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Weak wave turbulence scaling theory for diffusion and relative diffusion in turbulent surface waves

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Abstract. – We examine the applicability of the weak wave turbulence theory in explaining experimental scaling results obtained for the diffusion and relative diffusion of particles moving on turbulent surface waves. For capillary waves our theoretical results are shown to be in good agreement with experimental results, where a distinct crossover in diffusive behavior is observed at the driving frequency. For gravity waves our results are discussed in the light of ocean wave studies.

Introduction. – The study of particles moving on surface waves has shown that the particle motion often is far from being Brownian [1–5]. Similar results are found in ocean studies [6]. The turbulence observed in surface waves is strongly influenced by the presence of a dispersion relation (for deep-water surface waves) $\omega_k^2 = gk + \sigma k^3/\rho$ between the wave oscillation frequency and the wave vector k [7]. Here g is the gravitational acceleration, σ is the surface tension coefficient, and ρ is the density. Experimentally surface waves can be studied by vertically oscillating a fluid with a free surface. On the surface of the fluid, waves with a clearly discernible wavelength Λ are then formed if the amplitude of oscillations exceeds a critical value, the so-called Faraday instability [8]. The excitation of surface waves in the Faraday system is a parametric effect giving rise to a fundamental wave frequency Ω equal to half the driving frequency. The frequency Ω and wavelength Λ are associated through the dispersion relation. At small frequencies, $k < k_a = (g\rho/\sigma)^{1/2}$, the first term in the dispersion relation dominates, and the waves formed are called gravity waves (typical of ocean waves). At large frequencies ($k > k_a$) the effect of gravity can be neglected, and the second term dominates. The waves thus formed are called capillary waves. For water, the wavelength $a = 2\pi/k_a$, separating gravity and capillary waves, is about 2 cm.

Several studies have been carried out to measure the diffusivity of particles moving on capillary surface waves generated above the Faraday instability [1–5, 9]. Two particles separated

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by a distance R move more rapidly apart as R is increased. The relative diffusivity can be defined as

$$\left\langle \frac{\mathrm{d}R^2}{\mathrm{d}t} \right\rangle = 2R\langle \delta v \rangle \ , \tag{1}$$

where δv is the longitudinal component of the velocity difference. The variance σ^2 of the relative diffusivity $\mathrm{d}R^2/\mathrm{d}t$ is related to the velocity difference squared,

$$\left(\frac{\sigma}{R}\right)^2 \sim \langle |\delta v(R)|^2 \rangle \ . \tag{2}$$

It has been found experimentally that in capillary waves $\left(\frac{\sigma}{R}\right)^2 \propto 1 - a(R/\Lambda)^{-1/4}$, with $a = 0.66$ [9].

The purpose of the present investigation is to examine the applicability and limitations of the weak wave turbulence theory in explaining experimental scaling results obtained for the diffusion and relative diffusion of particles moving on turbulent surface waves. Diffusion theories based on random wave fields have likewise been considered [10], and Herterich and Hasselmann [11] particularly considered diffusion by surface gravity waves. For relative diffusion on capillary waves we find the exponent $-1/4$ as in experiments in the regime $R > \Lambda$. For gravity waves, we find an exponent $7/6 \sim 1.14$, a result which is intriguingly close to the value 1.15 found in ocean studies by Okubo [6]. We note that the relative-diffusion exponent 4/3 obtained in fully developed turbulence theory have been previously used in the discussion of experimental data [12].

To obtain the diffusivity, the particle displacement $\Delta x(t) = x(t_0+t) - x(t_0)$ over a time t is measured along an arbitrary axis for many initiation times t_0. From the resulting distribution the variance $V(t) = \langle [\Delta x(t)]^2 \rangle$ is found, and the diffusivity D can be extracted, $V(t) = 2Dt$. For Brownian motion the diffusivity D is a constant. For deep water capillary waves (we only consider deep-water surface waves) formed at high frequencies, the diffusion has been found experimentally to be anomalous:

$$V(t) \sim t^{\lambda} \ , \tag{3}$$

with $\lambda \geq 1$ (super-diffusive) [3–5, 9]. The exponent is observed to change drastically from a value $\lambda = 1.6$–1.9 at length scales below Λ to a value $\lambda = 1.0$–1.3 at length scales above Λ. The value of λ is observed to decrease with increasing drive; at large drive λ attains the Brownian motion value 1 [3, 4]. Assuming a connection between spatial and temporal correlations, we find for capillary waves $\lambda = 16/9$ for length scales below Λ, and $\lambda = 1$ for length scales above Λ, *i.e.* values rather close to the experimental values. Ocean studies have been performed using floaters [13], chemical tracers [6], and near-surface drifters. To what extent these studies relate to surface waves is unclear. Our results do, however, indicate a connection at least for the chemical tracers, where Okubo finds an exponent of 2.34 and from weak wave turbulence theory we find 8/3.

Spatial correlations and relative diffusion. – Spatial correlations between pairs of particles at $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{R}$ are related to the velocity difference squared

$$\langle |\delta v(R)|^2 \rangle = \langle |\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R}) - \boldsymbol{v}(\boldsymbol{r})|^2 \rangle = 2[\langle |\boldsymbol{v}(\boldsymbol{r})|^2 \rangle - \langle \boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R}) \cdot \boldsymbol{v}(\boldsymbol{r}) \rangle] \ . \tag{4}$$

Following the calculations of ref. [14], we have an expression for the space correlation of the velocity

$$\mathcal{C}(R) = \langle \boldsymbol{v}(\boldsymbol{r})\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{R}) \rangle \sim \int k\omega_k n_k e^{ikR\cos\theta} \mathrm{d}\boldsymbol{k} \sim \int_0^{\infty} k^2 \omega_k n_k J_0(kR) \mathrm{d}k \ , \tag{5}$$

where n_k is the isotropic Kolmogorov spectrum [15], and J_0 is the zeroth-order Bessel function.

For capillary waves, the dispersion relation is $\omega_k = (\sigma/\rho)^{1/2} k^{\alpha}$ with $\alpha = 3/2$, and the isotropic Kolmogorov spectrum (three-wave interactions) is [15,16]

$$n_k \sim P^{1/2} \rho^{3/4} \sigma^{-1/4} k^{-\beta} , \tag{6}$$

where $\beta = 17/4$ and P is the energy flux assumed to be constant. From measurements of the spectrum of wave amplitudes [17,18] we know that the spectrum levels off below $K = 2\pi/\Lambda$. In order to account for this change we introduce a scaling function $g(k/K)$ across the drive multiplying n_k. At $k < K$, $g(k/K)$ must rise from zero in such a way as to quench the decrease of n_k. At $k > K$, $g(k/K)$ can be assumed constant (up to the inverse dissipative scale k_{d}, where the theory breaks down). The motion at length scales below the dissipative scale is expected to become ballistic. Also the system size L is a relevant length scale. Obviously the diffusion change character near the system size since particles cannot travel longer distances than L. Also we shall consider the characteristic length scale a separating capillary from gravity waves.

The integral for the spatial correlations is given by

$$\mathcal{C}(R) \sim \int k^{2+\alpha-\beta} g(k/k_{\mathrm{F}}) J_0(kR) \mathrm{d}\boldsymbol{k} , \tag{7}$$

where the specific k-dependence for ω_k and n_k has been inserted. For capillary waves $2 + \alpha - \beta = -3/4$. This integral depends explicitly on R. Consider the value $K_R \simeq 2.4/R$, the lowest k-value at which $J_0(kR) = 0$. For $K_R > K$, the dominant contribution to the integral comes from the range $K < k < K_R$, where the scaling function g is constant,

$$\mathcal{C}(R) \sim \int_0^{\infty} k^{2+\alpha-\beta} g(k/K) J_0(kR) \mathrm{d}k \sim k^{3+\alpha-\beta}|_K^{K_R} \sim R^{-3-\alpha+\beta}. \tag{8}$$

The result for the velocity difference is

$$\langle |\delta v(R)|^2 \rangle \sim 1 - b(KR)^{-3-\alpha+\beta} \sim 1 - b(KR)^{-1/4} , \tag{9}$$

where b is a constant of order one [19].

For $K_R < K$, the Bessel function oscillations set in below K, while the integrand is still increasing in size due to the behavior of the scaling function $g(k/K)$. Therefore, the dominant contribution to the integral comes from a peak centered around K, it is oscillatory in nature with a vanishing envelope falling off like $R^{3/2}$. For large R-values we essentially have $\langle |\delta v(R)|^2 \rangle$ constant.

For gravity waves the dispersion relation is $\omega = g^{1/2} k^{\alpha}$ with $\alpha = 1/2$ (four-wave interaction) and we have an energy flow towards higher frequencies compared to Ω giving rise to the isotropic Kolmogorov spectrum [15,20]

$$n_k \sim P^{1/3} \rho^{2/3} k^{-\beta} , \tag{10}$$

where $\beta = 4$ and P is the energy flux assumed to be constant. Below Ω a constant wave number flux Q towards lower frequencies yields [21]

$$n_k \sim Q^{1/3} \rho^{2/3} g^{1/6} k^{-\beta+\alpha/3} , \tag{11}$$

with $\beta - \alpha/3 = 23/6$.

As above, we invoke a scaling function $g(k/K)$ describing the change in the behavior of n_k as k crosses the wave number K of the drive. According to the form of the spectra, we have

$$g(k/k_{\mathrm{F}}) \sim \begin{cases} 1\,, & \text{if } K < k < k_a, k_{\mathrm{d}}\,, \\ k^{\alpha/3}\,, & \text{if } k_L < k < K. \end{cases} \tag{12}$$

Other corrections are relevant at length scales outside the above regime. The motion at sufficiently large length scales ($k < k_L$) may be expected to become Brownian, as also found in studies of motion of drifters near the surface of the oceans [6].

For the velocity difference squared we have

$$\langle|\delta v(R)|^2\rangle \sim \int_0^\infty k^{2+\alpha-\beta} g(k/K)[1 - J_0(kR)]\mathrm{d}k \sim \int_{K_R}^\infty k^{-3/2} g(k/K)\mathrm{d}k\ . \tag{13}$$

For $K_R > K$, the scaling function is constant in the integration region above K_R, and we have

$$\langle|\delta v(R)|^2\rangle \sim K_R^{-1/2} \sim (KR)^{1/2}\ . \tag{14}$$

For $K_R < K$ the part of the integration, where $g(k/K) \sim k^{1/6}$, gives the R-dependent contribution, and

$$\langle|\delta v(R)|^2\rangle + \mathrm{const} \sim K_R^{-1/3} \sim (KR)^{1/3}\ . \tag{15}$$

The growth of $\langle|\delta v(R)|^2\rangle$ stops at length scale L. We note that the exponent 1/3 for the velocity difference squared implies an exponent 7/6 for the relative-diffusivity measure σ. It is interesting that Okubo [6] finds the relative-diffusion exponent to be 1.15 for oceanic diffusion, although we should emphasize that the relative-diffusion exponent 4/3 obtained in fully developed turbulence theory has generally been applied to explain experimental data [12].

Temporal correlations and capillary waves. – Next, we consider the single-particle diffusion in capillary waves. For arbitrary times, the mean-square displacement $\langle r^2\rangle = \langle|r(t_0 + t) - r(t_0)|^2\rangle$ is expressed in terms of the Lagrangian velocity correlation function $\mathcal{C}(\tau) = \langle \boldsymbol{v}(\boldsymbol{r}(t))\boldsymbol{v}(\boldsymbol{r}(t+\tau))\rangle$ as [22]

$$\langle r^2\rangle = 2t\int_0^t (1 - \tau/t)\mathcal{C}(\tau)\mathrm{d}\tau\ . \tag{16}$$

In the Taylor limit of very small t, $\mathcal{C}(\tau) \simeq \langle v_{\mathrm{L}}^2\rangle$, where $\langle v_{\mathrm{L}}^2\rangle$ is the mean-square Lagrangian velocity, to be determined by sampling along particle orbits. In this case

$$\langle r^2\rangle \approx \langle v_{\mathrm{L}}^2\rangle t^2\ . \tag{17}$$

For incompressible flows, the mean-square velocity obtained by Eulerian sampling, $\langle v_{\mathrm{E}}^2\rangle$, is identical to the mean-square velocity, $\langle v_{\mathrm{L}}^2\rangle$, obtained from Lagrangian measurements [23]. The subscripts on $\langle v^2\rangle$ will be omitted from here on. For large times, $t \gg \tau_{\mathrm{L}}$, where $\tau_{\mathrm{L}} = \frac{1}{\langle v^2\rangle}\int_0^\infty \mathcal{C}(\tau)\mathrm{d}\tau$ is the Lagrangian integral time scale, eq. (16) gives a diffusion-like dispersion

$$\langle r^2\rangle \sim 2\langle v^2\rangle\tau_{\mathrm{L}} t\ . \tag{18}$$

First, we consider capillary waves. If we invoke a scaling form $\langle r^2\rangle \sim t^\lambda$ at $t < T$, then eq. (16) gives

$$\mathcal{C}(\tau) \sim \tau^{\lambda-2}\ . \tag{19}$$

On the other hand, invoking the same scaling $\langle r^2 \rangle \sim t^\lambda$ and using eq. (8) yields

$$\mathcal{C}(\tau) \sim (\tau^{\frac{\lambda}{2}})^{-\frac{1}{4}} . \tag{20}$$

The result is a self-consistency condition for the exponent λ. We find

$$\lambda = \frac{4}{5+\alpha-\beta} = \frac{16}{9} . \tag{21}$$

This value obtained for λ is in good agreement with experimental observations which vary $\pm 10\%$ from this value [3,4]. From this the exponent for the time correlation function becomes $-2/9$.

Above, it has tacitly been assumed that the time scale t is larger than the one corresponding to the dissipative frequency ω_{d} corresponding to the inverse dissipative scale k_{d}. Experimentally, the sampled diffusion distances at small times may approach the dissipative scale at low values of the drive. Below this scale λ has the ballistic value 2, and the observed slope (1.8–1.9) slightly larger than the theoretical value 16/9 may be an effect hereof.

We have also assumed that the particles can follow the fluid velocity field, *i.e.* if $\boldsymbol{v}$ is the velocity field of the particles and $\boldsymbol{u}$ is that of the fluid, our assumption is that $\boldsymbol{v} = \boldsymbol{u}$, or at least that the weak ansatz $\mathcal{C}(\tau) = \langle \boldsymbol{u}(t+\tau)\boldsymbol{u}(t)\rangle$ is fulfilled. However, at large driving amplitudes the particles may not follow the fluid velocity. As an example, assume that corrections for the velocity correlation take the following form:

$$\langle \boldsymbol{v}(t)\boldsymbol{v}(t+\tau)\rangle \sim \tau^{-\delta}\langle \boldsymbol{u}(t)\boldsymbol{u}(t+\tau)\rangle , \tag{22}$$

where $\boldsymbol{v}$ and $\boldsymbol{u}$ are the velocity fields of the particles and of the flow, respectively. Following the calculation above, we then find

$$\lambda = \frac{4-2\delta}{5+\alpha-\beta} = \frac{16-8\delta}{9} , \tag{23}$$

which may explain why a slightly smaller exponent than 16/9 is observed for higher driving amplitudes (*e.g.*, $\delta = 0.2$ gives an exponent $\lambda = 1.6$).

Equivalently, the self-consistent condition for λ (eq. (21)) can be obtained in terms of the energy spectrum

$$E_k \sim k^{-\nu} , \tag{24}$$

where the energy per area is

$$E = \int E_{\boldsymbol{k}} \mathrm{d}\boldsymbol{k} = \int E_k \mathrm{d}k = \int k\omega_k n_k \mathrm{d}k , \tag{25}$$

we have the relationship $\nu = -(1+\alpha-\beta)$, and

$$\lambda = \frac{4}{4-\nu} . \tag{26}$$

The capillary-wave exponent is $\nu = 7/4$.

For $t > T$, the time correlation $\mathcal{C}(\tau)$ is oscillatory and therefore negligible for $t > T$. In this case, eq. (18) applies:

$$\langle r^2 \rangle \sim t , \tag{27}$$

and $\lambda = 1$. The Brownian-motion value obtained for λ at times τ above T is in agreement with experimental observations for large drives. The slightly larger values of λ at lower drives may arise from corrections to scaling. Experimentally, at sufficiently large times (above those considered in [3, 4]), the sampled diffusion distances are limited by the systems size L and eventually one must have $\lambda = 0$.

Temporal correlations and gravity waves. – For gravity waves, we can follow a similar assumption as the one proposed for capillary waves. However, in contrast to the situation for capillary waves, the scaling is given for $\langle|\boldsymbol{v}(t)-\boldsymbol{v}(t+\tau)|^2\rangle$. Thus, for $\tau < T$, we have from eq. (14)

$$\langle|\boldsymbol{v}(t)-\boldsymbol{v}(t+\tau)|^2\rangle \sim \tau^{\lambda/4}\,, \tag{28}$$

and for $\tau > T$, we get from eq. (15)

$$\langle|\boldsymbol{v}(t)-\boldsymbol{v}(t+\tau)|^2\rangle \sim \tau^{\lambda/6}\,. \tag{29}$$

Assuming

$$\sigma \sim \frac{R^2}{t}\,, \tag{30}$$

the exponents can now be calculated using eq. (2) from the self-consistent condition

$$t^{\lambda-1} \sim \begin{cases} t^{5\lambda/8}\,, & \text{small } t\,, \\ t^{7\lambda/12}\,, & \text{large } t\,. \end{cases} \tag{31}$$

The values obtained for λ are 8/3 for small t and 12/5 for large t. Interestingly, a similar situation is observed in ocean studies, where $\lambda = 2.34$ is found in several dye studies collected and analyzed by Okubo [6].

Conclusions. – Good agreement is found between theoretical and experimental scaling results for the diffusion and relative diffusion of particles on weakly turbulent surface waves. In the case of relative diffusion, for capillary waves the exponent $-1/4$ for the space correlation function is recovered. For gravity waves, a scaling behavior is derived for the velocity difference squared with the exponent changing from 1/2 at small distances to 1/3 at large distances. This corresponds respectively to scaling exponents 5/4 and 7/6 for the relative diffusivity. Intriguingly, Okubo finds a relative-diffusion exponent of 1.15 for oceanic diffusion [6]. This differs from the relative-diffusion exponent 4/3 obtained in fully developed turbulence theory, previously used in the discussion of experimental data [12].

For capillary waves the theory yields a crossover in the diffusivity at the wavelength Λ with a change in diffusion exponent λ from $\lambda = 16/9 \simeq 1.78$ to $\lambda = 1$, in good agreement with experimental findings. Particles suspended on the fluid surface may not follow the fluid far above the Faraday instability, and we suggest that this may explain the small discrepancy between weak-turbulence theory and experimental results in this regime. For gravity waves the weak-turbulence theory gives a diffusion exponent of $12/5 = 2.4$ for long times which compares favourably with the exponent 2.34 found by Okubo for dye diffusion. To what extent the weak wave turbulence theory really explains the oceanic diffusivity is unclear but it is fascinating to speculate that wave motion taking place mostly on a short scale could be responsible for the long-time diffusion in oceans. At very long time scales, *i.e.* above the Lagrangian time scale, ordinary Brownian motion is expected.

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REFERENCES

[1] RAMSHANKAR R., BERLIN D. and GOLLUB J. P., *Phys. Fluids A*, **2** (1990) 1955.
[2] RAMSHANKAR R. and GOLLUB J. P., *Phys. Fluids A*, **3** (1991) 1344.
[3] ALSTRØM P., ANDERSEN J. S., GOLDBURG W. I. and LEVINSEN M. T., *Chaos, Solitons and Fractals*, **5** (1995) 1455.
[4] SCHRÖDER E., LEVINSEN M.T. and ALSTRØM P., *Physica A*, **239** (1997) 314.
[5] HANSEN A. E., SCHRÖDER E., ALSTRØM P., ANDERSEN J. S. and LEVINSEN M. T., *Phys. Rev. Lett.*, **79** (1997) 1845.
[6] OKUBO A., *Deep-Sea Res.*, **18** (1971) 789 (see this reference for sources of data); OKUBO A., OZMIDOV R.V., *Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.*, **6** (1970) 534.
[7] See, *e.g.*, LANDAU L. D. and LIFSHITZ E. M., *Fluid Mechanics*, 2nd edition (Pergamon, Oxford) 1987, Chapt. VII.
[8] For a review of parametrically excited surface waves, see NEVOLIN V. G., *Inzh. Fiz. Zh.*, **47** (1984) 1028 (*J. Eng. Phys. (USSR)*, **47** (1984) 1028) or MILES J. and HENDERSON D., *Annu. Rev. Fluid Mech.*, **22** (1990) 143.
[9] SCHRÖDER E., ANDERSEN J. S., LEVINSEN M. T., ALSTRØM P. and GOLDBURG W. I., *Phys. Rev. Lett.*, **76** (1996) 4717.
[10] WEICHMAN P. B. and GLAZMAN R. E., *Phys. Rev. Lett.*, **83** (1999) 5011. WEICHMAN P. B. and GLAZMAN R. E., *J. Fluid Mech.*, **420** (2000) 147; KLYATSKIN V. I. and YAKUSHSKIN I. G., *JETP*, **91** (2000) 736; BALK A., *Phys. Lett. A*, **279** (2001) 370; BALK A. and MCLAUGHLIN R. M., *Phys. Lett. A*, **256** (1999) 299.
[11] HERTERICH K. and HASSELMANN R. J., *J. Phys. Oceanogr.*, **12** (1982) 704.
[12] MONIN A. S. and YAGLOM A. M., *Statistical Hydrodynamics*, Vol. **2** (MIT Press) 1975; GOLITSYN G. S., *Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.*, **37** (2001) 406.
[13] STOMMEL H., *J. Mar. Res.*, **8** (1949) 199.
[14] SCHRÖDER E. and ALSTRØM P., *Phys. Rev. E*, **57** (1998) 7329.
[15] ZAKHAROV V. E., L'VOV V. S. and FALKOVICH G., *Kolmogorov Spectra of Turbulence I, Wave Turbulence* (Springer-Verlag, Berlin) 1992.
[16] ZAKHAROV V. E. and FILONENKO N. N., *Zh. Prikl. Mekh. Tekh. Fiz*, **8** (1967) 62 (*J. Appl. Mech. Tech. Phys.*, **8** (1967) 37).
[17] HENRY E., ALSTRØM P. and LEVINSEN M.T., *Stochastic and Chaotic Dynamics in the Lakes*, edited by BROOMHEAD D. S., LUCHINSKAYA E. A., MCCLINTOCK P. V. E. and MULLIN T. (American Institute of Physics, Melville, NY, USA) 2000, p. 504.
[18] HENRY E., ALSTRØM P. and LEVINSEN M. T., *Europhys. Lett.*, **52** (2000) 27.
[19] We note that a similar result was derived in ref. [14], however, using incorrect limits on the integral.
[20] ZAKHAROV V. E. and FILONENKO N. N., *Dokl. Akad. Nauk SSSR*, **170** (1966) 1292 (*Sov. Phys. Dokl.*, **11** (1967) 881).
[21] ZAKHAROV V. E. and ZASLAVSKII M. M., *Izv. Akad. Nauk SSSR Fiz. Atmos. Okeana*, **18** (1982) 970 (*Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.*, **18** (1982) 747).
[22] HANSEN J. P. and MCDONALD I. R., *Theory of Simple Liquids* (Academic Press) 1976, Chapt. 7.
[23] TENNEKES H. and LUMLEY J. L., *A First Course on Turbulence* (MIT Press) 1972.