sigma_{AB} = -\Sigma_{BA}$ are the generator matrices of the $\mathrm{SL}(2,C)$ representation to which φ belongs. The asterisk on the derivative operators is a common notation used in WGT to distinguish these operators from their PGT counterparts (to which they reduce if w or B_μ vanishes). The corresponding commutators become

$$\left[\mathcal{D}^*_\mu, \mathcal{D}^*_\nu\right]\varphi = \tfrac{1}{2}\mathcal{R}^{AB}{}_{\mu\nu}\Sigma_{AB}\varphi + \mathcal{H}_{\mu\nu}w\varphi, \tag{6.3}$$

$$[\mathcal{D}^*_A, \mathcal{D}^*_B]\varphi = \tfrac{1}{2}\mathcal{R}^{CD}{}_{AB}\Sigma_{CD}\varphi - \mathcal{T}^{*C}{}_{AB}\,\mathcal{D}^*_C\varphi + \mathcal{H}_{AB}w\varphi, \tag{6.4}$$

where the field strengths have the forms

$$\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu}A^{AB}{}_{\nu]} + A^A{}_{E[\mu}A^{EB}{}_{\nu]}), \tag{6.5}$$

$$\mathcal{H}_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}, \tag{6.6}$$

$$\mathcal{T}^{*C}_{\ \ AB} = \mathcal{T}^{C}_{\ AB} + 2B_{[A}\delta^{C}_{B]}, \tag{6.7}$$

and $\mathcal{T}^{C}_{\ \mu\nu} = 2\mathcal{D}_{[\mu}b^{C}{}_{\nu]}$ is the usual expression for the translational gauge field strength in PGT. In the above expressions, Latin and Greek indices are related by $h_A{}^{\nu}$ and its inverse $b^A{}_{\nu}$, with the relation

$$g_{\mu\nu}h_A{}^{\mu}h_B{}^{\mu} = \eta_{AB}, \quad \eta_{AB}b^A{}_{\mu}b^B{}_{\mu} = g_{\mu\nu}. \tag{6.8}$$

By convention, we set $w(g_{\mu\nu}) = 2$, and one may show that the weights of the translational and rotational gauge fields are $w(h_A{}^{\mu}) = -1$ and $w(A^{AB}{}_{\mu}) = 0$, so that $w(b^A{}_{\mu}) = 1$ and the weight of its determinant is $w(b) = 4$, but the dilatational gauge field B_μ itself transforms inhomogeneously under dilations, as expected. The weights of the corresponding field strengths are $w(\mathcal{R}^{CD}{}_{AB}) = w(\mathcal{H}_{AB}) = -2$ and $w(\mathcal{T}^{*C}{}_{AB}) = -1$.

In the action $S = \int b\mathcal{L}\,\mathrm{d}^4x$, the Lagrangian $\mathcal{L}$ is the sum of terms corresponding to the free gravitational fields and terms containing the matter fields, respectively, and has the general form

$$\mathcal{L} = \mathcal{L}_{\mathrm{G}}(\mathcal{R}^{CD}{}_{AB}, \mathcal{T}^{*C}{}_{AB}, \mathcal{H}_{AB}) + \mathcal{L}_{\mathrm{M}}(\varphi, \mathcal{D}^*_A\varphi). \tag{6.9}$$

For S to be scale invariant (i.e. of weight 0), the weights of both $\mathcal{L}_{\mathrm{G}}$ and $\mathcal{L}_{\mathrm{M}}$ must be -4. Restricting our attention to terms in $\mathcal{L}_{\mathrm{G}}$ that are at most quadratic in the field strengths, these may thus be quadratic in $\mathcal{R}^{CD}{}_{AB}$ and $\mathcal{H}_{AB}$, or consist of the product of the two, but may not include terms linear in $\mathcal{R}^{CD}{}_{AB}$ or quadratic in $\mathcal{T}^{*C}{}_{AB}$.

One can, however, include further terms in the Lagrangian by introducing an additional massless scalar field (or fields) ϕ with Weyl weight $w(\phi) = -1$, often termed the compensator(s) [18], which is usually non-minimally (conformally) coupled to the field strength tensors of the gravitational gauge fields. For example, terms proportional to $\phi^2\mathcal{R}$ or $\phi^2\mathcal{L}_{\mathcal{T}^{*2}}$, where $\mathcal{L}_{\mathcal{T}^{*2}}$ consists of terms quadratic in $\mathcal{T}^{*C}{}_{AB}$, have weight $w = -4$ and so may be added to the total Lagrangian [66, 94–96]. One should also include a free kinetic term $(\mathcal{D}^*\phi)^2$ for the scalar field, and may also add a self-interaction term ϕ^4, but we shall not consider the latter here. Thus, also requiring parity-invariance, the Lagrangian for free WGT$^+$ has the form[1]

$$
\begin{aligned}
\mathcal{L}_{\mathrm{G}} = & -\lambda\phi^2\mathcal{R} + \tfrac{1}{6}(2r_1+r_2)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} + \tfrac{2}{3}(r_1-r_2)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD} \\
& + \tfrac{1}{6}(2r_1+r_2-6r_3)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB} + (r_4+r_5)\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_4-r_5)\mathcal{R}^{AB}\mathcal{R}_{BA} \\
& - c_1\mathcal{R}^{AB}\mathcal{H}_{AB} + \xi\mathcal{H}^{AB}\mathcal{H}_{AB} + \tfrac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi + \tfrac{1}{12}(4t_1+t_2+3\lambda)\phi^2\mathcal{T}^{*ABC}\mathcal{T}^*_{ABC} \\
& - \tfrac{1}{6}(2t_1-t_2+3\lambda)\phi^2\mathcal{T}^{*ABC}\mathcal{T}^*_{BCA} - \tfrac{1}{3}(t_1-2t_3+3\lambda)\phi^2\mathcal{T}^{*\ AB}_{B}\mathcal{T}^{*\ \ C}_{CA},
\end{aligned} \tag{6.10}
$$

where $\mathcal{R}^A{}_B = \mathcal{R}^{AC}{}_{BC}$, $\mathcal{R} = \mathcal{R}^A{}_A$ and $\mathcal{D}^*_A\phi = \partial_A\phi - B_A\phi$. The parameters in the Lagrangian are dimensionless and set in combinations that enable a straightforward comparison with our previous studies of PGT$^+$ in Chapters 4 and 5. Note that the Gauss–Bonnet identity has been used to remove the term proportional to $\mathcal{R}^2$.

Provided $\phi(x)$ does not vanish anywhere, one can use local scale invariance to set the field to a constant value ϕ_0, which is known as the Einstein gauge and is usually interpreted as breaking the scale symmetry. This interpretation is questioned in [28], however, since it is shown that if one rewrites the Lagrangian in terms of a set of scale-invariant variables [64], then the resulting equations of motion are the same as those of Einstein gauge, yet this approach involves no breaking of the scale symmetry. Relevant issues are recently discussed in [97, 98]. In any case, we will adopt the Einstein gauge $\phi = \phi_0$ here, the most significant effect of which is that the term $\frac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi$ in the Lagrangian becomes $\frac{1}{2}\nu\phi_0^2 B_A B^A$. We then absorb the ϕ_0^2 factor into the now dimensionful parameters λ, ν, t_1, t_2, and t_3, without loss of generality. Note that a potential term $\sim\phi^4$ for the compensator scalar field was not included in the Lagrangian, since it becomes a constant in the Einstein gauge, acting like an effective cosmological constant, which would be inconsistent with considering a Minkowski background.

Similar to PGT, WGT is most naturally interpreted as a field theory in Minkowski spacetime [23, 25, 28], in the same way as the gauge field theories describing the other fundamental interactions. It is more common, however, to reinterpret it geometrically in

[1] The minus signs before λ and c_1 are resulting from the sign difference of field strength $\mathcal{R}$ in the code and main text.

terms of a Weyl–Cartan spacetime (W_4), which generalises the Riemann–Cartan spacetime (U_4) underlying the geometric interpretation of PGT by incorporating local scale invariance [18].

Weyl–Cartan spacetime is a manifold with linear connection (Γ) and metric ($g_{\mu\nu}$) which satisfy

$$\mathcal{D}^*_\rho(\Gamma)g_{\mu\nu} = 0, \tag{6.11}$$

where the covariant derivative of a field φ with weight w is defined by

$$\mathcal{D}^*_\mu(\Gamma)\varphi \equiv \left(\mathcal{D}_\mu(\Gamma) + wB_\mu\right)\varphi, \tag{6.12}$$

in which $\mathcal{D}_\mu(\Gamma) = \partial_\mu + \Gamma^\sigma{}_{\rho\mu}X^\rho{}_\sigma$ is the U_4 covariant derivative. The semi-metricity condition (6.11) replaces the metricity condition in U_4. Since $w(g_{\mu\nu}) = 2$, the semi-metricity condition can also be written as $\mathcal{D}_\rho(\Gamma)g_{\mu\nu} = -2B_\rho g_{\mu\nu}$, from which one finds that the infinitesimal change of length of a parallel transported vector is proportional to the length itself, $\mathcal{D}_\rho(\Gamma)V^2 = -2B_\rho V^2$. One may solve for the connection Γ, which is given by

$$\Gamma^\mu{}_{\nu\rho} = \left\{{}^{\ \mu}_{\nu\rho}\right\} + \delta^\mu_\nu B_\rho + \delta^\mu_\rho B_\nu - g_{\nu\rho}B^\mu + K^\mu{}_{\nu\rho}\,. \tag{6.13}$$

In a local Lorentz frame, the Minkowski metric η_{AB} is invariant under Weyl transformation and satisfy $h_A{}^\mu h_B{}^\nu g_{\mu\nu} = \eta_{AB}$, so $w(\eta_{AB}) = 0$ and $w(h_A{}^\mu) = -1$. The local frame has a connection $A^{AB}{}_\mu$, and the covariant derivative $\mathcal{D}^*_A(A)$ has properties similar to (6.12), where

$$\mathcal{D}^*_\rho(A)\eta_{AB} = 0, \tag{6.14}$$

$$\mathcal{D}^*_\rho(A)\varphi \equiv \left(\mathcal{D}_\rho(A) + wB_\rho\right)\varphi, \tag{6.15}$$

and $\mathcal{D}_\rho(A)$ is the covariant derivative in U_4. One may also define the "total covariant derivative" $\mathcal{D}^*_\rho(\Gamma + A)$ to act on quantities with both coordinate and local Lorentz indices

$$\mathcal{D}^*_\rho(\Gamma + A)\varphi = \left(\mathcal{D}_\rho(\Gamma) + \mathcal{D}_\rho(A) - \partial_\rho - wB_\rho\right)\varphi. \tag{6.16}$$

Since the total covariant derivative $\mathcal{D}^*_\rho(\Gamma+A)V^A$ of the local Lorentz components of a vector is a coordinate tensor in Weyl–Cartan spacetime, the relation $\mathcal{D}^*_\rho(\Gamma+A)V^A = b^A{}_\mu \mathcal{D}^*_\rho(\Gamma+A)V^\mu$ should hold, from which one obtains the "tetrad postulate" , which is similar to (4.32) but with $\partial \to \partial^*$,

$$D^*_\mu(\Gamma+A)b^A{}_\nu \equiv \partial^*_\mu b^A{}_\nu + A^A{}_{B\mu} b^B{}_\nu - \Gamma^\sigma{}_{\nu\mu} b^A{}_\sigma = 0, \tag{6.17}$$

where $\partial^*_\mu \equiv \partial_\mu + wB_\mu$. One can therefore express the affine connection in the quantities corresponding to gauge fields as

$$\Gamma^\lambda{}_{\nu\mu} = h_A{}^\lambda (\partial^*_\mu b^A{}_\nu + A^A{}_{B\mu} b^B{}_\nu), \tag{6.18}$$

and hence show that the translational gauge field strength is equivalent to (minus) the geometric torsion tensor

$$\mathcal{T}^{*\rho}{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}, \tag{6.19}$$

in terms of which the contorsion is given by (4.35).

From (6.18), (6.19), and (4.35), one also obtains

$$A_{AB\mu} = \Delta^*_{AB\mu} + K_{AB\mu}, \tag{6.20}$$

where we define the quantities

$$\Delta^*_{AB\mu} \equiv \Delta_{AB\mu}|_{\partial\to\partial^*} = \Delta_{AB\mu} - B_A b_{B\mu} + B_B b_{A\mu}, \tag{6.21}$$

where $\Delta_{AB\mu}$ is given in Equations (4.37) and (4.38).

One then finds that, in contrast to the torsion, the geometric (Riemann) curvature tensor differs from the rotational gauge field strength $\mathcal{R}^\rho{}_{\sigma\mu\nu}$, so we denote the former by

$$\begin{aligned}\tilde{\mathcal{R}}^\rho{}_{\sigma\mu\nu} &= \mathcal{R}^\rho{}_{\sigma\mu\nu} + H_{\mu\nu}\delta^\rho_\sigma, \\ &= \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\rho{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\rho{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu}.\end{aligned} \tag{6.22}$$

Unlike $\mathcal{R}_{\rho\sigma\mu\nu}$, the curvature tensor $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$ is not antisymmetric in (ρ,σ), while both are antisymmetric in (μ,ν) [18, 28]. Indeed, one may take advantage of these symmetry properties by using $\mathcal{R}_{\rho\sigma\mu\nu}$ to perform calculations instead of $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$. One should note, however, that unlike the curvature tensor in Riemann spacetime V_4 familiar from general relativity, neither $\mathcal{R}_{\rho\sigma\mu\nu}$ nor $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$ is symmetric in $(\rho\sigma,\mu\nu)$.[2]

6.2 The “root” theory

We now apply the method described in Chapter 2 to the “root” theory (6.10), where none of the critical conditions is satisfied. We first linearise the Lagrangian around the Minkowski background using $A_{ABC} \sim O(t)$, $B_A \sim O(t)$, $h_A{}^\mu = \delta_A{}^\mu + f_A{}^\mu$, and $f_{AB} = \mathfrak{s}_{AB} - \mathfrak{a}_{AB} \sim O(t)$, where $\mathfrak{s}$ and $\mathfrak{a}$ denote the symmetric and antisymmetric parts of f, respectively.[3] Note that we cannot perturb ϕ as $\phi_0 + \varepsilon$, for some excitation ε, because we have already fixed the gauge on ϕ. The Lagrangian then becomes

$$b\mathcal{L}_{\mathrm{G}} = \left(2\lambda\partial_A A^{BA}{}_B\right) + \mathcal{O}\left(t^2\right), \tag{6.23}$$

where the linear term is just a total derivative. We then decompose the quadratic part into

$$b\mathcal{L}_{\mathrm{G}} = \sum_{J,P,i,j} a(J^P)_{ij}\hat{\zeta}^\dagger \cdot \hat{P}(J^P)_{ij} \cdot \hat{\zeta}, \tag{6.24}$$

using the spin projection operators (SPOs) $\hat{P}(J^P)_{ij}$ [52, 53, 73]. Section II of [70] contains a description of our notation. The SPOs for WGT$^+$ are given in Appendix 2.A. One then obtains the a-matrices:

[2] Unlike the case in PGT, even if we set the torsion to zero, neither $\mathcal{R}_{\rho\sigma\mu\nu}$ nor $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$ is symmetric in $(\rho\sigma,\mu\nu)$.

[3] Note that Eq. (52) in [70] contains a typographical error, and should read $f_{AB} = \mathfrak{s}_{AB} - \mathfrak{a}_{AB}$, as here. This correction does not affect the remaining contents in [70, 71].

$$a(0^-) = \begin{array}{cc} & A \\ A & \left(2\left(k^2 r_2 + t_2\right) \right) \end{array}, \tag{6.25}$$

$$a(0^+) = \begin{array}{cc} & \begin{array}{cccc} A & \mathfrak{s} & \mathfrak{s} & B \end{array} \\ \begin{array}{c} A \\ \mathfrak{s} \\ \mathfrak{s} \\ B \end{array} & \begin{pmatrix} 2\left(2k^2\left(r_1 - r_3 + 2r_4\right) + t_3\right) & 2i\sqrt{2}kt_3 & 0 & -2\sqrt{6}\left(t_3 - \lambda\right) \\ -2i\sqrt{2}kt_3 & 4k^2\left(t_3 - \lambda\right) & 0 & 4i\sqrt{3}k\left(t_3 - \lambda\right) \\ 0 & 0 & 0 & 0 \\ -2\sqrt{6}\left(t_3 - \lambda\right) & -4i\sqrt{3}k\left(t_3 - \lambda\right) & 0 & 4\left(3t_3 - 3\lambda + \frac{\nu}{4}\right) \end{pmatrix} \end{array}, \tag{6.26}$$

$$a(1^-) =$$

$$\begin{array}{cc} & \begin{array}{ccccc} A & A & \mathfrak{s} & \mathfrak{a} & B \end{array} \\ \begin{array}{c} A \\ A \\ \mathfrak{s} \\ \mathfrak{a} \\ B \end{array} & \begin{pmatrix} 2\left[k^2\left(r_1 + r_4 + r_5\right) + \frac{1}{6}\left(t_1 + 4t_3\right)\right] & -\frac{\sqrt{2}}{3}\left(t_1 - 2t_3\right) & -\frac{\sqrt{2}}{3}ik\left(t_1 - 2t_3\right) & \frac{\sqrt{2}}{3}ik\left(t_1 - 2t_3\right) & -c_1k^2 + 4t_3 - 4\lambda \\ -\frac{\sqrt{2}}{3}\left(t_1 - 2t_3\right) & \frac{2}{3}\left(t_1 + t_3\right) & \frac{2}{3}ik\left(t_1 + t_3\right) & -\frac{2}{3}ik\left(t_1 + t_3\right) & 2\sqrt{2}\left(t_3 - \lambda\right) \\ \frac{\sqrt{2}}{3}ik\left(t_1 - 2t_3\right) & -\frac{2}{3}ik\left(t_1 + t_3\right) & \frac{2}{3}k^2\left(t_1 + t_3\right) & -\frac{2}{3}k^2\left(t_1 + t_3\right) & -2i\sqrt{2}k\left(t_3 - \lambda\right) \\ -\frac{\sqrt{2}}{3}ik\left(t_1 - 2t_3\right) & \frac{2}{3}ik\left(t_1 + t_3\right) & -\frac{2}{3}k^2\left(t_1 + t_3\right) & \frac{2}{3}k^2\left(t_1 + t_3\right) & 2i\sqrt{2}k\left(t_3 - \lambda\right) \\ -c_1k^2 + 4t_3 - 4\lambda & 2\sqrt{2}\left(t_3 - \lambda\right) & 2i\sqrt{2}k\left(t_3 - \lambda\right) & -2i\sqrt{2}k\left(t_3 - \lambda\right) & 4\left(3t_3 - 3\lambda + \frac{\nu}{4} + k^2\xi\right) \end{pmatrix} \end{array}, \tag{6.27}$$

$$a(1^+) = \begin{array}{cc} & \begin{array}{ccc} A & A & \mathfrak{a} \end{array} \\ \begin{array}{c} A \\ A \\ \mathfrak{a} \end{array} & \begin{pmatrix} \frac{1}{3}\left(6k^2\left(2r_3 + r_5\right) + t_1 + 4t_2\right) & \frac{1}{3}\sqrt{2}\left(t_1 - 2t_2\right) & -\frac{1}{3}i\sqrt{2}k\left(t_1 - 2t_2\right) \\ \frac{1}{3}\sqrt{2}\left(t_1 - 2t_2\right) & \frac{2}{3}\left(t_1 + t_2\right) & -\frac{2}{3}ik\left(t_1 + t_2\right) \\ \frac{1}{3}i\sqrt{2}k\left(t_1 - 2t_2\right) & \frac{2}{3}ik\left(t_1 + t_2\right) & \frac{2}{3}k^2\left(t_1 + t_2\right) \end{pmatrix} \end{array}, \tag{6.28}$$

$$a(2^-) = \begin{array}{cc} & A \\ A & \left(2\left(k^2 r_1 + \frac{t_1}{2}\right) \right) \end{array}, \tag{6.29}$$

$$a(2^+) = \begin{array}{c|cc} & A & \mathfrak{s} \\ \hline A & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ \mathfrak{s} & & \end{array} \quad \begin{array}{cc} & \\ \end{array}$$

$$a(2^+) = \begin{array}{c} \\ A \\ \mathfrak{s} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \quad A \quad & \quad \mathfrak{s} \quad \end{array} \\ \begin{pmatrix} 2\left(k^2\left(2r_1 - 2r_3 + r_4\right) + \frac{t_1}{2}\right) & i\sqrt{2}kt_1 \\ -i\sqrt{2}kt_1 & 2k^2\left(t_1 + \lambda\right). \end{pmatrix} \end{array} \tag{6.30}$$

In general, if any of the matrices $a(J^P)$ in the decomposition (6.24) are singular, then the theory possesses gauge invariances. One may fix these gauges by deleting rows and columns of the a-matrices such that they become non-singular. The elements of the resulting matrices are usually denoted by $b_{ij}(J^P)$. For WGT$^+$, some of the a-matrices given above are indeed singular. In particular, one may delete the third row/column of $a(0^+)$, the third and fourth row/column of $a(1^-)$, and the third row/column of $a(1^+)$ to obtain the corresponding non-singular b-matrices. The singular nature of these three a-matrices results in them having both null right and left eigenvectors, which give us gauge invariance and source constraints respectively. For each spin-parity sector, the null left eigenvectors are given by

$$0^+ : (0,0,1,0) \tag{6.31}$$

$$1^- : (0,-ik,0,1,0)\,,(0,ik,1,0,0) \tag{6.32}$$

$$1^+ : (0,-ik,1)\,, \tag{6.33}$$

where one should note that the B-field is not involved, since the corresponding vector component is always zero, and the remaining components are the same as those found for PGT$^+$. This is no surprise, since the dilation gauge invariance has been fixed by adopting the Einstein gauge, and the remaining symmetry should indeed be local Poincaré invariance.

The null eigenvectors may be used to derive the form of the associated gauge invariances and the corresponding source constraints for WGT$^+$, which are found to be the same as those in PGT$^+$, as expected. The gauge invariances are given by

$$\delta h_{AB} = u_{[AB]} + ik_B v_A \tag{6.34}$$

$$\delta A_{ABC} = ik_C u_{[AB]}\,, \tag{6.35}$$

where $u_{[AB]}$ and v_A are some arbitrary fields, and the source constraints have the form

$$k^A \sigma_{AB} = 0 \tag{6.36}$$

$$ik^A \tau_{ABC} - \sigma_{[AB]} = 0, \tag{6.37}$$

where σ_{AB} is the source current of f_{AB}, and τ_{ABC} is the source current of A_{ABC}.

The requirement that a theory is free from ghosts and tachyons places conditions on the b-matrices, and one must consider the massless and massive particle sectors separately. For the massless modes, one requires only that there be no ghosts. As discussed in Chapter 2, this is determined by considering the coefficient matrices $\mathbf{Q}_{2n}$ in a Laurent series expansion of the saturated propagator about the origin in momentum space. For WGT$^+$, one finds that all of the entries $\mathbf{Q}_{2n}$ vanish identically for $n > 1$, and so the saturated propagator does not have a higher pole at $k^2 = 0$. The non-zero eigenvalues of $\mathbf{Q}_2$ are found to be

$$\frac{1+6|\vec{k}|^2}{\lambda}, \frac{1+8|\vec{k}|^2}{2\lambda}, \tag{6.38}$$

and so there are 2 degrees of freedom in the propagating massless particle sector.[4] The massless no-ghost condition is that all eigenvalues of $\mathbf{Q}_{2n}$ are non-negative, and so one requires simply that

$$\lambda > 0. \tag{6.39}$$

Turning to the massive particle sector, one must first determine the particle masses by calculating the determinants of the b-matrices:

$$\det\left[b\left(0^-\right)\right] = 2k^2 r_2 + 2t_2, \tag{6.40}$$

$$\det\left[b\left(0^+\right)\right] = 16\left(r_1 - r_3 + 2r_4\right)\left(t_3 - \lambda\right)\nu k^4 - 8\lambda\left[12\left(t_3-\lambda\right)\lambda + t_3\nu\right]k^2, \tag{6.41}$$

$$\det\left[b\left(1^-\right)\right] = -\tfrac{2}{3}\left(t_1+t_3\right)\left[c_1^2 - 8\left(r_1+r_4+r_5\right)\xi\right]k^4 + \tfrac{4}{3}\{6c_1t_1\left(t_3-\lambda\right) + \left(r_1+r_4+r_5\right)$$

[4]Note that the expression for the eigenvalues is not unique, but depends on the form chosen for the source constraints. To be precise, one can obtain another set of the null vectors $\mathbf{n}_i$ in (2.27) by linear combination.

$$[12(t_3-\lambda)(t_1+\lambda)+(t_1+t_3)\nu]+6t_1t_3\xi\}k^2+2t_1[12\lambda(t_3-\lambda)+t_3\nu], \tag{6.42}$$

$$\det[b(1^+)] = \frac{4}{3}(2r_3+r_5)(t_1+t_2)k^2+2t_1t_2, \tag{6.43}$$

$$\det[b(2^-)] = 2r_1k^2+t_1, \tag{6.44}$$

$$\det[b(2^+)] = 4(2r_1-2r_3+r_4)(t_1+\lambda)k^4+2t_1\lambda k^2, \tag{6.45}$$

from which one finds that there is no massive mode in the 0^+ sector, and the particle masses in the other sectors are given by

$$m^2(0^-) = -\frac{t_2}{r_2}, \tag{6.46}$$

$$m^2(0^+) = \frac{12\lambda^2(t_3-\lambda)+t_3\lambda}{2(r_1-r_3+2r_4)(t_3-\lambda)\nu}, \tag{6.47}$$

$$m^2(1^-) = (\text{the two roots of } \det[b(1^-)]), \tag{6.48}$$

$$m^2(1^+) = -\frac{3t_1t_2}{2(2r_3+r_5)(t_1+t_2)}, \tag{6.49}$$

$$m^2(2^-) = -\frac{t_1}{2r_1}, \tag{6.50}$$

$$m^2(2^+) = -\frac{t_1\lambda}{2(2r_1-2r_3+r_4)(t_1+\lambda)}. \tag{6.51}$$

The no-tachyon conditions are then simply $m^2(J^P) > 0$. We give the conditions for the 1^- sector in the end of this section because of the length of the expressions involved. Note also for the 1^- sector that one requires the two roots of (6.42) to be distinct in order to avoid a dipole ghost. Hence, in each sector, the masses are distinct, and so one can apply (2.42) directly to obtain the massive no-ghost conditions:

$$0^- : r_2 < 0, \tag{6.52}$$

$$\begin{aligned} 0^+ : &(r_1-r_3+2r_4)(t_3-\lambda)\lambda\nu^2\{24(t_3-\lambda)\lambda^3+12(r_1-r_3+2r_4)(t_3-\lambda)\lambda\nu \\ &+[(r_1-r_3+2r_4)t_3+t_3\lambda-\lambda^2]\nu^2\} > 0, \end{aligned} \tag{6.53}$$

$$1^+ : (2r_3+r_5) > 0, \tag{6.54}$$

$$2^- : r_1 < 0, \tag{6.55}$$

$$2^+ : \lambda\,(2r_1 - 2r_3 + r_4)\,(\lambda + t_1)\left[(2r_1 - 2r_3 + r_4)\,t_1 - \lambda^2 - \lambda t_1\right] < 0, \tag{6.56}$$

where again we do not write out the condition for 1^- because of its length, but instead give the relevant expression in the end of this section.

The combined no-ghost-and-tachyon conditions for each sector other than 1^- are then

$$0^- : t_2 > 0,\; r_2 < 0 \tag{6.57}$$

$$0^+ : r_1 + 2r_4 > r_3,\; (t_3 - \lambda)\,\lambda\,\nu\,[12\lambda\,(t_3 - \lambda) + t_3\nu] > 0 \tag{6.58}$$

$$1^+ : 2r_3 + r_5 > 0,\; t_1 t_2 (t_1 + t_2) < 0 \tag{6.59}$$

$$2^- : t_1 > 0, r_1 < 0 \tag{6.60}$$

$$2^+ : 2r_1 + r_4 > 2r_3,\; \lambda t_1 (\lambda + t_1) < 0. \tag{6.61}$$

Note that, except for the 0^+ and 1^- sectors, the combined condition in each of the other spin-parity sectors is exactly the same as originally found in [29] for PGT$^+$.

For the 1^- sector, to avoid tachyons and a dipole ghost, one requires the roots of (6.42) to be real and distinct, such that

$$\begin{aligned}&\{6c_1t_1\,(t_3 - \lambda) + (r_1 + r_4 + r_5)\,[12\,(t_3 - \lambda)\,(t_1 + \lambda) + (t_1 + t_3)\,\nu] + 6t_1t_3\xi\}^2\\ &\quad + 3t_1\,(t_1 + t_3)\,[12\,(t_3 - \lambda)\,\lambda + t_3\nu]\left[c_1^2 - 8\,(r_1 + r_4 + r_5)\,\xi\right] > 0.\end{aligned} \tag{6.62}$$

The no-tachyons conditions that both of the roots are positive then read

$$\begin{aligned}&(t_1 + t_3)\left[c_1^2 - 8\,(r_1 + r_4 + r_5)\,\xi\right]\{6c_1t_1\,(t_3 - \lambda) + (r_1 + r_4 + r_5)\,[12\,(t_3 - \lambda)\,(t_1 + \lambda)\\ &\quad + (t_1 + t_3)\,\nu] + 6t_1t_3\xi\} > 0,\end{aligned} \tag{6.63}$$

$$t_1\,(t_1 + t_3)\,[12\,(t_3 - \lambda)\,\lambda + t_3\nu]\left[c_1^2 - 8\,(r_1 + r_4 + r_5)\,\xi\right] < 0. \tag{6.64}$$

The no-ghost condition is

$$\begin{aligned}
&\left[c_1^2-8\left(r_1+r_4+r_5\right)\xi\right]\left\{3c_1\left(t_1-2t_3\right)\left(t_3-\lambda\right)-\left(r_1+r_4+r_5\right)\left[\left(t_1+t_3\right)^2+18\left(t_3-\lambda\right)^2\right]\right.\\
&\qquad\left.-3\left(t_1^2+2t_3^2\right)\xi\right\}<0, \qquad (6.65)\\
&\left(t_1+t_3\right)\left[c_1^2-8\left(r_1+r_4+r_5\right)\xi\right]\left\{9\left(t_1+t_3\right)\left\{2t_1\left(7t_3^2-12t_3\lambda+6\lambda^2\right)+t_1^2\left(14t_3-12\lambda+\nu\right)\right.\right.\\
&\qquad\left.+2t_3\left[12\left(t_3-\lambda\right)\lambda+t_3\nu\right]\right\}^2\left[c_1^2-8\left(r_1+r_4+r_5\right)\xi\right]-48t_1\left[12\left(t_3-\lambda\right)\lambda+t_3\nu\right]\\
&\qquad\left[-3c_1\left(t_1-2t_3\right)\left(t_3-\lambda\right)+\left(r_1+r_4+r_5\right)\left[\left(t_1+t_3\right)^2+18\left(t_3-\lambda\right)^2\right]+3t_1^2\xi+6t_3^2\xi\right]^2\\
&\qquad+16\left\{2t_1\left(7t_3^2-12t_3\lambda+6\lambda^2\right)+t_1^2\left(14t_3-12\lambda+\nu\right)+2t_3\left[12\left(t_3-\lambda\right)\lambda+t_3\nu\right]\right\}\\
&\qquad\left\{9c_1t_1\left(-t_3+\lambda\right)+\frac{3}{2}\left(r_1+r_4+r_5\right)\left[-12\left(t_3-\lambda\right)\left(t_1+\lambda\right)-\left(t_1+t_3\right)\nu\right]-9t_1t_3\xi\right\}\\
&\qquad\left\{3c_1\left(t_1-2t_3\right)\left(t_3-\lambda\right)-\left(r_1+r_4+r_5\right)\left[\left(t_1+t_3\right)^2+18\left(t_3-\lambda\right)^2\right]\right.\\
&\qquad\left.\left.-3\left(t_1^2+2t_3^2\right)\xi\right\}\right\}<0. \qquad (6.66)
\end{aligned}$$

Combining the requirements for no tachyons and no ghosts in the 1^- sector, there exists at least one parameter set satisfying all five conditions above, for example

$$c_1=-9, r_1=-1, r_4=0, r_5=0, t_1=\frac{1}{2}, t_3=-1, \lambda=-4, \nu=-142, \xi=-18, \qquad (6.67)$$

where the other parameters may take arbitrary values provided they do not make the theory a critical case.

Finally, if we consider all the no-tachyon and no-ghost conditions from all the massive sectors, we find that they cannot be satisfied simultaneously. Thus, the root theory must contain a massive ghost or tachyon.

6.3 Critical cases

If the parameters in the action satisfy certain "critical conditions", the particle masses (6.46)–(6.51) can become zero or infinite, and the resulting critical cases may possess additional gauge invariances, so one may have to re-evaluate the no-tachyon and no-ghost conditions for both the massless and massive sectors.

6.3.1 Unitarity

In attempting to apply the method in Chapter 2 to obtain all the critical cases of the root WGT$^+$ theory, one finds that some of the coefficients in Equations (6.41) and (6.42) cannot be factorized into linear combinations of the parameters. Consequently, the method in Chapter 2 cannot be applied straightforwardly to obtain all the critical cases, and one must check carefully where it is applicable. For example, one of the factors in the coefficient of the k^2 term in (6.41) is

$$12\,(t_3-\lambda)\,\lambda+t_3\nu, \tag{6.68}$$

which cannot be written as the product of factors that are linear in the Lagrangian parameters. Indeed, for (6.68) to equal zero, one has the two solutions:

$$\nu=-\frac{12\,(t_3-\lambda)\,\lambda}{t_3}\quad \text{with } t_3\neq 0, \tag{6.69}$$

$$t_3=\lambda=0. \tag{6.70}$$

It is therefore not as straightforward to apply the condition $12\,(t_3-\lambda)\,\lambda+t_3\nu=0$ by substitution. Moreover, the second solution (6.70) requires one to eliminate two degrees of freedom in the parameters simultaneously and thus breaks the hierarchy of the "tree" of critical cases discussed in Chapter 2.

In general, one finds that allowing any of the Lagrangian parameters ν, ξ, or c_1 in (6.10) to be non-zero introduces similar problems. It requires further improvement of our systematic method to accommodate such cases, and so here we set $\nu=\xi=c_1=0$ to avoid these difficulties. Thus, for the remainder of this section, the "root theory" refers to (6.10) with $\nu=\xi=c_1=0$. As we will show below, however, one may nevertheless construct a theory with $\nu\neq 0$ and/or $\xi\neq 0$ from a theory with $\nu=\xi=0$, provided its a-matrices are "non-mixing".

Starting from the "root" theory, we systematically identify 862 critical cases (excluding the "vanishing" Lagrangian, for which all parameters are zero). Of these critical cases, we find 168 are free of ghosts and tachyons, provided the parameters in each case satisfy some additional conditions that preclude them from generating another critical case; this general issue is discussed in detail in Appendix 6.A. The full set of results, displayed in an interactive form, can be found at: http://www.mrao.cam.ac.uk/projects/gtg/wgt/.

6.3.2 Comparison with previous results

We now compare our results with the only other example of a unitary WGT$^+$ theory of which we are aware in the literature [99]. This has the Lagrangian

$$\mathcal{L} = -\lambda\phi^2\mathcal{R} + a\mathcal{R}^2 - \tfrac{1}{4}H^{\mu\nu}H_{\mu\nu} + \tfrac{1}{2}\mathcal{D}^*_\mu\phi\mathcal{D}^{*\mu}\phi, \tag{6.71}$$

which on adopting the Einstein gauge becomes

$$\mathcal{L} = -\lambda\phi_0^2\mathcal{R} + a\mathcal{R}^2 - \tfrac{1}{4}H^{\mu\nu}H_{\mu\nu} + \tfrac{1}{2}\phi_0^2 B_\mu B^\mu. \tag{6.72}$$

Thus, the B-field is decoupled from the other gauge fields and so the theory can be viewed as the combination of PGT$^+$ with $\mathcal{L} = -\lambda\phi_0^2\mathcal{R} + a\mathcal{R}^2$ and Proca theory $\mathcal{L}_{\mathrm{Pr}} = -\frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}\phi_0^2 B_\mu B^\mu$ for a massive vector field. The Proca part is well-known to be unitary. Using the Gauss–Bonnet identity, the PGT$^+$ part may be shown to correspond to the critical case $r_1 = r_2 = 2r_3 - r_4 = 2r_3 + r_5 = t_1 + t_2 = t_1 + t_3 = t_1 + \lambda = 0, r_3 \neq 0, \lambda \neq 0$. This a type C critical case of the root PGT$^+$ theory with no massive mode and massless modes with 2 degrees of freedom; the no-ghost-and-tachyon condition is simply $\lambda > 0$. Therefore, provided this condition is satisfied, the theory (6.71) is indeed unitary.

One should note that the presence of the kinetic terms for the B- and ϕ-fields means that (6.71) is not a critical case of our redefined WGT$^+$ with $\nu = \xi = c_1 = 0$ in (6.10), but is a critical case of the "full" WGT$^+$ root theory without this constraint on the Lagrangian parameters. In particular, (6.71) belongs to an extended set of theories with $\nu \neq 0$ and $\xi \neq 0$

that can be separated into a PGT$^+$ part and a dilaton part, which we discuss below in the context of power-counting renormalisability. We note, however, that the PGT$^+$ part of (6.71) is not listed in Chapter 5 because one cannot obtain non-mixing b-matrices by deleting rows and columns from its a-matrices.

6.3.3 Power-counting renormalisability

In addition to possessing no ghosts or tachyons, a healthy physical theory should also be renormalisable. The first step in assessing whether this is possible is to determine whether the theory is power-counting renormalisable (PCR).

As discussed in Chapter 5, the key quantity for determining whether a theory is PCR is the propagator

$$\hat{D} = \sum_{J,P,i,j} b_{ij}^{-1}\hat{P}(J^P)_{ij}. \tag{6.73}$$

In particular, if the b-matrices are block diagonal, with each block containing only one of the fields A, $\mathfrak{s}$, $\mathfrak{a}$ and B, then there are no mixing terms in the (gauge-fixed) Lagrangian and it is straightforward to obtain the propagators for these fields separately from $\hat{D}$. Extending the original PCR criterion used by Sezgin in [29] would require the propagator of the A- and B-fields to decay at least as quickly as k^{-2} at high energy, and those of the $\mathfrak{s}$- and $\mathfrak{a}$-fields to fall off at least as k^{-4}. This can be shown by following the discussion in Section 5.1. The most general Lagrangian in the Einstein gauge with ϕ_0 absorbed into the coefficients is

$$\begin{aligned}
b\mathcal{L}_G &\sim b\left(\lambda\mathcal{R} + r\mathcal{R}^2 + t\mathcal{T}^{*2} + \xi\mathcal{H}^2 + c_1\mathcal{R}\mathcal{H} + \nu B^2\right)\\
&\sim \left(1+f+f^2+\ldots\right)\Big\{\lambda\,(1+f)^2\left(\partial A + A^2\right) + r\,(1+f)^4\left(\partial A + A^2\right)^2\\
&\quad + t\,(1+f)^2\left[\partial(f+f^2+\ldots) + (1+f+f^2+\ldots)(A+B)\right]^2 + \xi\,(1+f)^4(\partial B)^2\\
&\quad c_1(1+f+f^2+\ldots)\left(\partial A + A^2\right)\partial B + \nu\,(1+f)^2 B^2\Big\},
\end{aligned} \tag{6.74}$$

with $[\lambda]_{\mathrm{M}} = 2$, $[r]_{\mathrm{M}} = 0$, $[t]_{\mathrm{M}} = 2$, $[\xi]_{\mathrm{M}} = 0$, $[c_1]_{\mathrm{M}} = 0$, $[A]_{\mathrm{M}} = 1$, $[f]_{\mathrm{M}} = 0$, and $[B]_{\mathrm{M}} = 1$. Similarly, the PCR conditions for the WGT fields are $l_A \geq 2$, $l_h \geq 4$, and $l_B \geq 2$.

By contrast, our alternative PCR criterion used in Chapter 5 also permits the presence of non-propagating fields (for which the propagator decays no faster than $\sim k^0$), since these should completely decouple from the rest of the theory. The ultimate consistency of these two approaches in identifying particular theories as PCR is discussed at length in Chapter 5, although the second approach is preferred since it identifies further critical cases that reduce to those identified by Sezgin's criterion after integrating out any non-propagating modes. We therefore again adopt the latter method here, which is consistent with our previous work.

On performing this analysis, one finds that most of the critical cases identified as PCR are identical to those listed in Tables 5.2, 5.4 and 5.6, or are a PGT$^+$ without any propagating mode (which were not listed in Chapter 5) but with an additional propagating dilaton. One may understand the reason for this by first expanding the $\mathcal{T}^{*2}$ terms in (6.10) to obtain

$$\mathcal{T}^*_{ABC}\mathcal{T}^{*ABC} = \mathcal{T}_{ABC}\mathcal{T}^{ABC} + 4B_A T^{CA}_{C} + 6B^A B_A, \tag{6.75}$$

$$\mathcal{T}^*_{ABC}\mathcal{T}^{*BCA} = \mathcal{T}_{ABC}\mathcal{T}^{BCA} - 2B_A T^{CA}_{C} - 3B^A B_A, \tag{6.76}$$

$$\mathcal{T}^{*B}{}_{BA}\,\mathcal{T}^{*C}{}_{C}{}^{A} = \mathcal{T}^{B}{}_{BA}\,\mathcal{T}^{C}{}_{C}{}^{A} + 6B_A T^{CA}_{C} + 9B^A B_A. \tag{6.77}$$

The BT terms are the only possible origin for mixing terms containing the B-field after linearisation, and so there will be no mixing terms in the a-matrices if these terms vanish, for which the condition on the Lagrangian parameters is

$$t_3 = \lambda. \tag{6.78}$$

Moreover, the same condition ensures that the B^2 terms from $\mathcal{T}^{*2}$ also vanish. Hence, if $t_3 = \lambda$, the $\mathcal{R} + \mathcal{R}^2 + \mathcal{T}^{*2}$ part of the WGT$^+$ Lagrangian is identical to its PGT$^+$ counterpart with the replacement $\mathcal{T}^* \to \mathcal{T}$.

The PGT$^+$ critical cases identified as PCR in Chapter 5 and having $t_3 = \lambda$ are:

1. PGT$^+$ with 2 massless d.o.f. and a massive mode: Case 1, 3, 4, 6, and 7 in Table 5.2.

2. PGT$^+$ with only 2 massless d.o.f.: Case 9-13, 17, and 19 in Table 5.4.

3. PGT^+ with only massive mode(s): Cases 26-28, 30-36, and 38-40, 55, and 58 in Table 5.6. These cases all have 1 massive mode, either 0^- or 2^-.

If the PGT^+ part of a WGT^+ satisfying $t_3 = \lambda$ has no propagating mode, then the corresponding WGT^+ can at most have a propagating B-field. There are 37 critical cases of PGT^+ satisfying $t_3 = \lambda$ and containing no propagating mode (these are not listed in Chapters 4 and 5). Requiring $\xi \neq 0$ in the corresponding WGT^+ Lagrangian (6.10) ensures that they contain a propagating dilaton. The dilaton part of WGT^+ Lagrangians satisfying $t_3 = \lambda$ is simply

$$\mathcal{L}_B = \xi \mathcal{H}^{AB} \mathcal{H}_{AB}, \tag{6.79}$$

which is that of a massless 1^- vector.

For all cases for which the a-matrices are non-mixing, there are no cross terms of B and the other fields and so adding a mass term for B in the Lagrangian does not affect the other fields. Hence, if one adds the term $\frac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi$ to such a case, the only effect is either to make an already propagating B-field massive, or to add a non-propagating B-field. In the former (and more interesting) case, the corresponding dilaton Lagrangian is a Proca theory in the Einstein gauge ($\phi_0 = 1$)

$$\mathcal{L}_B = \xi \mathcal{H}^{AB} \mathcal{H}_{AB} + \tfrac{1}{2}\nu B_\mu B^\mu, \tag{6.80}$$

and the corresponding no-ghost-and-tachyon condition is $\xi < 0$ and $\nu > 0$. With these extensions, one can thus construct more tachyon and ghost free and PCR cases for WGT^+ from the PGT^+ cases with $t_3 = \lambda$.

There are, however, some PCR critical cases of WGT^+ that cannot be constructed directly from PGT^+ in the manner described above. These cases have non-mixing b-matrices, but their a-matrices contain mixing terms. In particular, this occurs when there are mixing terms $\sim BA$ in the linearised Lagrangian. Since the B-field can be fixed using the additional gauge invariance of the critical case, there are no BA terms in the b-matrices. We list these further PCR critical cases in Tables 6.1 and 6.2.

Table 6.1 Parameter conditions for the PC renormalisable critical cases that are ghost and tachyon free and cannot be constructed directly from PGT. The parameters listed in "Additional conditions" must be non-zero to prevent the theory becoming a different critical case.

#	Critical condition	Additional conditions	No-ghost-and-tachyon condition
59	$r_1, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$	$t_2 > 0, r_2 < 0, r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
60	$r_2, r_1 - r_3, r_4, t_1, t_2, \lambda = 0$	$r_1, r_1 + r_5, 2r_1 + r_5, t_3$	$r_1\,(r_1 + r_5)\,(2r_1 + r_5) < 0$
61	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$	$r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
62	$r_1, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$	$r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$	$r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
63	$r_1, r_2, \frac{r_3}{2} - r_4, t_1, \lambda = 0$	$r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$	$r_3\,(2r_3 + r_5)\,(r_3 + 2r_5) < 0$
64	$r_1, r_3, r_4, r_5, \lambda = 0$	$r_2, t_1, t_2, t_1 + t_2, t_3$	$t_2 > 0, r_2 < 0$
65	$r_1, r_3, r_4, r_5, t_1 + t_2, \lambda = 0$	r_2, t_1, t_3	$r_2 < 0, t_1 < 0$
66	$r_2, r_1 - r_3, r_4, r_1 + r_5, t_1 + t_2, \lambda = 0$	r_1, t_1, t_3	$t_1 > 0, r_1 < 0$
67	$r_1, r_3, r_4, r_5, t_1, \lambda = 0$	r_2, t_2, t_3	$t_2 > 0, r_2 < 0$
68	$r_1, r_3, r_4, t_1, \lambda = 0$	r_2, r_5, t_2, t_3	$t_2 > 0, r_2 < 0$
69	$r_1 - r_3, r_4, 2r_1 + r_5, t_1, \lambda = 0$	r_1, r_2, t_2, t_3	$t_2 > 0, r_2 < 0$
70	$r_1, \frac{r_3}{2} - r_4, 2r_3 + r_5, t_1, \lambda = 0$	r_2, r_3, t_2, t_3	$t_2 > 0, r_2 < 0$
71	$r_1, \frac{r_3}{2} - r_4, \frac{r_3}{2} + r_5, t_1, \lambda = 0$	r_2, r_3, t_2, t_3	$t_2 > 0, r_2 < 0$

Table 6.2 Particle content of the PC renormalisable critical cases that are ghost and tachyon free and cannot be constructed directly from PGT. The column "b-sectors" describes the diagonal elements in the b^{-1}-matrix of each spin-parity sector in the sequence $\{0^-,0^+,1^-,1^+,2^-,2^+\}$. Here it is notated as φ^n_v or φ^n_l, where φ is the field, $-n$ is the power of k in the element in the b^{-1}-matrix when k goes to infinity, v means massive pole, and l means massless pole. If $n=\infty$, it represents that the diagonal element is zero. If $n \leq 0$, the field is not propagating. The "|" notation denotes the different form of the elements of the b^{-1}-matrices in different choices of gauge fixing, and the "&" connects the diagonal elements in the same b^{-1}-matrix. The superscript "N" represents that there is non-zero off-diagonal term in the b^{-1}-matrix.

#	Massless mode d.o.f.	Massive mode	b-sector
1	2	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
2	2	$\times$	$\left\{\times, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\right\}$
3	2	$\times$	$\left\{\times, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\right\}$
4	2	$\times$	$\left\{A^2_{\mathrm{l}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\right\}$
5	2	$\times$	$\left\{A^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$
6	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^0\&A^0)^{\mathrm{N}} \mid (A^0\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&B^0)^{\mathrm{N}} \mid (\mathfrak{s}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}} \mid (\mathfrak{a}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}},\right.$ $\left.(A^0\&A^0)^{\mathrm{N}} \mid (A^0\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}}\right\}$
7	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^0\&A^0)^{\mathrm{N}} \mid (A^0\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&B^0)^{\mathrm{N}} \mid (\mathfrak{s}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}} \mid (\mathfrak{a}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}},\right.$ $\left.(A^\infty\&A^0)^{\mathrm{N}} \mid (A^\infty\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, A^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}}\right\}$
8	0	2^-	$\left\{A^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^0\&A^0)^{\mathrm{N}} \mid (A^0\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^0\&B^0)^{\mathrm{N}} \mid (\mathfrak{s}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}} \mid (\mathfrak{a}^2_{\mathrm{l}}\&B^0)^{\mathrm{N}},\right.$ $\left.(A^\infty\&A^{-2})^{\mathrm{N}} \mid (A^\infty\&\mathfrak{a}^0_{\mathrm{l}})^{\mathrm{N}}, A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}}\right\}$
9	0	0^-	$\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid \mathfrak{a}^2_{\mathrm{l}} \mid B^0, A^0 \mid \mathfrak{a}^2_{\mathrm{l}}, \times, \times\}$
10	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, \times\right\}$
11	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, A^0 \mid \mathfrak{a}^2_{\mathrm{l}}, A^2_{\mathrm{l}}, \times\right\}$
12	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{s}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&B^0_{\mathrm{l}})^{\mathrm{N}}, A^0 \mid \mathfrak{a}^2_{\mathrm{l}}, \times, A^2_{\mathrm{l}}\right\}$
13	0	0^-	$\left\{A^2_{\mathrm{v}}, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid B^0, A^0 \mid \mathfrak{s}^2_{\mathrm{l}} \mid \mathfrak{a}^2_{\mathrm{l}} \mid B^0, (A^2_{\mathrm{l}}\&A^0_{\mathrm{l}})^{\mathrm{N}} \mid (A^2_{\mathrm{l}}\&\mathfrak{a}^2_{\mathrm{l}})^{\mathrm{N}}, \times, A^2_{\mathrm{l}}\right\}$

6.4 Torsion-free WGT$^+$

As well as the general case of WGT$^+$, one may also consider the simpler cases with vanishing torsion or curvature, respectively, which are *not* merely special cases of the general WGT$^+$ action, because additional constraints are placed not only the coefficients, but also on the fields. In this section we consider the case of vanishing torsion.

If one sets the torsion $\mathcal{T}^{*\rho}{}_{\mu\nu}$ to zero, then one sees from (6.20) that the gauge fields $A^{AB}{}_{\mu}$, $h_a{}^{\mu}$ and B_μ are no longer independent. Indeed, (6.20) gives an explicit expression for the A-field in terms of the B- and b-fields. On making this substitution in the Lagrangian, one may then can apply the same method as in the previous section to investigate torsion-free WGT$^+$ and its critical cases. In this simpler theory, one need not set $\nu = \xi = c_1 = 0$, since one does not encounter critical conditions that are non-linear in the Lagrangian parameters. Hence, we do not adopt this restriction in this section.

6.4.1 The "root" theory

In this case, the a-matrices of the root theory (6.10) are

$$a(0^+) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{s} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{s} & B \end{array} \\ \begin{pmatrix} 8(r_1 - r_3 + 2r_4)k^4 - 4\lambda k^2 & 0 & 8i\sqrt{3}(r_1 - r_3 + 2r_4)k^3 \\ 0 & 0 & 0 \\ -8i\sqrt{3}(r_1 - r_3 + 2r_4)k^3 & 0 & 24k^2(r_1 - r_3 + 2r_4) + 12\lambda + \nu \end{pmatrix} \end{array}, \tag{6.81}$$

$$a(1^-) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{a} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{a} & B \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4k^2(c_1 + 2r_1 + 2r_4 + 2r_5 + \xi) + 12\lambda + \nu \end{pmatrix} \end{array}, \tag{6.82}$$

$$a(1^+) = \begin{array}{c} \\ \mathfrak{a} \end{array} \begin{array}{c} \mathfrak{a} \\ \begin{pmatrix} 0 \end{pmatrix} \end{array}, \tag{6.83}$$

$$a(2^{+}) = \begin{array}{cc} & \mathfrak{s} \\ \mathfrak{s} & \left(4\left(2r_1 - 2r_3 + r_4\right)k^4 + 2\lambda k^2 \right) \end{array}, \tag{6.84}$$

where the SPOs are obtained from those listed in Appendix 2.A by simply deleting the rows and columns corresponding to the A- and ϕ-fields. The a-matrices for 0^- and 2^- sectors have no element, so we do not list them. One can fix the gauge simply by removing the rows and columns whose elements are all zeros from the a-matrices, to obtain the corresponding b-matrices. These may then be inverted to obtain the saturated propagator.

Considering first the massless sector, the nonzero eigenvalues of the Laurent series coefficient matrix $\mathbf{Q}_2$ are

$$\frac{1}{\lambda}, \frac{1}{2\lambda}. \tag{6.85}$$

Thus, the theory has two massless d.o.f., and the no-ghost condition for the massless sector is simply

$$\lambda > 0. \tag{6.86}$$

Turning to the massive sector, the determinants of the b-matrices are

$$\det\left[b\left(0^{+}\right)\right] = 8\left(r_1 - r_3 + 2r_4\right)\nu k^4 - 4\lambda\left(12\lambda + \nu\right)k^2, \tag{6.87}$$

$$\det\left[b\left(1^{-}\right)\right] = 4\left(c_1 + 2r_1 + 2r_4 + 2r_5 + \xi\right)k^2 + 12\lambda + \nu, \tag{6.88}$$

$$\det\left[b\left(2^{+}\right)\right] = 4\left(2r_1 - 2r_3 + r_4\right)k^4 + 2\lambda k^2, \tag{6.89}$$

from which one obtains the masses

$$m^2\left(0^{+}\right) = \frac{\lambda\left(12\lambda + \nu\right)}{2\left(r_1 - r_3 + 2r_4\right)\nu}, \tag{6.90}$$

$$m^2\left(1^{-}\right) = \frac{-12\lambda - \nu}{4\left(c_1 + 2r_1 + 2r_4 + 2r_5 + \xi\right)}, \tag{6.91}$$

$$m^2\left(2^{+}\right) = -\frac{\lambda}{2\left(2r_1 - 2r_3 + r_4\right)}. \tag{6.92}$$

The no-tachyon conditions $m^2(J^P) > 0$ may then be read off from the above expressions. In each sector, the masses are distinct, and so one can again apply (2.42) directly to obtain the massive no-ghost conditions

$$0^+ : \frac{1}{4\lambda} + \frac{6\lambda^2}{(r_1 - r_3 + 2r_4)\,\nu^2} + \frac{3}{\nu} > 0, \tag{6.93}$$

$$1^- : c_1 + 2\,(r_1 + r_4 + r_5) + \xi < 0, \tag{6.94}$$

$$2^+ : \lambda < 0. \tag{6.95}$$

One thus finds that the combined no-ghost-and-tachyon conditions for the massive sector are

$$0^+ : r_1 + 2r_4 > r_3, \lambda\nu(12\lambda + \nu) > 0, \tag{6.96}$$

$$1^- : 12\lambda + \nu > 0, c_1 + 2\,(r_1 + r_4 + r_5) + \xi < 0, \tag{6.97}$$

$$2^+ : 2r_1 + r_4 > 2r_3, \lambda < 0. \tag{6.98}$$

Since the conditions in the massive 2^+ sector contradict the condition (6.86) in the massless sector, the theory must have a ghost or tachyon.

6.4.2 Critical cases

We now consider the critical cases of torsion-free WGT$^+$. As discussed in detail in Chapter 2, one finds all conditions that cause a theory to be a critical case. While some conditions may cause criticality in more than one way, one can still divide all the critical conditions into three categories, which we called type A, B and C conditions, respectively.

Considering first the root theory, it becomes critical and thereby loses one d.o.f in the Lagrangian parameter space if any of the following expressions vanishes:

$$\text{Type B: } \lambda, 12\lambda + \nu, \tag{6.99}$$

$$\begin{aligned}\text{Type C: } & 2r_1 - 2r_3 + r_4, r_1 - r_3 + 2r_4, \nu, \\ & c_1 + \xi + 2r_1 + 2r_4 + 2r_5.\end{aligned} \tag{6.100}$$

The two critical cases resulting from the type B conditions (6.99) of the root theory contain ghosts or tachyons, but some of their descendant critical cases, all of which result from type A or C conditions, are free from ghosts and tachyons. The critical cases resulting from type A and type B conditions of torsion-free WGT^{+} are shown in Figure 6.1, whereas those arising from type C critical conditions are listed in Table 6.3; those cases that are ghost-and-tachyon-free are indicated, as described in the captions. One sees that four cases in Figure 6.1 are free from ghosts and tachyons, and nine critical cases in Table 6.3 share this property. We also note that there are 15 critical cases of the root theory in total that result from type C conditions, which correspond to self-consistent combinations of those in (6.100). As is clear from (6.92), those critical cases resulting from type C conditions and for which $2r_1 - 2r_3 + r_4 = 0$ are free from ghosts and tachyons because the 2^{+} massive mode is not propagating.

6.4.3 Comparison with previous results

The particle spectrum of a subset of torsion-free Weyl-invariant higher-curvature gravity theories has been studied previously by [100], both in (anti-)de Sitter and Minkowski backgrounds (to our knowledge, this is the only other investigation of a torsionless WGT^{+} ground-state in the literature). For $n = 4$ spacetime dimensions, the coefficients $(\alpha, \beta, \gamma, \varepsilon, \sigma)$ in their Lagrangian (see equations (1), (7) and (14) in [100]) are related to those in our notation used in (6.10) by

$$
\begin{aligned}
\alpha &= -\tfrac{1}{2}r_1 + r_3 = \tfrac{1}{4}(r_4 - r_5), \\
\beta &= r_4 + r_5 = -\tfrac{1}{2}c_1, \\
\gamma &= \tfrac{1}{2}r_1, \\
\varepsilon &= \xi - (r_4 + r_5 + 2r_1), \\
\sigma &= \lambda,
\end{aligned}
\tag{6.101}
$$

together with the conditions

$$
r_1 = r_2, \qquad \nu = -1. \tag{6.102}
$$

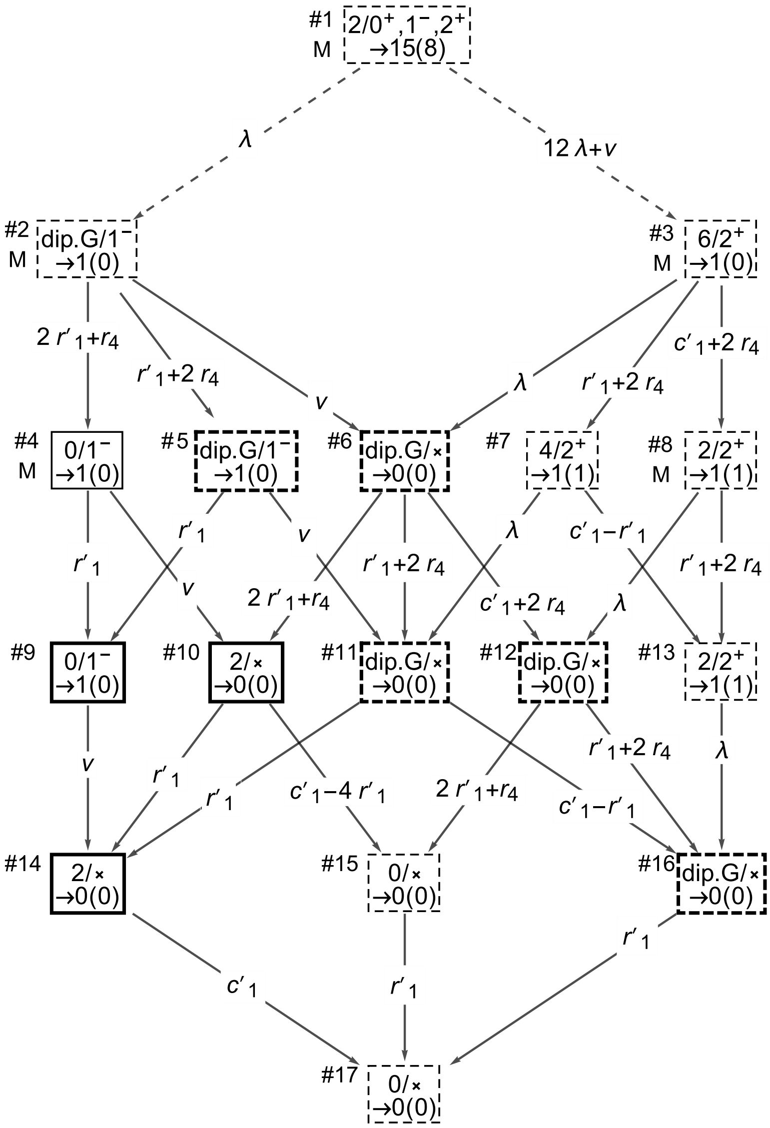


Fig. 6.1 Critical cases of torsionless WGT$^+$ resulting from type A or type B conditions. Each node represents a critical case, except the top and bottom nodes, which represent the root theory and the zero Lagrangian, respectively. Each arrow points from a node to one of its critical cases. A solid arrow represents type A critical condition, and a dashed arrow represents type B. The labels on the arrows are the critical parameters; for brevity, the variables $r_1' = r_1 - r_3$ and $c_1' = c_1 + 2r_1 + 2r_5 + \xi$ have been defined. The critical condition of a node can be obtained by setting all the critical parameters to zeros in the path from the root theory to that node, and the conditions are path independent. In each node, the first line is in the format "d.o.f. of massless mode or 'dip.G' if there are massless dipole ghosts/massive mode", and the second line is "number of child critical cases resulting form type C conditions (number of no-ghost-and-tachyon cases among them)", which are not shown but are listed in Table 6.3. The dashed/solid frames indicate those cases that contain any/no ghost or tachyon. The thick frames indicate PCR cases, and the thin frames indicate those that are non-PCR or have mixing b-matrices. The "M" under the number at the left of the nodes with mixing b-matrices.

Table 6.3 Critical cases of torsion-free WGT$^+$ resulting from type C conditions. The first numbers in the column "#" correspond to the numbers in Figure 6.1, and the corresponding nodes are the parent critical cases of the rows. The "Critical condition" column indicates the critical condition with respect to the parent case. For example, "#1-3" is the third critical case resulting from type C conditions of case #1. The symbols '$\circ$'/'$\times$' indicate whether the theory is possible to be free of ghosts and tachyons. The "–" symbols denote that there is no propagating mode, and the "M" symbols indicate the cases with mixing b-matrices.

#	Critical condition	Massive mode	No-ghost-and-tachyon	PCR
#1-1	ν	$1^-, 2^+$	$\times$	M
#1-2	$r_1' + 2r_4$	$1^-, 2^+$	$\times$	$\times$
#1-3	$r_1' + 2r_4, \nu$	$1^-, 2^+$	$\times$	$\times$
#1-4	$c_1' + 2r_4$	$0^+, 2^+$	$\times$	M
#1-5	$\nu, c_1' + 2r_4$	2^+	$\times$	M
#1-6	$r_1' + 2r_4, c_1' + 2r_4$	2^+	$\times$	$\times$
#1-7	$r_1' + 2r_4, \nu, c_1' + 2r_4$	2^+	$\times$	$\times$
#1-8	$2r_1' + r_4$	$0^+, 1^-$	$\circ$	M
#1-9	$2r_1' + r_4, \nu$	1^-	$\circ$	M
#1-10	$2r_1' + r_4, r_1' + 2r_4$	1^-	$\circ$	$\times$
#1-11	$2r_1' + r_4, r_1' + 2r_4, \nu$	1^-	$\circ$	$\times$
#1-12	$2r_1' + r_4, c_1' + 2r_4$	0^+	$\circ$	M
#1-13	$2r_1' + r_4, \nu, c_1' + 2r_4$	$\times$	$\circ$	M
#1-14	$2r_1' + r_4, r_1' + 2r_4, c_1' + 2r_4$	$\times$	$\circ$	$\times$
#1-15	$2r_1' + r_4, r_1' + 2r_4, \nu, c_1' + 2r_4$	$\times$	$\circ$	$\times$
#2-1	$c_1' + 2r_4$	$\times$	$\times$	M
#3-1	$2r_1' + r_4$	$\times$	$\times$	M
#4-1	$c_1' - 4r_1'$	$\times$	–	–
#5-1	$c_1' - r_1'$	$\times$	$\times$	$\circ$
#7-1	r_1'	$\times$	$\circ$	$\times$
#8-1	$2r_1' + r_4$	$\times$	$\circ$	M
#9-1	c_1'	$\times$	–	–
#13-1	r_1'	$\times$	$\circ$	$\times$

In particular, one should note that the Lagrangian in [100] is written in terms of the curvature tensor $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$. As discussed in Section 6.1, this has even fewer symmetry properties than the rotational gauge field strength tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ used in (6.10). Consequently, there are further quadratic combinations of $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$ that could appear in the Lagrangian in [100], but only three such terms are included. Consequently, there are fewer degrees of freedom in the parameters of their Lagrangian, as compared with our Lagrangian in (6.10), as is evident from the above parameter identifications. Moreover, since $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$ has many fewer symmetries than the standard curvature tensor in Riemannian spacetime V_4, the appropriate form of the Gauss–Bonnet identity differs from the usual formula that is assumed in Eq. (34) of [100] (see, for example [28, 101]); fortunately most of the conclusions presented in [100] do not depend on this expression.

The constraints on our parameters in (6.101)–(6.102) do not coincide with any of the critical conditions in any critical case, so the structure of our "criticality tree" of torsion-free WGT$^+$ is not affected. In [100], it is found that about a 4-dimensional Minkowski background, the WGTs considered are unitary provided (in terms of our parameters)

$$2(r_1 - r_3) + r_4 = 0, \tag{6.103}$$

$$r_1 - r_3 + 2r_4 = 0, \tag{6.104}$$

$$\lambda > 0. \tag{6.105}$$

Both equalities coincide with our type C critical conditions, and they eliminate 2^+ and 0^+ massive modes, leaving a 1^- massive mode. The condition on λ also matches ours, so their result is consistent with our critical case #1-10 of the root theory, listed in Table 6.3.

It is concluded in [100], however, that the theory has a massless spin-2 field and a massless spin-0 field, and so the massless sector has 3 d.o.f, whereas we find just 2. This difference may result from the fact that they employ a gauge fixing condition $\mathcal{D}^*_\mu B^\mu = 0$ on the B^μ-field (their A^μ-field), described in their Eq. (30), but then treat this field as if it is unconstrained when reading off the particle content from their Eq. (59). This situation is analogous to that in Stueckelberg theory, as discussed in Appendix 5.A. If one fixes the gauge by setting $\partial \cdot B = 0$, then the Lagrangian appears to describe a massive vector B and a massless scalar ϕ without interaction. Conversely, if one instead sets $\phi = 0$, the Lagrangian contains only a massive vector without constraint. Thus, one should interpret the theory as containing either a massive vector or a massive vector with a Stueckelberg ghost and a Faddeev–Popov ghost.

Also, it is claimed in [100] that unitarity requires both (6.103) and (6.104) to hold, whereas we require only the former condition, if no Type A or B critical condition is satisfied. The condition (6.104) is necessary in [100] because they do not adopt the Einstein gauge, and so require the higher-derivative Pais–Uhlenbeck term $(\bar{\Box}\Phi_L)^2$ to vanish, where Φ_L is the linearised ϕ. By contrast, all the higher-order poles in our saturated propagator vanish due to the source constraints, and so the condition (6.104) is not necessary in our case. This difference may be worthy of further investigation.

6.4.4 Power-counting renormalisability

We determine whether each critical case is PCR using the same method as discussed in Section 6.3.3. The results are presented in Figure 6.1 and Table 6.3. In particular, we find three critical cases in Figure 6.1 that are both PCR and contain no ghost or tachyon; these are indicated by nodes with thick, solid frames. We note that each of these theories can be gauge fixed to contain only the B gauge field. It is also worth highlighting that, perhaps as a consequence of this, there is no simultaneously unitary and PCR case in torsion-free PGT$^+$, and so these three theories may be worthy of further investigation. No critical case in Table 6.3 is both PCR and unitary.

6.5 Curvature-free WGT$^+$

In this section, we consider WGT$^+$ with vanishing curvature. This is a more subtle condition than the equivalent case in PGT$^+$, which was discussed in Chapter 4. As mentioned in Section 6.1, the geometric (Riemann) curvature tensor $\tilde{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu}$ in Weyl–Cartan spacetime differs from the rotational gauge field strength $\mathcal{R}^{\rho}{}_{\sigma\mu\nu}$, so it is unclear which should be set to zero. Here we consider only the case in which the latter vanishes, since this may imposed in the same way as in PGT by simply setting $A_{AB\mu} = 0$, since the expression for the rotational gauge field strength in terms of the rotational gauge field are identical in PGT and WGT. In this simpler theory, one sees from (6.10) that one requires only the Lagrangian parameters ξ, ν, t_1, t_2 and t_3, since one can set $\lambda = 0$ without loss of generality.

6.5.1 The "root" theory

In this case, the a-matrices of the root theory are

$$a\left(0^+\right) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{s} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{s} & B \end{array} \\ \begin{pmatrix} 4k^2 t_3 & 0 & 4i\sqrt{3}kt_3 \\ 0 & 0 & 0 \\ -4i\sqrt{3}kt_3 & 0 & 12t_3 + \nu \end{pmatrix} \end{array}, \tag{6.106}$$

$$a\left(1^-\right) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{a} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{a} & B \end{array} \\ \begin{pmatrix} \frac{2}{3}k^2\left(t_1 + t_3\right) & -\frac{2}{3}k^2\left(t_1 + t_3\right) & -2i\sqrt{2}kt_3 \\ -\frac{2}{3}k^2\left(t_1 + t_3\right) & \frac{2}{3}k^2\left(t_1 + t_3\right) & 2i\sqrt{2}kt_3 \\ 2i\sqrt{2}kt_3 & -2i\sqrt{2}kt_3 & 12t_3 + \nu + 4k^2\xi \end{pmatrix} \end{array}, \tag{6.107}$$

$$a\left(1^+\right) = \begin{array}{c} \\ \mathfrak{a} \end{array} \begin{array}{c} \mathfrak{a} \\ \begin{pmatrix} \frac{2}{3}k^2\left(t_1 + t_2\right) \end{pmatrix} \end{array}, \tag{6.108}$$

$$a\left(2^+\right) = \begin{array}{c} \\ \mathfrak{s} \end{array} \begin{array}{c} \mathfrak{s} \\ \begin{pmatrix} 2k^2 t_1 \end{pmatrix} \end{array}. \tag{6.109}$$

As in the torsion-free theory, the SPOs are obtained from those listed in Appendix 2.A by deleting the rows and columns corresponding to the A- and ϕ-fields, and the a-matrices for the 0^- and 2^- sectors contain no elements. After fixing the gauge by deleting rows and columns, one obtains the non-singular b-matrices, which may be inverted to obtain the saturated propagator.

Considering first the massless sector, one finds that the Laurent series coefficient matrix $\mathbf{Q}_4$ is non-zero in this case, and the condition for it to vanish is

$$\nu = -\frac{12t_1\left(t_1 - 2t_2\right)t_3}{t_1^2 - 2t_1t_2 + 4t_1t_3 + t_2t_3}. \tag{6.110}$$

One further finds that the Laurent coefficient matrix $\mathbf{Q}_2$ cannot be positive definite and contains eight nonzero eigenvalues, which are too complicated to be given here. Consequently, the root theory must contain ghosts in the massless sector.

One can, however, continue to analyze the massive sector. The determinants of the b-matrices are

$$\det\left[b\left(0^+\right)\right] = 4t_3\nu k^2, \tag{6.111}$$

$$\det\left[b\left(1^-\right)\right] = \tfrac{2}{3}\left[t_3\nu + t_1\left(12t_3 + \nu\right)\right]k^2 + \tfrac{8}{3}\left(t_1 + t_3\right)\xi k^4, \tag{6.112}$$

$$\det\left[b\left(1^+\right)\right] = \tfrac{2}{3}\left(t_1 + t_2\right)k^2, \tag{6.113}$$

$$\det\left[b\left(2^+\right)\right] = 2t_1k^2. \tag{6.114}$$

Only the 1^- sector contains a massive mode, with mass

$$m^2\left(1^-\right) = \frac{-12t_1t_3 - (t_1 + t_3)\nu}{4\left(t_1 + t_3\right)\xi}, \tag{6.115}$$

and the no-tachyon condition is $m^2\left(1^-\right) > 0$. Applying (2.42) directly, in this case the no-ghost condition is

$$1^- : \left(t_1 + t_3\right)\left[12t_1t_3 + \left(t_1 + t_3\right)\nu\right]\xi\left\{\left(t_1 + t_3\right)\left[12t_1t_3 + \left(t_1 + t_3\right)\nu\right] - 72t_3^2\xi\right\} < 0. \tag{6.116}$$

The combined no-ghost-and-tachyon conditions for the massive sector are thus

$$\xi < 0, \qquad \nu > -\frac{12t_1t_3}{t_1 + t_3}, \tag{6.117}$$

but one should recall that the massless sector always contains a ghost.

6.5.2 Critical cases

The critical cases of the root theory occur when any of the following expressions vanishes:

$$\text{Type A: } t_1, t_1+t_2, t_3, \nu, \tag{6.118}$$

$$\text{Type B: } 12t_1t_3+t_1\nu+t_3\nu, \tag{6.119}$$

$$\text{Type C: } t_1+t_3, \xi. \tag{6.120}$$

However, since $12t_1t_3+t_1\nu+t_3\nu$ cannot be factorized into a linear combination of the parameters, one cannot apply our algorithm to find all the critical cases directly. We therefore below consider the critical case $\nu=0$, which removes the kinetic term of the scalar field ϕ, as the simplified root theory and instead find its critical cases. Before turning to these, we note that the massless sector of this simplified root theory requires $t_1-2t_2=0$ to make its Laurent series coefficient matrix $\mathbf{Q}_4$ vanish, and thus prevent the presence of dipole ghosts, but in any case the matrix $\mathbf{Q}_2$ has seven nonzero eigenvalues and cannot be made be positive definite. Therefore, the massless sector must contain a ghost. The conditions for the massive sector of the simplified root theory to be ghost and tachyon free may be obtained from (6.115)–(6.117) by setting $\nu=0$.

Turning now to the critical cases of the simplified root theory, the critical conditions are given by (6.118)–(6.120) with $\nu=0$. One should note that this results in the simplified root theory containing no type B critical condition, since the resulting condition that t_1t_3 should vanish is trivially factorised and the separate requirements that t_1 or t_3 should vanish are already included in the type A critical conditions, and it turns out that there is no type B critical condition in the descendants. The critical cases resulting from type A or type C conditions are summarised in Figure 6.2 and Table 6.4, respectively. Cases that are ghost-and-tachyon-free are indicated, as described in the captions. In particular, we note that there are nine critical cases in Figure 6.2 that are free from ghosts and tachyons, and three such critical cases in Table 6.4.

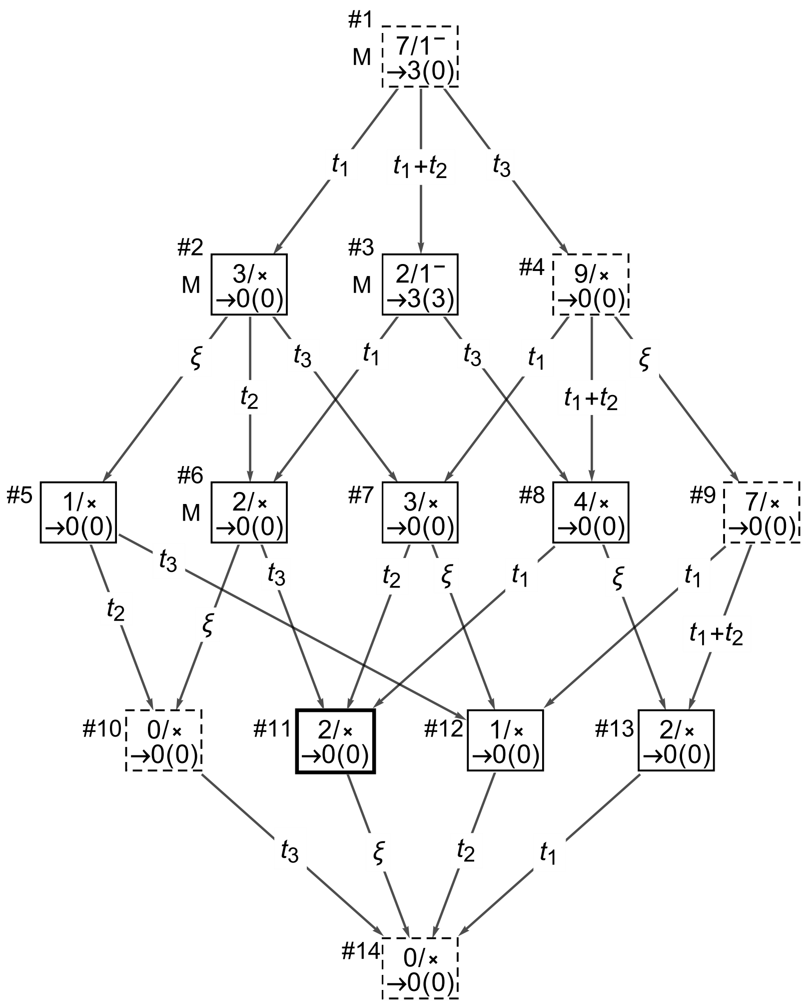


Fig. 6.2 Critical cases resulting from type A critical conditions of curvature-free WGT$^+$. The notation follows that of Figure 6.2.

Table 6.4 Critical cases resulting from type C critical conditions of curvature-free WGT$^+$. The notation follows that of Table 6.3.

#	Critical condition	Massive mode	No-ghost-and-tachyon	PCR
#1-1	ξ	$\times$	$\times$	M
#1-2	t_1+t_3	$\times$	$\times$	M
#1-3	t_1+t_3,ξ	$\times$	$\times$	M
#3-1	ξ	$\times$	$\circ$	M
#3-2	t_1+t_3	$\times$	$\circ$	M
#3-3	t_1+t_3,ξ	$\times$	$\circ$	M

6.5.3 Power-counting renormalisability

We determine whether each critical case is PCR using the same method as discussed in Section 6.3.3. The results are presented in Figure 6.2 and Table 6.4. In particular, we find that there is just a single critical case in Figure 6.2, which is just the pure dilaton Lagrangian $\mathcal{L} \sim \mathcal{H}^2$, that is both PCR and unitary; this is indicated by the node with a thick, solid frame. No such critical case is found in Table 6.4.

6.6 Conclusions

We have used the systematic method in Chapter 2 to determine the no-ghost-and-tachyon conditions for the most general WGT$^+$ (the root theory), and found it must contain a ghost or tachyon. For a subset of the theory, with the restriction $\nu = \xi = c_1 = 0$ on the parameters in the Lagrangian (6.10), which removes the kinetic terms for the scalar field ϕ and dilational gauge field B, respectively, and the only "cross term" $\mathcal{R}^{AB}\mathcal{H}_{AB}$ between gauge field strengths, we found and categorised all 862 critical cases, and identified 168 that are free from ghosts and tachyons. The full set of results displayed in an interactive form can be found at: http://www.mrao.cam.ac.uk/projects/gtg/wgt/. We compared our findings with the only other example of a unitary WGT$^+$ of which we are aware in the literature [99], and found the results to be consistent. We further identified those critical cases of WGT$^+$ that are also PCR. Most of these are identical to those in PGT$^+$ listed in Chapter 5, or are a PGT$^+$ without any propagating mode (which were not listed in Chapter 5). Nonetheless, we also identified a further 13 PCR and ghost-and-tachyon-free critical cases of WGT$^+$ that cannot be constructed directly from PGT$^+$.

We repeated our analysis for the simpler cases of torsion-free and curvature-free WGT$^+$, which are not merely special cases of the general WGT$^+$ action, because additional constraints are placed not only the coefficients, but also on the fields. For the torsion-free case, we found that the root theory (without any further conditions on the Lagrangian parameters) must contain a ghost or tachyon. Nonetheless, we identify 13 critical cases that are free from ghosts and tachyons. We also compare our results with the only other invesigation of the ground-state of a torsionless WGT^{+} of which we are aware in the literature. We find our results to be consistent, apart from a minor issue related to the number of propagating degrees of freedom in the massless sector, most probably resulting from the different approaches to gauge-fixing used in the two analyses. Of our 13 ghost-and-tachyon-free critical cases, we further identified three that are also PCR, each of which can be gauge fixed to contain only the B gauge field. This may explain the sharp contrast with torsion-free PGT^{+}, for which there is no unitary and PCR critical case, and suggests that these three theories may be worthy of further investigation.

For curvature-free WGT^{+}, we find that the massless sector of the root theory (again with no further conditions on the Lagrangian parameters) must contain a ghost. For the simplified root theory with $\nu = 0$, which has no kinetic term for the scalar field ϕ in the Lagrangian and is itself found to have a ghost in the massless sector, we find 13 critical cases that are free from ghosts and tachyons, of which just a single case is found also to be PCR, which corresponds to the pure dilaton Lagrangian $\mathcal{L} \sim \mathcal{H}^2$.

All the restrictions on Lagrangian parameters mentioned above are necessary to avoid critical conditions that cannot be written as the product of real linear terms, which is required by the systematic method in Chapter 2. We plan to improve our approach to accommodate such cases in future work, and also apply the method to more general gauge theories, such as metric affine gravities, whose unitarity was recently investigated by [102] using SPOs.

Appendix 6.A Completeness of the critical cases

In this appendix, we will review the completeness of the critical conditions and additional conditions introduced in Chapter 2. An "additional condition" is defined as the condition(s) to prevent a theory from being critical. In Chapter 2, the additional condition was the requirement that the "sibling critical conditions" should not be satisfied, and we will call this the "sibling additional condition". For example, consider a theory that has the critical conditions that the (linear) parameter combinations X, Y, and Z should vanish; we will call X, Y and Z the "critical parameters" of the theory. In the case, the sibling critical parameters for the critical case $X=0$ are Y and Z. To prevent a theory from being critical, one can require the "critical parameters" not equal to zeros. We will call this kind of condition a "child additional condition". In PGT^{+}, as discussed in Chapter 2, the "sibling additional condition" is identical to the "child additional condition", except for the root case. This occurs because we add only one linear condition at a time for cases resulting from type A or B critical conditions, but we attempt to use all possible combinations of conditions simultaneously for type C critical parameters (which we term "combining" the conditions). We then recursively find the child critical cases of cases resulting from type A and B critical conditions (the "uncombined" cases), but stop doing that for those from type C critical conditions (the "combined" cases). If type C critical conditions are treated in the same way as type A and type B, then the statement is not valid for PGT^{+}.

There are two situations in which the statement is invalid. The first is the occurence of "hidden" critical parameters. Consider a theory with only a 1×1 b-matrix $(XY+Zk^2)$. The theory has type B critical parameters, X and Y, and a type C one, Z. For the critical case $X=0$, the b-matrix becomes (Zk^2), so there is only one critical parameter Z. To prevent the theory being critical ("child additional condition"), one requires $Z\neq 0$. However, its sibling critical parameters are Y and Z, which are different. The critical parameter Y is hidden in this case. If there are "hidden" parameters and one is requiring only child additional conditions, then a point in the parameter space may belong to more than one critical case. For example, the critical case $X=0, Z\neq 0$ and the case $Y=0, Z\neq 0$ has the overlap $X=Y=0, Z\neq 0$, and they actually have the same b-matrix (Zk^2) and represent the same theory. If we use the sibling additional condition instead, the two cases become $X = 0, Y \neq 0, Z \neq 0$ and $Y = 0, X \neq 0, Z \neq 0$, and there is no overlap. "Hidden" parameters do not occur in PGT$^+$ or any of the critical cases discussed in this thesis, if we "combine" all the type C critical cases as in Chapter 2. While the overlapping and redundancy do no real harm to the correctness of our results, it may be worth modifying our algorithm to accommodate the situation for simplicity.

The second reason is the occurence of "emergent" critical parameters. Some critical parameters appear after a b-matrix becomes singular and a new b-matrix forms, which may happen in critical cases resulting from a type A critical parameter (it is worth noting that critical parameters of the root theory are always "emergent" because it has no parent or sibling critical cases). In PGT$^+$ and torsion-free or simplified curvature-free WGT$^+$, either the new b-matrix is 0×0, or its critical parameters are already included in the sibling critical parameters, and so there is no "emergent" critical parameter. However, in simplified full WGT$^+$, this is not the case. For example, the $b(0^+)$-matrix of the simplified root WGT$^+$ is

$$\begin{pmatrix} 2\left[2k^2\left(r_1 - r_3 + 2r_4\right) + t_3\right] & -2i\sqrt{2}kt_3 & 2\sqrt{6}\left(t_3 - \lambda\right) \\ 2i\sqrt{2}kt_3 & 4k^2\left(t_3 - \lambda\right) & 4i\sqrt{3}k\left(t_3 - \lambda\right) \\ 2\sqrt{6}\left(t_3 - \lambda\right) & -4i\sqrt{3}k\left(t_3 - \lambda\right) & 12\left(t_3 - \lambda\right) \end{pmatrix}, \tag{6.121}$$

which has $\det\left[b(0^+)\right] = -96\left(t_3 - \lambda\right)\lambda^2 k^2$. Its critical case $\lambda = 0$ has

$$\begin{pmatrix} 2\left[2k^2\left(r_1 - r_3 + 2r_4\right) + t_3\right] & -2i\sqrt{2}kt_3 \\ 2i\sqrt{2}kt_3 & 4k^2 t_3 \end{pmatrix} \tag{6.122}$$

with $\det\left[b(0^+)\right] = 16\left(r_1 - r_3 + 2r_4\right)t_3 k^4$. The critical parameter $\left(r_1 - r_3 + 2r_4\right)$ is neither a critical parameter of the root theory, nor among the sibling critical parameters of case $\lambda = 0$. However, the "emergent" parameters will not affect our algorithm if we apply the "child additional condition", which already includes the "emergent" parameters.

In conclusion, as long as there is no "hidden" critical parameter in critical cases resulting from type A and B critical parameters, and the cases resulting from type C critical parameters are "combined", then we can apply the child additional conditions for the "uncombined" cases and the sibling additional conditions for the "combined" cases as the "(extended) additional condition", respectively.[5] This is what the term "additional condition" actually means in this thesis. Our algorithm then holds, and each parameter set corresponds to one critical case. We have also checked that the all critical cases in Chapter 4 and Chapter 6 cover the entire parameter space and the critical cases have no overlap.

[5]This is also equivalent to combining the sibling and child additional conditions as the additional condition for all cases.

Chapter 7

Concluding remarks

Gauge theories are successful in providing a unified framework to describe fundamental interactions. It is natural also to consider gravitation as a gauge theory. In this thesis, we investigated the unitarity and renormalisability of Poincaré gauge theory (PGT) and Weyl gauge theory (WGT) with a systematic method.

In Chapter 2, we presented the systematic approach to investigate the no-ghost-and-tachyon conditions for general gauge theories of gravitation. The systematic method first linearises the free gravitational Lagrangian and then decomposes the Fourier transformed linearised quadratic Lagrangian with the spin projection operators (SPOs). The Lagrangian is then represented by matrices, and each matrix corresponds to a specific spin-parity J^P. After fixing gauges by deleting rows and columns in the singular matrices to make them non-singular, we then obtain the saturated propagator by sandwiching the non-singular matrices by source currents. The source currents should satisfy some constraints to maintain the gauge invariance. The no-tachyon condition requires the masses to be real, and it can be obtained by requiring the non-zero zeros of the determinants of the non-singular matrices to be positive. The no-ghost condition requires the residues of the saturated propagator on the poles to be positive definite. For the massive poles, the condition can be obtained by requiring the residue of the trace of the non-singular matrices to be positive (or negative). However, it requires more work to deal with the massless poles because the SPOs are not well-defined when $k^2 = 0$. We expand the source constraints into tensor components and express the components by some free variables. After expanding the saturated propagator into tensor components and replacing the source current components with the free variables, the source constraints are satisfied. We then require its residue to be positive definite. If the Lagrangian parameters satisfy some critical conditions, the theory may change qualitatively. We classify the critical conditions into three categories: those introducing any additional gauge invariance (type A), those not type A and making any massive pole massless (type B), and those not type A or B and removing any massive pole (type C). By applying one condition (one equation) and removing one degree of freedom in the parameter space at a time, we can recursively find all the critical cases if the critical conditions contain only linear combinations of the Lagrangian parameters. The method was implemented with MATHEMATICA, and we have shown some details of the implementation in Chapter 3.

In Chapter 4, we constructed PGT by gauging the Poincaré symmetry. The matter Lagrangian is made invariant with the minimal coupling method, with the covariant derivative constructed with gauge fields ${A^{AB}}_{\mu}$ and ${h_A}^{\mu}$, and the free gravitational Lagrangian should be constructed with the field strengths. We applied the method to investigate the 9-parameter most general parity-preserving PGT with up to two derivatives, as well as all of the critical cases. We found 450 critical cases that are free of ghosts and tachyons within the 1918 critical cases in total. The full set of results displayed in an interactive form can be found at: http://www.mrao.cam.ac.uk/projects/gtg/pgt/. We also examined torsion-free and curvature-free PGT$^+$, which are not subsets of the critical cases of the full theory, in the same way. We identified and compared our results with the literature for the (small) subset of critical cases that have been analysed previously. We found that they are basically consistent, although there are a few minor differences most probably due to typographical errors in previous works. We also listed a subset of the source constraints in the critical cases carrying additional gauge invariance.

In Chapter 5, we reviewed the original criterion of PCR from the superficial degrees of divergence argument. We then proposed the alternative PCR criterion, which allows non-propagating modes. With the alternative PCR criterion, we found 58 critical cases of PGT that are both PCR and free of ghosts and tachyons. Within these theories, seven have two massless d.o.f. in propagating modes and a massive 0^- or 2^- mode, 12 have only two massless d.o.f., and 39 have only massive mode(s). We have also clarified the role of non-propagating modes in the PCR criterion. Appendix 5.A discussed this issue further by applying the methods used in this chapter to the simpler cases of the Proca and Stueckelberg theories.

In Chapter 6 we applied the systematic method to study the most general parity-preserving WGT with up to two derivatives and with the compensator under the Einstein gauge. The root theory contains 2 d.o.f. of propagating massless mode, which is the same as its counterpart of PGT. It also contains one 0^-, 0^+, 1^+, 2^-, and 2^+ massive mode, as well as two 1^- ones. However, the no-ghost-and-tachyon conditions for all spin-parity sectors cannot be satisfied at the same time, and therefore there must be ghosts or tachyons. Because some critical conditions are not linear in the Lagrangian parameters when applying the algorithm to find all critical cases derived from the root theory, we simplify the root theory by setting $\nu = \xi = c_1 = 0$ and investigate a subset of the original critical cases. Within the subset, we found and categorised all 862 critical cases, and identified 168 that are free from ghosts and tachyons. The full set of results displayed in an interactive form can be found at: http://www.mrao.cam.ac.uk/projects/gtg/wgt/. We found that most of the critical cases that are PCR and ghost-and-tachyon-free can be directly constructed from some PGT critical cases. We listed a further 13 critical cases with both of the properties which cannot be directly constructed from critical cases of PGT. We also analysed torsion-free and curvature-free WGT$^+$. For the torsion-free case, the root theory must contain a ghost or tachyon, but there are 13 critical cases that are free from ghosts and tachyons. Three of them are also PCR, but each of which can be gauge fixed to contain only the B gauge field. For curvature-free WGT$^+$, the root theory must contain a massless ghost. We investigated the critical cases of the simplified root theory with $\nu = 0$ because of the same reason as full WGT$^+$. There are 13 critical cases free from ghosts and tachyons, and only one of them is also PCR, which corresponds to the pure dilaton Lagrangian $\mathcal{L} \sim \mathcal{H}^2$.

In future work, we plan to improve the shortcomings of our systematic method. Our method can only determine whether dipole ghost exists, but it does not yield the number of degrees of freedom nor the spin-parities of the massless particles if there are dipole ghosts. Even if there is no dipole ghost, it only yields the total degrees of freedom of the massless particles, but not their spins or parities. When finding all critical cases, the method requires all the critical conditions are linear combinations of the Lagrangian parameters. If the restriction is violated, then the degrees of freedom of parameter space eliminated by applying one equation may be greater than one, and the hierarchy structure of the critical cases breaks down.

Since our method linearises the Lagrangian, it cannot include non-linear effects. The authors of [49, 50, 83, 84] showed that linearizing a theory can change its structure qualitatively and particle content. By performing a full non-linear analysis, one may further constrain the valid cases listed in this thesis.

It is also worth investigating all the PCR and ghost-and-tachyon-free theories we found in this thesis further, especially those containing massless propagating particles. There are a few papers which examined some of the critical cases in Section 5.2. Barker et al. [103] systematically categorised 33 of the PGT critical cases we presented in Section 5.2 into 14 classes and investigated cosmological solutions of them. It turns out that Class ${}^{3}\mathrm{C}^{*}$ of PGT in their paper mimics the cosmology of GR and may ease the current Hubble constant tension [104, 105] via an effective dark radiation period. Cases 14 and 16 in Section 5.2, which may contain massless 2^{+} particles, belong to the class. In their latest paper [106], they further found Class ${}^{2}\mathrm{A}^{*}$ of PGT not only reproduces ΛCDM cosmology, but also has dark radiation and its own dark energy (the massive 0^{-} particle). The authors also showed that Class ${}^{2}\mathrm{A}^{*}$ is an analogue to bi-scalar-tensor theories (or generalised bi-galileon theories [107], which are multi-scalar versions of Horndeski theory [108]) described by a metric tensor and scalars, which provides the broader community with a unified framework for future investigation.

We have always performed perturbation around the flat spacetime in this thesis, but this is not the correct background space for our actual universe. It is possible that the particle spectrum of a gauge theory changes qualitatively [109]. It may worth making our method compatible with de Sitter or cosmological spacetime and find what change this causes.

We also plan to apply the method to more general gauge theories. Lasenby and Hobson [28] proposed the extended Weyl gauge theory (eWGT), which treats the Lorentz field strength ($\mathcal{R}$) and the translational field strength ($\mathcal{T}$) in a more balanced manner with regard to transformation properties by "extending" the transformation rule for the A-field. It has been shown that eWGT produces the same cosmology as PGT, up to a linear map of the parameters [103]. The unitarity of the affine gauge theories was recently investigated by Percacci and Sezgin [102] using the SPO framework, and our method may be applied to perform a complete search. Overall the interest in gauge theories of gravity is increasing, and we believe our methods are well placed to contribute to the study of their possible quantum properties.

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