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Gauge Theories, Duality Relations and the Tensor Hierarchy

Eric A. Bergshoeff †, **Jelle Hartong** ⋆, **Olaf Hohm** †
Mechthild Hübscher ‡, and **Tomás Ortín** ‡

† *Centre for Theoretical Physics, University of Groningen,*
Nijenborgh 4, 9747 AG Groningen, The Netherlands

email: `E.A.Bergshoeff@rug.nl`, `O.Hohm@rug.nl`

⋆ *Center for Research and Education in Fundamental Physics,*
Institute for Theoretical Physics,
Sidlerstrasse 5, CH-3012 Bern, Switzerland

email: `hartong@itp.unibe.ch`

‡ *Instituto de Física Teórica UAM/CSIC Facultad de Ciencias C-XVI,*
C.U. Cantoblanco, E-28049-Madrid, Spain

email: `Mechthild.Huebscher@uam.es`, `Tomas.Ortin@uam.es`

Abstract

We compute the complete 3- and 4-dimensional tensor hierarchies, i.e. sets of p-form fields, with $1 \leq p \leq D$, which realize an off-shell algebra of bosonic gauge transformations. We show how these tensor hierarchies can be put on-shell by introducing a set of duality relations, thereby introducing additional scalars and a metric tensor. These so-called duality hierarchies encode the equations of motion of the bosonic part of the most general gauged supergravity theories in those dimensions, including the (projected) scalar equations of motion.

We construct gauge-invariant actions that include all the fields in the tensor hierarchies. We elucidate the relation between the gauge transformations of the p-form fields in the action and those of the same fields in the tensor hierarchy.

Contents

1 Introduction

The bosonic degrees of freedom of a generic supergravity theory are described by a metric tensor field and a set of (electric) p-form potentials with $p \geq 0$. In order to describe the correct number of degrees of freedom these fields must satisfy second-order differential equations. In general one may realize the supersymmetry algebra on a larger set of p-form potentials as long as this does not upset the counting of degrees of freedom. Such potentials are expected to exist in order to allow for the coupling of various types of branes. Examples of such potentials are the (magnetic) $(D-p-2)$-forms. Whereas the p-form couples to an (electric) $(p-1)$-brane, the $(D-p-2)$-form potential couples to a (magnetic) $(D-p-3)$-brane. The magnetic $(D-p-2)$-forms do not describe new degrees of freedom since they are related to the electric p-forms via a first-order duality relation. By virtue of the Bianchi identities that the curvatures of the electric and magnetic potentials satisfy, the second-order equations can be derived as integrability conditions of the duality relations:

$$\text{Bianchi identities} \;\;\&\;\; \text{duality relations} \;\Leftrightarrow\; \text{equations of motion}\,. \tag{1.1}$$

For instance, in the case of IIA/IIB supergravity the supersymmetry algebra can be realized on all p-forms $(0 \leq p \leq 10)$ with p odd (IIA) or p even (IIB). The Bianchi identities and duality relations then lead to all equations of motion (except the Einstein equation). This is often referred to as the "democratic formulation" of IIA/IIB supergravity [1].

The idea of deriving the equations of motion of supergravity from an underlying set of Bianchi identities and first-order differential equations has been pursued in several contexts in the literature. It already occurs in the work of [2] for the case of maximal supergravity including massive IIA supergravity [3]. Similar duality relations are natural in the E_{11}-approach to supergravity [4–7]. Duality relations also play an important role in encoding the integrability of a system, for instance in maximal two-dimensional supergravity [8].

Recently, it has been shown that dual potentials are not only relevant to describe the coupling to branes but play also a crucial role in the construction of a supersymmetric action for certain gauged supergravity theories. A systematic way to study the most general gaugings of a supergravity theory is provided by the embedding tensor approach [9–13], which is a powerful technique to construct in a unified way gauged supergravity theories for different gauge groups. Usually, supersymmetric actions involve besides the metric tensor only electric potentials. However, using the embedding tensor approach, it has been shown that to describe a magnetic gauging in $D = 4$, i.e. a gauging involving a magnetic vector field, the action must also contain a dual 2-form potential via a Chern-Simons-like topological coupling.[1] In general dimensions p-form potentials of even higher rank are introduced. For instance, the action corresponding to certain gaugings in $D = 6$ requires magnetic 2-form and 3-form potentials [17]. This led to the notion of a *tensor hierarchy*, which consists of a system of potentials of all degrees $p = 1, \ldots, D$ and their respective curvatures, which are related by Bianchi identities. Note that the tensor hierarchy does

[1] In the context of $N = 2, D = 4$ supergravity it has been shown how the local supersymmetry algebra can be closed on some of these dual 2-form fields [16].

not contain 0-form potentials, i.e. scalars, and the metric tensor. These are introduced at a later stage, see below.

We wish to stress that for theories in specific dimensions generically not the full tensor hierarchy is used or needed in the construction of an action. Moreover, the field equations for the new (magnetic) potentials take the form of *projected* duality relations and, therefore, do not encode the full set of second-order equations via their integrability conditions. It is the purpose of this paper to investigate gauged supergravities from the point of view that all bosonic field equations (except the Einstein equation and part of the scalar equations of motion) should be derivable from first-order duality relations. This will naturally include the *full* tensor hierarchy, which is required by consistency. We will focus on the bosonic gauge symmetries that are realized by the $D = 3$ and (non-anomalous[2]) $D = 4$ tensor hierarchy independent of any supersymmetry. Our results apply for any number of supersymmetries, not just the maximal or half-maximal cases. Hence we obtain an off-shell formulation[3] of all bosonic symmetries that act in the bosonic sector of any (non-anomalous) $D = 3, 4$ gauged supergravity theory.

In the $D = 4$ case we use as our staring point Ref. [11]. We use the same formalism, impose the same constraints on the embedding tensor and follow the same steps up to the 2-form level reproducing exactly the same results, but we carry out the program to its completion, determining explicitly all the 3- and 4-forms and their gauge transformations. Here we find already a surprise in the sense that in $D = 4$ we find more top-form potentials than follow from the expectations formulated in Refs. [13,18][4]. Our results and the general results and conjectures of these references[5] cannot be straightforwardly compared, though, since in these works on the general structure of tensor hierarchies only one possible constraint on the embedding tensor (the *standard* quadratic constraint) is considered, while in the 4-dimensional setup of Ref. [11] the embedding tensor is subject to two additional constraints, one quadratic and one linear. They are ultimately responsible for the existence of additional 4-forms, which we find to be in one-to-one correspondence with the constraints[6].

Next, we will make precise how a set of dynamical equations can be defined by the

[2]By a non-anomalous tensor hierarchy we refer to a specific form of the so-called representation (or linear) constraint imposed on the embedding tensor. This constraint is such that the classical action of the corresponding gauged supergravity is gauge invariant.

[3]By "off-shell formulation" we mean that the commutator algebra of gauge symmetries closes without the need to impose constraints on the fields. In this sense an off-shell formulation is not related to any particular action.

[4]For instance, we find in $D = 4$ not only top-forms that correspond to quadratic constraints of the embedding tensor but also top-forms that are related to certain linear constraints, see subsection 3.4.

[5]There are no direct computations of tensor hierarchies up to the 4-form level in the literature. All we know about them, up to now, is based on general arguments.

[6]Note added in proof: it has recently been shown in Ref. [19] that the introduction of these additional 4-forms is consistent with $N = 1, D = 4$ supergravity. Furthermore, it has been shown that the gauging of particular classes of theories (e.g. $N = 1, D = 4$ supergravity with a non-vanishing superpotential) may require additional constraints on the embedding tensor, which lead to extensions of the tensor hierarchy and, in particular, to additional 4-forms related to the new constraints.

introduction of first-order duality relations. Besides the p-form potentials these duality relations also contain the scalars and the metric tensor defining the theory. The set of dynamical equations not only contains the equations of motion putting all electric potentials on-shell but it also involves the (projected) scalar equations of motion. The tensor hierarchy supplemented by this set of duality relations will be called the *duality hierarchy*. This set of duality relations cannot be derived from an action, though the relation to a possible action will be elucidated in a last step.

For the readers' convenience we briefly outline our program, which can be summarized by the following 3-step procedure. The first step consists of the general construction of the tensor hierarchy, which is an *off-shell* system. The structure in generic dimension has been given in [12, 13]. The explicit form, however, of the *complete* $D = 4$ tensor hierarchy is not available in the literature since it was constructed in [11] only up to the 2-form level. (For the construction of the tensor hierarchy of maximal and half-maximal 4-dimensional supergravities, see [20] and references therein.) The complete $D = 3$ tensor hierarchy has been discussed in [12, 21]. To construct the tensor hierarchy one usually starts from the p-form potential fields of all degrees $p = 1, \ldots, D$ and then constructs the gauge-covariant field strengths of all degrees $p = 2, \ldots, D$. These field strengths are related to each other via a set of Bianchi identities of all degrees $p = 3, \ldots, D$. Usually, one starts with the construction of the covariant field strength for 1-form potentials which, for general gaugings, requires the introduction of 2-form potentials. The corresponding 3-form Bianchi identity relates the 2-form field strength to a 3-form field strength for the 2-form potential, whose construction requires the introduction of a 3-form potential, etc. This bootstrap procedure ends with the introduction of the top-form potentials. The only input required for this construction is the number of electric $p \geq 1$-form potentials, the global symmetries of the theory and the representations of this group under which the p-forms transform. Changing these data leads to different theories that can be seen as different *realizations* of the low-rank sector of the same tensor hierarchy.

A trick that simplifies the construction outlined above and which makes the construction of the complete $D = 4$ tensor hierarchy feasible is to first construct the set of all Bianchi identities relating the $(p + 1)$-form field strengths to the $(p + 2)$-field strengths. This systematic construction of the Bianchi identities can be carried out *even if we do not know explicitly the transformation rules of the potentials*. These can be found afterwards by using the covariance of the different field strengths. The resulting gauge transformations form an algebra that closes off-shell: at no stage of the calculation equations of motions are involved.

The second step is to complement the tensor hierarchy with a set of duality relations and as such to promote it to what we have called duality hierarchy. The duality relations contain more 'external' information about the particular theory we are dealing with. It will introduce the scalars and the metric tensor field that were not involved in the construction of the tensor hierarchy[7]. More precisely, some of the duality relations contain the scalar fields via functions that define all scalar couplings, i.e. the Noether currents, the (scalar

[7]The dual scalars, i.e. the $(D - 2)$-form potentials, are included in the tensor hierarchy.

derivative of the) scalar potential and functions that define the scalar-vector couplings. In this way the duality hierarchy contains all the information about the particular realization of the tensor hierarchy as a field theory.

The duality hierarchy leads to a set of dynamical equations that not only contains the equations of motion for the electric potentials but it also involves the (projected) scalar equations of motion according to the rule:

$$\text{Tensor hierarchy} \ \ \& \ \ \text{duality relations} \quad \Leftrightarrow \quad \text{dynamical equations}\,. \tag{1.2}$$

The gauge algebra of the tensor hierarchy closes off-shell even in the presence of the duality relations. However, in the context of the duality hierarchy this is a basis-dependent statement. We are free to modify the gauge transformations by adding terms that are proportional to the duality relations. Of course, in this new basis the gauge algebra will close on-shell, i.e. up to terms that are proportional to the duality relations. We will call the original basis with off-shell closed algebra the off-shell basis.

The last and third step is the construction of a gauge-invariant action for all p-form potentials, scalars and metric.[8] In this last step we encounter a few subtleties that we will clarify. In particular, we will answer the following questions:

1. How are the equations of motion that follow from the gauge-invariant action related to the set of dynamical equations defined by the duality hierarchy?

2. How are the gauge transformations of the p-form potentials occurring in the action related to the gauge transformations that follow from the tensor hierarchy?

It turns out that the construction of a gauge-invariant action requires that the gauge transformations of the duality hierarchy are given in a particular basis that can be obtained from the off-shell basis by a change of basis that will be described in this paper. To be specific, the two sets of transformation rules (those corresponding to the off-shell tensor hierarchy and those that leave the action invariant) differ by terms that are proportional to the duality relations. It is important to note that once a gauge-invariant action is specified the gauge transformations that leave this action invariant are not anymore related to the off-shell basis by a legitimate basis transformation *from the action point of view*. This is due to the fact that from the action point of view one is not allowed to remove terms that are *not* proportional to one of the equations of motion that follow from this action[9]. However, although some projected duality relations follow by extremizing the action, this is not the case for all duality relations of the duality hierarchy. Therefore, from the action point of view, the gauge transformations that leave the action invariant are not equivalent to the gauge transformations of the duality hierarchy in the off-shell basis. Indeed, the gauge transformations in the off-shell basis do not leave the action invariant.

This work is organized as follows. In section 2 we briefly review a few basic facts about the embedding tensor formalism that will be needed later on. In section 3 we construct

[8]Strictly speaking, in $D = 4$ not all 2-forms enter the action, see sec. 5.

[9]One may only change the gauge transformations by adding so-called "equations of motion symmetries".

the complete $D = 4$ tensor hierarchy for non-anomalous supergravities. We introduce the setup of our procedure in subsection 3.1, present the standard construction of the vector 2-form field strengths in subsection 3.2 and the construction of the 3-form and 4-form field strengths in subsections 3.3 and 3.4. In section 4 we add to the tensor hierarchy duality relations and construct the duality hierarchy. We show how the set of dynamical equations that follows from this duality hierarchy not only contains the equations of motion of the different potentials but also the (projected) scalar equations of motion. Finally, in section 5 we construct a gauge-invariant action for all the fields of the $D = 4$ tensor hierarchy and show how this result is related to the duality hierarchy. The general analysis of theories in $D = 3$ is presented in Section 6, which completes the investigations [12, 21] discussed in the literature so far. The $D = 4$ and $D = 3$ cases may be studied independently, and the latter serves as a toy model which elucidates some (but not all) of the subtleties of the four-dimensional analysis. Our conclusions are contained in section 7 and the three appendices contain a summary of the 4-dimensional results.

2 The embedding tensor formalism

We start by giving a brief review of the the embedding tensor formalism [9, 10, 12, 13]. Readers familiar with this technique may skip this part.

The embedding tensor formalism is a convenient tool to study gaugings of supergravity theories in a universal and general way, that does not require a case-by-case analysis. This technique formally maintains covariance with respect to the global invariance group G of the ungauged theory, even though in general G will ultimately be broken by the gauging to the subgroup that is gauged. It turns out that all couplings that deform an ungauged supergravity into a gauged one, as Yukawa couplings, scalar potentials, etc., can be given in terms of a special tensor, called the embedding tensor. Thus, gauged supergravities are classified by the embedding tensor, subject to a number of algebraic or group-theoretical constraints, some of which we will discuss below.

To be more precise, the embedding tensor $\Theta_M{}^\alpha$ pairs the generators t_α of the group G with the vector fields $A_\mu{}^M$ used for the gauging. The indices $\alpha, \beta, \ldots$ label the adjoint representation of G and the indices $M, N, \ldots$ label the representation $\mathcal{R}_V$ of G, in which the vector fields that will be used for the gauging transform. Thus, the choice of $\Theta_M{}^\alpha$, which generally will not have maximal rank, determines which combinations of vectors

$$A_\mu{}^M \Theta_M{}^\alpha \,, \tag{2.1}$$

can be seen as the gauge fields associated to (a subset of) the generators t_α of the group G, and, simultaneously, or alternatively, which combinations of group generators

$$X_M \;=\; \Theta_M{}^\alpha \, t_\alpha \tag{2.2}$$

can be seen as the generators of the gauge group. Consequently, the embedding tensor can be used to define covariant derivatives

$$D_\mu \;=\; \partial_\mu - A_\mu{}^M\,\Theta_M{}^\alpha\,t_\alpha \;=\; \partial_\mu - A_\mu{}^M\,X_M\;, \tag{2.3}$$

which shows that the embedding tensor can also be interpreted as a set of gauge coupling constants[10] of the theory. Even though $\Theta_M{}^\alpha$ has been introduced as a tensor of the duality group G, it is not taken to transform according to its index structure, i.e. in the tensor product $\mathcal{R}_V \otimes \mathrm{Adj}^*$, but must be inert under G for consistency. This requirement leads to the so-called quadratic constraints, which state that the embedding tensor is invariant under the gauge group. If we denote the generators of G (with structure constants $f_{\alpha\beta}{}^\gamma$) in the representation $\mathcal{R}_V$ by $(t_\alpha)_M{}^N$, this amounts to the condition

$$\delta_P\Theta_M{}^\alpha \;=\; \Theta_P{}^\beta t_{\beta M}{}^N\Theta_N{}^\alpha + \Theta_P{}^\beta f_{\beta\gamma}{}^\alpha\Theta_M{}^\gamma \;=\; 0\;. \tag{2.4}$$

Therefore, seemingly G-covariant expressions actually break the duality group to the subgroup which is gauged.

In the next sections we will frequently make use of the objects

$$X_{MN}{}^P \;\equiv\; \Theta_M{}^\alpha t_{\alpha N}{}^P \;=\; X_{[MN]}{}^P + Z^P{}_{MN}\;, \tag{2.5}$$

with $Z^P{}_{MN}$ denoting the symmetric part of $X_{MN}{}^P$, in terms of which the quadratic constraints read

$$\Theta_P{}^\alpha Z^P{}_{MN} \;=\; 0\;. \tag{2.6}$$

Thus, the antisymmetry of the 'structure constants' of the gauge group holds only upon contraction with the embedding tensor. Similar relations, that are familiar from ordinary gauge theories but hold in the present context only upon contraction with Θ, will be encountered at several places in the next sections. Note that standard closure of the gauge group follows from (2.4) in that

$$[X_M, X_N] \;=\; -X_{MN}{}^P X_P \;=\; -X_{[MN]}{}^P X_P \tag{2.7}$$

by virtue of (2.6).

So far, the discussion has been quite general. In the remaining part of this paper we are going to discuss the $D=4$ and $D=3$ tensor hierarchies in full detail. For these cases the embedding tensor can be specialized according to the known representation of the vector fields. Also, our notation for the indices will slightly differ from the general case to accord with the literature. In the $D=4$ case we will work with electric vectors $A^\Lambda{}_\mu$, with $\Lambda = 1,\ldots,\bar{n}$, and magnetic vectors $A_{\Lambda\mu}$. Together, these vectors will be combined into a symplectic contravariant vector $A^M{}_\mu$ with M labeling the fundamental representation

[10] G may have a product structure and each factor may have a different coupling constant, which is contained in the embedding tensor. We, therefore, do not write any other explicit coupling constants apart from $\Theta_M{}^\alpha$.

of $Sp(2\bar{n},\mathbb{R})$. Also the adjoint index of the global symmetry group will be denoted by A instead of α. This leads to the following notation for the $D=4$ embedding tensor:

$$D=4\,:\qquad \Theta_M{}^{\alpha} \quad\rightarrow\quad \Theta_M{}^{A}\,. \tag{2.8}$$

On the other hand, in the $D=3$ case the representation $\mathcal{R}_V$ of the vector fields is equal to the adjoint representation of the global symmetry group G. Therefore, the $D=3$ embedding tensor carries two adjoint indices and this leads to the following notation:[11]

$$D=3\,:\qquad \Theta_M{}^{\alpha} \quad\rightarrow\quad \Theta_{MN}\,. \tag{2.9}$$

We now discus the $D=4$ tensor hierarchy in sections 3, 4 and 5 and, next, the $D=3$ tensor hierarchy in section 6.

3 The $D=4$ tensor hierarchy

In this section we will construct the complete $D=4$ tensor hierarchy extending the results of Ref. [11] following the outline of Ref. [12]. We will follow closely the notation and conventions used in these references.

3.1 The setup

The (bosonic) electric fields of any 4-dimensional field theory are the metric, scalars and (electric) vectors. Only the latter are needed in the construction of the tensor hierarchy. We denote them by $A^{\Lambda}{}_{\mu}$ where $\Lambda,\Sigma,\ldots=1,\cdots,\bar{n}$. In 4-dimensional ungauged theories one can always introduce their magnetic duals which we denote by a similar index in lower position $A_{\Lambda\,\mu}$.

The symmetries of the equations of motion of 4-dimensional theories that act on the electric and magnetic vectors are always subgroups of $Sp(2\bar{n},\mathbb{R})$ [22] . Thus, it is convenient to define the symplectic contravariant vector

$$A^{M}{}_{\mu} \;=\; \begin{pmatrix} A^{\Lambda}{}_{\mu} \\ A_{\Lambda\,\mu} \end{pmatrix}\,. \tag{3.1}$$

It is also convenient to define the symplectic metric Ω_{MN} by

$$\Omega_{MN} = \begin{pmatrix} 0 & \mathbb{I}_{\bar{n}\times\bar{n}} \\ -\mathbb{I}_{\bar{n}\times\bar{n}} & 0 \end{pmatrix}\,, \tag{3.2}$$

and its inverse Ω^{MN} by

$$\Omega^{MN}\Omega_{NP} = -\delta^{M}{}_{P}\,. \tag{3.3}$$

[11] We assume that G carries an invariant Cartan-Killing form, such that the indices can be freely raised and lowered. This assumption is satisfied for the duality groups of three-dimensional supergravity.

They will be used, respectively, to lower and raise symplectic indices, *e.g.*[12]

$$A_M \equiv \Omega_{MN} A^N = (A_\Lambda \, , -A^\Lambda) \, , \qquad A^M = A_N \Omega^{NM} \, . \tag{3.4}$$

The contraction of contravariant and covariant symplectic indices is, evidently, equivalent to the symplectic product: $A^M B_M = A^M \Omega_{MN} B^N = -A_M B^M$.

We denote the global symmetry group of the theory by G and its generators by T_A, $A, B, C, \ldots = 1, \cdots, \mathrm{rank}\, G$. These satisfy the commutation relations

$$[T_A, T_B] = -f_{AB}{}^C T_C \, . \tag{3.5}$$

G can actually be larger than $Sp(2\bar{n}, \mathbb{R})$ and/or not be contained in it[13], but, according to the above discussion, it will always act on A^M as a subgroup of it, i.e. infinitesimally

$$\delta_\alpha A^M = \alpha^A T_{A\,N}{}^M A^N \, , \qquad \delta_\alpha A_M = -\alpha^A T_{A\,M}{}^N A_N \, , \tag{3.6}$$

where

$$T_{A\,[MN]} \equiv T_{A\,[M}{}^P \Omega_{N]P} = 0 \, . \tag{3.7}$$

This is an important general property of the 4-dimensional case. It is implicit in this formalism that some of the matrices $T_{A\,M}{}^N$ may act trivially on the vectors, i.e. they may vanish. Otherwise we could only deal with $G \subset Sp(2\bar{n}, \mathbb{R})$.

Apart from its global symmetries, an ungauged theory containing $\bar{n}$ Abelian vector fields will always be invariant under the $2\bar{n}$ Abelian gauge transformations

$$\delta_\Lambda A^M{}_\mu = -\partial_\mu \Lambda^M \, , \tag{3.8}$$

where $\Lambda^M(x)$ is a symplectic vector of local gauge parameters.

To gauge a subgroup of the global symmetry group G we must promote the global parameters α^A to arbitrary spacetime functions $\alpha^A(x)$ and make the theory invariant under these new transformations. This is achieved by identifying these arbitrary functions with a subset of the (Abelian) gauge parameters Λ^M of the vector fields and subsequently using the corresponding vectors as gauge fields. This identification is made through the embedding tensor $\Theta_M{}^A \equiv (\Theta_\Lambda{}^A \, , \Theta^{\Lambda\, A})$:

$$\alpha^A(x) \equiv \Lambda^M(x) \Theta_M{}^A \, . \tag{3.9}$$

The embedding tensor allows us to keep treating all vector fields, used for gaugings or not, on the same footing. It hence allows us to formally preserve the symplectic invariance even after gauging.

As discussed in section 2 the embedding tensor must satisfy a number of constraints which guarantee the consistency of the theory. Some of these constraints have already been

[12] In what follows we will mostly use differential-form language and suppress the spacetime indices.
[13] The symmetries of a set of scalars decoupled from the vectors are clearly unconstrained.

discussed in section 2. In total we have three constraints which we list below. First of all, in the $D = 4$ case we must impose the following quadratic constraint

$$Q^{AB} \equiv \tfrac{1}{4}\Theta^{M\,[A}\Theta_M{}^{B]} = 0\,, \tag{3.10}$$

which guarantees that the electric and magnetic gaugings are mutually local [11]. Observe that the antisymmetry of Ω^{MN} and the above constraint imply $\Theta^{M\,A}\Theta_M{}^B = 0$. This constraint is a particular feature of the 4-dimensional case.

As mentioned in section 2 there is a second quadratic constraint which encodes the fact that the embedding tensor has to be itself invariant under gauge transformations. If the gauge transformations of objects with contravariant and covariant symplectic indices are

$$\delta_\Lambda \xi^M = \Lambda^N \Theta_N{}^A T_{A\,P}{}^M \xi^P\,, \qquad \delta_\Lambda \eta_M = -\Lambda^N \Theta_N{}^A T_{A\,M}{}^P \xi_P\,, \tag{3.11}$$

and the gauge transformations of objects with contravariant and covariant adjoint indices are written in the form

$$\delta_\Lambda \pi^A = \Lambda^M \Theta_M{}^B f_{BC}{}^A \pi^C\,. \qquad \delta_\Lambda \zeta_A = -\Lambda^M \Theta_M{}^B f_{BA}{}^C \zeta_C\,, \tag{3.12}$$

then

$$\delta_\Lambda \Theta_M{}^A = -\Lambda^N Q_{NM}{}^A\,, \qquad Q_{NM}{}^A \equiv \Theta_N{}^A T_{A\,M}{}^P \Theta_P{}^A - \Theta_N{}^A \Theta_M{}^B f_{AB}{}^A\,, \tag{3.13}$$

and the second quadratic constraint reads

$$Q_{NM}{}^A = 0\,. \tag{3.14}$$

The third constraint applies to all 4-dimensional supergravity theories that are free of gauge anomalies [30] and can be expressed using the X generators introduced in section 2, see Eq. (2.5):

$$X_M \equiv \Theta_M{}^A T_A\,, \qquad X_{MN}{}^P \equiv \Theta_M{}^A T_{A\,N}{}^P\,. \tag{3.15}$$

This constraint (the so-called *representation constraint*) is linear in $\Theta_M{}^A$ and reads as follows [11]:

$$L_{MNP} \equiv X_{(MNP)} = X_{(MN}{}^Q \Omega_{P)Q} = 0\,. \tag{3.16}$$

The three constraints that the embedding tensor has to satisfy are not independent, but are related by

$$Q_{(MN)}{}^A - 3L_{MNP}Z^{PA} - 2Q^{AB}T_{BMN} = 0\,. \tag{3.17}$$

This relation can be used to show that the constraint $Q^{AB} = 0$ follows from the constraint $Q_{(MN)}{}^A = 0$ when the linear constraint $L_{MNP} = 0$ is explicitly solved, whenever the action of the global symmetry group on the vectors is faithful. We will neither solve explicitly the linear constraint by choosing to work only with representations allowed by it, nor we will assume the action of the global group on the vectors to be faithful, since there are many interesting situations in which this is not the case and we aim to be as general as possible. In (half-) maximal supergravities, though, the global symmetry group always acts faithfully on the vector fields.

These two choices, which differ from those made in the explicit examples found in the literature (see e.g. Ref. [20]) will have important consequences in the field content of the tensor hierarchy and are the reason why our results also differ from those obtained in them. Before we go on we wish to collect a few properties of the X generators $X_{MN}{}^{P}$ in a separate subsection.

3.1.1 The X generators and their properties

We first discuss the symmetry properties of the X generators. By their definition, and due to the symplectic property of the $T_{A\,N}{}^{P}$ generators, see Eq. (3.7), we have

$$X_{MNP} = X_{MPN}\,. \tag{3.18}$$

From the definition of the quadratic constraint Eq. (3.14) it follows that

$$X_{(MN)}{}^{P}\Theta_{P}{}^{C} = Q_{(MN)}{}^{C}\,, \tag{3.19}$$

and so it will vanish[14], although, in general, we will have

$$X_{(MN)}{}^{P} \neq 0\,. \tag{3.20}$$

This implies, in particular

$$X_{(MN)P} = -\tfrac{1}{2}X_{PMN} + \tfrac{3}{2}L_{MNP} \;\;\Rightarrow X_{(MN)}{}^{P} = Z^{PA}T_{AMN} + \tfrac{3}{2}L_{MN}{}^{P}\,, \tag{3.21}$$

where we have defined

$$Z^{PA} \equiv -\tfrac{1}{2}\Omega^{NP}\Theta_{N}{}^{A} = \begin{cases} +\tfrac{1}{2}\Theta^{\Lambda A}\,, \\ \\ -\tfrac{1}{2}\Theta_{\Lambda}{}^{A}\,, \end{cases}\,. \tag{3.22}$$

Z^{PA} will be used to project in directions orthogonal to the embedding tensor since, due to the first quadratic constraint Eq. (3.10), we find that

$$Z^{MA}\Theta_{M}{}^{B} = -\tfrac{1}{2}Q^{AB}\,. \tag{3.23}$$

We next discuss some properties of the products of two X generators. From the commutator of the T_A generators and the definition of the generators X_M and the matrices $X_{MN}{}^{P}$ we find the commutator of the X_M generators to be

[14] Here we will keep the terms proportional to constraints for later use, including the linear constraints in (3.21).

$$[X_M, X_N] = Q_{MN}{}^C T_C - X_{MN}{}^P X_P\,. \tag{3.24}$$

This reduces to (cf. to Eq. (2.7))

$$[X_M, X_N] = -X_{[MN]}{}^P X_P\,, \tag{3.25}$$

upon use of the above constraint and $Q_{MN}{}^C = 0$. From the commutator Eq. (3.24) one can derive the analogue of the Jacobi identities

$$\begin{aligned} X_{[MN]}{}^Q X_{[PQ]}{}^R + X_{[NP]}{}^Q X_{[MQ]}{}^R + X_{[PM]}{}^Q X_{[NQ]}{}^R \;&= \\ = -\tfrac{1}{3}\{X_{[MN]}{}^Q X_{(PQ)}{}^R + X_{[NP]}{}^Q X_{(MQ)}{}^R &+ X_{[PM]}{}^Q X_{(NQ)}{}^R\} \\ -Q_{[MN|}{}^C T_{C\,|P]}{}^R\,. \end{aligned} \tag{3.26}$$

We finally present two more useful identities that can be derived from the commutators:

$$X_{(MN)}{}^Q X_{PQ}{}^R - X_{PN}{}^Q X_{(MQ)}{}^R - X_{PM}{}^Q X_{(NQ)}{}^R \;=\; -Q_{P(M|}{}^C T_{C\,|N)}{}^R\,, \tag{3.27}$$

$$X_{[MN]}{}^Q X_{PQ}{}^R - X_{PN}{}^Q X_{[MQ]}{}^R + X_{PM}{}^Q X_{[NQ]}{}^R \;=\; Q_{P[M|}{}^C T_{C\,|N]}{}^R\,. \tag{3.28}$$

3.2 The vector field strengths F^M

We now return to the construction of the field strengths of the different p-form potentials. In what follows we will set all the constraints explicitly to zero in order to simplify the expressions. In this section we consider the vector field strengths.

To construct the vector field strength it is convenient to start from the covariant derivative. This derivative acting on objects transforming according to $\delta\phi = \Lambda^M \delta_M \phi$ is defined by

$$\mathfrak{D}\phi = d\phi + A^M \delta_M \phi\,. \tag{3.29}$$

For instance, the covariant derivative of a contravariant symplectic vector

$$\mathfrak{D}\xi^M = d\xi^M + X_{NP}{}^M A^N \xi^P\,, \tag{3.30}$$

transforms covariantly provided that

$$\delta A^M = -\mathfrak{D}\Lambda^M + \Delta A^M\,, \qquad \Theta_M{}^A \Delta A^M = 0\,. \tag{3.31}$$

The Ricci identity of the covariant derivative on Λ^N can be written in the form

$$\mathfrak{D}\mathfrak{D}\Lambda^M \;=\; X_{NP}{}^M F^N \Lambda^P\,, \tag{3.32}$$

for some 2-form F^M. Since this expression is gauge-covariant, F^M, contracted with the embedding tensor, will automatically be gauge-covariant, whatever it is and it is natural to identify it with the gauge-covariant vector field strength. The above expression defines it up to a piece ΔF^M which is projected out by the embedding tensor, just like ΔA^M in δA^M. An explicit calculation gives

$$F^M = dA^M + \tfrac{1}{2} X_{[NP]}{}^M A^N \wedge A^P + \Delta F^M \,, \qquad \Theta_M{}^A \Delta F^M = 0 \,. \tag{3.33}$$

The possible presence of ΔF^M is a novel feature of the embedding tensor formalism. Its gauge transformation rule can be found by using the gauge covariance of F^M. Under Eq. (3.31), using $\Theta_M{}^A \Delta F^M = 0$, we find that

$$\delta F^M = \Lambda^P X_{PN}{}^M F^N + \mathfrak{D}\Delta A^M - 2X_{(NP)}{}^M (\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P) + \delta\Delta F^M \,, \tag{3.34}$$

so that F^M transforms covariantly provided that we take

$$\delta\Delta F^M = -\mathfrak{D}\Delta A^M + 2Z^{MA} T_{ANP} (\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P) \,, \tag{3.35}$$

where we have used Eq. (3.21). Since both ΔA^M and ΔF^M are annihilated by the embedding tensor, we conclude that in the generic situation we are considering here[15] $\Delta F^M = Z^{MA} B_A$ where B_A is some 2-form field in the adjoint of G and $\Delta A^M = -Z^{MA}\Lambda_A$ where Λ_A is a 1-form gauge parameter in the same representation. Then

$$F^M = dA^M + \tfrac{1}{2} X_{[NP]}{}^M A^N \wedge A^P + Z^{MA} B_A \,, \tag{3.36}$$

$$\delta A^M = -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A \,, \tag{3.37}$$

$$\delta B_A = \mathfrak{D}\Lambda_A + 2T_{ANP}[\Lambda^N F^P + \tfrac{1}{2} A^N \wedge \delta A^P] + \Delta B_A \,, \tag{3.38}$$

where ΔB_A is a possible additional term which is projected out by Z^{MA}, i.e. $Z^{MA}\Delta B_A = 0$, and can be determined by studying the construction of a gauge-covariant field strength H_A for the 2-form B_A.

3.3 The 3-form field strengths H_A

We continue to determine the form of H_A using the Bianchi identity for F^M just as we used the Ricci identity to find an expression for F^M. An explicit computation using Eq. (3.36) gives

[15]The only information we have about the embedding tensor in a generic situation is provided by the three constraints $Q_{NP}{}^E = 0$, $Q^{AB} = 0$, $L_{MNP} = 0$. There is only one which we can write in the form $\Theta_M{}^A \times \text{Something}^M = 0$, which is the constraint $Q^{AB} = 0$ and that uniquely identifies $\text{Something}^M = Z^{MB}$ up to a proportionality constant.

$$\mathfrak{D}F^M = Z^{MA}\{\mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P]\}\,. \tag{3.39}$$

It is clear that the expression in brackets must be covariant and it defines a 3-form field strength H_A up to terms ΔH_A that are projected out by Z^{MA}, i.e.

$$\mathfrak{D}F^M = Z^{MA}H_A\,, \tag{3.40}$$

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + \Delta H_A \tag{3.41}$$

with $Z^{MA}\Delta H_A = 0$. Both ΔB_A and ΔH_A are determined by requiring gauge covariance of H_A. An explicit calculation gives

$$\begin{aligned}\delta H_A \;=\; & -\Lambda^M \Theta_M{}^B f_{BA}{}^C H_C \\ & -Y_{AM}{}^C[\Lambda^M H_C - \delta A^M \wedge B_C - F^M \wedge \Lambda_C - \tfrac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta A^P] \\ & +\mathfrak{D}\Delta B_A + \delta\Delta H_A\,.\end{aligned} \tag{3.42}$$

We have defined the Y-tensor as

$$Y_{AM}{}^C \equiv \Theta_M{}^B f_{AB}{}^C - T_{A\,M}{}^N\Theta_N{}^C\,. \tag{3.43}$$

and it satisfies the condition

$$Z^{MA}Y_{AN}{}^C = \tfrac{1}{2}\Omega^{PM}Q_{PN}{}^C = 0\,. \tag{3.44}$$

The 3-form field strengths H_A transform covariantly provided that the last two lines in Eq. (3.42) vanish. A natural solution is to take

$$\Delta B_A \equiv -Y_{AM}{}^C\Lambda_C{}^M\,, \qquad \Delta H_A \equiv Y_{AM}{}^C C_C{}^M\,, \tag{3.45}$$

where $\Lambda_C{}^M$ is a 2-form gauge parameter and $C_C{}^M$ is a 3-form field about which we will not make any assumptions for the moment. In particular, we will not assume it to satisfy any constraints in spite of the fact that we expect it to be “dual” to the embedding tensor, which is a constrained object. We are going to see that, actually, we are not going to need any such explicit constraints to construct a fully consistent tensor hierarchy. On the other hand, we are going to find Stückelberg shift symmetries acting on $C_C{}^M$ whose role is, precisely, to compensate for the lack of explicit constraints and, potentially, allow us to remove the same components of $C_C{}^M$ which would be eliminated by imposing those constraints. We anticipate that those Stückelberg shift symmetries require the existence of 4-forms in order to construct gauge-covariant 4-form field strengths $G_C{}^M$. It should come as no surprise after this discussion, that the 4-forms are in one-to-one correspondence with the constraints of the embedding tensor. Working with unconstrained fields is simpler and it is one of the advantages of our approach.

We then, find

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + Y_{AM}{}^C C_C{}^M\,, \tag{3.46}$$

$$\delta B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2}A^N \wedge \delta A^P] - Y_{AM}{}^C \Lambda_C{}^M\,, \tag{3.47}$$

$$\begin{aligned} \delta C_C{}^M &= \mathfrak{D}\Lambda_C{}^M + \Lambda^M H_C - \delta A^M \wedge B_C - F^M \wedge \Lambda_C \\ &\quad -\tfrac{1}{3}T_{C\,NP}A^M \wedge A^N \wedge \delta A^P + \Delta C_C{}^M\,, \end{aligned} \tag{3.48}$$

where we have introduced a possible additional term $\Delta C_C{}^M$ analogous to ΔA^M and ΔB_A which now is projected out by $Y_{AM}{}^C$

$$Y_{AM}{}^C \Delta C_C{}^M = 0\,, \tag{3.49}$$

and which will be determined by requiring gauge covariance of the 4-form field strength $G_C{}^M$.

3.4 The 4-form field strengths $G_C{}^M$

To determine the 4-form field strengths $G_C{}^M$ we use the Bianchi identity of H_A. We can start by taking the covariant derivative of both sides of the Bianchi identity of F^M Eq. (3.40) and then using the Ricci identity. We thus get

$$Z^{MA}\mathfrak{D}H_A = X_{NP}{}^M F^N \wedge F^P = Z^{MA}T_{A\,NP}F^N \wedge F^P\,. \tag{3.50}$$

This implies that $\mathfrak{D}H_A = T_{A\,MN}F^M \wedge F^N + \Delta\mathfrak{D}H_A$ where $Z^{MA}\Delta\mathfrak{D}H_A = 0$, suggesting that $\Delta\mathfrak{D}H_A \sim Y_{AM}{}^C G_C{}^M$. A direct calculation yields the result

$$\begin{aligned} G_C{}^M &= \mathfrak{D}C_C{}^M + F^M \wedge B_C - \tfrac{1}{2}Z^{MA}B_A \wedge B_C \\ &\quad +\tfrac{1}{3}T_{C\,SQ}A^M \wedge A^S \wedge (F^Q - Z^{QA}B_A) \\ &\quad -\tfrac{1}{12}T_{C\,SQ}X_{NT}{}^Q A^M \wedge A^S \wedge A^N \wedge A^T \\ &\quad +\Delta G_C{}^M\,, \end{aligned} \tag{3.51}$$

where

$$Y_{AM}{}^C \Delta G_C{}^M = 0\,. \tag{3.52}$$

The Bianchi identity then takes the form

$$\mathfrak{D}H_A = Y_{AM}{}^{C}G_C{}^{M} + T_{A\,MN}F^M \wedge F^N\,. \tag{3.53}$$

$\Delta C_C{}^M$ and $\Delta G_C{}^M$ must now be determined by using the gauge covariance of the full field strength $G_C{}^M$. It is tempting to repeat what we did in the previous cases. However, the calculation is, now, much more complicated and it would be convenient to have some information about the new tensor(s) orthogonal to $Y_{AM}{}^C$ that we may expect.

Given that the projectors arise naturally in the computation of the Bianchi identities, we are going to "compute" the Bianchi identity of $G_C{}^M$ obviating the fact that it is already a 4-form, and in $D = 4$ its Bianchi identity is trivial. We have not used the dimensionality of the problem so far (except in the existence of magnetic vector fields that gives rise to the symplectic structure and in the assignment of adjoint indices to the 2-forms) and, in any case, our only goal in performing this computation is to find the relevant invariant tensor(s).

Thus, we apply $\mathfrak{D}$ to both sides of Eq. (3.53) using the Bianchi identity of F^M Eq. (3.40) and the Ricci identity. This leads to the following identity

$$Y_{AM}{}^{C}\{\mathfrak{D}G_C{}^M - F^M \wedge H_C\} = 0\,, \tag{3.54}$$

from which it follows that

$$\mathfrak{D}G_C{}^M = F^M \wedge H_C + \Delta\mathfrak{D}G_C{}^M\,, \qquad Y_{AM}{}^{C}\Delta\mathfrak{D}G_C{}^M = 0\,. \tag{3.55}$$

Acting again with $\mathfrak{D}$ on both sides of this last equation and using the Ricci and Bianchi identities, we get in an straightforward manner

$$\begin{aligned} \mathfrak{D}\Delta\mathfrak{D}G_C{}^M &= W_C{}^{MAB}H_A \wedge H_B \\ &\quad + W_{CNPQ}{}^{M}F^N \wedge F^P \wedge F^Q \\ &\quad + W_{CNP}{}^{EM}F^N \wedge G_E{}^P\,, \end{aligned} \tag{3.56}$$

where

$$W_C{}^{MAB} \equiv -Z^{M[A}\delta_C{}^{B]}\,, \tag{3.57}$$

$$W_{CNPQ}{}^{M} \equiv T_{C\,(NP}\delta_{Q)}{}^{M}\,, \tag{3.58}$$

$$W_{CNP}{}^{EM} \equiv \Theta_N{}^D f_{CD}{}^E \delta_P{}^M + X_{NP}{}^M \delta_C{}^E - Y_{CP}{}^E \delta_N{}^M\,. \tag{3.59}$$

We thus found the desired new tensors. The Y-tensor annihilates the three new W tensors in virtue of the 3 constraints satisfied by the embedding tensor

$$Y_{AM}{}^{C}W_C{}^{MAB} = Y_{AM}{}^{C}W_{CNPQ}{}^{M} = Y_{AM}{}^{C}W_{CNP}{}^{EM} = 0\,, \tag{3.60}$$

as expected. Note that the first and third W-tensors are linear in Θ but that the second W-tensor is independent of Θ. Other important sets of identities satisfied by these W-tensors can be found in Appendix A.

Coming back to our original problem of determining the form of $\Delta G_C{}^M$ and $\Delta C_C{}^M$, we conclude from the previous analysis that

$$\Delta C_C{}^M = -W_C{}^{MAB}\Lambda_{AB} - W_{CNPQ}{}^M\Lambda^{NPQ} - W_{CNP}{}^{EM}\Lambda_E{}^{NP}\,, \tag{3.61}$$

$$\Delta G_C{}^M = W_C{}^{MAB}D_{AB} + W_{CNPQ}{}^M D^{NPQ} + W_{CNP}{}^{EM}D_E{}^{NP}\,, \tag{3.62}$$

where $\Lambda_{AB}, \Lambda^{NPQ}, \Lambda_E{}^{NP}$ are 3-form gauge parameters and $D_{AB}, D^{NPQ}, D_E{}^{NP}$ are possible 4-forms whose presence will be justified in $G_C{}^M$ if their gauge transformations are non-trivial in order to make the 4-form field strengths gauge covariant. Taking into account the symmetries of the W-tensors, it is easy to see that $D_{AB} = D_{[AB]}$, $D^{NPQ} = D^{(NPQ)}$ and analogously for the gauge parameters $\Lambda_{AB}, \Lambda^{NPQ}$. $D_E{}^{NP}$ and $\Lambda_E{}^{NP}$ have no symmetries.

We observe that the three 4-form D-potentials seem to be associated to the three constraints Q^{AB}, L_{NPQ}, $Q_{NP}{}^E$ given in Eqs. (3.10), (3.14) and (3.16) in the sense that they carry the same representations. Only the last one was expected according to the general formalism developed in Ref. [12] and the specific study of the top forms performed in Ref. [13,18]. We find that in 4 dimensions there are more top-form potentials due to the additional structures (e.g. the symplectic one) and properties of 4-dimensional theories.

Knowing the different W tensors it is now a relatively straightforward task to obtain by a direct calculation the expression for $\delta G_C{}^M$, collect the terms proportional to the three W-structures and determine the gauge transformations of the three different 4-form D-potentials by requiring gauge-covariance of $G_C{}^M$. An explicit calculation gives

$$\begin{aligned}\delta D_{AB} &= \mathfrak{D}\Lambda_{AB} + \alpha B_{[A}\wedge Y_{B]P}{}^E\Lambda_E{}^P + \mathfrak{D}\Lambda_{[A}\wedge B_{B]} - 2\Lambda_{[A}\wedge H_{B]}\\ &\quad +2T_{[A|NP}[\Lambda^N F^P - \tfrac{1}{2}A^N\wedge\delta A^P]\wedge B_{|B]}\,,\end{aligned} \tag{3.63}$$

$$\begin{aligned}\delta D_E{}^{NP} &= \mathfrak{D}\Lambda_E{}^{NP} - [F^N - \tfrac{1}{2}(1-\alpha)Z^{NA}B_A]\wedge\Lambda_E{}^P + C_E{}^P\wedge\delta A^N\\ &\quad +\tfrac{1}{12}T_{EQR}A^N\wedge A^P\wedge A^Q\wedge\delta A^R + \Lambda^N G_E{}^P\,,\end{aligned} \tag{3.64}$$

$$\begin{aligned}\delta D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 2A^{(N}\wedge(F^P - Z^{PA}B_A)\wedge\delta A^{Q)}\\ &\quad +\tfrac{1}{4}X_{RS}{}^{(N}A^{P|}\wedge A^R\wedge A^S\wedge\delta A^{|Q)} - 3\Lambda^{(N}F^P\wedge F^{Q)}\,,\end{aligned} \tag{3.65}$$

where α is an arbitrary real constant. We hence find that there is a 1-parameter family of solutions to the problem of finding a gauge-covariant field strength for the 3-form. The origin of this freedom resides in the presence of a Stückelberg-type symmetry which we discuss in the next subsection.

3.4.1 Stückelberg symmetries

Differentiating (3.17) with respect to $\Theta_Q{}^C$ using Eqs. (A.7)-(A.9) gives the following identity among the W tensors:

$$W_{C(MN)}{}^{AQ} - 3W_{CMNP}{}^{Q}Z^{PA} - 2W_C{}^{QAB}T_{B\,MN} = \tfrac{3}{2}L_{MN}{}^{Q}\delta_C{}^{A}\,. \tag{3.66}$$

The relation (3.66) gives rise to symmetries under Stückelberg shifts of the 4-forms in the 4-form field strength $G_C{}^M$

$$\begin{aligned}
\delta D_E{}^{NP} &= \Xi_E{}^{(NP)}\,,\\
\delta D_{AB} &= -2\Xi_{[A}{}^{MN}T_{B]MN}\,,\\
\delta D^{NPQ} &= -3Z^{(N|A}\Xi_A{}^{|PQ)}\,.
\end{aligned} \tag{3.67}$$

This shift symmetry, which allows us to remove the part symmetric in NP of $D_E{}^{NP}$, also leaves the 4-form field strengths $G_C{}^M$ invariant.

If we multiply (3.17) by Z^{NE} we find another relation between constraints

$$Q^{AB}Y_{BP}{}^{E} - \tfrac{1}{2}Z^{NA}Q_{NP}{}^{E} = 0\,. \tag{3.68}$$

Differentiating it again with respect to the embedding tensor we find the following relation between W-tensors[16]:

$$W_C{}^{MAB}Y_{BP}{}^{E} - \tfrac{1}{2}Z^{NA}W_{CNP}{}^{EM} = \tfrac{1}{4}Q^{M}{}_{P}{}^{E}\delta_C{}^{A} - Q^{AB}[\delta_P{}^{M}f_{BC}{}^{E} - T_{B\,P}{}^{M}\delta_C{}^{E}]\,, \tag{3.69}$$

which implies that the Stückelberg shift

$$\begin{aligned}
\delta D_E{}^{NP} &= \tfrac{1}{2}Z^{NB}\Xi_{BE}{}^{P}\,,\\
\delta D_{AB} &= Y_{[A|P}{}^{E}\Xi_{B]E}{}^{P}\,,
\end{aligned} \tag{3.70}$$

leaves invariant the 4-form field strength $G_C{}^M$ up to terms proportional to the quadratic constraints, which are taken to vanish identically in the tensor hierarchy. This shift symmetry is associated to the arbitrary parameter α in the gauge transformations of D_{AB} and $D_E{}^{NP}$. Observe that, even though it is based on the identity Eq. (3.69) which we can get from Eq. (3.66), this symmetry is genuinely independent from that in Eq. (3.67).

This finishes the construction of the 4-dimensional tensor hierarchy. The field strengths, Bianchi identities and gauge transformations of the hierarchy's p-form fields are collected in Appendix B. By construction the algebra of all bosonic gauge transformations closes off-shell on all p-form potentials. No equations of motion are needed at this stage.

[16]This identity can also be obtained multiplying Eq. (3.66) by Z^{NE}.

4 The $D = 4$ duality hierarchy

In this section we are going to introduce dynamical equations for the tensor hierarchy via the introduction of first-order duality relations, see Eq. (1.2). This promotes the tensor hierarchy to a *duality hierarchy*. We will see that the dynamical equations will not only contain the equations of motions of the p-form potentials but also the (projected) scalar equations of motion. These scalars, together with the metric, will be introduced via the duality relations. In particular, the scalar couplings enter into the duality relations via functions that can be identified with the Noether currents, the (scalar derivative of the) scalar potential and the kinetic matrix describing the coupling of the scalars to the vectors. In this way the duality hierarchy puts the tensor hierarchy on-shell and establishes a link with a Yang-Mills-type gauge field theory containing a metric, scalars and p-form potentials. This field theory can be viewed as the bosonic part of a gauged supergravity theory. We stress that at this point we only compare equations of motion. It is only in the last and third step that we consider an action for the fields of the hierarchy. We will assume that the Yang-Mills-type gauge field theory has an action but we will only consider its equations of motion in order to properly identify in the duality relations the Noether current, scalar potential and the scalar-vector kinetic function.

In the next subsection we will first consider a Yang-Mills-type gauge field theory with purely electric gaugings, i.e. only electric 1-forms are involved in the gauging. In particular we will compare the equations of motion of this field theory with the dynamical equations of the duality hierarchy. This example shows us how to introduce the metric and scalars in the duality hierarchy. In the next subsection we will first consider a formally symplectic-covariant generalization of the equations of motion with purely electric gaugings. This generalization necessarily involves electric and magnetic gaugings. We will see that this generalization does not lead to gauge-invariant answers unless we also include the equations of motion corresponding to the magnetic 2-form potentials. In this way we recover the observation of [11–15] that magnetic gaugings require the introduction of magnetic 2-form potentials in the action of the field theory.

4.1 Purely electric gaugings

Having $N = 1, D = 4$ supergravity in mind, we consider complex scalars Z^i $(i = 1, \cdots, n)$ with Kähler metric $\mathcal{G}_{ij^*}$ admitting holomorphic Killing vectors $K_A = k_A{}^i\partial_i + \text{c.c.}$. The index A of the Killing vectors must be associated to those of the generators of the global symmetry group G. In general, not all the global symmetries will act on the scalars. Therefore, we assume that some of the K_A may be identically zero just as some of the matrices $T_{A\,M}{}^N$ can be zero for other values of A. The action for the electrically gauged theory is

$$S_{\text{elec}}[g, Z^i, A^\Lambda] = \int \left\{\star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^i \wedge \star\mathfrak{D}Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V\right\}\,, \tag{4.1}$$

where $\mathfrak{D}Z^i$ is given by

$$\mathfrak{D}Z^i = dZ^i + A^\Lambda \Theta_\Lambda{}^A k_A{}^i\,, \tag{4.2}$$

and where G_Λ denotes the combination of scalars and electric vector field strengths defined by

$$G_\Lambda{}^+ = f_{\Lambda\Sigma}(Z) F^{\Sigma\,+}\,, \tag{4.3}$$

where $F^{\Sigma\,+} = \frac{1}{2}(F^\Sigma + i \star F^\Sigma)$. It is assumed that the scalar-dependent kinetic matrix $f_{\Lambda\Sigma}(Z)$ is invariant under the global symmetry group, i.e.[17]

$$\pounds_A f_{\Lambda\Sigma} = 2 T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega}\,, \tag{4.4}$$

where $\pounds_A$ stands for the Lie derivative with respect to K_A, since this is a pre-condition to gauge the theory. However, the potential needs only be invariant under the gauge transformations, because the gauging usually adds to the globally-invariant potential of the ungauged theory another piece. Thus, we must have

$$\pounds_A V = Y_{A\Lambda}{}^C \frac{\partial V}{\partial \Theta_\Lambda{}^C}\,, \tag{4.5}$$

where $Y_{A\Lambda}{}^C$ is the electric component of the tensor defined in Eq. (3.43). Indeed, using this property, one can show that under the gauge transformations

$$\begin{aligned} \delta Z^i &= \Lambda^\Lambda \Theta_\Lambda{}^A k_A{}^i\,, \\ \delta A^\Lambda &= -\mathfrak{D}\Lambda^\Lambda\,, \end{aligned} \tag{4.6}$$

the scalar potential V is gauge invariant:

$$\delta V = \Lambda^\Sigma \Theta_\Sigma{}^A \pounds_A V = \Lambda^\Sigma Q_\Sigma{}^{\Lambda C} \frac{\partial V}{\partial \Theta_\Lambda{}^A} = 0\,, \tag{4.7}$$

on account of the quadratic constraint.

The equations of motion (plus the Bianchi identity for F^Λ) corresponding to the action (4.1) are given by

[17]Here we are only considering a restricted type of perturbative symmetries of the theory, excluding Peccei-Quinn-type shifts of the kinetic matrix for simplicity. We will consider these shifts together with the possible non-perturbative symmetries in the general gaugings' section.

$$
\begin{aligned}
\mathcal{E}_{\mu\nu} &\equiv -\star\frac{\delta S}{\delta g^{\mu\nu}} = G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] \\
&\quad -4\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda\,+}{}_\mu{}^\rho F^{\Sigma\,-}{}_{\nu\rho} + \tfrac{1}{2}g_{\mu\nu}V\,, \qquad (4.8)
\end{aligned}
$$

$$
\mathcal{E}_i \equiv \tfrac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_\Sigma{}^+ \wedge F^{\Sigma +} - \star\tfrac{1}{2}\partial_i V\,, \qquad (4.9)
$$

$$
\begin{aligned}
\mathcal{E}_\Lambda &\equiv -\tfrac{1}{4}\star\frac{\delta S}{\delta A^\Lambda} = \mathfrak{D}G_\Lambda - \tfrac{1}{4}\Theta_\Lambda{}^A \star j_A\,, \\
\mathcal{E}^\Lambda &\equiv \mathfrak{D}F^\Lambda\,, \qquad (4.10)
\end{aligned}
$$

where

$$
j_A \equiv 2k^*_{Ai}\mathfrak{D}Z^i + \text{c.c.}\,, \qquad (4.11)
$$

is the covariant Noether current.

According to the second Noether theorem there is an off-shell relation between equations of motion of a theory associated to each gauge invariance. For instance, associated to general covariance we find the well-known identity

$$
\nabla^\mu \mathcal{E}_{\mu\nu} - (\mathfrak{D}_\nu Z^i \mathcal{E}_i^* + \text{c.c.}) + 2F^\Lambda{}_{\nu\rho}(\star\mathcal{E}_\Lambda)^\rho = 0\,, \qquad (4.12)
$$

which implies the on-shell covariant conservation of the energy-momentum tensor. Similarly, the identity associated to the Yang-Mills-type gauge invariance of the theory is given by

$$
\mathfrak{D}\mathcal{E}_\Lambda + \tfrac{1}{2}\Theta_\Lambda{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) = 0\,. \qquad (4.13)
$$

Using the Ricci identity for the covariant derivative and Eqs. (4.4) and (4.5) we find that this equation is indeed satisfied because the Noether current satisfies the identity

$$
\mathfrak{D}\star j_A = -2(k_A{}^i\mathcal{E}_i + \text{c.c.}) + 4T_{A\,\Sigma}{}^\Gamma F^\Sigma \wedge G_\Gamma + \star Y_{A\Lambda}{}^C\frac{\partial V}{\partial \Theta_\Lambda{}^C}\,. \qquad (4.14)
$$

We are now going to establish a relation between the tensor hierarchy and the equations of motion for the vector fields, their Bianchi identities and the following projected scalar equations of motion:

$$\mathfrak{D}G_\Lambda - \tfrac{1}{4}\Theta_\Lambda{}^A \star j_A \;=\; 0\,, \tag{4.15}$$

$$\mathfrak{D}F^\Lambda \;=\; 0\,, \tag{4.16}$$

$$k_A{}^i\Big[\mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_\Sigma{}^+ \wedge F^{\Sigma+} - \star\tfrac{1}{2}\partial_i V\Big] + \text{c.c.} \;=\; 0\,. \tag{4.17}$$

Note that, unlike the tensor hierarchy, these equations contain not only p-form potentials but also the metric and scalars.

In order to derive the above equations of motion from the tensor hierarchy we must complement the tensor hierarchy with a set of duality relations that reproduces the scalar and metric dependence of these equations. Besides the usual $\mathfrak{D}^2 Z$ term in the last equation the scalar dependence of (4.15)-(4.17) resides in the magnetic 2-forms G_Λ, the Noether currents j_A and the derivatives $\partial_i V$ of the scalar potential V. The latter derivative is equivalently represented, via the invariance property (4.5), by the derivative $\dfrac{\partial V}{\partial\Theta_\Lambda{}^A}$ of the scalar potential with respect to the embedding tensor. These are precisely the objects that occur in the following set of duality relations that we introduce:

$$\begin{aligned} G_\Lambda &= F_\Lambda\,, \\ j_A &= -2\star H_A\,, \\ \frac{\partial V}{\partial\Theta_\Lambda{}^A} &= -2\star G_A{}^\Lambda\,, \end{aligned} \tag{4.18}$$

where the magnetic 2-form field strengths F_Λ, the 3-form field strengths H_A and the 4-form field strengths $G_A{}^\Lambda$ are those of the tensor hierarchy. The tensor hierarchy, together with the above duality relations, forms the duality hierarchy. Upon hitting the duality relations (4.18) with a covariant derivative and next applying one of the Bianchi identities of the tensor hierarchy we precisely obtain the equations of motion (4.15)-(4.17). In the case of the scalar equations of motion we first obtain the identity

$$\mathfrak{D}\star j_A - 4T_{A\,\Sigma}{}^\Gamma F^\Sigma \wedge G_\Gamma - \star Y_{A\Lambda}{}^C\frac{\partial V}{\partial\Theta_\Lambda{}^A} = 0\,. \tag{4.19}$$

Next, by comparing this equation with the Noether identity (4.14) we derive the projected scalar equations of motion (4.17), i.e.

$$k_A{}^i\mathcal{E}_i + \text{c.c.} = 0. \tag{4.20}$$

It also works the other way around. By substituting the duality relations into the equations of motion the scalar and metric dependence of these equations can be eliminated and one recovers the hierarchy's Bianchi identities for a purely electric embedding tensor

$\Theta^{\Sigma A} = 0$. To be precise, Eqs. (4.15) and (4.16) are mapped into the 3-form Bianchi identities (3.40). Furthermore, Eq. (4.19), which is equivalent to (4.17) upon use of the Noether identity (4.14), is mapped into the 4-form Bianchi identities (3.53).

We conclude that, at least in this case, the duality hierarchy encodes precisely the vector equations of motion and the projected scalar equations of motion via the duality rules (4.18).

4.2 General gaugings

In this subsection we wish to consider the more general case of electric and magnetic gaugings. Our starting point is the formally symplectic-covariant generalization of the equations of motion (4.15)-(4.17)[18]

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho} + \tfrac{1}{2}g_{\mu\nu}V\,, \\
\mathcal{E}_i &= \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\,, \qquad (4.21)\\
\mathcal{E}_M &\equiv \mathfrak{D}G_M - \tfrac{1}{4}\Theta_M{}^A \star j_A\,,
\end{aligned}$$

where we have defined

$$(G^M) \equiv \begin{pmatrix} F^\Sigma \\ G_\Sigma \end{pmatrix}, \qquad G_\Sigma{}^+ = f_{\Sigma\Gamma}(Z)F^{\Gamma+}\,, \tag{4.22}$$

and where the electric and magnetic field strengths F^M are defined as in the tensor hierarchy, i.e. including the 2-form B_A for which we do not want to have an independent equation of motion to preserve the original number of degrees of freedom.

The requirement that the kinetic matrix is invariant under the global symmetry group G and that the potential is gauge-invariant leads to the conditions

$$\pounds_A f_{\Lambda\Sigma} = -T_{A\,\Lambda\Sigma} + 2T_{A\,(\Lambda}{}^\Omega f_{\Sigma)\Omega} - T_A{}^{\Omega\Gamma} f_{\Omega\Lambda} f_{\Gamma\Sigma}\,, \tag{4.23}$$

$$\pounds_A V = Y_{AM}{}^C \frac{\partial V}{\partial \Theta_M{}^C}\,, \tag{4.24}$$

from which it follows that

$$k_A{}^i \partial_i G_M{}^+ \wedge G^{M+} = k_A{}^i \partial_i f_{\Lambda\Sigma} F^{\Lambda+} \wedge F^{\Sigma+} = -T_{A\,MN} G^M \wedge G^N\,. \tag{4.25}$$

A direct computation using the above properties leads to the following identity for the covariant Noether current:

[18] The Einstein and scalar equations of motion are just a rewriting of the original ones, which are already symplectic-invariant.

$$
\mathfrak{D}\star j_A = -2(k_A{}^i\mathcal{E}_i + \text{c.c.}) - 2T_{A\,MN}G^M \wedge G^N + \star Y_A{}^{\Lambda C}\frac{\partial V}{\partial \Theta_\Lambda{}^C}\,. \tag{4.26}
$$

On the other hand, the Ricci identity gives

$$
\mathfrak{D}\mathfrak{D}G_M = -X_{NM}{}^P F^N \wedge G_P = X_{NPM}F^N \wedge G^P\,. \tag{4.27}
$$

Taking the covariant derivative of the full $\mathcal{E}_M$ and using Eqs. (4.26) and (4.27) we find

$$
\mathfrak{D}\mathcal{E}_M + \tfrac{1}{2}\Theta_M{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) = X_{NPM}(F^N - G^N)\wedge G^P = \Theta^{\Sigma A}(F_\Sigma - G_\Sigma)\wedge T_{A\,PM}G^P\,. \tag{4.28}
$$

This is the gauge identity associated to the standard electric and magnetic gauge transformations of the vectors and scalars

$$
\begin{aligned}
\delta Z^i &= \Lambda^M \Theta_M{}^A k_A{}^i\,,\\
\delta A^M &= -\mathfrak{D}\Lambda^M\,,
\end{aligned} \tag{4.29}
$$

provided that the right-hand side of the equation vanishes. Since this is not the case we conclude that the equations of motion are not gauge-invariant. Hence, a naive symplectic covariantization of the electric gauging case is not enough to obtain a gauge-invariant answer involving magnetic gaugings.

In order to re-obtain gauge invariance we extend the set of equations of motion, adding, arbitrarily, as equation of motion of the 2-forms B_A

$$
\mathcal{E}^A \equiv \Theta^{MA}(F_M - G_M) = -\Theta^{\Sigma A}(F_\Sigma - G_\Sigma)\,, \tag{4.30}
$$

so that the above identity becomes again a relation between equations of motion

$$
\mathfrak{D}\mathcal{E}_M + \tfrac{1}{2}\Theta_M{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) + T_{A\,MP}\mathcal{E}^A \wedge G^P = 0\,, \tag{4.31}
$$

that we can interpret as the gauge identity associated to an off-shell gauge invariance of the extended set of equations of motion.

The price we may have to pay for doing this is the possible modification of the equations of motion of the vector fields: the above gauge identities are associated to the gauge transformations of B_A

$$
\delta B_A = 2T_{A\,MP}\Lambda^M G^P + 2R_{A\,M}\wedge \delta A^M\,, \tag{4.32}
$$

where $R_{A\,M}$ is a 1-form that is cancelled in the above gauge identity by an extra term in the equation of motion of the vector fields:

$$
\mathcal{E}'_M = \mathcal{E}_M + R_{A\,M}\mathcal{E}^A \wedge A^M\,. \tag{4.33}
$$

The 1-forms $R_{A\,M}$ must be such that the infinitesimal gauge transformations form a closed algebra. The gauge identity takes now the form

$$\mathfrak{D}\mathcal{E}'_M + \tfrac{1}{2}\Theta_M{}^A(k_A{}^i\mathcal{E}_i + \text{c.c.}) + T_{A\,MP}\mathcal{E}^A \wedge G^P - \mathfrak{D}(R_{A\,M}\mathcal{E}^A \wedge A^M) = 0\,. \tag{4.34}$$

In order to make contact with the tensor hierarchy we take $R_{A\,M} = \frac{1}{2}X^P{}_{MN}A^N \wedge (F_P - G_P)$.

We observe that the equations of motion also satisfy the relation

$$\mathfrak{D}\mathcal{E}^A - \tfrac{1}{2}T_{B\,MN}\Theta^{PA}A^N \wedge \mathcal{E}^B + \Theta^{MA}\mathcal{E}_M = 0\,, \tag{4.35}$$

which can be interpreted as the gauge identity associated to the symmetry

$$\begin{aligned} \delta A^M &= Z^{MA}\Lambda_A\,, \\ \delta B_A &= \mathfrak{D}\Lambda_A - \tfrac{1}{2}T_{AMN}\Theta^{NB}A^M \wedge \Lambda_B\,. \end{aligned} \tag{4.36}$$

As we did in the electric gauging case, we are now going to establish a relation between the tensor hierarchy and the following equations of motion:

$$\mathcal{E}'_M = \mathfrak{D}G_M - \tfrac{1}{4}\Theta_M{}^A \star j_A + \tfrac{1}{2}T_{A\,MN}A^N \wedge \Theta^{PA}(F_P - G_P) = 0\,, \tag{4.37}$$

$$\mathcal{E}^A = \Theta^{MA}(F_M - G_M) = 0\,, \tag{4.38}$$

$$k_A{}^i\mathcal{E}_i = k_A{}^i\Big[\mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\Big] = 0\,. \tag{4.39}$$

These equations are invariant under the gauge transformations

$$\delta_a Z^i = \delta_h Z^i\,, \tag{4.40}$$

$$\delta_a A^M = \delta_h A^M\,, \tag{4.41}$$

$$\delta_a B_A = \delta_h B_A - 2T_{A\,NP}\Lambda^N(F^P - G^P)\,, \tag{4.42}$$

where we have denoted by δ_a the gauge transformations that leave this system of equations invariant and by δ_h those derived in the construction of the 4-dimensional tensor hierarchy (summarized in Appendix B). $\delta_a B_A$ is, therefore, just $\delta_h B_A$ with F^P replaced by G^P.

Following the electric gauging case, in order to derive the above equations of motion from the tensor hierarchy, we introduce the following set of duality relations:

$$\begin{aligned} G^M &= F^M\,, \\ j_A &= -2\star H_A\,, \\ \frac{\partial V}{\partial \Theta_M{}^A} &= -2\star G_A{}^M\,. \end{aligned} \tag{4.43}$$

We note that the gauge-covariance of the first duality relation is more subtle in that G^M transforms not only covariantly, but also into $G^M - F^M$, see [30]. Note that the equation of motion of the magnetic 2-form potentials, $\mathcal{E}^A = 0$, is identified as a projected duality relation. To recover the other equations of motion we have to again hit the duality relations (4.43) with a covariant derivative and next apply one of the Bianchi identities of the tensor hierarchy. To derive the projected scalar equations of motion we first obtain the identity

$$\mathfrak{D} \star j_A + 2T_{A\,MN} G^M \wedge G^N - \star Y_A{}^{\Lambda C} \frac{\partial V}{\partial \Theta_\Lambda{}^A} \;=\; 0 \tag{4.44}$$

from the duality hierarchy and, next, apply the Noether identity (4.26).

The gauge identities guarantee the existence of a gauge-invariant action from which the equations of motion $\mathcal{E}'_M$ and $\mathcal{E}^A$ can be derived. This action has actually been constructed in Ref. [11]. In our conventions, it is given by

$$\begin{aligned} S[g_{\mu\nu}, Z^i, A^M, B_A] \;=\; & \int \big\{ \star R - 2\mathcal{G}_{ij^*} \mathfrak{D} Z^i \wedge \star \mathfrak{D} Z^{*\,j^*} + 2F^\Sigma \wedge G_\Sigma - \star V \\ & -4Z^{\Sigma A} B_A \wedge \big(F_\Sigma - \tfrac{1}{2} Z_\Sigma{}^B B_B\big) \\ & -\tfrac{4}{3} X_{[MN]\Sigma} A^M \wedge A^N \wedge \big(F^\Sigma - Z^{\Sigma B} B_B\big) \\ & -\tfrac{2}{3} X_{[MN]}{}^\Sigma A^M \wedge A^N \wedge \big(dA_\Sigma - \tfrac{1}{4} X_{[PQ]\Sigma} A^P \wedge A^Q\big) \big\} \;. \end{aligned} \tag{4.45}$$

A general variation of the above action gives

$$\delta S = \int \left\{ \delta g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S}{\delta Z^i} + \text{c.c.} \right) - \delta A^M \wedge \star \frac{\delta S}{\delta A^M} + 2\delta B_A \wedge \star \frac{\delta S}{\delta B_A} \right\} , \tag{4.46}$$

where

$$\frac{\delta S}{\delta g^{\mu\nu}} \;=\; \star \mathbb{I} \mathcal{E}_{\mu\nu} \,, \tag{4.47}$$

$$-\tfrac{1}{2} \frac{\delta S}{\delta Z^i} \;=\; \mathcal{E}_i \,, \tag{4.48}$$

$$-\tfrac{1}{4} \star \frac{\delta S}{\delta A^M} \;=\; \mathcal{E}'_M \,, \tag{4.49}$$

$$\star \frac{\delta S}{\delta B_A} \;=\; \mathcal{E}^A \,. \tag{4.50}$$

4.3 The unconstrained case

In this subsection we briefly comment on the meaning of the top-form and next to top-form potentials. Experience shows that these higher-rank potentials can be related to constraints: the constancy of $\Theta_M{}^A$, $\mathfrak{D}\Theta_M{}^A = 0$, can be associated to the 3-form potential, and the quadratic and linear constraints $Q_{NP}{}^E = 0$, $Q^{AB} = 0$, $L_{NPQ} = 0$ can be associated to the 4-form potentials $D_E{}^{NP}, D_{AB}, D^{NPQ}$ that we have providentially found. We would like to stress, however, that prior to relaxing the constraints one is forced to introduce these potentials if one requires that the field equations are derivable as compatibility conditions from the duality relations, as we showed in the previous section.

In view of the discussion of an action principle with Lagrange multipliers in the next section, we reconsider the gauge identities of the equations $\mathcal{E}'_M, \mathcal{E}^A$ defined in the previous subsections assuming that those constraints are not satisfied. We then denote the embedding tensor by $\vartheta_M{}^A = \vartheta_M{}^A(x)$ in order to indicate that it is now space-time dependent. Evidently, we are going to get extra terms proportional to the constraints which we will reinterpret as equations of motion of the 3- and 4-form potentials, obtaining new gauge identities that involve the equations of motion of all fields. Thus, off-shell gauge invariance will have been preserved by the same mechanism used in the previous case. The price that we will have to pay is the same: modifying the gauge transformations and the equations of motion.

This procedure is too complicated in this case, though. As an example, let us take the covariant derivative of $\mathcal{E}^A$:

$$\mathfrak{D}\mathcal{E}^A = -\mathfrak{D}\vartheta_M{}^A \wedge (F^M - G^M) + \vartheta^{MA}(\mathfrak{D}F_M - \mathfrak{D}G_M)\,. \tag{4.51}$$

The unconstrained Bianchi identity for F^M is

$$\begin{aligned}\mathfrak{D}F^M \;=\;& Z^{MB}[H_B - Y_{BN}{}^C C_C{}^N] + L^M{}_{RS}[\tfrac{3}{2}A^R \wedge dA^S + \tfrac{1}{2}X_{NP}{}^S A^R \wedge A^N \wedge A^P]\\ &+\mathfrak{D}\vartheta_N{}^A \wedge [\tfrac{1}{2}\Omega^{NM}B_A + \tfrac{1}{2}T_{A\,P}{}^M A^N \wedge A^P] + \tfrac{1}{3}Q_{NP}{}^E T_{E\,R}{}^M A^N \wedge A^P \wedge A^R\,,\end{aligned} \tag{4.52}$$

and, using the equation of motion $\mathcal{E}'_M$ we can write the following gauge identity

$$\begin{aligned}&\mathfrak{D}\mathcal{E}^A - \tfrac{1}{2}T_{B\,MN}\vartheta^{MA}A^N \wedge \mathcal{E}^B + \vartheta^{MA}\mathcal{E}'_M + Q^{AB}[2(H_B + \tfrac{1}{2}\star j_B) - 2Y_{BN}{}^C C_C{}^N]\\ &\quad+\mathfrak{D}\vartheta_M{}^B \wedge [\tfrac{1}{2}\vartheta^{MA}B_B + \tfrac{1}{2}T_{B\,P}{}^Q\vartheta_Q{}^A A^M \wedge A^P + \delta_B{}^A(F^M - G^M)]\\ &\quad+L_{MRS}\vartheta^{MA}[-\tfrac{3}{2}A^R \wedge dA^S - \tfrac{1}{2}X_{NP}{}^S A^R \wedge A^N \wedge A^P]\\ &\quad-\tfrac{1}{3}Q_{NP}{}^E T_{E\,R}{}^M \vartheta_M{}^A A^N \wedge A^P \wedge A^R \;=\; 0\,.\end{aligned} \tag{4.53}$$

It is very difficult to infer directly from this and similar identities all the gauge transformations of the fields and the modifications of the equations of motion. Thus, we are going to adopt a different strategy in the next section: we are going to construct directly a gauge-invariant action.

5 The $D = 4$ action

In this section we perform the third and last step of our procedure: the construction of an action for the fields of the tensor hierarchy[19]. Our starting point is the action Eq. (4.45), which we will denote by S_0 in what follows and which includes, besides the metric, only scalars, 1-forms and 2-forms and which is invariant under the gauge transformations Eqs. (4.40)-(4.42). We now want to add to it 3- and 4-forms as Lagrange multipliers enforcing the covariant constancy of the embedding tensor (which we promote to an unconstrained scalar field $\Theta_M{}^A(x)$) and the three algebraic constraints Q^{AB}, L_{NPQ}, $Q_{NP}{}^E$ that we have imposed on the embedding tensor. The new terms must be metric-independent ("topological") and scalar-independent in order to leave unmodified the scalar and Einstein equations of motion (4.21) which are derived from the action S_0 given in Eq. (4.45).

Thus, we add to S_0 the following piece ΔS given by[20]

$$\Delta S = \int \left\{ \mathfrak{D}\vartheta_M{}^A \wedge \tilde{C}_A{}^M + Q_{NP}{}^E \tilde{D}_E{}^{NP} + Q^{AB}\tilde{D}_{AB} + L_{NPQ}\tilde{D}^{NPQ} \right\} . \tag{5.1}$$

The tildes in $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ indicate that these 3- and 4-form fields need not be identical to those found in the hierarchy, although we expect them to be related by field redefinitions.

The action S_0 is no longer gauge invariant under the gauge transformations involving 0- and 1-form gauge parameters Λ^M, Λ_A, without imposing any constraints on the embedding tensor, but the non-vanishing terms in the transformation can only be proportional to the l.h.s.'s of the constraints $\mathfrak{D}\vartheta_M{}^C = 0$, $Q_{NP}{}^E = 0$, $Q^{AB} = 0$ and $L_{NPQ} = 0$ and, by choosing appropriately the gauge transformations of $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ we can always make the variation of the action $S \equiv S_0 + \Delta S$ vanish. Having done that we would like to relate the tilded fields with the untilded ones in the hierarchy.

Let us start by computing the general variation of the action. Taking into account the fact that the fields $g_{\mu\nu}$, Z^i and $B_{A\,\mu\nu}$ only occur in S_0, that the field $A^M{}_\mu$ occurs in S_0 and in the term $\mathfrak{D}\vartheta_M{}^A\tilde{C}_A{}^M$ in ΔS and that the new fields $\tilde{C}_C{}^M$, $\tilde{D}_{AB}$, $\tilde{D}^{NPQ}$ and $\tilde{D}_E{}^{NP}$ only occur in ΔS, we find

[19] Actually, not all the 2-forms B_A will appear in the action but only $\Theta^{\Lambda A}B_A$.

[20] Observe that $\mathfrak{D}\Theta_M{}^A = d\Theta_M{}^A - Q_{NM}{}^A A^N$ and, therefore, the covariant constancy of the embedding tensor plus the quadratic constraint $Q_{NP}{}^E = 0$ imply $d\Theta_M{}^A = 0$.

$$
\begin{aligned}
\delta S \;=\; & \int \Big\{ \delta g^{\mu\nu} \frac{\delta S_0}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S_0}{\delta Z^i} + \text{c.c.} \right) - \delta A^M \wedge \star \frac{\delta S_0}{\delta A^M} + 2\delta B_A \wedge \star \frac{\delta S_0}{\delta B_A} \\
& + \mathfrak{D}\vartheta_M{}^A \wedge \delta \tilde{C}_A{}^M + Q_{NP}{}^E (\delta \tilde{D}_E{}^{NP} - \delta A^N \wedge \tilde{C}_E{}^P) + Q^{AB} \delta \tilde{D}_{AB} \\
& + L_{NPQ} \delta \tilde{D}^{NPQ} + \delta \vartheta_M{}^A \frac{\delta S}{\delta \vartheta_M{}^A} \Big\} \,.
\end{aligned} \tag{5.2}
$$

The scalar and Einstein equations of motion are as in Eqs. (4.21) and (4.47),(4.48). The variations of the old action S_0 with respect to A^M and B_A are modified by terms proportional to the constraints. We can write them in the form

$$
\begin{aligned}
-\tfrac{1}{4}\star \frac{\delta S_0}{\delta A^M} \;=\; & \mathfrak{D} F_M - \tfrac{1}{4} \vartheta_M{}^A \star j_A - \tfrac{1}{3} dX_{[PQ]M} \wedge A^P \wedge A^Q - \tfrac{1}{2} Q_{(NM)}{}^E A^N \wedge B_E \\
& - L_{MNP} A^N \wedge \left(dA^P + \tfrac{3}{8} X_{[RS]}{}^P A^R \wedge A^S \right) + \tfrac{1}{8} Q_{NP}{}^A T_{AQM} A^N \wedge A^P \wedge A^Q \\
& - d(F_M - G_M) - X_{[MN]}{}^P A^N \wedge (F_P - G_P) \,,
\end{aligned} \tag{5.3}
$$

$$
\star \frac{\delta S_0}{\delta B_A} \;=\; \vartheta^{PA} (F_P - G_P) + Q^{AB} B_B \,. \tag{5.4}
$$

In deriving these equations we have used the unconstrained Bianchi identity for F^Λ, given by the upper component of Eq. (4.52), to replace H_A in the equation of motion of A_Λ. This has allowed us to write a symplectic-covariant expression for the equation of motion of A^M.

The only non-trivial variation that remains to be computed in Eq. (5.2) is the equation of motion of the embedding tensor. We get

$$
\begin{aligned}
\frac{\delta S}{\delta \vartheta_M{}^A} \;=\; & -\mathfrak{D} \tilde{C}_A{}^M + Z^{MB} B_B \wedge B_A - 2(F^M - G^M) \wedge B_A - \star \frac{\partial V}{\partial \vartheta_M{}^A} \\
& + W_{ANP}{}^{EM} \tilde{D}_E{}^{NP} + W_A{}^{BCM} \tilde{D}_{BC} + W_{ANPQ}{}^M \tilde{D}^{NPQ} \\
& + A^M \wedge \Big\{ - \star j_A + Y_{AN}{}^C \tilde{C}_C{}^N - T_{AN}{}^P A^N \wedge (F_P - G_P) \\
& - \tfrac{4}{3} T_{ANR} A^N \wedge \left[dA^R + \tfrac{3}{8} X_{[PQ]}{}^R A^P \wedge A^Q + \tfrac{3}{2} Z^{RB} B_B \right] \Big\} \,.
\end{aligned} \tag{5.5}
$$

We are going to use this equation to find the relation between the tilded fields and the hierarchy fields. Using Eqs. (4.43) and the definitions of the tensor hierarchy's field strengths H_A and $G_A{}^M$, we are left with

$$
\begin{aligned}
\tfrac{1}{2}\frac{\delta S}{\delta \vartheta_M{}^A} \;&=\; \mathfrak{D}(-\tfrac{1}{2}\tilde{C}_A{}^M - C_A{}^M - A^M \wedge B_A) \\
&\quad + Y_{AP}{}^C A^M \wedge (\tfrac{1}{2}\tilde{C}_C{}^P + C_C{}^P + A^P \wedge B_C) + W_A{}^{BCM}(\tfrac{1}{2}\tilde{D}_{BC} - D_{BC}) \\
&\quad + W_{ANP}{}^{EM}(\tfrac{1}{2}\tilde{D}_E{}^{NP} - D_E{}^{NP} + \tfrac{1}{2}A^N \wedge A^P \wedge B_E) \\
&\quad + W_{ANPQ}{}^M(\tfrac{1}{2}\tilde{D}^{NPQ} - D^{NPQ}) \,,
\end{aligned}
\tag{5.6}
$$

which is satisfied if we identify

$$
\begin{aligned}
\tilde{C}_A{}^M \;&=\; -2(C_A{}^M + A^M \wedge B_A)\,, \qquad & \tilde{D}_E{}^{NP} \;&=\; 2D_E{}^{NP} - A^N \wedge A^P \wedge B_E\,, \\
\tilde{D}_{BC} \;&=\; 2D_{BC}\,, & \tilde{D}^{NPQ} \;&=\; 2D^{NPQ}\,.
\end{aligned}
\tag{5.7}
$$

Using these identifications ΔS reads

$$
\begin{aligned}
\Delta S \;&=\; \int \big\{-2\mathfrak{D}\vartheta_M{}^A \wedge (C_A{}^M + A^M \wedge B_A) + 2Q_{NP}{}^E(D_E{}^{NP} - \tfrac{1}{2}A^N \wedge A^P \wedge B_E) \\
&\quad + 2Q^{AB}D_{AB} + 2L_{NPQ}D^{NPQ}\big\}\;,
\end{aligned}
\tag{5.8}
$$

and a general variation of the total action $S = S_0 + \Delta S$ is given by

$$
\begin{aligned}
\delta S \;&=\; \int \Bigg\{\delta g^{\mu\nu}\frac{\delta S_0}{\delta g^{\mu\nu}} + \left(\delta Z^i \frac{\delta S_0}{\delta Z^i} + \text{c.c.}\right) - \delta A^M \wedge \star\frac{\delta S_0}{\delta A^M} + 2\delta B_A \wedge \star\frac{\delta S_0}{\delta B_A} \\
&\quad + \mathfrak{D}\vartheta_M{}^A \wedge [-2\delta C_A{}^M - 2\delta A^M \wedge B_A - 2A^M \wedge \delta B_A] + Q^{AB}[2\delta D_{AB}] \\
&\quad + Q_{NP}{}^E[2\delta D_E{}^{NP} + 2\delta A^N \wedge C_E{}^P + 2\delta A^{(N} \wedge A^{P)} \wedge B_E - A^N \wedge A^P \wedge \delta B_E] \\
&\quad + L_{NPQ}[2\delta D^{NPQ}] + \delta\vartheta_M{}^A \frac{\delta S}{\delta \vartheta_M{}^A}\Bigg\}\,.
\end{aligned}
\tag{5.9}
$$

The first variation of the total action S with respect to $\vartheta_M{}^A$ can be written in the form

$$
\begin{aligned}
\tfrac{1}{2}\frac{\delta S}{\delta \vartheta_M{}^A} \;&=\; (G_A{}^M - \tfrac{1}{2}\star \partial V/\partial \vartheta_M{}^A) - A^M \wedge (H_A + \tfrac{1}{2}\star j_A) \\
&\quad - \tfrac{1}{2}T_{AN}{}^P A^M \wedge A^N \wedge (F_P - G_P) - (F^M - G^M) \wedge B_A\,.
\end{aligned}
\tag{5.10}
$$

We can now check the gauge invariance of the total action S. We are going to use for the gauge transformations of all the fields (except for the scalars and vectors) the Ansatz $\delta_a = \delta_h + \Delta$ where Δ is a piece to be determined. If we assume that the embedding tensor is exactly invariant[21], i.e. $\delta\vartheta_M{}^A = 0$, we find

$$\Delta B_A = -2T_{A\,NP}\Lambda^N(F^P - G^P)\,, \tag{5.11}$$

$$\Delta C_A{}^M = \Lambda_A \wedge (F^M - G^M) - \Lambda^M(H_A + \tfrac{1}{2}\star j_A)\,, \tag{5.12}$$

$$\Delta D_{AB} = 2\Lambda_{[A} \wedge (H_{B]} + \tfrac{1}{2}\star j_{B]}) - 2T_{[A|\,NP}\Lambda^N(F^P - G^P)\wedge B_{|B]}\,, \tag{5.13}$$

$$\Delta D_E{}^{NP} = -\Lambda^N(G_E{}^P - \tfrac{1}{2}\star \partial V/\partial\vartheta_P{}^E) + (F^N - G^N)\wedge \Lambda_E{}^P\,, \tag{5.14}$$

$$\begin{aligned}\Delta D^{NPQ} &= -3\delta A^{(N}\wedge A^P \wedge (F^{Q)} - G^{Q)}) + 6\Lambda^{(N}F^P \wedge (F^{Q)} - G^{Q)}) \\ &\quad -3\Lambda^{(N}(F^P - G^P)\wedge(F^{Q)} - G^{Q)})\,,\end{aligned} \tag{5.15}$$

where we have used in this calculation the non-trivial Ricci identities[22]

$$\begin{aligned}\vartheta_M{}^C\mathfrak{D}\mathfrak{D}\Lambda_C{}^M &= \mathfrak{D}\vartheta_M{}^A \wedge (-Y_{AP}{}^E A^M \wedge \Lambda_E{}^P) + Q_{NP}{}^E[(F^N - Z^{NA}B_A)\wedge\Lambda_E{}^P \\ &\quad -\tfrac{1}{2}Y_{EQ}{}^C A^N \wedge A^P \wedge \Lambda_C{}^Q]\,,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\mathfrak{D}\mathfrak{D}F_M &= X_{NPM}F^N \wedge F^P - 2Q^{AB}T_{A\,PM}F^P \wedge B_B + dX_{NPM}\wedge A^N \wedge F^P \\ &\quad -\tfrac{1}{2}Q_{NP}{}^E T_{E\,MQ}A^N \wedge A^P \wedge F^Q\,,\end{aligned} \tag{5.17}$$

and the variations of the kinetic matrix and the potential Eqs. (4.23) and (4.24).

We observe that all terms in the extra variations Δ vanish when we use the duality relations (4.43). Actually, all of them, except for just one term in ΔD^{NPQ}, are such that the variations δ_a are obtained from the tensor hierarchy variations δ_h simply by replacing the scalar-independent field strengths $F^M, H_A, G_A{}^M$ by the corresponding scalar-dependent objects $G^M, j_A, \dfrac{\partial V}{\partial\vartheta_\Lambda{}^A}$ via the duality relations (4.43).

[21]One could also allow ϑ_M^A to transform according to its indices as $\delta\vartheta_M^A = -Q_{NM}{}^A\Lambda^N$. This is like adding a term proportional to an equation of motion, that of $D_A{}^{NM}$, to the zero variation.

[22]If the constraints are satisfied, $\vartheta_M{}^C\mathfrak{D}\mathfrak{D}\Lambda_C{}^M = \mathfrak{D}\mathfrak{D}(\vartheta_M{}^C\Lambda_C{}^M) = dd(\vartheta_M{}^C\Lambda_C{}^M) = 0$. Therefore, when they are not satisfied, $\vartheta_M{}^C\mathfrak{D}\mathfrak{D}\Lambda_C{}^M$ must be proportional to them.

Finally, we note that the variations δ_a and δ_h are equivalent from the point of view of the duality hierarchy. The two sets of transformation rules differ by terms that are proportional to the duality relations. The only difference is that the commutator algebra corresponding to δ_h closes off-shell whereas the algebra corresponding to δ_a closes up to terms that are proportional to the duality relations. The two sets of transformation rules are not equivalent from the action point of view in the sense that only one of them, the one with transformation rules δ_a, leaves the action invariant, whereas the other, with transformations δ_h, does not.

6 The 3-dimensional case

As an illustration of our general procedure we will construct in this section the complete $D = 3$ tensor and duality hierarchy corresponding to a generic $D = 3$ gauged supergravity theory, extending the analysis of the maximally supersymmetric case [12, 21]. The $D = 3$ hierarchy is sufficiently short in order to allow for a straightforward analysis and nevertheless captures the features expected to appear in general dimensions.

6.1 Generalities on $D = 3$

Three-dimensional gauged supergravity has been constructed in [9, 10] for the maximal case and subsequently generalized to lower supersymmetries in [23, 24].

$D = 3$ (ungauged) supergravities are particularly simple theories because their only physical bosonic degrees of freedom are described by scalar fields, since in $D = 3$ the metric and p-forms with $p \geq 2$ have no dynamics and vectors are dual to scalars. The number of scalar fields as well as the rigid symmetry group G is ultimately constrained by supersymmetry. For instance, in case of maximal supersymmetry there are 128 scalars, which parameterize the coset space $E_{8(8)}/SO(16)$, and thus we have $G = E_{8(8)}$. However, for the general construction of the tensor hierarchy to be discussed here supersymmetry does not play any role, and so for the moment we will leave the group G completely generic, thereby capturing the most general situation in $D = 3$.

The original formulation [9, 10] of maximal gauged $D = 3$ supergravity requires the introduction of gauge vectors ${A_\mu}^M$ transforming in the adjoint representation of G which do not describe new degrees of freedom but are dual to scalars. Owing to this fact, the embedding tensor carries in three dimensions two adjoint indices and thus reads Θ_{MN}. More precisely, the gauge vectors enter via a topological Chern-Simons term, whose invariant tensor is precisely given by Θ_{MN} (cf. (6.22) below). In this case, the embedding tensor is symmetric, $\Theta_{MN} = \Theta_{NM}$, and the tensors defined in (2.5) read

$$ {X_{MN}}^P \;=\; \Theta_{MK} {f^{KP}}_N \;=\; {X_{[MN]}}^P + {Z^P}_{MN}\,, \qquad {Z^P}_{MN} \;=\; \Theta_{K(M} {f^{KP}}_{N)}\,, \tag{6.1} $$

with the structure constants of G satisfying $[t^M, t^N] = -{f^{MN}}_K t^K$. As in (2.6), the quadratic constraint states that the symmetric part Z vanishes upon contraction with the embedding tensor, $\Theta_{PK} Z^P{}_{MN} = 0$. Ultimately, supersymmetry requires in addition a linear constraint. However, for the bosonic gauge covariance of the tensor hierarchy this constraint is immaterial and thus it is sufficient to impose only the quadratic constraints.

For the present purpose it suffices to inspect the equations of motion of the gauge vectors. By virtue of the Chern-Simons term they take the form of first-order duality relations,

$$e^{-1} \varepsilon^{\mu\nu\rho} \Theta_{MN} F_{\nu\rho}{}^N \;=\; -2\Theta_{MN} J^{\mu N} \; . \tag{6.2}$$

Here, the current $J^{\mu M}$ corresponds to the Noether current of the ungauged theory, which can be written in terms of the Killing vector fields $k_i{}^M(\phi)$ generating G as

$$J_\mu{}^M \;=\; D_\mu \phi^i \, k_i{}^M \; , \tag{6.3}$$

where $i, j, \ldots$ are the coordinate labels of the scalar manifold. The field strength takes the standard form

$$F_{\mu\nu}{}^M \;=\; \partial_\mu A_\nu{}^M - \partial_\nu A_\mu{}^M + X_{NP}{}^M A_{[\mu}{}^N A_{\nu]}{}^P \; , \tag{6.4}$$

where the quadratic term has to be antisymmetrized explicitly due to the lack of antisymmetry of the 'structure constant' $X_{NP}{}^M$.

At this stage the situation is very similar to the four-dimensional case discussed in the previous section. Due to the simplicity in $D = 3$, it is instructive to repeat below a few of the remarks we already made in the previous section. First, one may wonder whether it is possible to obtain the scalar equations of motion from the duality relation (6.2) by acting on it with a derivative D_μ. This turns out not to be the case, since (6.2) is only a projected version of the naive duality relation in that both sides appear contracted with the embedding tensor. In fact, in gauged supergravity there is a scalar potential, whose contributions to the scalar field equations are invisible upon contraction with the embedding tensor. Thus, the duality relation obtained from the action does not imply the scalar field equations, though it is nevertheless compatible with them. One might be tempted to impose the unprojected duality relations by dropping the contraction with Θ_{MN}, in order to obtain the full field equations. However, there are two immediate obstacles. First, the naive Bianchi identity $D_{[\mu} F_{\nu\rho]}{}^M = 0$ required for deriving second-order equations as integrability conditions holds for the field strength in (6.4) only upon contraction with Θ_{MN}. Second, it is clear that the contributions from a scalar potential cannot be reproduced in this way, due to the fact that one cannot 'pull out a derivative' of the scalar potential. It turns out that the resolution of these two problems is related and naturally suggested by the structure of the tensor hierarchy. Specifically, this will introduce higher-rank tensor fields that allow for covariant field strengths satisfying consistent Bianchi identities. Moreover, these additional tensor fields will be accompanied by further duality relations which encode, in particular, the scalar potential. This set of first-order field equations defines the duality hierarchy which will be discussed in the next subsection.

6.2 The $D = 3$ tensor and duality hierarchy

As in the $D = 4$ case, the tensor hierarchy can be systematically introduced by requiring that the field strengths satisfy Bianchi identities and transform covariantly according to their index structure. First, we modify the field strength (6.4) by a Stückelberg-like coupling involving a 2-form gauge potential $B^{NK} = B^{KN}$,

$$H_{\mu\nu}{}^{M} \;=\; F_{\mu\nu}{}^{M} - 2Z^{M}{}_{NK} B_{\mu\nu}{}^{NK} \;. \tag{6.5}$$

By virtue of the quadratic constraint (2.6) the extra term vanishes upon contraction with the embedding tensor. Thus, all non-covariant terms in the variation of the (unprojected) $F_{\mu\nu}{}^{M}$ can be absorbed into a suitable variation of the 2-form potential. Specifically, under the standard form of the gauge transformation

$$\delta A_{\mu}{}^{M} \;=\; D_{\mu}\Lambda^{M} \;=\; \partial_{\mu}\Lambda^{M} + X_{NP}{}^{M} A_{\mu}{}^{N} \Lambda^{P} \,, \tag{6.6}$$

one finds

$$\delta F_{\mu\nu}{}^{M} \;=\; X_{NP}{}^{M} F_{\mu\nu}{}^{N} \Lambda^{P} - 2Z^{M}{}_{NP} A_{[\mu}{}^{N} \delta A_{\nu]}{}^{P} \;. \tag{6.7}$$

We note that upon contraction with Θ_{MN} the second term vanishes and the structure constant in the first term is antisymmetric. In particular the latter property is required in order to derive the standard covariant form of the gauge transformation. The lack of covariance for the unprojected field strength can now be compensated by assigning gauge transformations to the 2-form in (6.5). Requiring the covariant variation

$$\delta H_{\mu\nu}{}^{M} \;=\; -\Lambda^{N} X_{NP}{}^{M} H_{\mu\nu}{}^{P} \,, \tag{6.8}$$

determines the 2-form gauge variation, with parameter $\Lambda^{MN} = \Lambda^{NM}$, to be

$$\delta B_{\mu\nu}{}^{MN} \;=\; D_{[\mu}\Lambda_{\nu]}{}^{MN} - A_{[\mu}{}^{\langle M} \delta A_{\nu]}{}^{N\rangle} + \Lambda^{\langle M} H_{\mu\nu}{}^{N\rangle} + \cdots \,, \tag{6.9}$$

up to terms that vanish upon contraction with $Z^{P}{}_{MN}$. Here, the brackets $\langle\ \rangle$ a priori denote ordinary (unit-strength) symmetrization. However, Eq. (6.9) and all subsequent relations directly generalize to the case, where a linear constraint has been imposed, for which $\langle\ \rangle$ has to be interpreted as the corresponding projector onto the surviving representations. We have also added the variation of the 2-form under its own gauge parameter $\Lambda_{\mu}{}^{MN}$. Invariance of (6.5) then requires that $A_{\mu}{}^{M}$ transforms (as a shift) under this symmetry, i.e., the gauge variation (6.6) has to be modified by $\delta' A_{\mu}{}^{M} = Z^{M}{}_{NP} \Lambda_{\mu}{}^{NP}$.

In a next step one can introduce a 3-form field strength $G_{\mu\nu\rho}{}^{MN}$ for the 2-form gauge potential by requiring gauge covariance. It turns out, however, to be more convenient to determine the leading terms of this field strength by requiring that the modified field strength for the original gauge vector satisfies a Bianchi identity,

$$D_{[\mu} H_{\nu\rho]}{}^{M} \;=\; -2Z^{M}{}_{NP} G_{\mu\nu\rho}{}^{NP} \;. \tag{6.10}$$

This uniquely determines the field strength up to terms that vanish by contraction with $Z^P{}_{MN}$. Ultimately, we want to write covariant duality relations involving the uncontracted $G_{\mu\nu\rho}{}^{MN}$. As before, this can be achieved via introducing a new potential, which is a 3-form, and assigning appropriate gauge transformations to it. Without repeating the detailed steps of the derivation, we simply state the results. (For more details we refer the reader to [12].) The 3-form field strength reads

$$\begin{aligned} G_{\mu\nu\rho}{}^{MN} &= D_{[\mu}B_{\nu\rho]}{}^{MN} - A_{[\mu}{}^{\langle M}\partial_\nu A_{\rho]}{}^{N\rangle} - \frac{1}{3}X_{KL}{}^{\langle M}A_{[\mu}{}^{N\rangle}A_\nu{}^K A_{\rho]}{}^L \\ &\quad -\frac{2}{3}Y^{MN}{}_{P,KL}\, C_{\mu\nu\rho}{}^{P,KL}\,. \end{aligned} \tag{6.11}$$

Here, we have introduced the intertwining Y-tensor [12]

$$Y^{MN}{}_{P,KL} \;=\; Z^{\langle M}{}_{KL}\delta^{N\rangle}{}_P - X_{P\langle K}{}^{\langle M}\delta^{N\rangle}{}_{L\rangle}\,, \tag{6.12}$$

which relates the irreducible representation in which $B_{\mu\nu}{}^{MN}$ transforms to the irreducible representation of the 3-form.

Summarizing, we find that the 2-from field strengths (6.5) and the 3-form field strengths (6.11) transform covariantly under the following gauge transformations of the $D = 3$ tensor hierarchy:

$$\delta A_\mu{}^M \;=\; D_\mu\Lambda^M + Z^M{}_{NP}\Lambda_\mu{}^{NP} \tag{6.13}$$

$$\delta B_{\mu\nu}{}^{MN} \;=\; D_{[\mu}\Lambda_{\nu]}{}^{MN} - A_{[\mu}{}^{\langle M}\delta A_{\nu]}{}^{N\rangle} + \Lambda^{\langle M}H_{\mu\nu}{}^{N\rangle} + \frac{2}{3}Y^{MN}{}_{P,KL}\Lambda_{\mu\nu}{}^{P,KL} \tag{6.14}$$

$$\begin{aligned} \delta C_{\mu\nu\rho}{}^{P,MN} &= D_{[\mu}\Lambda_{\nu\rho]}{}^{P,MN} - 3\,\delta A_{[\mu}{}^{\langle P}B_{\nu\rho]}{}^{MN\rangle} + A_{[\mu}{}^{\langle P}A_\nu{}^M\delta A_{\rho]}{}^{N\rangle} \\ &\quad -\frac{3}{2}H_{[\mu\nu}{}^{\langle P}\Lambda_{\rho]}{}^{MN\rangle} - 3\Lambda^{\langle P}G_{\mu\nu\rho}{}^{MN\rangle}\,. \end{aligned} \tag{6.15}$$

Again, the brackets $\langle\,\rangle$ generically impose the constraints on the 2-form and, via (6.12), also the corresponding constraints on the 3-form. As in $D = 4$ the above gauge transformations of the $D = 3$ tensor hierarchy close off-shell. In three dimensions the tensor hierarchy terminates at this point, as there are no higher-rank tensor fields and no further non-trivial Bianchi identities beyond the 3-form identity (6.10).

Now we are in a position to impose manifestly gauge-covariant duality relations, whose compatibility conditions with the Bianchi identities reproduce the supergravity equations of motion (up to the Einstein equation). First, we introduce the unprojected form of the duality relation (6.2), in which the field strength gets modified according to (6.5),

$$\mathcal{E}^{\mu\,M} \;\equiv\; e^{-1}\varepsilon^{\mu\nu\rho}H_{\nu\rho}{}^M + 2J^{\mu M} \;=\; 0\,. \tag{6.16}$$

Next, we define a duality relation for the 2-form potential, which introduces the derivative of the scalar potential with respect to Θ,

$$\mathcal{E}^{MN} \;\equiv\; e^{-1}\varepsilon^{\mu\nu\rho}G_{\mu\nu\rho}{}^{MN} + \frac{1}{4}G^{MN,KL}\Theta_{KL} \;=\; 0\;. \tag{6.17}$$

Here, $G^{MN,KL}$ is a (scalar-dependent) matrix fixed by supersymmetry (for the explicit form in case of $\mathcal{N}=16$ see [21]), which determines the potential according to

$$V \;=\; \frac{1}{32}G^{MN,KL}\Theta_{MN}\Theta_{KL}\;. \tag{6.18}$$

The (formal) G-invariance implies the following identity

$$k^{iM}\frac{\partial V}{\partial\phi^i} - 2Z^M{}_{NP}\frac{\partial V}{\partial\Theta_{NP}} \;=\; 0\;. \tag{6.19}$$

The claim is that the $D=3$ duality hierarchy (6.16) and (6.17) encodes the equations of motion up to the Einstein equations. In this example there are just two equations of motion: the vector equations (6.2), resulting from (6.16) by contracting with Θ_{MN}, and the scalar equations of motion,

$$D_\mu\left(g_{ij}D^\mu\phi^j\right) \;=\; -2\,\frac{\partial V}{\partial\phi^i}\;, \tag{6.20}$$

where g_{ij} is the metric on the scalar manifold. By acting with D_μ on (6.16) and using the second duality relation (6.17) one obtains as a consequence of the Bianchi identity (6.10)

$$D_\mu J^{\mu M} \;=\; -2k^{iM}\frac{\partial V}{\partial\phi^i}\;. \tag{6.21}$$

Alternatively, these equations are identical to the 3-form Bianchi identity (6.10) after replacing in (6.21) the scalar-dependent Noether current $J^{\mu M}$ by the scalar-independent 2-form field strength H^M via the duality relation (6.16) and after replacing the scalar-dependent (derivative of) the scalar potential V by the scalar-independent 3-form field strength G^{MN} via the duality relation (6.17). These second-order 'conservation equations' can be viewed as projected scalar equations of motion in the sense that (6.21) results from (6.20) by contracting with the Killing vector $k_i{}^M$.

6.3 The $D=3$ action

An action including all fields of the $D=3$ tensor hierarchy has already been constructed in [12, 21] (see [27] for the case of global supersymmetry). It reads

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{4}eR + \frac{1}{4}eP^{\mu a}P_{\mu a} - eV \\
& -\frac{1}{4}\varepsilon^{\mu\nu\rho}A_\mu{}^M\vartheta_{MN}\left(\partial_\nu A_\rho{}^N + \frac{1}{3}X_{KL}{}^N A_\nu{}^K A_\rho{}^L\right) + \mathcal{L}_{\text{fermions}} \\
& +\frac{1}{4}\varepsilon^{\mu\nu\rho}D_\mu\vartheta_{MN}B_{\nu\rho}{}^{MN} + \frac{1}{6}\vartheta_{PK}Z^K{}_{MN}\varepsilon^{\mu\nu\rho}C_{\mu\nu\rho}{}^{P,MN}\;.
\end{aligned}
\tag{6.22}
$$

Here we used the definition

$$
P_\mu{}^a \;=\; D_\mu\phi^i\, e_i{}^a(\phi)\;, \tag{6.23}
$$

where $e_i{}^a$ denotes the vielbein on the scalar manifold with flat indices $a, b, \ldots$. We denote the embedding tensor by $\vartheta_{MN} = \vartheta_{MN}(x)$ in order to indicate that it is now a space-time dependent field. As long as the precise form of the scalar potential and the fermionic couplings is not specified, this form of the action is completely general and applies to all gauged supergravities in $D = 3$. In particular, the scalar kinetic term represents a generic non-linear sigma model.

In (6.22) we have made use of the fact that the 2-form potentials emerging in the tensor hierarchy carry the same G-representation as the embedding tensor. This follows from the fact that the tensor $Z^M{}_{NK}$ contracting the 2-forms in (6.5) can be viewed as a G-rotation of ϑ_{MN} and thus satisfies the same representation constraint (if any) as the embedding tensor. The space-time dependent embedding tensor $\vartheta_{MN}(x)$ in (6.22) is set to a constant satisfying the quadratic constraints by the field equations for the 2- and 3-forms.

Like in $D = 4$, in principle, it is also possible to enforce linear constraints via additional top-form Lagrange multipliers. However, since for the action in $D = 3$ the linear constraint is immaterial for bosonic gauge invariance, this would be redundant and so we will not follow this route here. This is in contrast to the $D = 4$ case where linear constraints do play a role for bosonic gauge invariance. In that case we did introduce a Lagrange multiplier for the linear constraint.

In this reformulation with dynamical embedding tensor the original invariance of the action is violated by terms proportional to $\partial_\mu\vartheta_{MN}$ and the quadratic constraint. This can be compensated by assigning appropriate gauge transformations to the 2- and 3-form, as has been done in [21]. The corresponding gauge variations will be denoted by δ_a in order to distinguish them from the gauge transformations δ_h of the tensor hierarchy. As in $D = 4$ we find that δ_a and δ_h differ by terms that are proportional to the duality relations (6.16) and (6.17):

$$
\begin{aligned}
\delta_a B_{\mu\nu}{}^{MN} \;&=\; \delta_h B_{\mu\nu}{}^{MN} + \frac{1}{2}e\varepsilon_{\mu\nu\rho}\Lambda^{\langle M}\mathcal{E}^{\rho\, N\rangle}\;, \\
\delta_a C_{\mu\nu\rho}{}^{P,MN} \;&=\; \delta_h C_{\mu\nu\rho}{}^{P,MN} - \frac{3}{4}e\varepsilon_{\sigma[\mu\nu}\mathcal{E}^{\sigma\langle P}\Lambda_{\rho]}{}^{MN\rangle} - \frac{1}{2}e\varepsilon_{\mu\nu\rho}\Lambda^{\langle P}\mathcal{E}^{MN\rangle}\;,
\end{aligned}
\tag{6.24}
$$

as can be inferred from [21] by comparing the δ_a variations with the tensor hierarchy.[23] We note that the variations of the original vectors and scalars remain unchanged. This modification is precisely such that all field strengths in the transformation rules get replaced by dual (matter) contributions, as Noether currents, etc.

Let us stress again that (6.24) is not equivalent to the original gauge transformations of the tensor hierarchy. First of all, (6.24) does not represent a modification by an equations-of-motion symmetry, since this would have to act on all fields and not just the 2- and 3-forms. Moreover, the modified gauge transformations are not even on-shell equivalent to the tensor hierarchy, due to the fact that neither the duality relation (6.16) nor (6.17) follows from the action. More precisely, the field equations are

$$\frac{\delta S}{\delta A_\mu{}^M} \;=\; -\frac{1}{4}\vartheta_{MN}\,\mathcal{E}^{\mu\,N} \;=\; 0\;, \tag{6.25}$$

$$\frac{\delta S}{\delta \vartheta_{MN}} \;=\; -\frac{1}{4}\left(\mathcal{E}^{MN} + A_\mu{}^{\langle M}\mathcal{E}^{\mu\,N\rangle}\right) \;=\; 0\;. \tag{6.26}$$

Thus, the first duality relation appears only in a contracted version. Once its unprojected form (6.16) has been imposed by hand, the field equations for the embedding tensor (6.26) turn out to be equivalent to (6.17). As a consequence, the field equations obtained from the action are not manifestly gauge-covariant but rather rotate under the gauge transformations in a highly intricate way into the other field equations (including second-order matter equations) [21]. Moreover, the off-shell closure of the gauge algebra characteristic for the abstract tensor hierarchy is violated in that closure requires the validity of *all* field equations (except the Einstein equation).

7 Conclusions

In this paper we have showed how the second-order p-form equations of motion and the projected scalar equations of motion of general $D = 3, 4$ gauged supergravity theories[24] can be derived by a *duality hierarchy*, i.e. a set of first-order duality relations between p-form curvatures.

Our starting point has been the complete *tensor hierarchy* of the embedding tensor formalism which we have used to derive the off-shell gauge algebra for a set of p-form potentials, not including the scalars and the metric tensor. Next, in a second step we have put the tensor hierarchy on-shell by introducing duality relations between the curvatures of the tensor hierarchy. These duality relations contain the metric tensor and all the information about the scalar couplings via natural objects, like the Noether current, the derivative of the scalar potential with respect to the embedding tensor and, in the case of

[23]Strictly speaking, only the maximally supersymmetric case has been investigated in [21]. However, as far as invariance of the bosonic Lagrangian is concerned, this is no restriction.

[24]Actually, our results should apply, unmodified, to more general $D = 3, 4$ theories with no supersymmetry.

four dimensions, a function describing the scalar-vector couplings. We have showed how the duality relations, together with the Bianchi identities of the tensor hierarchy, lead to the desired second-order equations of motion for the p-form potentials and to the projected equations of motion for the scalars.

In a third and final step we have constructed a gauge-invariant action for all the fields of the tensor hierarchy. Here a subtlety occurred. We find that the gauge transformations of the action, with on-shell closed gauge algebra, are not the same as the gauge transformations of the tensor hierarchy, with off-shell closed gauge algebra. They differ by (unprojected) duality relations some of which do not follow from extremizing the action although they are part of the duality hierarchy. We find that the transformation rules that leave the action invariant are obtained from the transformation rules of the tensor hierarchy by replacing everywhere curvatures by dual curvatures via the duality relations except in one term in the gauge transformations of the 4-forms D^{NPQ}, associated to the linear constraint. This exception to the almost-general rule disappears if one solves the linear constraint at the beginning and uses only the allowed field representations. It is reasonable to conjecture that the same will be true in other dimensions and, if true, it would be interesting to find an explanation for this general pattern. It would also be interesting to find out how the general $D = 4$ tensor hierarchy is modified if one relaxes the linear constraint as in Ref. [30], in which the classical lack of gauge invariance can be compensated by a quantum anomaly.

It is natural to ask under which circumstances the duality hierarchy can give rise to the *full* set of scalar equations of motion. For this to be the case, the Killing vector fields need to be 'left-invertible'. For instance, in the $D = 3$ example this means that (6.21) implies (6.20). A necessary condition is that the dimension of the isometry group is larger or equal to the dimension of the scalar manifold. This is satisfied for coset manifolds G/H. In order to see this, let $\mathcal{V}$ be G-valued and $P_\mu{}^a = [\mathcal{V}^{-1}D_\mu\mathcal{V}]^a$ the coset part of the G-invariant Maurer-Cartan forms. The Noether current results from $P_\mu{}^a$ by converting the flat index to a curved or rigid one by means of the coset vielbein $\mathcal{V}$,

$$J_\mu{}^M \;=\; \mathcal{V}^M{}_a P_\mu{}^a \,, \tag{7.1}$$

where the contraction is only over the 'coset directions'. Comparing with (6.3) one infers

$$k_i{}^M \;=\; \mathcal{V}^M{}_a \, e_i{}^a \,. \tag{7.2}$$

Since the vielbeine e and $\mathcal{V}$ are both invertible the desired result follows. Thus, in case of supergravity theories based on coset manifolds, the entire set of field equations (except the Einstein equations) are encoded in first-order duality relations.

It is tempting to conjecture that this pattern will persist in general dimensions $D > 4$. In the context of higher dimensions it is noteworthy that to construct an action not always all fields of the tensor hierarchy are involved. Apart from low-rank forms, which are required for consistent gaugings, and the $(D-1)$- and D-forms, which can be interpreted as Lagrange multipliers, there appears a gap 'in between'. For instance, the $D = 5$ gauged supergravity actions of [25, 26] do not contain a 3-form. In contrast, at the level of the duality hierarchy one is forced to introduce this 3-form in order to recover the correct second-order field equations [7].

One may wonder whether it is possible to also obtain the Einstein equations as compatibility conditions from duality relations. Remarkably, this turns out to be possible upon introducing the dual graviton transforming in the mixed-Young tableaux representation $(D-3,1)$, as has been shown recently [28]. At first sight one would think that one cannot write first-order duality relations since it is not possible to 'pull out a derivative' of the energy-momentum tensor [29]. This is similar to the scalar equations of motion discussed in this paper, where it was not possible to pull out a derivative of the scalar potential. The resolution to this obstruction is in precise analogy to the scalar equations: it requires the introduction of an extra higher rank tensor field, which in this case contains the $(D-2,1)$ Young tableaux. Thus, like in (6.17), a second duality relation has to be imposed, that explicitly contains the energy-momentum tensor. It is intriguing that, therefore, *all* supergravity equations can be written as first-order duality relations (assuming a sufficiently large symmetry in the scalar sector).

Finally, it is interesting to contemplate the possible relation of our findings to the E_{11} approach to supergravity [4–7]. In that context the formulation in terms of duality relations seems to be more natural and thus the present analysis may be of relevance. In this context we note the different status of the higher p-forms in the action and the duality hierarchy. For instance, the incorporation of the top-form and next-to-top form potentials in an action leads to complicated gauge transformation rules with an on-shell closed gauge algebra [21]. It is unlikely that such a structure has a direct Kac-Moody origin. In contrast, the gauge symmetries realized on the duality relations close off-shell in agreement with the tensor hierarchy, and therefore a possible connection to Kac-Moody algebras appears to be more promising. The Kac-Moody approach to supergravity has only been developed so far for supergravities whose scalar sector is given by a coset manifold. It is precisely for these cases that the duality hierarchy reproduces the full set of scalar equations of motion and not just the projected ones. It would be of interest to extend both the Kac-Moody approach as well as the duality hierarchy to supergravities whose scalar sector is given by more general manifolds.

Note added: We would like to mention ref. [31], which was brought to our attention after this paper has been submitted to the bulletin board. Section 4 of [31] also deals with the $D=4$ tensor hierarchy and has some overlap with our section 3.

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A Properties of the W tensors

The W tensors defined in Eqs. (3.57)-(3.59) satisfy the following properties:

$$\Theta_M{}^C W_C{}^{MAB} = 2Q^{AB}\,, \tag{A.1}$$

$$\Theta_M{}^C W_{CNPQ}{}^M = L_{NPQ}\,, \tag{A.2}$$

$$\Theta_M{}^C W_{CNP}{}^{EM} = 2Q_{NP}{}^E\,, \tag{A.3}$$

$$\frac{\partial Q^{AB}}{\partial \Theta_M{}^C} = W_C{}^{MAB}\,, \tag{A.4}$$

$$\frac{\partial L_{NPQ}}{\partial \Theta_M{}^C} = W_{CNPQ}{}^M\,, \tag{A.5}$$

$$\frac{\partial Q_{NP}{}^E}{\partial \Theta_M{}^C} = W_{CNP}{}^{EM}\,, \tag{A.6}$$

$$\delta\Theta_M{}^C W_C{}^{MAB} = \Theta_M{}^C \delta W_C{}^{MAB} = \tfrac{1}{2}\delta(\Theta_M{}^C W_C{}^{MAB}) = \delta Q^{AB}\,, \tag{A.7}$$

$$\delta\Theta_M{}^C W_{CNPQ}{}^M = \delta L_{NPQ}\,, \tag{A.8}$$

$$\delta\Theta_M{}^C W_{CNP}{}^{EM} = \Theta_M{}^C \delta W_{CNP}{}^{EM} = \tfrac{1}{2}\delta(\Theta_M{}^C W_{CNP}{}^{EM}) = \delta Q_{NP}{}^E\,, \tag{A.9}$$

where Q^{AB}, $Q_{NP}{}^E$ and L_{NPQ} are the quadratic and linear constraints Eqs. (3.10), (3.13) and (3.16) imposed on the embedding tensor and where we have not used the constraints themselves.

B Transformations and field strengths in the $D = 4$ tensor hierarchy

The gauge transformations of the different fields of the tensor hierarchy are

$$\delta_h A^M = -\mathfrak{D}\Lambda^M - Z^{MA}\Lambda_A\,, \tag{B.1}$$

$$\delta_h B_A = \mathfrak{D}\Lambda_A + 2T_{A\,NP}[\Lambda^N F^P + \tfrac{1}{2}A^N \wedge \delta_h A^P] - Y_{AM}{}^C\Lambda_C{}^M\,, \tag{B.2}$$

$$\begin{aligned}
\delta_h C_A{}^M &= \mathfrak{D}\Lambda_A{}^M - F^M \wedge \Lambda_A - \delta_h A^M \wedge B_A - \tfrac{1}{3}T_{A\,NP}A^M \wedge A^N \wedge \delta_h A^P + \Lambda^M H_A \\
&\quad -W_A{}^{MAB}\Lambda_{AB} - W_{ANPQ}{}^M\Lambda^{NPQ} - W_{ANP}{}^{EM}\Lambda_E{}^{NP}\,, \\
\delta_h D_{AB} &= \mathfrak{D}\Lambda_{AB} + \alpha B_{[A} \wedge Y_{B]P}{}^E\Lambda_E{}^P + \mathfrak{D}\Lambda_{[A} \wedge B_{B]} - 2\Lambda_{[A} \wedge H_{B]} \\
&\quad +2T_{[A|NP}[\Lambda^N F^P - \tfrac{1}{2}A^N \wedge \delta_h A^P] \wedge B_{|B]}\,,
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
\delta_h D_E{}^{NP} &= \mathfrak{D}\Lambda_E{}^{NP} - [F^N - \tfrac{1}{2}(1-\alpha)Z^{NA}B_A] \wedge \Lambda_E{}^P \\
&\quad +C_E{}^P \wedge \delta_h A^N + \tfrac{1}{12}T_{EQR}A^N \wedge A^P \wedge A^Q \wedge \delta_h A^R + \Lambda^N G_E{}^P\,,
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
\delta_h D^{NPQ} &= \mathfrak{D}\Lambda^{NPQ} - 2A^{(N} \wedge dA^P \wedge \delta_h A^{Q)} - \tfrac{3}{4}X_{RS}{}^{(N}A^{P|} \wedge A^R \wedge A^S \wedge \delta_h A^{|Q)} \\
&\quad -3\Lambda^{(N}F^P \wedge F^{Q)}\,,
\end{aligned} \tag{B.5}$$

and their gauge-covariant field strengths are

$$F^M = dA^M + \tfrac{1}{2}X_{[NP]}{}^M A^N \wedge A^P + Z^{MA}B_A\,, \tag{B.6}$$

$$H_A = \mathfrak{D}B_A + T_{A\,RS}A^R \wedge [dA^S + \tfrac{1}{3}X_{NP}{}^S A^N \wedge A^P] + Y_{AM}{}^C C_C{}^M\,, \tag{B.7}$$

$$\begin{aligned}
G_C{}^M &= \mathfrak{D}C_C{}^M + [F^M - \tfrac{1}{2}Z^{MA}B_A] \wedge B_C + \tfrac{1}{3}T_{C\,SQ}A^M \wedge A^S \wedge dA^Q \\
&\quad +\tfrac{1}{12}T_{C\,SQ}X_{NT}{}^Q A^M \wedge A^S \wedge A^N \wedge A^T \\
&\quad +W_C{}^{MAB}D_{AB} + W_{CNPQ}{}^M D^{NPQ} + W_{CNP}{}^{EM}D_E{}^{NP}\,.
\end{aligned} \tag{B.8}$$

These field strengths are related by the following hierarchical Bianchi identities

$$\mathfrak{D}F^{M} = Z^{MA}H_{A}\,, \tag{B.9}$$

$$\mathfrak{D}H_{A} = Y_{AM}{}^{C}G_{C}{}^{M} + T_{A\,MN}F^{M}\wedge F^{N}\,. \tag{B.10}$$

C Gauge transformations in the $D = 4$ duality hierarchy and action

In hierarchy variables, the total action takes the form

$$\begin{aligned} S = \int \Big\{ & \star R - 2\mathcal{G}_{ij^*}\mathfrak{D}Z^{i}\wedge\star\mathfrak{D}Z^{*j^*} + 2F^{\Sigma}\wedge G_{\Sigma} - \star V \\ & -4Z^{\Sigma A}B_{A}\wedge\left(F_{\Sigma} - \tfrac{1}{2}Z_{\Sigma}{}^{B}B_{B}\right) - \tfrac{4}{3}X_{[MN]\Sigma}A^{M}\wedge A^{N}\wedge\left(F^{\Sigma} - Z^{\Sigma B}B_{B}\right) \\ & -\tfrac{2}{3}X_{[MN]}{}^{\Sigma}A^{M}\wedge A^{N}\wedge\left(dA_{\Sigma} - \tfrac{1}{4}X_{[PQ]\Sigma}A^{P}\wedge A^{Q}\right) \\ & -2\mathfrak{D}\vartheta_{M}{}^{A}\wedge(C_{A}{}^{M} + A^{M}\wedge B_{A}) + 2Q_{NP}{}^{E}(D_{E}{}^{NP} - \tfrac{1}{2}A^{N}\wedge A^{P}\wedge B_{E}) \\ & +2Q^{AB}D_{AB} + 2L_{NPQ}D^{NPQ}\Big\}\,. \end{aligned} \tag{C.1}$$

A general variation of this action is given by

$$\begin{aligned} \delta S = \int \Bigg\{ & \delta g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}} + \left(\delta Z^{i}\frac{\delta S}{\delta Z^{i}} + \text{c.c.}\right) - \delta A^{M}\wedge\star\frac{\delta S}{\delta A^{M}} + 2\delta B_{A}\wedge\star\frac{\delta S}{\delta B_{A}} \\ & -2\mathfrak{D}\vartheta_{M}{}^{A}\wedge\delta C_{A}{}^{M} + 2Q_{NP}{}^{E}\delta D_{E}{}^{NP} + 2Q^{AB}\delta D_{AB} + 2L_{NPQ}\delta D^{NPQ} \\ & +\delta\vartheta_{M}{}^{A}\frac{\delta S}{\delta\vartheta_{M}{}^{A}}\Bigg\}\,, \end{aligned} \tag{C.2}$$

where

$$
\begin{aligned}
\frac{\delta S}{\delta g^{\mu\nu}} &= \star\mathbb{I}\left\{G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_\mu Z^i \mathfrak{D}_\nu Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*\,j^*}] - G^M{}_{(\mu|}{}^\rho \star G_{M|\nu)\rho}\right. \\
&\quad \left. +\tfrac{1}{2}g_{\mu\nu}V\right\}\,, \qquad \text{(C.3)}
\end{aligned}
$$

$$
\tfrac{1}{2}\frac{\delta S}{\delta Z^i} = \mathcal{G}_{ij^*}\mathfrak{D}\star\mathfrak{D}Z^{*\,j^*} - \partial_i G_M{}^+ \wedge G^{M+} - \star\tfrac{1}{2}\partial_i V\,, \qquad \text{(C.4)}
$$

$$
\begin{aligned}
-\tfrac{1}{4}\star\frac{\delta S}{\delta A^M} &= \mathfrak{D}F_M - \tfrac{1}{4}\vartheta_M{}^A \star j_A - \tfrac{1}{3}dX_{[PQ]M}\wedge A^P \wedge A^Q + \tfrac{1}{2}Q_{MP}{}^E C_E{}^P - \tfrac{1}{2}Q_{(NM)}{}^E A^N \wedge B_E \\
&\quad -L_{MNP}A^N \wedge \left(dA^P + \tfrac{3}{8}X_{[RS]}{}^P A^R \wedge A^S\right) + \tfrac{1}{8}Q_{NP}{}^E T_{EQM} A^N \wedge A^P \wedge A^Q \\
&\quad -d(F_M - G_M) - X_{[MN]}{}^P A^N \wedge (F_P - G_P) + \tfrac{1}{2}\mathfrak{D}\vartheta_M{}^A \wedge B_A\,, \qquad \text{(C.5)}
\end{aligned}
$$

$$
\star\frac{\delta S}{\delta B_A} = \vartheta^{PA}(F_P - G_P) + Q^{AB}B_B - \mathfrak{D}\vartheta_M{}^A \wedge A^M - \tfrac{1}{2}Q_{NP}{}^A A^N \wedge A^P\,, \qquad \text{(C.6)}
$$

$$
\begin{aligned}
\tfrac{1}{2}\frac{\delta S}{\delta \vartheta_M{}^A} &= (G_A{}^M - \tfrac{1}{2}\star\partial V/\partial\vartheta_M{}^A) - A^M \wedge (H_A + \tfrac{1}{2}\star j_A) \\
&\quad +\tfrac{1}{2}T_{ANP}A^M \wedge A^N \wedge (F^P - G^P) - (F^M - G^M)\wedge B_A\,, \qquad \text{(C.7)}
\end{aligned}
$$

and vanishes, up to total derivatives, for the gauge transformations

$$\delta_a \vartheta_M{}^A = 0\,, \tag{C.8}$$

$$\delta_a Z^i = \Lambda^M \vartheta_M{}^A k_A{}^i\,, \tag{C.9}$$

$$\delta_a A^M = \delta_h A^M\,, \tag{C.10}$$

$$\delta_a B_A = \delta_h B_A - 2T_{A\,NP}\Lambda^N(F^P - G^P)\,, \tag{C.11}$$

$$\delta_a C_A{}^M = \delta_h C_A{}^M + \Lambda_A \wedge (F^M - G^M) - \Lambda^M(H_A + \tfrac{1}{2}\star j_A)\,, \tag{C.12}$$

$$\delta_a D_{AB} = \delta_h D_{AB} + 2\Lambda_{[A} \wedge (H_{B]} + \tfrac{1}{2}\star j_{B]}) - 2T_{[A|\,NP}\Lambda^N(F^P - G^P) \wedge B_{|B]}\,, \tag{C.13}$$

$$\delta_a D_E{}^{NP} = \delta_h D_E{}^{NP} - \Lambda^N(G_E{}^P - \tfrac{1}{2}\star \partial V/\partial\vartheta_P{}^E) + 2(F^N - G^N) \wedge \Lambda_E{}^P\,, \tag{C.14}$$

$$\begin{aligned}\delta_a D^{NPQ} = {}& \delta_h D^{NPQ} - 3\delta A^{(N} \wedge A^P \wedge (F^{Q)} - G^{Q)}) + 6\Lambda^{(N} F^P \wedge (F^{Q)} - G^{Q)}) \\ & -3\Lambda^{(N}(F^P - G^P) \wedge (F^{Q)} - G^{Q)})\,.\end{aligned} \tag{C.15}$$

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