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The Adapted Ordering Method for Lie Algebras and Superalgebras and their Generalizations

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ABSTRACT

In 1998 the Adapted Ordering Method was developed for the representation theory of the superconformal algebras in two dimensions. It allows: to determine maximal dimensions for a given type of space of singular vectors, to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams. In this article we present the Adapted Ordering Method for general Lie algebras and superalgebras, and their generalizations, provided they can be triangulated. We also review briefly the results obtained for the Virasoro algebra and for the $N = 2$ and Ramond $N = 1$ superconformal algebras.

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1 Introduction and Notation

In 1998 the Adapted Ordering Method was developed by M. Dörrzapf and B. Gato-Rivera[1], for the study of the representation theory of the superconformal algebras in two dimensions, also known as super Virasoro algebras. These are infinite-dimensional Lie superalgebras which contain the Virasoro algebra as a subalgebra. They were first constructed three decades ago independently by Kac, along with his classification of Lie superalgebras[2], and by Ademollo et al. as the symmetry algebras of the supersymmetric strings[3]. At present, although several research lines make use of the superconformal algebras, their main relevance in physics is still the fact that they provide the underlying symmetries of Superstring Theory. The superconformal symmetries have a number N of fermionic anticommuting currents, corresponding to N supersymmetries. Their mode decomposition provide the N infinite sets of anticommuting generators of the superconformal algebras, whereas the Virasoro operators provide the infinite set of commuting generators, together with some other infinite sets of commuting generators which exist for $N > 1$ and arise as symmetries between the supercurrents. The Adapted Ordering Method was applied successfully to the $N = 2$ superconformal algebras[1, 4] (topological, Neveu-Schwarz, Ramond and twisted) and to the Ramond $N = 1$ superconformal algebra[5], allowing to obtain rigorous proofs for several conjectured results, as well as many new results, especially for the case of the twisted $N = 2$ superconformal algebra and the case of the Ramond $N = 1$ superconformal algebra.

An obvious question now is whether the Adapted Ordering Method can be generalized and can be applied to the study of algebras different than the superconformal ones. The answer is positive and the purpose of this article is precisely to provide the general description of the Adapted Ordering Method for Lie algebras and superalgebras, and their generalizations, provided they have a triangular decomposition, as is the case for many of them[6].

Let us introduce some basic concepts and notation which will be used throughout this article. For a given algebra or superalgebra one defines freely generated modules over a highest weight (h.w.) vector, denoted as *Verma modules*. The *annihilation operators* of the algebra are the generators which annihilate the h.w. vectors of the Verma modules, whereas the *creation operators* are the generators directly involved in the construction of the Verma modules by acting on the h.w. vectors. A Verma module is in general irreducible, but in some degenerate cases it contains submodules which are freely generated over, at least, one h.w. vector different from the h.w. vector of the Verma module. These vectors are annihilated by all the annihilation operators of the algebra, consequently, and are usually referred to as *singular vectors*. The irreducible h.w. representations are then obtained as the quotients of the Verma modules divided by all their submodules. Surprisingly, the complete set of singular vectors do not generate all the submodules in the case of Verma modules which contain *subsingular vectors*. The reason is that subsingular vectors are singular vectors of the quotient space, but not of the Verma module itself[11−14]. In this case one has to divide further by the submodules generated by the subsingular vectors, repeating this division procedure successively, if necessary.

On the Verma modules one introduces a hermitian contravariant form, known as Shapovalov form. The vanishing of the corresponding determinant indicates the existence of at least one singular vector. The determinant may not detect the whole set of singular vectors, however, neither does it give the dimension of the space of singular vectors with some given weights. There could be in fact more than one linearly independent singular vectors with the same weights. Therefore, the dimensions of the spaces of singular vectors have to be found by an independent procedure. The Adapted Ordering Method provides such a procedure since it puts upper limits on these dimensions, allowing to determine the maximal dimension for a given space of singular vectors. For most weights of a Verma module these upper limits on the dimensions of the spaces of singular vectors are found equal to zero and, as a consequence, one obtains a rigorous proof that there cannot exist any singular vectors for these weights. For some weights, however, one finds that spaces of singular vectors are allowed to exist, either only one-dimensional, as is the case for the Virasoro algebra, or even higher dimensional spaces, as it happens for the $N = 2$ and Ramond $N = 1$ superconformal algebras[4, 5, 14–16]. As we will see, the Adapted Ordering Method also allows to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams, as a result.

The idea for developing the Adapted Ordering Method originated, in rudimentary form, from a procedure due to A. Kent for the study of the representations of the Virasoro algebra[7]. For this purpose the author analytically continued the Virasoro Verma modules, yielding 'generalised' Verma modules, where he constructed 'generalised' singular vectors in terms of analytically continued Virasoro operators. This analytical continuation is not necessary, however, for the Adapted Ordering Method, nor is it necessary to construct singular vectors in order to apply it. The underlying idea is the concept of *adapted orderings* for all the possible terms of the 'would be' singular vectors. An adapted ordering is a criterion, satisfying certain requirements, to decide which of two given terms is the bigger one. To be more specific, a total ordering will be called *adapted* to a subset of terms provided some conditions are met. The complement of that subset will be the *ordering kernel* and will play a crucial rôle since its size puts un upper limit on the dimension of the space of singular vectors.

In what follows, in section 2 we will describe the Adapted Ordering Method for a general Lie algebra or superalgebra with a triangular decomposition and, as an example, we will apply this method to the Virasoro algebra. In section 3 we will review briefly the results obtained for the $N = 2$ and the Ramond $N = 1$ superconformal algebras, as an illustration of the possibilities of this method. Section 4 is devoted to conclusions.

2 The Adapted Ordering Method

Let $\mathcal{A}$ denote a Lie algebra or superalgebra with a triangular decomposition: $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{H}_\mathcal{A} \oplus \mathcal{A}^+$, where $\mathcal{A}^-$ is the set of *creation operators*, $\mathcal{A}^+$ is the set of *annihilation operators*, and $\mathcal{H}_\mathcal{A}$ is the *Cartan subalgebra*. In general, an eigenvector with respect to the Cartan subalgebra with *relative weights* given by the set $\{l_i\}$, in particular a singular vector $\Psi_{\{l_i\}}$, can be expressed as a sum of products of creation operators with total weights $\{l_i\}$ acting on a h.w. vector with weights $\{\Delta_i\}$:

$$\Psi_{\{l_i\}} \quad = \sum_{m_1,m_2,....\in\mathbb{N}_0} \sum_{a,b,c,...} k_{a_{-1}^{m_1},a_{-2}^{m_2},...b_{-1}^{n_1},b_{-2}^{n_2},.....} X_{\{l_i\}}^{a_{-1}^{m_1},a_{-2}^{m_2},...b_{-1}^{n_1},b_{-2}^{n_2},.....} |\{\Delta_i\}\rangle \ , \tag{1}$$

where $a_{-1}, a_{-2}, ...b_{-1}, b_{-2}, ...$ are the creation operators of the algebra, $X_{\{l_i\}}^{a_{-1}^{m_1},a_{-2}^{m_2},...b_{-1}^{n_1},b_{-2}^{n_2},.....}$ are the products of the creation operators: $a_{-1}^{m_1} a_{-2}^{m_2} .....b_{-1}^{n_1} b_{-2}^{n_2}.....$, with total weights $\{l_i\}$, which will be denoted simply as *terms*, and $k_{a_{-1}^{m_1},a_{-2}^{m_2},...b_{-1}^{n_1},b_{-2}^{n_2},.....} \in \mathbb{C}$ are coefficients which depend on the given term. A non-trivial term Y then refers to a term with non-trivial coefficient k_Y. Observe that the weights of $\Psi_{\{l_i\}}$ are given by $\{l_i + \Delta_i\}$, it is however customary to label the vectors in the Verma modules by their *relative weights* $\{l_i\}$.

Now let us define the set $\mathcal{C}_{\{l_i\}}$ as the set of all the terms with weights $\{l_i\}$:

$$\mathcal{C}_{\{l_i\}} \quad = \quad \{X_{\{l_i\}}^{a_{-1}^{m_1}, a_{-2}^{m_2}, \ldots b_{-1}^{n_1}, b_{-2}^{n_2}, \ldots\ldots}\,, \; m_1, m_2, \ldots . n_1, n_2, \ldots .. \in \mathbb{N}_0\}\,, \tag{2}$$

and let $\mathcal{O}$ denote a total ordering on $\mathcal{C}_{\{l_i\}}$, that is an ordering such that any two different terms in $\mathcal{C}_{\{l_i\}}$ are ordered with respect to each other. Thus $\Psi_{\{l_i\}}$ in Eq. (1) needs to contain an $\mathcal{O}$-*smallest* $X_0 \in \mathcal{C}_{\{l_i\}}$ with $k_{X_0} \neq 0$ and $k_Y = 0$ for all $Y \in \mathcal{C}_{\{l_i\}}$ with $Y <_{\mathcal{O}} X_0$ and $Y \neq X_0$. We define an *adapted ordering* on $\mathcal{C}_{\{l_i\}}$ as follows:

Definition 2.A *A total ordering $\mathcal{O}$ on $\mathcal{C}_{\{l_i\}}$ is called adapted to the subset $\mathcal{C}^A_{\{l_i\}} \subset \mathcal{C}_{\{l_i\}}$ in the Verma module $\mathcal{V}_{\{\Delta_i\}}$ if for any element $X_0 \in \mathcal{C}^A_{\{l_i\}}$ at least one annihilation operator Γ exists for which $\Gamma\, X_0\, |\{\Delta_i\}\rangle$ contains a non-trivial term $\tilde{X}$*

$$\Gamma\, X_0\, |\{\Delta_i\}\rangle \quad = \quad (k_{\tilde{X}} \tilde{X} + \ldots\ldots.)\; |\{\Delta_i\}\rangle \tag{3}$$

which is absent, however, for all $\Gamma\, X\, |\{\Delta_i\}\rangle$, where X is any term $X \in \mathcal{C}_{\{l_i\}}$ which is $\mathcal{O}$-larger than X_0, that is such that $X_0 <_{\mathcal{O}} X$. The complement of $\mathcal{C}^A_{\{l_i\}}$, $\mathcal{C}^K_{\{l_i\}} = \mathcal{C}_{\{l_i\}} \setminus \mathcal{C}^A_{\{l_i\}}$ is the kernel with respect to the ordering $\mathcal{O}$ in the Verma module $\mathcal{V}_{\{\Delta_i\}}$.

Now we will see that the coefficients with respect to the terms of the ordering kernel $\mathcal{C}^K_{\{l_i\}}$ uniquely identify a singular vector $\Psi_{\{l_i\}}$. Since the size of the ordering kernels are in general small, it turns out that just a few coefficients completely determine a singular vector no matter its size, what allows to find easily product expressions for descendant singular vectors. For example, in the case of the conformal and N=1,2 superconformal algebras the ordering kernels found for most weights have zero or one term, for some weights they have two terms and for some other weights they have three terms. This property is summarized in the following theorem:

Theorem 2.B *Let $\mathcal{O}$ denote an ordering adapted to $\mathcal{C}^A_{\{l_i\}}$ at weights $\{l_i\}$ with kernel $\mathcal{C}^K_{\{l_i\}}$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. If two singular vectors $\Psi^1_{\{l_i\}}$ and $\Psi^2_{\{l_i\}}$ with the same weights have $k^1_X = k^2_X$ for all $X \in \mathcal{C}^K_{\{l_i\}}$, then*

$$\Psi^1_{\{l_i\}} \quad \equiv \quad \Psi^2_{\{l_i\}}\,. \tag{4}$$

Proof of Theorem 2.B: Let us consider the singular vector $\Psi_{\{l_i\}} = \Psi^1_{\{l_i\}} - \Psi^2_{\{l_i\}}$, which does not contain any terms of the ordering kernel $\mathcal{C}^K_{\{l_i\}}$, simply because $k^1_X = k^2_X$ for all $X \in \mathcal{C}^K_{\{l_i\}}$. As $\mathcal{C}_{\{l_i\}}$ is a totally ordered set with respect to $\mathcal{O}$, the non-trivial terms of $\Psi_{\{l_i\}}$, provided $\Psi_{\{l_i\}}$ is non-trivial, need to have a $\mathcal{O}$-smallest $X_0 \in \mathcal{C}^A_{\{l_i\}}$. Thus the coefficient k_{X_0} of X_0 in $\Psi_{\{l_i\}}$ must be non-trivial. As $\mathcal{O}$ is adapted to $\mathcal{C}^A_{\{l_i\}}$ one can find an annihilation operator Γ such that $\Gamma X_0\, |\{\Delta_i\}\rangle$ contains a non-trivial term that cannot be created by Γ acting on any other term of $\Psi_{\{l_i\}}$ which is $\mathcal{O}$-larger than X_0. But X_0 was chosen to be the $\mathcal{O}$-smallest term of $\Psi_{\{l_i\}}$. Therefore, $\Gamma X_0\, |\{\Delta_i\}\rangle$ contains a non-trivial term that cannot be created from any other term of $\Psi_{\{l_i\}}$. The coefficient of this term is obviously given by $c k_{X_0}$ with c a non-trivial complex number. But $\Psi_{\{l_i\}}$ is a singular vector and therefore must be annihilated by any annihilation operator, in particular by Γ. It follows that $k_{X_0} = 0$, contrary to our original assumption. Thus, the set of non-trivial terms of $\Psi_{\{l_i\}}$ is empty and therefore $\Psi_{\{l_i\}} = 0$. This results in $\Psi^1_{\{l_i\}} = \Psi^2_{\{l_i\}}$. $\square$

Theorem 2.B states, therefore, that if two singular vectors with the same weights, in the same Verma module, agree on the coefficients of the ordering kernel, then they are identical. A crucial point now is that the size of the kernel puts an upper limit on the dimension of the corresponding space of singular vectors, as stated in the following theorem:

Theorem 2.C *Let $\mathcal{O}$ denote an ordering adapted to $\mathcal{C}^{A}_{\{l_i\}}$ at weights $\{l_i\}$ with kernel $\mathcal{C}^{K}_{\{l_i\}}$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. If the ordering kernel $\mathcal{C}^{K}_{\{l_i\}}$ has n elements, then there are at most n linearly independent singular vectors $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with relative weights $\{l_i\}$.*

Proof of Theorem 2.C:

Suppose there were more than n linearly independent singular vectors $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with relative weights $\{l_i\}$. We choose $n+1$ linearly independent singular vectors among them $\Psi_1,\ldots,\Psi_{n+1}$. The ordering kernel $\mathcal{C}^{K}_{\{l_i\}}$ has the n elements $X_1,\ldots,X_n$. Let k_{jk} denote the coefficient of the term X_j in the vector Ψ_k in a suitable basis decomposition. The coefficients k_{jk} thus form a n by $n+1$ matrix M. The homogeneous system of linear equations $M\lambda = 0$ thus has a non-trivial solution $\lambda^0 = (\lambda^0_1, \ldots, \lambda^0_{n+1})$ for the vector λ. We then form the linear combination $\Psi = \sum_{i=1}^{n+1} \lambda^0_i \Psi_i$. Obviously, the coefficient of X_j for the vector Ψ is just given by the j-th component of the vector $M\lambda$ which is trivial for $j = 1, \ldots, n$. Hence, the coefficients of Ψ are trivial on the ordering kernel. On the other hand, Ψ is a linear combination of singular vectors and therefore it is also a singular vector. Due to theorem 2.B one immediately finds that $\Psi \equiv 0$ and therefore $\sum_{i=1}^{n+1} \lambda_i \Psi_i = 0$. This, however, contradicts the assumption that $\Psi_1,\ldots, \Psi_{n+1}$ are linearly independent. $\Box$

Therefore, one needs to find suitable orderings in order to obtain the smallest possible kernels. Observe that the maximal possible dimension n does not imply that all the singular vectors of the corresponding type are n-dimensional. From this theorem one deduces that if $\mathcal{C}^{K}_{\{l_i\}} = \emptyset$ for a given Verma module, then there are no singular vectors with relative weights $\{l_i\}$ in it. That is:

Theorem 2.D *Let $\mathcal{O}$ denote an ordering adapted to $\mathcal{C}^{A}_{\{l_i\}}$ at weights $\{l_i\}$ with trivial kernel $\mathcal{C}^{K}_{\{l_i\}} = \emptyset$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. A singular vector $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with relative weights $\{l_i\}$ must be therefore trivial.*

Although this theorem is deduced straightforwardly from theorem 2.C, which is exactly proven, there is another interesting proof using theorem 2.B

Proof of Theorem 2.D: The trivial vector 0 satisfies any annihilation conditions for any weights. As the ordering kernel is trivial, $\mathcal{C}^{K}_{\{l_i\}} = \emptyset$, the components of the vectors 0 and $\Psi_{\{l_i\}}$ agree on the ordering kernel and using theorem 2.B we obtain $\Psi_{\{l_i\}} = 0$. $\Box$

As a simple example of the Adapted Ordering Method we will see now the application of this method to the Virasoro algebra V, which has been extensively studied in the literature[8–10]. This algebra is given by the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}\,, \quad [C, L_m] = 0\,, \quad m,n \in \mathbb{Z}\,, \tag{5}$$

where C commutes with all operators of V and can hence be taken to be constant $c \in \mathbb{C}$. V can be written in its *triangular decomposition*: $\mathsf{V} = \mathsf{V}^- \oplus \mathcal{H}_{\mathsf{V}} \oplus \mathsf{V}^+$, where $\mathsf{V}^- = \mathrm{span}\{L_{-m} : m \in \mathbb{N}\}$

is the set of *creation operators*, $\mathsf{V}^+ = \mathrm{span}\{L_m : m \in \mathbf{N}\}$ is the set of *annihilation operators*, and the *Cartan subalgebra* is given by $\mathcal{H}_{\mathsf{V}} = \mathrm{span}\{L_0, C\}$. For elements of V that are eigenvectors of L_0 with respect to the adjoint representation the L_0-eigenvalue is usually called the *level* l. The *terms* are obviously given by the products of the form $L_{-p_I} \dots L_{-p_1}$, $p_q \in \mathbf{N}$ for $q = 1, \dots, I$, $I \in \mathbf{N}$, with level $l = \sum_{q=1}^{I} p_q$. Note that annihilation operators $L_m \in \mathsf{V}^+$ have negative level $l = -m$, $m \in \mathbf{N}$.

A representation with L_0-eigenvalues bounded from below contains a highest weight (h.w.) vector $|\Delta\rangle$, with L_0-eigenvalue Δ, which is annihilated by the set of annihilation operators V^+ :

$$\mathsf{V}^+ \, |\Delta\rangle = 0 \,, \quad L_0 \, |\Delta\rangle = \Delta \, |\Delta\rangle \ . \tag{6}$$

The Verma module $\mathcal{V}_\Delta$ built on $|\Delta\rangle$ is L_0-graded in a natural way. The corresponding L_0-eigenvalue is called the *conformal weight* and is written for convenience as $\Delta + l$, where l is the *level.* Any proper submodule of $\mathcal{V}_\Delta$ needs to contain a *singular vector* Ψ_l that is not proportional to the h.w. vector $|\Delta\rangle$ but still satisfies the h.w. conditions with conformal weight $\Delta + l$:

$$\mathsf{V}^+ \Psi_l = 0 \,, \quad L_0 \Psi_l = (\Delta + l) \Psi_l \,. \tag{7}$$

Now we will see the total ordering on the set of terms $\mathcal{C}_l$ at level l defined by Kent[7] for the Virasoro algebra. One has to take into account, however, that Kent used the following ordering in order to show that, in his generalised Verma modules, the generalised singular vectors at level 0 satisfying the h.w. conditions are actually proportional to the h.w. vector. Using the Adapted Ordering technology, though, one deduces that this ordering already implies that all Virasoro singular vectors are unique at their levels up to proportionality, simply because the ordering kernel for each level $l \in \mathbf{N}$ has just one element: L_{-1}^l.

Definition 2.E *On the set $\mathcal{C}_l$ of terms of Virasoro operators at level l one introduces the total ordering $\mathcal{O}_{\mathsf{V}}$ for $l \in \mathbf{N}$: For any two terms $X_1, X_2 \in \mathcal{C}_l$, $X_1 \neq X_2$, with $X_i = L_{-m^i_{I_i}} \dots L_{-m^i_1} L_{-1}^{n^i}$, $n^i = l - m^i_{I_i} \dots - m^i_1$, or $X_i = L_{-1}^l$, $i = 1, 2$ one defines*

$$X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2 \quad \text{if} \quad n^1 > n^2 \,. \tag{8}$$

If, however, $n^1 = n^2$ one computes the index $j_0 = \min\{j : m^1_j - m^2_j \neq 0, j = 1, \dots, \min(I_1, I_2)\}$. One then defines

$$X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2 \quad \text{if} \quad m^1_{j_0} < m^2_{j_0} \,. \tag{9}$$

For $X_1 = X_2$ one sets $X_1 <_{\mathcal{O}_{\mathsf{V}}} X_2$ and $X_2 <_{\mathcal{O}_{\mathsf{V}}} X_1$.

The index j_0 describes the first mode, read from the right to the left, for which the generators in X_1 and X_2 (L_{-1} excluded) are different. For example, in $\mathcal{C}_8$ one has $L_{-2}L_{-2}L_{-2}L_{-1}^2 <_{\mathcal{O}_{\mathsf{V}}} L_{-4}L_{-2}L_{-1}^2$ with index $j_0 = 2$. Observe that $L_{-1}^l \in \mathcal{C}_l$ is the $\mathcal{O}_{\mathsf{V}}$-smallest term in $\mathcal{C}_l$. Now using the Adapted Ordering Method one finds the following theorem[1].

Theorem 2.F *The ordering $\mathcal{O}_{\mathsf{V}}$ is adapted to $\mathcal{C}^A_l = \mathcal{C}_l \setminus \{L_{-1}^l\}$ for each level $l \in \mathbf{N}$ and for all Verma modules $\mathcal{V}_\Delta$. The ordering kernel is given by the single element set $\mathcal{C}^K_l = \{L_{-1}^l\}$.*

For example let us consider the set of terms at level 3, $\mathcal{C}_3 = \{L_{-1}^3, L_{-2}L_{-1}, L_{-3}\}$. One finds the total ordering $L_{-1}^3 <_{\mathcal{O}_V} L_{-2}L_{-1} <_{\mathcal{O}_V} L_{-3}$, which is adapted to $\mathcal{C}_3^A = \{L_{-2}L_{-1}, L_{-3}\}$ with the ordering kernel $\mathcal{C}_3^K = \{L_{-1}^3\}$. To see this one has to compute the action of the annihilation operators $\Gamma \in \{L_1, L_2, L_3\}$ on the three terms. In fact, the action of L_1 already reveals the structure of $\mathcal{C}_3^A$, as $L_1 L_{-2}L_{-1} |\Delta\rangle$ contains the term L_{-1}^2 that is absent in $L_1 L_{-3} |\Delta\rangle$. The action of the three annihilation operators on $L_{-1}^3 |\Delta\rangle$, however, produce terms that are also created by the action of these operators on $L_{-2}L_{-1} |\Delta\rangle$ and/or $L_{-3} |\Delta\rangle$.

Finally, from the previous theorem one now deduces the known result about the uniqueness of the Virasoro singular vectors[7].

Theorem 2.G *If the Virasoro Verma module $\mathcal{V}_\Delta$ contains a singular vector Ψ_l at level l, $l \in \mathbf{N}$, then Ψ_l is unique up to proportionality.*

3 Results for the superconformal algebras

As an illustration of the possibilities of the Adapted Ordering Method, in this section we will review briefly the results obtained for the $N = 2$ and Ramond $N = 1$ superconformal algebras. This method has been applied to the topological, to the Neveu-Schwarz and to the Ramond $N = 2$ algebras in Ref. 1, to the twisted $N = 2$ algebra in Ref. 4 and to the Ramond $N = 1$ algebra in Ref. 5. As the representation theory of these superconformal algebras has different types of Verma modules, one has to introduce different adapted orderings for each type and the corresponding kernels also allow different degrees of freedom.

Let us start with the topological $N = 2$ superconformal algebra T. It contains the Virasoro generators $\mathcal{L}_m$ with trivial central extension, a Heisenberg algebra $\mathcal{H}_m$ corresponding to a U(1) current, and the fermionic generators $\mathcal{G}_m$ and $\mathcal{Q}_m$ corresponding to two anticommuting fields with conformal weights 2 and 1 respectively. T satisifies the (anti-)commutation relations[17]

$$
\begin{array}{ll}
[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}\,, & [\mathcal{H}_m, \mathcal{H}_n] = \frac{C}{3} m \delta_{m+n}\,, \\
[\mathcal{L}_m, \mathcal{G}_n] = (m-n)\mathcal{G}_{m+n}\,, & [\mathcal{H}_m, \mathcal{G}_n] = \mathcal{G}_{m+n}\,, \\
[\mathcal{L}_m, \mathcal{Q}_n] = -n\mathcal{Q}_{m+n}\,, & [\mathcal{H}_m, \mathcal{Q}_n] = -\mathcal{Q}_{m+n}\,, \\
[\mathcal{L}_m, \mathcal{H}_n] = -n\mathcal{H}_{m+n} + \frac{C}{6}(m^2+m)\delta_{m+n}\,, & \\
\{\mathcal{G}_m, \mathcal{Q}_n\} = 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{C}{3}(m^2+m)\delta_{m+n}\,, & \\
\{\mathcal{G}_m, \mathcal{G}_n\} = \{\mathcal{Q}_m, \mathcal{Q}_n\} = 0\,, & m,\ n \in \mathbf{Z}\,.
\end{array}
\tag{10}
$$

The set of *annihilation operators* which is common for all the Verma modules, T^+, is spanned by the generators with positive index, the set of *creation operators* which is common for all the Verma modules, T^-, is spanned by the generators with negative index, and the *zero modes* are given by $\mathsf{T}^0 = \mathrm{span}\{\mathcal{L}_0, \mathcal{H}_0, C, \mathcal{G}_0, \mathcal{Q}_0\}$. The Cartan subalgebra is generated by $\mathcal{H}_\mathsf{T} = \mathrm{span}\{\mathcal{L}_0, \mathcal{H}_0, C\}$, where C can be taken to be constant $c \in \mathbf{C}$, and the fermionic generators $\{\mathcal{G}_0, \mathcal{Q}_0\}$ can act as annihilation or as creation operators, classifying the different types of Verma modules in this way.

A h.w. vector $|\Delta, h\rangle^{\mathcal{N}}$ is an eigenvector of $\mathcal{H}_\mathsf{T}$ with $\mathcal{L}_0$-eigenvalue Δ, $\mathcal{H}_0$-eigenvalue h, and vanishing T^+ action. Additional vanishing conditions $\mathcal{N}$ are possible with respect to the operators $\mathcal{G}_0$ and $\mathcal{Q}_0$, resulting as follows[14]. One can distinguish four different types of h.w. vectors $|\Delta, h\rangle^{\mathcal{N}}$ labeled by a superscript $\mathcal{N} \in \{G, Q, GQ\}$, or no superscript at all: h.w. vectors $|\Delta, h\rangle^G$ annihilated by $\mathcal{G}_0$ but not by $\mathcal{Q}_0$ ($\mathcal{G}_0$-closed), h.w. vectors $|\Delta, h\rangle^Q$ annihilated by $\mathcal{Q}_0$ but not by $\mathcal{G}_0$ ($\mathcal{Q}_0$-closed), h.w. vectors $|0, h\rangle^{GQ}$ annihilated by both $\mathcal{G}_0$ and $\mathcal{Q}_0$ (*chiral*), with zero conformal weight necessarily, and finally undecomposable h.w. vectors $|0,h\rangle$ that are neither annihilated by $\mathcal{G}_0$ nor by $\mathcal{Q}_0$ (*no-label*), also with zero conformal weight. Hence we have four different types of Verma modules[14]: $\mathcal{V}^{G}_{\Delta,h}$, $\mathcal{V}^{Q}_{\Delta,h}$, $\mathcal{V}^{GQ}_{0,h}$ and $\mathcal{V}_{0,h}$, built on the four different types of h.w. vectors.

For elements X of T which are eigenvectors of $\mathcal{H}_{\mathsf{T}}$ with respect to the adjoint representation one defines the *level* l as $[\mathcal{L}_0, X] = lX$ and the *charge* q as $[\mathcal{H}_0, X] = qX$. In particular, elements of the form $X = \mathcal{L}_{-l_L} \dots \mathcal{L}_{-l_1}\mathcal{H}_{-h_H} \dots \mathcal{H}_{-h_1}\mathcal{Q}_{-q_Q} \dots \mathcal{Q}_{-q_1}\mathcal{G}_{-g_G} \dots \mathcal{G}_{-g_1}$, and any reorderings of it, have level $l = \sum_{j=1}^{L} l_j + \sum_{j=1}^{H} h_j + \sum_{j=1}^{Q} q_j + \sum_{j=1}^{G} g_j$ and charge $q = G - Q$. The Verma modules are naturally $\mathbf{N}_0 \times \mathbf{Z}$ graded with respect to the $\mathcal{H}_{\mathsf{T}}$ eigenvalues relative to the eigenvalues (Δ, h) of the h.w. vector. For a vector labeled as $\Psi_{l,q}$ in $\mathcal{V}^{N}_{\Delta,h}$ the $\mathcal{L}_0$-eigenvalue is $\Delta + l$ and the $\mathcal{H}_0$-eigenvalue is $h + q$ with the level $l \in \mathbf{N}_0$ and the relative charge $q \in \mathbf{Z}$.

The singular vectors are annihilated by T^+ and may also satisfy additional vanishing conditions with respect to the operators $\mathcal{G}_0$ and $\mathcal{Q}_0$. Therefore one also distinguishes singular vectors of the types[14] $\Psi^{G}_{l,q}$, $\Psi^{Q}_{l,q}$, $\Psi^{GQ}_{l,q}$ and $\Psi_{l,q}$. As there are 4 types of Verma modules and 4 types of singular vectors one might think of 16 different combinations of singular vectors in Verma modules. However, no-label and chiral singular vectors do not exist neither in *chiral* Verma modules $\mathcal{V}^{GQ}_{0,h}$ nor in *no-label* Verma modules[14] $\mathcal{V}_{0,h}$ (with one exception: chiral singular vectors exist at level 0 in no-label Verma modules). Using the Adapted Ordering Method one has to introduce adapted orderings for the remaining 12 combinations, whose kernels give upper limits for the dimensions of the corresponding spaces of singular vectors. One finds that for most charges q singular vectors do not exist. For the case of the Verma modules $\mathcal{V}^{G}_{\Delta,h}$ built on $\mathcal{G}_0$-closed h.w. vectors $|\Delta,h\rangle^{G}$, for $c \neq 3$, the maximal dimensions for the spaces of singular vectors $\Psi^{G}_{l,q}$, $\Psi^{Q}_{l,q}$ $\Psi^{GQ}_{l,q}$ and $\Psi_{l,q}$ are given as follows[1]:

	$q=-2$	$q=-1$	$q=0$	$q=1$	$q=2$
$\Psi^{G}_{l,q,\|\Delta,h\rangle^{G}}$	0	1	2	1	0
$\Psi^{Q}_{l,q,\|\Delta,h\rangle^{G}}$	1	2	1	0	0
$\Psi^{GQ}_{l,q,\|-l,h\rangle^{G}}$	0	1	1	0	0
$\Psi_{l,q,\|-l,h\rangle^{G}}$	0	1	1	0	0

TAB. a Maximal dimensions for singular vectors spaces in $\mathcal{V}^{G}_{\Delta,h}$.

Charges q that are not given have dimension 0 and hence do not allow any singular vectors. The maximal dimensions for the case of the Verma modules $\mathcal{V}^{Q}_{\Delta,h}$, for $c \neq 3$, are obtained simply by interchanging $G \leftrightarrow Q$ and $q \leftrightarrow -q$ in the previous table.

For the case of singular vectors in chiral Verma modules $\mathcal{V}^{GQ}_{0,h}$ and in no-label Verma modules $\mathcal{V}_{0,h}$, for $c \neq 3$, one obtains the following maximal dimensions[1]:

	$q=-2$	$q=-1$	$q=0$	$q=1$	$q=2$
$\Psi^{G}_{l,q,\|0,h\rangle^{GQ}}$	0	0	1	1	0
$\Psi^{Q}_{l,q,\|0,h\rangle^{GQ}}$	0	1	1	0	0
$\Psi^{G}_{l,q,\|0,h\rangle}$	0	1	3	3	1
$\Psi^{Q}_{l,q,\|0,h\rangle}$	1	3	3	1	0

TAB. b Maximal dimensions for singular vectors spaces in $\mathcal{V}^{GQ}_{0,h}$ and in $\mathcal{V}_{0,h}$.

Tables Tab. a and Tab. b prove the conjecture made in Ref. 14, using the algebraic mechanism denoted *the cascade effect*, about the possible existing types of topological singular vectors. In addition, low level examples were constructed[14] for all these types, what proves that all of them exist already at level 1. The four types of two-dimensional spaces of singular vectors of Tab. a also exist starting at level 2, and four examples at level 3 were constructed[14] as well. For the case of the three-dimensional spaces of singular vectors in no-label Verma modules in Tab. b, the corresponding types of singular vectors have been constructed at level 1 generating one-dimensional[14] as well as two-dimensional[1] spaces, but no further search has been done for the three-dimensional spaces.

Transferring the dimensions we have found in tables Tab. a and Tab. b to the Neveu-Schwarz $N=2$ algebra[3, 19–22] is straightforward as this algebra is related to the topological $N=2$ algebra through the topological twists $T_W^{\pm}$: $\mathcal{L}_m = L_m \pm 1/2 H_m$, $\mathcal{H}_m = \pm H_m$, $\mathcal{G}_m = G^{\pm}_{m+1/2}$ and $\mathcal{Q}_m = G^{\mp}_{m-1/2}$, where $G^{\pm}_{m+1/2}$ are the half-integer moded fermionic generators. As a result, the standard Neveu-Schwarz h.w. vectors correspond to $\mathcal{G}_0$-closed topological h.w. vectors, whereas the chiral (antichiral) Neveu-Schwarz h.w. vectors, annihilated by $G^{+}_{-1/2}$ ($G^{-}_{-1/2}$), correspond to chiral topological h.w. vectors. This implies[12, 14] that the standard and chiral and antichiral Neveu-Schwarz singular vectors correspond to topological singular vectors of the types $\Psi^{G}_{l,q,|\Delta,h\rangle^G}$ and $\Psi^{GQ}_{l,q,|\Delta,h\rangle^G}$, whereas the Neveu-Schwarz singular vectors built in chiral or antichiral Verma modules correspond to topological singular vectors of only the type $\Psi^{G}_{l,q,|\Delta,h\rangle^{GQ}}$. As a consequence, by untwisting the first row of table Tab. a one recovers the results[15, 16] that in Verma modules of the Neveu-Schwarz $N=2$ algebra singular vectors only exist for charges $q=0,\pm 1$ and two-dimensional spaces only exist for uncharged singular vectors. By untwisting the third row of table Tab. a one gets a proof for the conjecture[14] that chiral singular vectors in Neveu-Schwarz Verma modules only exist for $q=0,1$ whereas antichiral singular vectors only exist for $q=0,-1$. The untwisting of the first row of table Tab. b, finally, proves the conjecture[12, 14] that in chiral Neveu-Schwarz Verma modules $\mathcal{V}^{NS,ch}_{h/2,h}$ singular vectors only exist for $q=0,-1$, whereas in antichiral Verma modules $\mathcal{V}^{NS,a}_{-h/2,h}$ singular vectors only exist for $q=0,1$.

As to the representations of the Ramond $N=2$ algebra[19–22], they are exactly isomorphic to the representations of the topological $N=2$ algebra. Namely, combining the topological twists $T_W^{\pm}$ and the spectral flows one constructs a one-to-one mapping between the Ramond singular vectors and the topological singular vectors, at the same levels and with the same charges[18]. Therefore the results of tables Tab. a and Tab. b can be transferred to the Ramond singular vectors simply by exchanging the labels $G \to (+)$, $Q \to (-)$, where the helicity $(\pm)$ denotes the vectors annihilated by the fermionic zero modes $G_0^{\pm}$, and by taking into account that the chiral and undecomposable *no-helicity* Ramond vectors[14, 13, 18], require conformal weight $\Delta + l = c/24$.

The twisted $N=2$ superconformal algebra[19–22] is not related to the other three $N=2$ algebras. It has mixed modes, integer and half-integer, for the fermionic generators, and the eigenvectors have no charge, as the U(1) current generators are half-moded, but they have *fermionic parity*. The Adapted Ordering Method was worked out for the twisted $N=2$ algebra in Ref. 4. The maximal dimension for the spaces of singular vectors in standard Verma modules was found to be two and these two-dimensional singular spaces were shown to exist by explicit computation, starting at level 3/2. In Verma modules built on G_0-closed h.w. vectors, however, the singular vectors were found to be only one-dimensional. This method also allowed to propose a reliable conjecture for the coefficients of the relevant terms of all singular vectors, i.e. for the coefficients with respect to the ordering kernels, what made possible to identify all the cases of two-dimensional spaces of singular vectors for all levels, as well as to identify all G_0-closed singular vectors. The resulting expressions, in turn, led to the discovery of subsingular vectors for this algebra, and several explicit examples were also computed. Finally, the multiplication rules for singular vectors operators were derived using the ordering kernel coefficients, what set the basis for the analysis of the twisted $N = 2$ embedding diagrams.

Finally let us consider the $N = 1$ superconformal algebras[8, 22–24]. The structure of the h.w. representations of the Neveu-Schwarz $N = 1$ algebra has been completely understood in Ref. 25. The corresponding Verma modules do not contain two-dimensional singular vector spaces neither subsingular vectors. In the case of the Ramond $N = 1$ algebra, however, the application of the Adapted Ordering Method in Ref. 5 has shown that its representations have a much richer structure than previously suggested in the literature. In particular, it was found that standard Verma modules may contain two-dimensional spaces of singular vectors and also subsingular vectors. Moreover, the two-dimensional ordering kernels allowed to derive multiplication rules for singular vector operators and led to expressions for the two-dimensional spaces of singular vectors. Using these multiplication rules descendant singular vectors were studied and embedding diagrams were derived for the rational models. In addition, this allowed to conjecture the ordering kernel coefficients of all singular vectors and therefore identify these vectors uniquely.

4 Conclusions and Final Remarks

We have presented the Adapted Ordering Method for general Lie algebras and superalgebras, and their generalizations, provided they can be triangulated, as is the case in many interesting examples. This method is based on the concept of adapted orderings, leading to the definition of the ordering kernels, which play a crucial rôle since their sizes limit the dimensions of the corresponding spaces of singular vectors. As a result the adapted orderings must be chosen such that the ordering kernels are as small as possible. Weights for which the ordering kernels are trivial do not allow any singular vectors in the corresponding weight spaces. On the other hand, non-trivial ordering kernels give us the maximal dimensions of the possible spaces of singular vectors and uniquely define all singular vectors through the coefficients with respect to them, allowing to set the basis for constructing embedding diagrams.

The Adapted Ordering Method follows from the Definition 2.A plus the Theorems 2.B, 2.C and 2.D, which are rigorously proven. There is nothing in the Definition 2.A, neither in the three theorems, that restricts the application of this method to infinite-dimensional algebras. For the same reason, it seems clear that the Adapted Ordering Method should be useful also for generalized Lie algebras and superalgebras such as affine Kac-Moody algebras, non-linear W-algebras, superconformal W-algebras, loop Lie algebras, Borcherds algebras, F-Lie algebras for $F > 2$ ($F = 1$ are Lie algebras and $F = 2$ are Lie superalgebras), etc.

One may wonder whether there are any prescriptions in order to construct the most suitable orderings with the smallest kernels.The answer to this question is that there are no general prescriptions or recipes as the orderings depend entirely on the given algebras. The way to proceed is a matter of trial and error. That is, one constructs one total ordering first, that can always be done since a total ordering is simply a definition establishing which of two given terms is the bigger one, then one computes the kernel and decides whether this kernel is small enough. In the case it is not, then one constructs a second ordering and repeats the procedure until one finds a suitable ordering. It may also happen, for a given algebra, that this procedure does not give any useful information because all the total orderings one can construct are adapted only to the empty subset, in which case the ordering kernel is the whole set of terms: $\mathcal{C}^K_{\{l_i\}} = \mathcal{C}_{\{l_i\}}$.

The Adapted Ordering Method has been applied so far to the $N = 2$ and Ramond $N = 1$ superconformal algebras, allowing to prove several conjectured results as well as to obtain many new results, as we have reviewed. For example, this method allowed to discover subsingular vectors and two-dimensional spaces of singular vectors for the twisted $N = 2$ and Ramond $N = 1$ algebras[4, 5]. (For the other three isomorphic $N = 2$ algebras two-dimensional singular spaces had been discovered[14–16], as well as subsingular vectors[11–14], before the Adapted Ordering Method was applied to them). We are convinced therefore that this method should be of very much help for the study of the representation theory of many other algebras, in particular the $N > 2$ superconformal algebras, and some (at least) of the generalized Lie algebras listed above.

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References

[1] M. Dörrzapf and B. Gato-Rivera. Singular dimensions of the $N = 2$ superconformal algebras. I. *Commun. Math. Phys.* 206 (1999) 493.

[2] V.G. Kac. Lie superalgebras. *Adv. Math.* 26 (1977) 8.

[3] M. Ademollo, L. Brink, A. d'Adda, R. d'Auria, E. Napolitano, S. Sciuto, E. del Giudice, P. di Vecchia, S. Ferrara, F. Gliozzi, R. Musto, and R. Pettorino. Supersymmetric strings and colour confinement. *Phys. Lett.* B62 (1976) 105.

[4] M. Dörrzapf and B. Gato-Rivera. Singular dimensions of the $N = 2$ superconformal algebras. II: the twisted $N = 2$ algebra. *Commun. Math. Phys.* 220 (2001) 263.

[5] M. Dörrzapf. Highest weight representations of the $N = 1$ Ramond algebra. *Nucl. Phys.* B595 (2001) 605.

[6] R.V. Moody and A. Pianzola. *Lie algebras with triangular decomposition.* John Wiley, New York 1995.

[7] A. Kent. Singular vectors of the Virasoro algebra. *Phys. Lett.* B273 (1991) 56.

[8] V.G. Kac. *Contravariant form for infinite-dimensional Lie algebras and superalgebras.* Lecture Notes in Physics, Vol 94 (1979) 441.

[9] B.L. Feigin and D.B. Fuchs. *Representations of Lie groups and related topics.* A.M. Vershik and A.D. Zhelobenko eds., Gordon and Breach, 1990.

[10] A. Rocha-Caridi and N.R. Wallace. Characters of irreducible representations of the Virasoro algebra. *Math. Z.* 185 (1984) 1.

[11] B. Gato-Rivera and J.I. Rosado. Interpretation of the determinant formulae for the chiral representations of the $N = 2$ superconformal algebra. IMAFF-FM-96/38, NIKHEF-96-007, hep-th/9602166 (1996).

[12] B. Gato-Rivera and J.I. Rosado. Chiral determinant formulae and subsingular vectors for the $N = 2$ superconformal algebras. *Nucl. Phys.* B503 (1997) 447.

[13] M. Dörrzapf and B. Gato-Rivera. Transmutations between singular and subsingular vectors of the $N = 2$ superconformal algebras. *Nucl. Phys.* B557 (1999) 517.

[14] B. Gato-Rivera and J.I. Rosado. Families of singular and subsingular vectors of the topological $N = 2$ superconformal algebra. *Nucl. Phys.* B514 (1998) 477.

[15] M. Dörrzapf. *Superconformal field theories and their representations.* PhD thesis, University of Cambridge, 1995.

[16] M. Dörrzapf. Analytic expressions for singular vectors of the $N = 2$ superconformal algebra. *Commun. Math. Phys.* 180 (1996) 195.

[17] R. Dijkgraaf, E. Verlinde, and H. Verlinde. Topological strings in $d < 1$. *Nucl. Phys.* B352 (1991) 59.

[18] M. Dörrzapf and B. Gato-Rivera. Determinant formula for the topological $N = 2$ superconformal algebra. *Nucl. Phys.* B558 (1999) 503.

[19] W. Boucher, D. Friedan and A. Kent. Determinant formulae and unitarity for the $N = 2$ superconformal algebras in two dimensions or exact results on string compactification. *Phys. Lett.* B172 (1986) 316.

[20] V.K. Dobrev. Characters of the unitarizable highest weight modules over the $N = 2$ superconformal algebras. *Phys. Lett.* B186 (1987) 43.

[21] Y. Matsuo. Character Formula of $c < 1$ unitary representation of $N = 2$ superconformal algebra. *Prog.Theor. Phys.* 77 (1987) 793.

[22] E. Kiritsis. Character formula and the structure of the representations of the $N = 1$, $N = 2$ superconformal algebras. *Int. J. Mod. Phys.* A3 (1988) 1871.

[23] A. Meurman and A. Rocha-Caridi. Highest weight representations of the Neveu-Schwarz and Ramond algebras. *Commun. Math. Phys.* 107 (1986) 263.

[24] V.K. Dobrev. Multiplet classification of the indecomposable highest weight modules over the Neveu-Schwarz and Ramond superalgebras. *Lett. Math. Phys* 11 (1986) 225.

[25] A.B. Astashkevich. On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras. *Commun. Math. Phys.* 186 (1997) 531.