A brief introduction to Loop Quantum Cosmology

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In recent years, Loop Quantum Gravity has emerged as a solid candidate for a non-perturbative quantum theory of General Relativity. It is a background independent theory based on a description of the gravitational field in terms of holonomies and fluxes. In order to discuss its physical implications, a lot of attention has been paid to the application of the quantization techniques of Loop Quantum Gravity to symmetry reduced models with cosmological solutions, a line of research that has been called Loop Quantum Cosmology. We summarize its fundamentals and the main differences with respect to the more conventional quantization approaches employed in cosmology until now. In addition, we comment on the most important results that have been obtained in Loop Quantum Cosmology by analyzing simple homogeneous and isotropic models. These results include the resolution of the classical big-bang singularity, which is replaced by a quantum bounce.

1. MOTIVATION

Gravity is the only fundamental physical interaction which is not yet satisfactorily described quantum mechanically. Even without adhering to the belief that all fundamental interactions should finally be unified in a single theory, a strong motivation to search for a quantum theory of gravity comes from the very own results of General Relativity. The classical singularity theorems that arise in Einstein theory [1] imply that (in a variety of physically relevant situations) the predictability breaks down, so that the regime of applicability of General Relativity has been surpassed. Therefore, a new and more

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fundamental theory is needed for a correct physical description.

In trying to quantize General Relativity, the first obstacle that one finds is that Einstein theory is not renormalizable as a quantum field theory, so that a conventional perturbative quantization cannot be performed. In this context, an alternative quantization program, known as Loop Quantum Gravity (LQG), has recently been proposed for General Relativity [2, 3, 4]. LQG is an attempt to construct a nonperturbative quantum theory of gravity using techniques similar to those of gauge field theories (e.g., Yang-Mills). The application of these nonperturbative quantization techniques to simple gravitational models with application in cosmology, such as homogeneous and isotropic spacetimes with different types of matter content, has given rise to a new branch of gravitational physics called Loop Quantum Cosmology (LQC) [5].

2. HAMILTONIAN FORMULATION OF GENERAL RELATIVITY AND ASHTEKAR VARIABLES

LQG is a nonperturbative canonical quantization of General Relativity; therefore, it is constructed starting from a Hamiltonian formulation of Einstein theory [6]. Let us review very briefly this formulation.

We consider globally hyperbolic four-dimensional spacetimes $(g_{\alpha\beta}, M = R \times \Sigma)$, where $g_{\alpha\beta}$ is a Lorentzian metric, Greek indices are spacetime indices, and Σ is a three-dimensional manifold. For General Relativity, once Σ is given, the physically relevant information to determine the classical solutions is contained in the spatial three-metric h_{ab} induced on Σ, and in the corresponding extrinsic curvature $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$, where Latin indices from the beginning of the alphabet denote spatial indices, n is the unit normal to Σ, and $\mathcal{L}$ is the Lie derivative. Equivalently, we can adopt co-triads e_a^i (rather than spatial metrics) to describe the system. This allows the coupling to fermionic matter fields. With η_{ij} being the Euclidean three-metric, we have the relations

$$h_{ab} = e_a^i \eta_{ij} e_b^j, \qquad K_a^i = K_{ab} e_j^b \eta^{ij}. \tag{1}$$

Here, the triad e_j^b is the inverse of the co-triad, $e_a^i e_i^b = \delta_a^b$ and $e_a^i e_j^a = \delta_j^i$, and Latin indices from the middle of the alphabet are internal SU(2) indices, corresponding to the symmetries of the Euclidean metric η_{ij} under linear transformations.

A set of canonical variables for General Relativity (in the sense that their Poisson bracket is proportional to the identity) is given then by the densitized triad E_i^a and the extrinsic curvature in triadic form K_a^i:

$$E_i^a := \sqrt{\det h}\, e_i^a, \quad K_a^i \quad \rightarrow \quad \left\{ K_a^i(x), E_j^b(y) \right\} = \delta_a^b \delta_j^i \delta^{(3)}(x - y). \qquad (2)$$

In this expression, x and y are two generic points in Σ, and we have chosen units such that $8\pi G = 1$, where G is Newton constant.[1]

Actually, we can replace the triadic extrinsic curvature by a connection valued 1-form taking values on su(2), and still obtain a canonical set of variables. For this, it suffices to realize that the co-triad determines an su(2)-connection compatible with it, Γ_a^i, and notice that the sum of this connection with any vector (both from the internal and spatial viewpoints) provides again an su(2)-connection valued 1-form. Therefore, at the classical level, we can simply replace K_a^i with $A^{(\gamma)\,i}_{\quad a} = \Gamma_a^i + \gamma K_a^i$. Here, γ is a nonzero constant called the Immirzi parameter, and its presence can be seen to lead to an ambiguity in the quantization [7, 8] which is usually resolved in LQG by appealing to the recovery of the Bekenstein-Hawking law for the entropy of black holes [9]. For simplicity, we will set it equal to one from now on. The calculations for general γ can be easily reproduced along the lines explained below.

We will thus adopt as canonical variables the set formed by $A_a^i = \Gamma_a^i + K_a^i$ and E_i^a. In General Relativity, these variables are subject to three types of constraints [4, 6]. First, there is a Gauss constraint which generates SU(2)-transformations,

$$\mathcal{G}_i := \partial_a E_i^a + \epsilon_{ij}{}^k A_a^j E_k^a = 0. \qquad (3)$$

In addition, the invariance of the theory under spatial diffeomorphisms is reflected in the so-called vector or diffeomorphism constraint,

$$\mathcal{V}_a := F_{ab}^i E_i^b = 0, \qquad (4)$$

where F_{ab}^i is the curvature of the connection A_a^i, namely

$$F_{ab}^i = 2\partial_{[a} A_{b]}^i + \epsilon^i{}_{jk} A_a^j A_b^k. \qquad (5)$$

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[1] In the following, we also set $\hbar = c = 1$.

Here, ϵ_{ijk} is the totally antisymmetric symbol. Finally, the invariance of General Relativity under time reparametrizations leads to a scalar constraint, also called Hamiltonian constraint, which in vacuo takes the expression

$$\mathcal{S} := E_i^a E_j^b \left(\epsilon^{ij}{}_k F_{ab}^k - 4K_{[a}^i K_{b]}^j \right) = 0. \tag{6}$$

Given the four-dimensional covariance of Einstein theory, General Relativity is a completely constrained system, i.e., the total Hamiltonian which generates the dynamics is just a(n integrated) linear combination of constraints. In particular, apart from boundary terms, the Hamiltonian vanishes on classical solutions. On the other hand, it is worth pointing out that General Relativity is formulated in terms of connections and densitized triads without introducing any metric background structure. This background independence plays a fundamental role in the theory and will be a basic guideline for the selection of a quantization procedure in the construction of LQG.

3. HOLONOMY AND FLUX ALGEBRA

Since SU(2)-transformations are symmetries of our gravitational systems, only the gauge invariant information about the connection is physically relevant. Taking into account that this information is captured by the Wilson loops [10, 11], we can then replace the connection by SU(2)-holonomies. More specifically, from now on we will consider holonomies along piecewise analytic[2] edges e, where we understand that an edge is an embedding of the interval [0,1] in Σ [4, 12]. We call h_e the corresponding holonomy,

$$h_e = \mathcal{P} \exp \int_e A_a^i \tau_i dx^a. \tag{7}$$

Here, the symbol $\mathcal{P}$ denotes path ordering, and $\{\tau_j = -\frac{i}{2}\sigma_j; \ j = 1, 2, 3\}$ is a basis in the algebra su(2), with σ_j being the Pauli matrices. Let us notice that the line integral appearing in the holonomies implies a one-dimensional smearing of the connection, and that no use of background structures has been made in the definition of the holonomy.

Since the most relevant field divergences in our theory are expected to come from the appearance of the three-dimensional delta function in the basic Poisson brackets between

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[2] The restriction of piecewise analyticity ensures that the intersection between edges, as well as the intersection of an edge with a (piecewise analytic) surface, occurs in a finite number of points [12].

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our variables, and we have already smeared the connection over one dimension, it seems natural to try to smear now E_i^a over two dimensions. Once again, we want to carry out this smearing without employing any background structure. Remarkably, this requirement can be fulfilled because E_i^a is a vector density. Hence, for any piecewise analytic surface S and any su(2)-valued smooth function f^i on it, we introduce the associated flux of the densitized triad,

$$E(S, f) = \int_S E_i^a f^i \epsilon_{abc} \, dx^b dx^c. \tag{8}$$

The defined holonomies and fluxes form an algebra under Poisson brackets. In the following, we take this algebra as our algebra of elementary phase space variables. From this perspective, the quantization of the system amounts to constructing a representation of this algebra. A keystone result in LQG is a uniqueness representation theorem known as the LOST theorem (after the initials of its authors [13]). The LOST theorem states that there exists only one cyclic representation of the holonomy-flux algebra with a diffeomorphism-invariant state (interpretable as a "vacuum"). Therefore, the choice of the algebra of elementary variables, motivated by background independence, together with the identification of diffeomorphism invariance as a fundamental symmetry suffice to pick out a unique quantization (up to unitary equivalence).

In order to gain insight into the kind of quantization adopted in LQG, let us first call cylindrical those complex functions of the connection that depend on it only via the holonomies along a finite number of edges (forming a graph [12]). We can identify the commutative unital ∗-algebra of these functions as the algebra of configuration variables. By completing it with respect to the sup-norm[3] (i.e. the supremum norm), we obtain a commutative C^*-algebra with identity. Gel'fand theory ensures then that this algebra is (isomorphic to) that of continuous functions on a certain compact space, $\bar{A}$, which is usually called the spectrum [14]. Smooth connections are dense in this space $\bar{A}$ of quantum generalized connections. Besides, the Hilbert space of any representation of the C^*-configuration algebra is of the form $L^2(\bar{A}, \mu)$ for some measure μ. The LOST theorem guarantees that there is a unique Hilbert space $L^2(\bar{A}, \mu_{AL})$ supporting a representation not just of the holonomies, but of the whole holonomy-flux algebra, and such that μ_{AL}

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[3] The use of this norm is motivated by the fact that, in a representation in which configuration variables acted by multiplication, the operator norm would coincide with the sup-norm.

(the so-called the Ashtekar-Lewandowski measure) is a diffeomorphism-invariant, regular Borel measure. This representation turns out not to be continuous and, as an important consequence, the connection itself cannot be defined as an operator valued distribution [12].

4. LOOP QUANTUM COSMOLOGY: FLAT FRW MODEL

LQC confronts the quantum analysis of cosmological systems by applying similar quantization techniques to those described for LQG. Here, we will focus our discussion on a simple but physically relevant model, namely, the case of homogeneous and isotropic flat (FRW) cosmologies. As the matter content, we will consider a massless minimally coupled scalar field.

The spatial manifold Σ is topologically $I\!\!R^3$, endowed with the action of the Euclidean group. One can introduce a fiducial flat co-triad, ${}^0e_a^i$, with the corresponding fiducial metric and triad, ${}^0h_{ab}$ and ${}^0e_i^a$. Given the non-compactness of Σ, we also choose a reference cell adapted to the fiducial triad in order to integrate homogenous quantities, such as the symplectic structure or the Hamiltonian, without introducing infinities in our formalism. We use the symbol V_0 to denote the fiducial volume of this cell. Actually, physical results can be proven independent of these choices under a suitable definition of the elementary variables [15, 16]. In more detail, one can fix the gauge and diffeomorphism freedom so that

$$A_a = c\, V_0^{-1/3}\ {}^0e_a^i \tau_i, \quad \text{and} \quad E^a = p\, V_0^{-2/3} \sqrt{\det {}^0h}\ \ {}^0e_i^a \tau^i. \tag{9}$$

Here, c and p are constant on Σ (but not under evolution), and describe the only remaining degrees of freedom in our basic variables. In the following, we call Γ^S the subspace of the gravitational phase space (for full General Relativity) defined in this way.

The gravitational symplectic structure induced on Σ is just

$$\Omega^S = 3\mathbf{d}c \wedge \mathbf{d}p, \quad \text{so that} \quad \{c, p\} = \frac{1}{3}. \tag{10}$$

The variables c and p are hence canonical, apart from the factor of 1/3. Holonomies along straight edges $\mu\, {}^0e_i^a$ in the fiducial directions suffice to separate symmetric connections, i.e., given two different connections, there always exists an edge of this kind for which the

corresponding holonomies differ [14]. We thus restrict our attention to those holonomies, $h_{\circ e_i}(\mu)$, which have the form

$$h_{\circ e_i}(\mu) = \cos\left(\frac{\mu c}{2}\right)\mathbf{1} + 2\sin\left(\frac{\mu c}{2}\right)\tau_i. \tag{11}$$

Similarly, densitized triads can now be smeared just across squares with edges parallel to the fiducial directions,

$$E(S, f) = p\,^0A(S, f)V_0^{-2/3}. \tag{12}$$

The factor $^0A(S, f)$ measures only the fiducial area of S weighted with an orientation factor. In this sense, fluxes are totally determined by p, which therefore plays the role of a momentum. The configuration algebra, on the other hand, is generated by sums of products of matrix elements of holonomies. Thus, it is the linear space of continuous and bounded complex functions in $\mathbb{R}$ provided by finite sums of the form $f(c) = \sum_n f_n\, e^{i\mu_n c}$. Its completion with respect to the sup-norm is known to be (isomorphic to) the Bohr C^*-algebra of almost periodic functions [14].

5. BOHR COMPACTIFICATION AND POLYMER REPRESENTATION

As we have commented, the configuration C^*-algebra is the algebra of almost periodic functions. The (Gel'fand) spectrum of this algebra is the Bohr compactification of the real line, $\mathbb{R}_{Bohr}$ [14]. This compactification can be understood as the set of group homomorphisms from the group $\mathbb{R}$ (with the sum) to the multiplicative group T of complex numbers with unit norm. So, every $x \in \mathbb{R}_{Bohr}$ is a map $x : \mathbb{R} \to T$ which satisfies

$$x(0) = 1, \qquad x(p_1 + p_2) = x(p_1)x(p_2) \quad \forall p_1, p_2 \in \mathbb{R}. \tag{13}$$

Since T is a commutative group, the operation $x\tilde{x}(p) := x(p)\tilde{x}(p)$ provides a commutative group structure in $\mathbb{R}_{Bohr}$. This group is compact with respect to the Tychonoff product topology. We recall that the Tychonoff topology is the weakest topology for which the functions $F_p : \mathbb{R}_{Bohr} \to T$ given by evaluation at p [i.e. $F_p(x) := x(p)$] are all continuous for any $p \in \mathbb{R}$ [14]. Besides, the real line is actually dense in $\mathbb{R}_{Bohr}$. This result follows from the fact that the algebra of functions $f(c)$ considered at the end of the previous section separates points $c \in \mathbb{R}$ [14].

The compact group $\mathbb{R}_{Bohr}$ is equipped with a normalized invariant measure under the group operation, namely, the Haar measure μ_H. The representation of the holonomy-flux

algebra for LQC is given precisely by the Hilbert space $L^2(\mathbb{R}_{Bohr}, \mu_H)$. In addition, since μ_H is invariant under multiplication in the group, we get that, $\forall \tilde{x} \in \mathbb{R}_{Bohr}$,

$$[1 - \tilde{x}(p)] \int_{\mathbb{R}_{Bohr}} F_p(x) d\mu_H(x) = 0, \tag{14}$$

from where it follows that

$$\int_{\mathbb{R}_{Bohr}} F_p(x) d\mu_H(x) = \delta_p^0. \tag{15}$$

Taking into account that, from our definitions, $F_{p_1} F_{p_2} = F_{p_1 + p_2}$ and $F_p^* = F_{-p}$, it is straightforward to conclude that the set $\{F_p, p \in \mathbb{R}\}$ is orthonormal (hence, the Hilbert space $L^2(\mathbb{R}_{Bohr}, \mu_H)$ is nonseparable). One can also see that this set is dense [14]. As a consequence, the Hilbert space $L^2(\mathbb{R}_{Bohr}, \mu_H)$ is isomorphic to the so-called "polymer" space of functions of $p \in \mathbb{R}$ that are square integrable with respect to the discrete measure. The isomorphism is given by $I : F_p \to |p\rangle \; \forall p \in \mathbb{R}$.

Employing then the orthonormal basis

$$\left\{ |p\rangle \; ; \; p \in \mathbb{R}, \langle \tilde{p}|p\rangle = \delta_p^{\tilde{p}} \right\}, \tag{16}$$

and introducing the notation $N_\mu := \exp(i\mu c/2)$, the polymer "momentum" representation is determined by the following action of the holonomy and flux operators:

$$\hat{p} |p\rangle = \frac{p}{6} |p\rangle, \qquad \hat{N}_\mu |p\rangle = |p + \mu\rangle. \tag{17}$$

In this representation, states take the general form

$$|\psi\rangle := \sum_{p \in \mathbb{R}} \psi(p) |p\rangle \; ; \qquad \sum_{p \in \mathbb{R}} |\psi(p)|^2 < \infty. \tag{18}$$

Note that normalizable states $\psi(p)$ differ from zero only on a countable subset of the real line for the label p, because the sequence $\{\psi(p)\}$ is square summable. On the other hand, it is worth noticing that the representation (of N_μ) fails to be continuous in μ, as can be easily seen by realizing that the state $|p\rangle$ is always orthogonal to $|p + \mu\rangle$ regardless of the value of $\mu \neq 0$. Therefore, the connection operator $\hat{c}$ is not well defined, in total parallelism with the situation discussed for LQG. The failure of continuity makes evident that the representation is inequivalent to the standard Schrdinger one of geometrodynamics [19] (often called the Wheeler-DeWitt representation). This lack of continuity explains why the Stone-von Neumann uniqueness theorem of Quantum Mechanics does not apply [17], allowing the results of LQC to differ radically from those –physically unsatisfactory– attained in geometrodynamics.

6.   QUANTUM FRW MODEL

With our symmetry reduction to the flat FRW model and our choice of fiducial structures, the triad adopts the expression $e_i^a = \text{sign}(p)|p|^{-1/2}V_0^{1/3}\, {}^0e_i^a$. This triad diverges at the big-bang singularity, corresponding to $p = 0$. In the quantum theory, on the other hand, $\hat{p}$ has just a point spectrum [18] which coincides with the whole real line, since the basis states $|p\rangle$ have unit norm $\forall p \in \mathbb{R}$. Since zero is included in this point spectrum, the related (inverse) operator $|\hat{p}|^{-1}$ is not well defined. However, it is actually possible to define a triad operator in terms of our elementary ones [20]. Classically, we have the following identity $\forall \bar{\mu} \in \mathbb{R}$:

$$\frac{\text{sign}(p)}{\sqrt{|p|}} = \frac{4}{\bar{\mu}}\text{tr}\left(\sum_{i=1}^{3}\tau^i h_{{}^0e_i}(\bar{\mu})\left\{h_{{}^0e_i}^{-1}(\bar{\mu}), \sqrt{|p|}\right\}\right). \tag{19}$$

Here, $h_{{}^0e_i}$ is again the holonomy along the edge ${}^0e_i$, and the symbol tr denotes the trace. Then, replacing Poisson brackets with $-i$ times commutators, we obtain

$$\frac{\bar{\mu}}{6}\left[\widehat{\frac{\text{sign}(p)}{\sqrt{|p|}}}\right] = \hat{N}_{-\bar{\mu}}|\hat{p}|^{1/2}\hat{N}_{\bar{\mu}} - \hat{N}_{\bar{\mu}}|\hat{p}|^{1/2}\hat{N}_{-\bar{\mu}}. \tag{20}$$

It is not difficult to check that this operator is diagonal in the p-basis. Furthermore, it is bounded from above, so that the classical divergence at $p = 0$ disappears quantum mechanically with this regularization of the triad [20]. In fact, this triad operator is such that it annihilates the state $|p = 0\rangle$.

Since we have already fixed the gauge and diffeomorphism freedom, the only constraint remaining in the system is the Hamiltonian one. For flat FRW spacetimes with a massless scalar field ϕ (and unit lapse), this constraint can be obtained from the evaluation of the following expression in the symmetry reduced model [15, 16]

$$H := \frac{1}{2}\int_{\mathbb{R}^3}|\det E|^{-1/2}\left(P_\phi^2 - \epsilon^{ij}{}_k E_i^a E_j^b F_{ab}^k\right) = 0. \tag{21}$$

To define the operator corresponding to $|\det E|^{-1/2}$ (or to $|\det E|^{-1/2}\epsilon^{ij}{}_k E_i^a E_j^b$ in the gravitational part of the constraint [20]) we proceed as we have explained above when discussing the triad operator. In addition, to introduce an operator representation for the curvature F_{ab}^k, we first recall the classical relation

$$F_{ab}^k = -2\lim_{\bar{\mu}\to 0}\text{tr}\left(\frac{h_{[ij]}(\bar{\mu}) - 1}{\bar{\mu}^2 V_0^{2/3}}\tau^k\, {}^0e_a^i\, {}^0e_b^j\right), \tag{22}$$

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which is valid for any real value of $\bar{\mu}$ and where

$$h_{[ij]}(\bar{\mu}) := h_{0e_i}(\bar{\mu}) h_{0e_j}(\bar{\mu}) h^{-1}_{0e_i}(\bar{\mu}) h^{-1}_{0e_j}(\bar{\mu}). \tag{23}$$

Nonetheless, after substituting classical holonomies by their quantum counterparts, the limit of zero regulator $\bar{\mu}$ cannot be taken in the resulting curvature operator. This circumstance is interpreted as a manifestation of the fact that, in LQG, the area spectrum is discrete with a minimum nonzero eigenvalue [4, 21], so that the square with edges $\bar{\mu}\,^0 e_i$ and $\bar{\mu}\,^0 e_j$, employed to define $h_{[ij]}(\bar{\mu})$, cannot be shrunk to zero. The regulator is then fixed by demanding that the physical area of this square equals the minimum nonvanishing eigenvalue allowed in LQG, which we call Δ from now on. Hence, one gets the operator relation $\bar{\mu}^2|\hat{p}| = \Delta$.

At this stage, it is convenient to relabel the p-basis by introducing the affine parameter associated with the vector field $\frac{1}{6}\bar{\mu}\partial_p$ [16]. This vector field can be regarded as that corresponding to the exponent $\frac{1}{2}\bar{\mu}c$ in the holonomy $N_{\bar{\mu}}$. Taking into account that the physical volume of the fiducial cell is given by the operator $\hat{V} = |\hat{p}|^{3/2}$, the above relabeling leads to a basis of volume eigenstates $|\nu\rangle$, where $\nu = 4\,\mathrm{sign}(p)|p|^{3/2}/\sqrt{\Delta}$. The operator $\hat{N}_{\bar{\mu}}$ is then defined to produce a constant unit shift in the new label [16],

$$\hat{N}_{\bar{\mu}}|\nu\rangle := |\nu + 1\rangle. \tag{24}$$

Using the standard Schrdinger representation for the matter field, so that the total Hilbert space is the tensor product of the polymeric one and of $L^2(\mathbb{R}, d\phi)$, and adopting a suitable factor ordering, one finally arrives at the following quantum Hamiltonian constraint:

$$\hat{H} := \frac{1}{2}\left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{3/2}\left(-6\widehat{\Omega}^2 + \hat{P}_\phi^2\right)\left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{3/2}, \tag{25}$$

$$\widehat{\Omega} := \frac{1}{4\sqrt{\Delta}i}\left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{-1/2}\widehat{\sqrt{|p|}}\left[\left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right)\widehat{\mathrm{sign}(p)} + \widehat{\mathrm{sign}(p)}\left(\hat{N}_{2\bar{\mu}} - \hat{N}_{-2\bar{\mu}}\right)\right]$$

$$\cdot\ \widehat{\sqrt{|p|}}\left[\widehat{\frac{1}{\sqrt{|p|}}}\right]^{-1/2}. \tag{26}$$

The symmetric factor ordering adopted for $\widehat{\Omega}$ arises naturally from the consideration of homogeneous but anisotropic models of Bianchi I type, where the ordering is well motivated, regarding the flat FRW cosmologies as a special case with vanishing anisotropies

[22]. It is straightforward to check that the above quantum constraint annihilates the state $|p=0\rangle$ (or equivalently $|\nu=0\rangle$) and leaves invariant its orthogonal complement. In the search for nontrivial solutions to the constraint, one can then restrict all considerations to this orthogonal complement, so that the classical singularity, corresponding to $p=0$, can be removed from the kinematical (gravitational) Hilbert space [22, 23]. In this sense, the big-bang singularity is already resolved quantum mechanically (see also [24]).

7.   DENSITIZED CONSTRAINT

Once the state $|\nu=0\rangle$ has been removed, let us call $Cyl_S^{\neq}$ the linear span of the nonzero volume eigenstates $\{|\nu\rangle\,;\nu\neq 0,\nu\in I\!\!R\}$. Based on previous experience with gravitational models, we expect nontrivial solutions to the constraint to live in the algebraic dual of $Cyl_S^{\neq}$. Since the operator $[\widehat{1/\sqrt{|p|}}]$ is invertible in the orthogonal complement of $|\nu=0\rangle$, it is easy to check that one gets a bijection between the considered solutions and those of the alternative "densitized" constraint [22]

$$\hat{C} := -6\widehat{\Omega}^2 + \hat{P}_\phi^2. \tag{27}$$

The operator $\widehat{\Omega}^2$ (with domain $Cyl_S^{\neq}$) has the following action:

$$\widehat{\Omega}^2\,|\nu\rangle = -\; f_+(\nu)f_+(\nu+2)\,|\nu+4\rangle + \left[f_+^2(\nu)+f_-^2(\nu)\right]|\nu\rangle$$
$$-\; f_-(\nu)f_-(\nu-2)\,|\nu-4\rangle\,, \tag{28}$$

where

$$f_\pm(\nu) = \frac{1}{4\sqrt{6}\,\Delta^{1/4}}g(\nu\pm 2)s_\pm(\nu)g(\nu), \qquad s_\pm(\nu) = \text{sign}(\nu\pm 2)+\text{sign}(\nu), \tag{29}$$

and

$$g(\nu) = \left|\left|1+\frac{1}{\nu}\right|^{\frac{1}{3}} - \left|1-\frac{1}{\nu}\right|^{\frac{1}{3}}\right|^{-\frac{1}{2}} \qquad \text{if}\quad \nu\neq 0, \tag{30}$$

while $g(\nu=0)=0$. We notice that $\widehat{\Omega}^2$ relates only states $|\nu\rangle$ whose label differ by a multiple of four. Moreover, owing to the linear combination of signs in $s_\pm(\nu)$, one can see that the real function $f_+(\nu)f_+(\nu+2)$ has a remarkable property, namely, it vanishes in the whole interval [-4,0]. Something similar happens with $f_-(\nu)f_-(\nu-2)$, which vanishes in [0,4]. As a consequence, for the label ν, the action of the operator $\widehat{\Omega}^2$ does not mix any

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of the semilattices $\mathcal{L}_\varepsilon^\pm := \{\pm(\varepsilon + 4n), \ n \in I\!N\}$, with $\varepsilon \in (0, 4]$ –but otherwise unspecified. In the following, we call $H_\varepsilon^\pm$ the corresponding Hilbert subspaces of states with support in these semilattices (i.e., the completion of the linear span of ν-states with $\nu \in \mathcal{L}_\varepsilon^\pm$). Each of these subspaces can be considered a superselection sector for the quantum theory, inasmuch as they provide irreducible representations for the physically relevant operators of the model [15, 16].

On the other hand, it is possible to prove that (up to a global multiplicative factor) $\widehat{\Omega}^2$, restricted to $H_\varepsilon^+ \cup H_{4-\varepsilon}^-$, differs by a symmetric, trace-class operator from an operator which is unitarily related with the Hamiltonian of a point particle in a Psch-Teller potential [22, 25]. From the properties of this Hamiltonian and Kato perturbation theory [26], it then follows that $\widehat{\Omega}^2$ is essentially self-adjoint and that its absolutely continuous spectrum[4] is $I\!R^+$. The rest of the spectrum can be proven empty [22]. Moreover, $\widehat{\Omega}^2$ commutes with the projections to H_ε^+ and $H_{4-\varepsilon}^-$. One can then see that the operator on any of these Hilbert spaces is positive with an absolutely continuous spectrum of unit degeneracy [22].

Hence, on any superselection sector $H_\varepsilon^\pm$, one obtains a spectral decomposition of the identity of the form

$$\mathbf{1}_{\pm\epsilon} = \int_0^\infty d\lambda |e_\lambda^{\pm\epsilon}\rangle\langle e_\lambda^{\pm\epsilon}|, \tag{31}$$

where $|e_\lambda^{\pm\epsilon}\rangle$ is a generalized eigenstate of $\widehat{\Omega}^2$ with eigenvalue equal to λ. Finally, let us comment that, expressing $|e_\lambda^{\pm\epsilon}\rangle$ in the ν-basis, the corresponding generalized eigenfunctions $e_\lambda^{\pm\epsilon}(\nu)$ can always be chosen real.

8. PHYSICAL STATES

Employing the above spectral decomposition associated with $\widehat{\Omega}^2$, elements of the polymer space $H_\varepsilon^\pm$ can be identified with elements of the Hilbert space $L^2(I\!R^+, d\lambda)$. It is now straightforward to solve the densitized constraint $-6\widehat{\Omega}^2 + \hat{P}_\phi^2 = 0$. Starting from the kinematical Hilbert space $H_\varepsilon^\pm \otimes L^2(I\!R, d\phi)$, the solutions adopt the form

$$\psi(\nu, \phi) = \int_0^\infty d\lambda \, e_\lambda^{\pm\varepsilon}(\nu) \left[\psi_+(\lambda) e^{i\sqrt{6\lambda}\phi} + \psi_-(\lambda) e^{-i\sqrt{6\lambda}\phi} \right]. \tag{32}$$

Physical states can be identified with positive frequency solutions, and hence with wavefunctions in $L^2(I\!R^+, d\lambda)$. A complete set of Dirac observables (acting on physical

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[4] See, e.g., Appendix C in reference [18] for a summary on operator theory.

states) is given by $\hat{P}_\phi$ and, e.g., $|\hat{\nu}|_{\phi_0}$, the latter being defined by the action of $|\hat{\nu}|$ when $\phi = \phi_0$. [5] In this way, the LQC approach succeeds in achieving a complete quantization of the flat FRW model with massless scalar field.

Rather than in general physical states, one is usually interested in states which display a semiclassical behavior in the region of large spatial volumes and matter fields, so that they can be regarded as potential candidates to explain the properties of universes like the one which we observe. With this motivation, we can concentrate our considerations on positive frequency states which, for a fixed large value of the scalar field $\phi = \phi_0 \gg 1$, are peaked on certain values $P_\phi = P_\phi^0$ and $\nu = \nu^0$ of the Dirac observables such that $|\nu^0| \gg 1$ and $|P_\phi^0| \gg 1$ [16]. In more detail, one analyzes Gaussians of the form

$$\psi_+ \left(\lambda = \frac{\omega^2}{6} \right) \propto e^{-(\omega + P_\phi^0)^2/(2\sigma^2)} e^{-i\omega\phi_1}, \quad \text{with} \quad \phi_1 = \phi_0 - \sqrt{\frac{2}{3}} \ln |\nu^0|. \tag{33}$$

Numerical integration of the quantum evolution dictated by the densitized Hamiltonian constraint shows that the state remains peaked on a trajectory which coincides with the union of a contracting and a expanding classical solutions except in the region where the matter energy density becomes comparable to the Planck density in order of magnitude [16]. At that moment, the effective trajectory on which the state is concentrated passes from a contraction to an expansion phase in such a way that the classical singularity is avoided. This quantum phenomenon which allows the resolution of the big bang singularity is usually called big bounce [15, 16].

9. CONCLUSION

We have seen that the quantization techniques of LQG prove to be successful in achieving a rigorous and complete quantum theory of simple cosmological models, like e.g. the case of flat FRW spacetimes provided with a minimally coupled scalar field. The resulting quantization adopted in LQC is inequivalent to the standard Schrdinger (or Wheeler-DeWitt) quantization which has been traditionally employed in geometrodynamics, a fact that explains why the emerging physics is radically different and allows LQC to supply

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[5] Recalling expression (32), this suffices to determine the action of the operator on positive frequency solutions for all values of ϕ.

satisfactory answers to fundamental problems that had remained open in Quantum Cosmology. In particular, this explains why, while the standard quantization fails to solve the cosmological singularities, these are cured in LQC. Actually, the singularities are resolved already at the kinematical level. Nonetheless, the resolution is much stronger. For physical states with good semiclassical behavior, numerical simulations show that the universe suffers a big bounce before reaching the big bang. This bounce occurs when the energy density $\rho = P_\phi^2/(2|p|^3)$ approaches a critical density of the order of the Planck density. Away from the bounce, states are peaked on classical solutions. Quantum corrections are strong close to the bounce, but even there the state remains peaked on a certain, modified trajectory.

Acknowledgments

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