Chern-Simons forms associated to homogeneous pseudo-Riemannian structures *

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Abstract

Forms of Chern-Simons type associated to homogeneous pseudo-Riemannian structures are considered. The corresponding secondary classes are a measure of the lack of a homogeneous pseudo-Riemannian space to be locally symmetric. Explicit computations are done for some pseudo-Riemannian Lie groups and their compact quotients.

1 Introduction

The characterization by É. Cartan of Riemannian locally symmetric spaces as those Riemannian manifolds whose curvature tensor is parallel was extended by Ambrose and Singer in [1]. They proved that a complete, simply connected Riemannian manifold is homogeneous if and only if it admits a $(1,2)$ tensor field S satisfying certain equations. If $S=0$ then (M,g) is Riemannian locally symmetric.

The purpose of the present paper is to provide forms of Chern-Simons type for each pseudo-Riemannian manifold (M,g) endowed with a homogeneous pseudo-Riemannian structure S. This construction furnishes odd-dimensional differential forms of degree greater than 1, which are null if $S=0$. Under certain conditions, these forms are closed and define secondary classes. Each of such triples (M,g,S) has thus a number (depending on the dimension of M) of these forms, and roughly speaking (when the corresponding group of real cohomology of the manifold is non-zero), the more non-vanishing classes of that kind a manifold has, the less symmetric it is.

We give several examples of such forms on some Lie groups equipped with left-invariant metrics: The 3-dimensional unimodular Lie groups, so having instances of Abelian, nilpotent, solvable and simple Lie groups; and the five-dimensional generalized Heisenberg group $H(1,2)$, which is nilpotent. Further, we consider the corresponding secondary classes of the compact quotients of the

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previous groups, identifying them in the real cohomology spaces of the quotient spaces. In [6], we also studied the oscillator group.

2 Preliminaries

Ambrose and Singer [1] proved that a connected, simply connected and complete Riemannian manifold (M, g) is homogeneous if and only if there exists a $(1, 2)$ tensor field S on M—called a homogeneous Riemannian structure—satisfying certain equations (see (2.1) below). In [4] we have extended that characterization to pseudo-Riemannian manifolds. Specifically, let (M, g) be a connected C^∞ pseudo-Riemannian manifold of dimension n and signature $(k, n-k)$. Let ∇ be the Levi-Civita connection of g and R its curvature tensor field. A *homogeneous pseudo-Riemannian structure* on (M, g) is a tensor field S of type $(1, 2)$ on M such that the connection $\widetilde{\nabla} = \nabla - S$ satisfies

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} R = 0, \quad \widetilde{\nabla} S = 0. \tag{2.1}$$

If g is a Lorentzian metric $(k = 1)$, we say that S is a homogeneous Lorentzian structure. In [4] we proved that if (M, g) is connected, simply connected and geodesically complete then it admits a homogeneous pseudo-Riemannian structure if and only if it is a reductive homogeneous pseudo-Riemannian manifold.

Let (P, M, G) be a principal fibre bundle over the n-dimensional C^∞ manifold M. Let $\mathcal{I}^r(G)$ be the real vector space of $\mathrm{Ad}(G)$-invariant polynomials of degree r. Let D be a connection in P, with connection 1-form ω and curvature form $\Omega = d\omega + \omega \wedge \omega$. Let $I \in \mathcal{I}^r(G)$ be an invariant polynomial. One can consider for each r the $2r$-form $I(\Omega^r) = I(\Omega, \ldots, \Omega)$, which is a $2r$-form on P, and projects to a (unique) $2r$-form on M, say again $I(\Omega^r)$. This form is closed and determines a cohomology class in $H^{2r}(M, \mathbb{R})$. Let $\widetilde{D}$ be another connection in P with connection 1-form $\widetilde{\omega}$ and curvature form $\widetilde{\Omega}$. Consider the connection given, for a $t \in [0, 1]$, by $\omega_t = \widetilde{\omega} + t(\omega - \widetilde{\omega})$, with curvature form $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$. Then we have the *transgression formula*

$$I(\Omega^r) - I(\widetilde{\Omega}^r) = dQ(\omega, \widetilde{\omega}), \tag{2.2}$$

where

$$Q(\omega, \widetilde{\omega}) := r \int_0^1 I(\omega - \widetilde{\omega}, \underbrace{\Omega_t, \ldots, \Omega_t}_{r-1})\, dt. \tag{2.3}$$

The Chern-Simons $(2r - 1)$-form $Q(\omega, \widetilde{\omega})$ on M defines, if $I(\Omega^r) = I(\widetilde{\Omega}^r)$, a secondary class.

3 Chern-Simons forms associated to a homogeneous pseudo-Riemannian structure

We consider the bundle of pseudo-orthonormal frames $p\colon \mathcal{O}_{k,n-k}(M) \to M$ over the pseudo-Riemannian n-manifold (M,g), where g is a metric of signature $(k,n-k)$. We define $\mathrm{Ad}(O(k,n-k))$-invariant polynomial functions $f_1,\dots,f_n$ on the Lie algebra $\mathfrak{o}(k,n-k)$ by

$$f(t,X) = \det(tI+X) = \sum_{r=0}^{n} f_r(X)\, t^{n-r}, \qquad X \in \mathfrak{o}(k,n-k).$$

Let Ω be the curvature form of a connection ω in $\mathcal{O}_{k,n-k}(M)$. Then, for each f_r, $r=1,\dots,n$, there exists a unique closed $2r$-form υ_r on M such that $p^*(\upsilon_r) = f_r(\Omega)$. One has $\det(I+\Omega) = p^*(1+\upsilon_1+\cdots+\upsilon_n)$, so having characteristic forms υ_r of degree $2r$, and a total form $\Upsilon(\mathcal{O}_{k,n-k}(M),\omega) = 1+\sum_{r=1}^{n}\upsilon_r$. The forms $f_r(\Omega)$ are the elementary symmetric functions $s_r(\Omega)$, $r=1,\dots,n$, of the eigenvalues of Ω, so that $\det(I+\Omega) = 1+s_1(\Omega)+s_2(\Omega)+\cdots+s_n(\Omega)$. By using Newton's recursive formulas, one can further compute the functions $s_r(\Omega)$ in terms of the traces of the powers of Ω from the expressions

$$\begin{aligned}\mathrm{tr}\,(\Omega^r) - s_1(\Omega)\,\mathrm{tr}\,(\Omega^{r-1}) + s_2(\Omega)\,\mathrm{tr}\,(\Omega^{r-2}) - \cdots \\ + (-1)^{r-1}s_{r-1}(\Omega)\,\mathrm{tr}\,(\Omega) + (-1)^r r\, s_r(\Omega) = 0, \qquad r=1,\dots,n,\end{aligned}$$

and since $\mathrm{tr}\,\Omega = 0$, we have after computation that

$$\det(I+\Omega) = 1 - \frac{1}{2}\mathrm{tr}\,(\Omega^2) + \frac{1}{3}\mathrm{tr}\,(\Omega^3) + \frac{1}{4}\Big(\frac{1}{2}(\mathrm{tr}\,(\Omega)^2)^2 - \mathrm{tr}\,(\Omega^4)\Big) + \cdots$$

Now, we consider here the Levi-Civita connection ∇ and the linear connection $\widetilde{\nabla} = \nabla - S$, with connection form $\widetilde{\omega}$ and curvature form $\widetilde{\Omega}$ (as in the previous section), where S is a homogeneous pseudo-Riemannian structure on (M,g), so that the general equation (2.2) can be written in this case as

$$s_r(\Omega) - s_r(\widetilde{\Omega}) = dQ(\omega,\widetilde{\omega}). \tag{3.1}$$

If $s_r(\Omega) = s_r(\widetilde{\Omega})$, then $Q(\omega,\widetilde{\omega})$ is closed, so determining a secondary class. In particular, if $r=2,3$, then this happens if $\mathrm{tr}\,(\Omega^r) = \mathrm{tr}\,(\widetilde{\Omega}^r)$. We shall denote by $Q^S_{2r-1}(M,g)$, or simply by Q^S_{2r-1}, the form $Q(\omega,\widetilde{\omega})$ in (3.1).

Definition 3.1. Let (M,g) be a pseudo-Riemannian manifold and let S be a homogeneous pseudo-Riemannian structure on M. We shall call the forms $Q^S_{2r-1}(M,g)$, for each $3 \le 2r-1 \le \dim M$, *Chern-Simons forms of pseudo-Riemannian homogeneity* (or simply *forms of homogeneity*) on (M,g,S). We shall call the corresponding real cohomology classes $[Q^S_{2r-1}](M,g)$ *secondary classes of pseudo-Riemannian homogeneity* (or simply *secondary classes of homogeneity*).

The case $r = 1$ in (3.1) is trivial, as the forms $\omega - \widetilde{\omega}$, Ω, and $\widetilde{\Omega}$ take values in $\mathfrak{o}(k, n-k)$. For $r = 2$, we get the formula

$$Q_3^S = -\frac{1}{2}\,\mathrm{tr}\,(2\,\sigma \wedge \widetilde{\Omega} + \sigma \wedge d\sigma + 2\,\sigma \wedge \widetilde{\omega} \wedge \sigma + \frac{2}{3}\,\sigma \wedge \sigma \wedge \sigma), \tag{3.2}$$

where $\sigma = \omega - \widetilde{\omega}$. One can obtain similar formulas for any r with $2r \leq \dim M$. We give also the formula for $r = 3$:

$$\begin{aligned} Q_5^S = \frac{1}{3}\,\mathrm{tr}\,\big\{&3\,\sigma \wedge \widetilde{\Omega}^2 + \frac{3}{2}(\sigma^2 \wedge \widetilde{\Omega} \wedge \widetilde{\omega} + 2\,\sigma \wedge \widetilde{\Omega} \wedge \sigma \wedge \widetilde{\omega} + \sigma \wedge \widetilde{\Omega} \wedge d\sigma \\ &+ \sigma^2 \wedge \widetilde{\omega} \wedge \widetilde{\Omega} + \sigma \wedge d\sigma \wedge \widetilde{\Omega} + 2\,\sigma^4 \wedge \widetilde{\omega} + \sigma^3 \wedge d\sigma) \\ &+ 2\,\sigma^3 \wedge \widetilde{\Omega} + 3\,\sigma^2 \wedge \widetilde{\omega} \wedge \sigma \wedge \widetilde{\omega} + 2\,\sigma \wedge \widetilde{\omega} \wedge \sigma \wedge d\sigma + \sigma^3 \wedge \widetilde{\omega}^2 \\ &+ \sigma^2 \wedge \widetilde{\omega} \wedge d\sigma + \sigma^2 \wedge d\sigma \wedge \widetilde{\omega} + \sigma \wedge (d\sigma)^2 + \frac{3}{5}\,\sigma^5\big\}, \end{aligned} \tag{3.3}$$

where $A^j = A \wedge \overset{(j}{\cdots} \wedge A$ for any matrix A. We now give some general results for the forms Q_{2r-1}^S.

Proposition 3.2. *If $S = 0$ then $Q_{2r-1}^S = 0$, for each r.*

Proof. Immediate from (2.3). □

Let S_1 and S_2 be homogeneous pseudo-Riemannian structures on (M_1, g_1) and (M_2, g_2) respectively. We recall [11, pp. 33–34] that an isomorphism between S_1 and S_2 is an isometry $\varphi\colon (M_1, g_1) \to (M_2, g_2)$ which is also an affine transformation with respect to the connections $\widetilde{\nabla}_1 = \nabla_1 - S_1$ and $\widetilde{\nabla}_2 = \nabla_2 - S_2$. Then we have the following proposition.

Proposition 3.3. *If $\varphi\colon (M_1, g_1) \to (M_2, g_2)$ is an isometry between S_1 on (M_1, g_1) and S_2 on (M_2, g_2), then $\varphi^*\big(Q_{2r-1}^{S_2}\big) = Q_{2r-1}^{S_1}$, for each r.*

Proof. According to the previous definition, we have that $\varphi^*\omega_2 = \omega_1$ and $\varphi^*\widetilde{\omega}_2 = \widetilde{\omega}_1$. Thus we have that $\varphi^*((\omega_t)_2) = \varphi^*(\widetilde{\omega}_2 + t(\omega_2 - \widetilde{\omega}_2)) = (\omega_t)_1$, and so $\varphi^*((\Omega_t)_2) = \varphi^*(d(\omega_t)_2 + (\omega_t)_2 \wedge (\omega_t)_2) = (\Omega_t)_1$. Hence for any invariant polynomial I we have that

$$\varphi^*\{I(\sigma_2, (\Omega_t)_2, \ldots, (\Omega_t)_2)\} = I(\sigma_1, (\Omega_t)_1, \ldots, (\Omega_t)_1).$$

As I is multilinear, we conclude. □

Proposition 3.4.

$$\mathrm{tr}\,(\Omega^r) - \mathrm{tr}\,(\widetilde{\Omega}^r) = \mathrm{tr}\,\Big\{\sum_{l=0}^{r-1} \binom{r}{l} \Omega^l \wedge (3[S,S] - \mathcal{A}S_S)^{r-l}\Big\},$$

where $\mathcal{A}S_S$ is defined by $(\mathcal{A}S_S)(X,Y) = S_{S(X,Y)-S(Y,X)}$.

Proof. First we recall that $d^\nabla S$ is defined [7, p. 22] by

$$(d^\nabla S)(X,Y) = \nabla_X S_Y - \nabla_Y S_X - S_{[X,Y]}, \tag{3.4}$$

and we put

$$[S,S](X,Y) = S_X S_Y - S_Y S_X = [S_X, S_Y]. \tag{3.5}$$

On the other hand, Ambrose-Singer's third equation (2.1) can be written as

$$(\nabla_X S)(Y,Z) = [S_X, S_Y](Z) - S_{S(X,Y)}Z. \tag{3.6}$$

Since ∇ is torsionless, by (3.4) we can write $(d^\nabla S)(X,Y)(Z) = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)$, and thus from (3.6) one has that

$$(d^\nabla S)(X,Y)(Z) = \{2[S_X, S_Y] - S_{S(X,Y)-S(Y,X)}\}(Z). \tag{3.7}$$

Hence, on account of (3.5) we have that $(d^\nabla S)(X,Y) = (2[S,S] - \mathcal{A}S_S)(X,Y)$. Substituting now (3.7) in Koszul's formula $\widetilde{\Omega} = \Omega + [S,S] + d^\nabla S$ (see [7, p. 22]), we obtain that $\widetilde{\Omega} = \Omega + 3[S,S] - \mathcal{A}S_S$. Finally, calculation of $\operatorname{tr}(\widetilde{\Omega}^r) = \operatorname{tr}(\Omega + 3[S,S] - \mathcal{A}S_S)^r$ gives us, on account of the property $\operatorname{tr}(\Phi \wedge \Psi) = \operatorname{tr}(\Psi \wedge \Phi)$ for any two $\mathrm{End}(TM)$-valued 2-forms Φ, Ψ, the expression in the statement. □

In particular, if $3[S,S] = \mathcal{A}S_S$, then $\operatorname{tr}(\Omega^r) - \operatorname{tr}(\widetilde{\Omega}^r) = 0$, and Q^S_{2r-1} defines, for $r = 2, 3$, a secondary class $[Q^S_{2r-1}]$.

4 Examples of forms Q^S_{2r-1} associated to homogeneous pseudo-Riemannian structures

4.1 The 3-dimensional unimodular Lie groups

Let G be a connected unimodular Lie group (with Lie algebra $\mathfrak{g}$) endowed with a left-invariant Riemannian metric g. We consider the homogeneous Riemannian structure S on (G,g) defined [11, p. 83] by

$$2g(S_X Y, Z) = g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y), \quad X,Y,Z \in \mathfrak{g}. \tag{4.1}$$

If $\dim G = 3$ there exists [8] an orthonormal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ such that

$$[E_1, E_2] = \lambda_3 E_3, \qquad [E_2, E_3] = \lambda_1 E_1, \qquad [E_3, E_1] = \lambda_2 E_2. \tag{4.2}$$

If ∇ is the Levi-Civita connection of G then $\nabla_{E_i} E_i = S_{E_i} E_i = 0$ and the remaining components of ∇ and S are given by

$$\nabla_{E_1} E_2 = S_{E_1} E_2 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3)E_3, \quad \nabla_{E_1} E_3 = S_{E_1} E_3 = \frac{1}{2}(\lambda_1 - \lambda_2 - \lambda_3)E_2,$$
$$\nabla_{E_2} E_1 = S_{E_2} E_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 - \lambda_3)E_3, \quad \nabla_{E_2} E_3 = S_{E_2} E_3 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3)E_1,$$
$$\nabla_{E_3} E_1 = S_{E_3} E_1 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3)E_2, \quad \nabla_{E_3} E_2 = S_{E_3} E_2 = \frac{1}{2}(-\lambda_1 - \lambda_2 + \lambda_3)E_1.$$

Let $\{\theta^1, \theta^2, \theta^3\}$ be the basis dual to $\{E_1, E_2, E_3\}$. We obtain for ω, $\widetilde{\omega}$ and $\widetilde{\Omega}$ defined as in Section 3, that $\widetilde{\omega} = 0$, $\widetilde{\Omega} = 0$,

$$\omega = \frac{1}{2} \begin{pmatrix} 0 & (-\lambda_1 - \lambda_2 + \lambda_3)\theta^3 & (\lambda_1 - \lambda_2 + \lambda_3)\theta^2 \\ (\lambda_1 + \lambda_2 - \lambda_3)\theta^3 & 0 & (\lambda_1 - \lambda_2 - \lambda_3)\theta^1 \\ (-\lambda_1 + \lambda_2 - \lambda_3)\theta^2 & (-\lambda_1 + \lambda_2 + \lambda_3)\theta^1 & 0 \end{pmatrix},$$

and then from (3.2), after some calculations, the next proposition.

Proposition 4.1. *The Chern-Simons form associated to the homogeneous Riemannian structure S on G, for arbitrarily fixed $\lambda_1, \lambda_2, \lambda_3$ as in* (4.2)*, is given by*

$$Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3}, g) = -\frac{1}{2}\big(\sum \lambda_i^3 - \sum_{i\neq j} \lambda_i \lambda_j^2 + 4\lambda_1\lambda_2\lambda_3\big)\, \theta^1 \wedge \theta^2 \wedge \theta^3. \tag{4.3}$$

If $S = 0$ then $\lambda_i = 0$ $(1 \leq i \leq 3)$ and the group G is commutative; in this case $Q_3^S(G_{0,0,0}, g) = 0$. Since $S_{E_1}E_1 = S_{E_2}E_2 = S_{E_3}E_3 = 0$, one has $c_{12}(S) = 0$ and hence S is of type $\mathcal{S}_2 \oplus \mathcal{S}_3$ (see [11, p. 84], [5]). In particular, S is of type $\mathcal{S}_2$, that is $\mathfrak{S}_{XYZ} S_{XYZ} = 0$ for every $X, Y, Z \in \mathfrak{g}$, if and only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$; and S is of type $\mathcal{S}_3$, that is $S_X Y + S_Y X = 0$ for $X, Y \in \mathfrak{g}$, if and only if $\lambda_1 = \lambda_2 = \lambda_3$. By [8] (see also [11, p. 84]), if $S \neq 0$ is of type $\mathcal{S}_2$ then the Lie algebra $\mathfrak{g}$ of G is either the Lie algebra $\mathfrak{e}(1,1)$ of the Lie group of rigid motions of the Minkowski plane or $\mathfrak{sl}(2,\mathbb{R})$, and we have that

$$Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3}, g) = -\frac{1}{2}\,(\lambda_1^3 - \lambda_2^3 - \lambda_3^3 + 4\lambda_1\lambda_2\lambda_3)\, \theta^1 \wedge \theta^2 \wedge \theta^3,$$

with $\sum \lambda_i = 0$. If $S \neq 0$ is of type $\mathcal{S}_3$ we may suppose $\lambda_i = 1$, $i = 1, 2, 3$; then $\mathfrak{g} = \mathfrak{su}(2)$, and we have that

$$Q_3^S(SU(2), g) = -\frac{1}{2}\, \theta^1 \wedge \theta^2 \wedge \theta^3.$$

As a consequence of Milnor's classification [8] of 3-dimensional unimodular Lie algebras, if S is neither of type $\mathcal{S}_2$ nor $\mathcal{S}_3$ then $\mathfrak{g}$ is either the Heisenberg Lie algebra $\mathfrak{h}_3$ or the Lie algebra $\mathfrak{e}_2$ of the Lie group of rigid motions of the Euclidean space. If $\mathfrak{g}$ is the Lie algebra of the Heisenberg group H_3 we may suppose that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$; in this case,

$$Q_3^S(H_3, g) = -\frac{1}{2}\, \theta^1 \wedge \theta^2 \wedge \theta^3.$$

If $\mathfrak{g} = \mathfrak{e}_2$, then one of the constants, suppose λ_3, is null; in this case,

$$Q_3^S(E(2)_{\lambda_1,\lambda_2}, g) = -\frac{1}{2}\,(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2\, \theta^1 \wedge \theta^2 \wedge \theta^3.$$

4.2 The Heisenberg group

Consider again the Heisenberg group H_3, that is, the simply connected Lie group corresponding to the Lie algebra $\mathfrak{h}_3 = \langle a, x, y \rangle$ with non-zero bracket $[x, y] = a$. We now endow H_3 with the left-invariant pseudo-Riemannian metric defined at $\mathfrak{h}_3$ by the diagonal matrix $g = \mathrm{diag}\,(\varepsilon, 1, 1)$ with respect to the given basis, where $\varepsilon = \pm 1$. Let $\{\tau, \alpha, \beta\}$ be the basis dual to $\{a, x, y\}$. Then, integrating Ambrose-Singer's equations (2.1), we obtain [5, 11] the 1-parameter family of homogeneous pseudo-Riemannian structures

$$S_\lambda = \lambda\, \tau \otimes (\alpha \wedge \beta) + \frac{1}{2}\, \varepsilon\, \beta \otimes (\tau \wedge \alpha) - \frac{1}{2}\, \varepsilon\, \alpha \otimes (\tau \wedge \beta), \qquad \lambda \in \mathbb{R}. \tag{4.4}$$

From this we have that

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & -\beta & \alpha \\ \varepsilon\beta & 0 & \varepsilon\tau \\ -\varepsilon\alpha & -\varepsilon\tau & 0 \end{pmatrix}, \text{ and letting } \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}:$$

$$\widetilde{\omega} = \Big(\frac{\varepsilon}{2} + \lambda\Big)\tau\, A, \qquad \widetilde{\Omega} = \Big(\frac{\varepsilon}{2} + \lambda\Big)\alpha \wedge \beta\, A,$$

and then, after some computations from (3.2), we obtain the following proposition.

Proposition 4.2. *The form of homogeneity on (H_3, g_ε) corresponding to the homogeneous pseudo-Riemannian structure S_λ is given by*

$$Q_3^{S_\lambda}(H_3, g_\varepsilon) = -\frac{1}{2}\Big(\frac{1}{2} - 2\lambda(\lambda + \varepsilon)\Big)\tau \wedge \alpha \wedge \beta.$$

Notice that in the Riemannian case (that is, when $\varepsilon = 1$) and if $\lambda = -\frac{1}{2}$, then S_λ is the homogeneous Riemannian structure on H_3 obtained in Section 4.1, where the Heisenberg group was considered as a particular case of 3-dimensional unimodular Lie group.

4.3 The generalized Heisenberg group $H(1, 2)$

Consider [3] a 2-nilpotent Lie group N with the left-invariant metric induced by a (not necessarily positive definite) inner product in their Lie algebra $\mathfrak{n}$. If $\mathfrak{n}$ is a Lie algebra with inner product $\langle\ ,\ \rangle$ and $\mathfrak{z}$ is the center of $\mathfrak{n}$, one considers a decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z} = \mathfrak{U} \oplus \mathfrak{Z}$, $\mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}$, $\mathfrak{U}$ stands for the null subspace of $\mathfrak{z}$, and $\mathfrak{V} \subset \mathfrak{v}$ for a complementary null subspace. An example of the construction in [3] is the generalized Heisenberg group $H(1, 2)$ of dimension 5, whose Lie algebra is $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E} = \langle\{u, z, v, e_1, e_2\}\rangle$, where $\mathfrak{U} = \langle\{u\}\rangle$, $\mathfrak{Z} = \langle\{z\}\rangle$, $\mathfrak{V} = \langle\{v\}\rangle$ and $\mathfrak{E} = \langle\{e_1, e_2\}\rangle$, with non-vanishing brackets $[e_1, e_2] = z$, $[v, e_2] = u$, and non-trivial inner products

$$\langle u, v\rangle = 1, \quad \langle z, z\rangle = \varepsilon, \quad \langle e_1, e_1\rangle = \bar{\varepsilon}_1, \quad \langle e_2, e_2\rangle = \bar{\varepsilon}_2,$$

where each ε-symbol is ± 1 independently, so that the pseudo-Riemannian metric on $H(1,2)$ defined by $\langle\ ,\ \rangle$ has signature $(k, 5-k)$, $1 \leq k \leq 4$. Let $\{\eta, \theta, \tau, \alpha^1, \alpha^2\}$ denote the dual basis to $\{u, z, v, e_1, e_2\}$. Then integration of Ambrose-Singer's equations (2.1) gives us [5] the only homogeneous pseudo-Riemannian structure

$$(4.5)\quad S = \frac{\varepsilon}{2}\,\alpha^2 \otimes (\theta \wedge \alpha^1) - \frac{\varepsilon}{2}\,\alpha^1 \otimes (\theta \wedge \alpha^2) - \frac{\varepsilon}{2}\,\theta \otimes (\alpha^1 \wedge \alpha^2) - \tau \otimes (\tau \wedge \alpha^2).$$

We obtain that

$$\omega = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 2\tau \\ 0 & 0 & 0 & -\alpha^2 & \alpha^1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon\bar{\varepsilon}_1\alpha^2 & 0 & 0 & \varepsilon\bar{\varepsilon}_1\theta \\ 0 & -\varepsilon\bar{\varepsilon}_2\alpha^1 & -2\bar{\varepsilon}_2\tau & -\varepsilon\bar{\varepsilon}_2\theta & 0 \end{pmatrix}, \qquad \widetilde{\omega} = 0, \qquad \widetilde{\Omega} = 0,$$

and by means of some computations from (3.2) and (3.3), we have the following proposition.

Proposition 4.3. *The forms of homogeneity on* $(H(1,2),\, g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *corresponding to the homogeneous pseudo-Riemannian structure* S *are*

$$Q_3^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = -\frac{1}{2}\,\bar{\varepsilon}_1\,\bar{\varepsilon}_2\,\theta \wedge \alpha^1 \wedge \alpha^2, \qquad Q_5^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = 0.$$

5 Secondary classes $[Q^S_{2r-1}]$ of compact quotients of Lie groups

Now, we determine the secondary classes $[Q^S_{2r-1}]$ of the compact quotients of the spaces considered in Section 4. For this, we first note that given a left-invariant form α on a Lie group G, then it is invariant under the action of a discrete subgroup Γ of G, so that there exists a form $\widehat{\alpha}$ on the quotient $\Gamma\backslash G$ such that $\pi^*(\widehat{\alpha}) = \alpha$, where π denotes the natural projection $\pi\colon G \to \Gamma\backslash G$. In the sequel, we shall denote by $\widehat{\alpha}$ such a projected form of a left-invariant form α on G onto a compact quotient $\Gamma\backslash G$. If g is a left-invariant metric on G then it projects to a metric $\widehat{g}$ on $\Gamma\backslash G$ such that the map $\pi\colon (G, g) \to (\Gamma\backslash G, \widehat{g})$ is a local pseudo-Riemannian isometry. Moreover, the Levi-Civita connection ∇ projects to the Levi-Civita connection $\widehat{\nabla}$ on $\Gamma\backslash G$ and each homogeneous pseudo-Riemannian structure S projects to a homogeneous pseudo-Riemannian structure $\widehat{S}$ on $\Gamma\backslash G$, where Γ is a uniform discrete subgroup of G.

5.1 The 3-dimensional unimodular groups

We first recall that for a compact orientable 3-dimensional manifold M one has $H^3(M, \mathbb{R}) \approx \mathbb{R}$. On the other hand, the compact quotients of the 3-dimensional unimodular Lie groups G were classified in [10], and such manifolds are orientable. Thus, $H^3(\Gamma\backslash G, \mathbb{R}) \approx \mathbb{R}$ in all the cases, which we now recall:

The Abelian group $\mathbb{R}^3$ has vanishing Chern-Simons form, so its only compact quotient, the 3-torus T^3, has no non-trivial corresponding secondary class.

The compact quotients of the Heisenberg group are the S^1-bundles over the torus T^2 with Euler class $m \in H^2(T^2, \mathbb{Z})$. One has such a bundle for each $m \in \mathbb{Z}$.

Let $\widetilde{E^0}(2)$ be the universal covering of the identity component $E^0(2) = SO(2) \ltimes \mathbb{R}^2$ of the Euclidean group $E(2)$. The compact quotients of $\widetilde{E^0}(2)$ are the 2-torus bundles over S^1, which are flat manifolds with cyclic holonomy equal to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\mathbb{Z}_6$ or 1.

The compact quotients of the group $E(1,1)$ of rigid motions of the Minkowski plane are torus bundles over S^1 satisfying a supplementary condition.

The group $SU(2) \approx S^3$ is compact. Their quotients as above are either lens spaces when Γ is a cyclic group (one for each $m \in \mathbb{Z}$, $m > 1$), or well the quotient spaces by Γ, where Γ is either the binary dihedral group, or the binary tetrahedral group, or the binary octahedral group, or the binary icosahedral group.

The compact quotients of the universal covering $\widetilde{SL}(2,\mathbb{R})$ of the Lie group $SL(2,\mathbb{R})$ are defined by a Fuchsian group Γ of the first kind satisfying certain conditions. We have the following proposition.

Proposition 5.1. *For any* 3*-dimensional unimodular Lie group* G*, the Chern-Simons form* $Q_3^S(G_{\lambda_1,\lambda_2,\lambda_3}, g)$ *in* (4.3) *defines the secondary class*

$$-\frac{1}{2}\big(\sum \lambda_i^3 - \sum_{i\neq j} \lambda_i\lambda_j^2 + 4\lambda_1\lambda_2\lambda_3\big)\,[\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3],$$

associated to the homogeneous pseudo-Riemannian structure $\widehat{S}$ *induced on any of the compact quotients* $(\Gamma\backslash G, \widehat{g})$ *by the homogeneous pseudo-Riemannian structure* S *in* (4.1)*. If* $G = H_3, SU(2)$*, the secondary class is given by*

$$-\frac{1}{2}\,[\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3].$$

For $G = E(1,1), \widetilde{SL}(2,\mathbb{R})$*, we have the class*

$$-\frac{1}{2}\,(\lambda_1^3 - \lambda_2^3 - \lambda_3^3 + 4\lambda_1\lambda_2\lambda_3)\,[\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3], \qquad \textstyle\sum\lambda_i = 0.$$

If $G = \widetilde{E^0}(2)$*, one has the class*

$$-\frac{1}{2}\,(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2\,[\widehat{\theta}^1 \wedge \widehat{\theta}^2 \wedge \widehat{\theta}^3].$$

5.2 The Heisenberg group

The cohomology of the compact quotient of the Heisenberg group H_3 by a discrete subgroup Γ is known to be [2], as a consequence of Nomizu's Theorem

[9], equal to

$$H^0(\Gamma\backslash H_3,\mathbb{R}) = \{[1]\}, \qquad H^1(\Gamma\backslash H_3,\mathbb{R}) = \{[\widehat{\alpha}],[\widehat{\beta}]\},$$
$$H^2(\Gamma\backslash H_3,\mathbb{R}) = \{[\widehat{\tau}\wedge\widehat{\alpha}],[\widehat{\tau}\wedge\widehat{\beta}]\}, \qquad H^3(\Gamma\backslash H_3,\mathbb{R}) = \{[\widehat{\tau}\wedge\widehat{\alpha}\wedge\widehat{\beta}]\}.$$

Then we have the following proposition.

Proposition 5.2. *The Chern-Simons form* $Q_3^{S_\lambda}(H_3, g_\varepsilon)$ *in* Proposition 4.2 *determines the secondary class* $-\frac{1}{2}\big(\frac{1}{2}-2\lambda(\lambda+\varepsilon)\big)[\widehat{\tau}\wedge\widehat{\alpha}\wedge\widehat{\beta}]$ *associated to the homogeneous pseudo-Riemannian structure* $\widehat{S}_\lambda$ *induced on the compact quotient* $(\Gamma\backslash H_3, \widehat{g}_\varepsilon)$ *by the homogeneous pseudo-Riemannian structure* S_λ *in* (4.4)*.*

5.3 The generalized Heisenberg group $H(1,2)$

We can compute, again as a consequence of Nomizu's Theorem, the cohomology of the compact quotient $\Gamma\backslash H(1,2)$ of the generalized Heisenberg group $H(1,2)$ by a discrete subgroup Γ, obtaining:

$$\begin{aligned}
&H^0(\Gamma\backslash H(1,2),\mathbb{R}) = \langle 1\rangle, \qquad H^1(\Gamma\backslash H(1,2),\mathbb{R}) = \langle[\widehat{\tau}],[\widehat{\alpha}^1],[\widehat{\alpha}^2]\rangle,\\
&H^2(\Gamma\backslash H(1,2),\mathbb{R}) = \langle[\widehat{\eta}\wedge\widehat{\tau}],[\widehat{\eta}\wedge\widehat{\alpha}^1+\widehat{\theta}\wedge\widehat{\tau}],[\widehat{\eta}\wedge\widehat{\alpha}^2],\\
&\qquad [\widehat{\theta}\wedge\widehat{\alpha}^1],[\widehat{\theta}\wedge\widehat{\alpha}^2],[\widehat{\tau}\wedge\widehat{\alpha}^1]\rangle,\\
&H^3(\Gamma\backslash H(1,2),\mathbb{R}) = \langle[\widehat{\eta}\wedge\widehat{\theta}\wedge\widehat{\alpha}^2],[\widehat{\eta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^1],[\widehat{\eta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^2],[\widehat{\eta}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2],\\
&\qquad [\widehat{\theta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^1],[\widehat{\theta}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2]\rangle,\\
&H^4(\Gamma\backslash H(1,2),\mathbb{R}) = \langle[\widehat{\eta}\wedge\widehat{\theta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^1],[\widehat{\eta}\wedge\widehat{\theta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^2],[\widehat{\eta}\wedge\widehat{\theta}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2]\rangle,\\
&H^5(\Gamma\backslash H(1,2),\mathbb{R}) = \langle[\widehat{\eta}\wedge\widehat{\theta}\wedge\widehat{\tau}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2]\rangle.
\end{aligned}$$

Then we have the following proposition.

Proposition 5.3. *The Chern-Simons form* $Q_3^S(H(1,2), g_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *in* Proposition 4.3 *determines the secondary class* $-\frac{1}{2}\,\bar{\varepsilon}_1\,\bar{\varepsilon}_2\,[\widehat{\theta}\wedge\widehat{\alpha}^1\wedge\widehat{\alpha}^2]$ *associated to the homogeneous pseudo-Riemannian structure* $\widehat{S}$ *induced on the compact quotient* $(\Gamma\backslash H(1,2), \widehat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2})$ *by the homogeneous pseudo-Riemannian structure* S *in* (4.5)*.*

6 Final remarks

For the class of pseudo-Riemannian homogeneity in Proposition 5.2, we have

$$[Q_3^{S_\lambda}](\Gamma\backslash H_3, \widehat{g}_\varepsilon) = 0, \text{ for } \varepsilon = 1,\ \lambda = \frac{\pm\sqrt{2}-1}{2}, \text{ or } \varepsilon = -1,\ \lambda = -\frac{1}{2},$$

so that in these cases the pseudo-Riemannian compact quotient of the Heisenberg group, endowed with that homogeneous pseudo-Riemannian structure, is "more symmetric" (although they are never symmetric in the usual sense) than the spaces corresponding to the rest of values of λ. Consider a compact quotient $\Gamma\backslash H(1,2)$ of the generalized Heisenberg group. By Proposition 5.3, we have that

$$[Q_3^S](\Gamma\backslash H(1,2),\widehat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) \neq 0, \qquad [Q_5^S](\Gamma\backslash H(1,2),\widehat{g}_{\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2}) = 0.$$

Hence this compact quotient, endowed with that homogeneous pseudo-Riemannian structure, is "more symmetric" than other pseudo-Riemannian manifold of the same dimension whose classes of pseudo-Riemannian homogeneity are nonnull.

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