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Bell Polynomials and Brownian Bridge in Spectral Gravity Models on Multifractal Robertson–Walker Cosmologies

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Abstract. We obtain an explicit formula for the full expansion of the spectral action on Robertson–Walker spacetimes, expressed in terms of Bell polynomials, using Brownian bridge integrals and the Feynman–Kac formula. We then apply this result to the case of multifractal Packed Swiss Cheese Cosmology models obtained from an arrangement of Robertson–Walker spacetimes along an Apollonian sphere packing. Using Mellin transforms, we show that the asymptotic expansion of the spectral action contains the same terms as in the case of a single Robertson–Walker spacetime, but with zeta-regularized coefficients, given by values at integers of the zeta function of the fractal string of the radii of the sphere packing, as well as additional log-periodic correction terms arising from the poles (off the real line) of this zeta function.

Contents

1. Introduction

The spectral action was proposed in the 1990s by Chamseddine and Connes [5] as a possible action functional for gravity coupled to matter that extends to non-commutative spaces. It was successfully applied to the construction of particle physics models [13], where its asymptotic expansion reconstructs the Lagrangian of the Standard Model with right-handed neutrinos and Majorana masses, [9], see also [42]. It was also shown in [9] that, in the gravity sector, the asymptotic expansion of the spectral action gives rise to a modified gravity model that includes, in addition to the Einstein–Hilbert action and the cosmological term of General Relativity, also a conformal gravity term (Weyl curvature) and a Gauss–Bonnet gravity term (which is non-dynamical and topological in dimension four).

It was shown in [4,35,36] that one can incorporate in the spectral action functional a scalar field, seen as a perturbation of the Dirac operator. The action functional then determines a potential for this scalar field that has the shape of a slow–roll potential, suitable for a cosmological inflation scenario. This slow–roll potential was used in [4,35,36] to study how an action functional model of gravity can address the cosmic topology question. For an overview of spectral action models in cosmology, see [34].

The spectral action, with its full asymptotic expansion, has been computed explicitly for various types of solutions of the Einstein equations, including Robertson–Walker metrics [7,22,23] and Bianchi IX gravitational instantons, [19–21].

The starting point of this paper is the computation of [7] of the spectral action for (Euclidean) Robertson–Walker metrics and the results of [23], showing that the coefficients of the asymptotic expansion of the spectral action for these metrics are recursively given by rational functions of the scaling factor of the metric and its derivatives, with $\mathbb{Q}$-coefficients.

We obtain here a different and more explicit derivation of the full expansion of the spectral action for Robertson–Walker metrics with a general expansion factor $a(t)$. As in [7], it is based on Brownian bridge integrals, but a more convenient choice of variables leads to more easily computable integrals and to a completely explicit form of the coefficients in terms of Bell polynomials.

This explicit form suggests that a deeper algebraic and combinatorial structure is present in the asymptotic expansion of the spectral action, at least for very regular geometries like the Robertson–Walker metrics. This structure is closely related to the Hopf algebra structure of renormalization in quantum field theories, manifested here through the Faà di Bruno Hopf algebra and its relation to the Bell polynomials.

We then consider the case where, instead of a single Robertson–Walker cosmology with spatial sections given by a sphere S^3, we have a multifractal arrangement in the form of a Robertson–Walker cosmology over an Apollonian packing of spheres. This type of multifractal cosmology models is known as "Packed Swiss Cheese Cosmology", [38]. They model spacetimes that are isotropic but non-homogeneous, based on a construction originally introduced in [39]. A model of the spectral action for Packed Swiss Cheese Cosmologies was developed in [3], based on a simplified static model with constant scaling factor. We extend the results here to the full model with an arbitrary underlying Robertson–Walker metric.

We consider an Apollonian packing of spheres S^3 with a sequence of radii $\mathcal{L} = \{a_{n,k}\}$. We endow each 4-dimensional spacetime $\mathbb{R} \times S^3$ with a Robertson–Walker metric $\mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2$, scaled by the corresponding radius $a_{n,k}^2$ in two possible ways, see (5.2) and (5.1). For a particular choice of a scaling factor of the form $a(t) = \sin(t)$, this general setting includes the case of packings of four-spheres.

We first illustrate a lower-dimensional example based on a special class of Apollonian circle packings, the Ford circles, where we show explicitly the terms arising in the spectral action that detect the fractal structure, which are expressible in terms of zeros of the Riemann zeta function. This example also illustrates the fact that the very restrictive condition on the sphere packing used in the simplified model of [3], based on an approximation by self-similar fractal strings with lattice property, is too strong for the general setting we need to consider here. The method we use in this paper to obtain the full asymptotic expansion of the spectral action is independent of this approximation assumption and only requires a milder condition on the fractal string $\mathcal{L} = \{a_{n,k}\}$ of the sphere packing, namely the property that the zeta function $\zeta_{\mathcal{L}}(z)$ admits analytic continuation to a meromorphic function on $\mathbb{C}$ with simple poles located away from the set of integers less than or equal to 4.

We obtain the full expansion of the spectral action on these multifractal Packed Swiss Cheese Cosmologies in terms of the expansion for a single Robertson–Walker metric obtained in the first part of the paper, using a Mellin transform argument. The resulting expansion has two series of terms, one that corresponds to the terms in the expansion of a single Robertson–Walker metric, where the coefficients are modified by a zeta-regularized sum of powers of the packing radii, so that the coefficients are no longer rational numbers but they contain the zeta values $\zeta_{\mathcal{L}}(4-2M)$, for $M \in \mathbb{N}$. The second series of terms corresponds to the poles of the fractal string zeta function $\zeta_{\mathcal{L}}(z)$ and give rise to a series of log-periodic terms as already observed in the simpler model of [3]. In this case, the coefficients are values of the zeta function of the Dirac operator of the underlying model Robertson–Walker metric.

2. Spectral Gravity and Robertson–Walker Metrics

We discuss here some general preliminary facts about Robertson–Walker metrics and the spectral action functional, which will be useful in the following sections.

2.1. Spectral Gravity

The spectral action functional can be defined on ordinary manifolds or more generally on non-commutative geometries, described in terms of the spectral triple formalism [12]. The functional is defined in terms of a regularized trace of a Dirac operator

$$\mathcal{S}_\Lambda = \operatorname{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) = \sum_{\lambda \in \operatorname{Spec}(D)} f\left(\frac{\lambda}{\Lambda}\right), \tag{2.1}$$

where $\Lambda \in \mathbb{R}_+^*$ is an energy scale and $f(x)$ is a smooth test function (which one can think of as a smooth approximation to a cutoff function). In the commutative case, one assumes that the underlying manifold is Riemannian and compact, so that the Dirac operator has compact resolvent; hence, the series in (2.1) makes sense. The general spectral triple axioms in the non-commutative case [12] are modeled on the analytic properties of Dirac operators on compact Riemannian spin manifolds. While definition (2.1) does not directly extend to Lorentzian geometries, it is often the case, including the case of Robertson–Walker metrics considered in the present paper, that the local terms in the asymptotic expansion of the spectral action may admit Wick rotations to Lorentzian signature and can be used in gravity and cosmology models [1,2,46].

Throughout this paper, we will work only with spacetimes with Euclidean signature, and in particular, with Robertson–Walker metrics of the form $\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2$ on $\mathbb{R}\times S^3$, where the geometry is given as a warped product of the flat metric on $\mathbb{R}$ and the round metric $\mathrm{d}\sigma^2$ on the 3-dimensional sphere S^3 of radius one. Here the real line $\mathbb{R}$ is used to parametrize the cosmic time t, and the spatial section of the universe is a 3-sphere, expanded by the scaling factor $a(t)$.

The asymptotic expansion of the spectral action is obtained from the heat kernel expansion for the square D^2 of the Dirac operator. Suppose that the heat kernel has a small time asymptotic expansion at $\tau \to 0^+$ of the form

$$\mathrm{Tr}(\mathrm{e}^{-\tau D^2}) \sim \sum_\alpha c_\alpha \tau^\alpha,$$

where we assume that the terms $\alpha > 0$ are integers, then, using a test function of the form $f(x) = \int_0^\infty \mathrm{e}^{-\tau x^2}\mathrm{d}\mu(\tau)$ for some measure μ with normalization $f(0) = \int_0^\infty \mathrm{d}\mu(\tau)$, we obtain an expansion (see [5,7,42])

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\alpha<0} f_\alpha\, c_\alpha\, \Lambda^{-\alpha} + a_0 f(0) + \sum_{\alpha>0} f_\alpha\, c_\alpha\, \Lambda^{-\alpha}, \tag{2.2}$$

where the coefficients f_α are given by:

$$f_\alpha = \begin{cases} \int_0^\infty f(v)v^{-\alpha-1}\mathrm{d}v & \alpha < 0 \\ (-1)^\alpha f^{(\alpha)}(0) & \alpha > 0,\ \alpha\in\mathbb{N} \end{cases} \tag{2.3}$$

Thus, the problem of computing the full spectral action expansion amounts to computing the heat kernel expansion coefficients.

In the case of $\mathbb{R}\times S^3$ with a Riemannian Robertson–Walker metric of the form

$$\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2 = \mathrm{d}t^2 + a(t)^2\left(\mathrm{d}\chi^2 \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right),$$

the square D^2 of the Dirac operator has the explicit form (see [7])

$$D^2 = -\left(\frac{\partial}{\partial t} + \frac{3a'(t)}{2a(t)}\right)^2 + \frac{1}{a(t)^2}(\gamma^0 D_3)^2 - \frac{a'(t)}{a^2(t)}\gamma^0 D_3,$$

with

$$D_3 = \gamma^1\left(\frac{\partial}{\partial\chi} + \cot\chi\right) + \gamma^2\frac{1}{\sin\chi}\left(\frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\right) + \gamma^3\frac{1}{\sin\chi\sin\theta}\frac{\partial}{\partial\phi}.$$

Using a basis of eigenfunctions of the Dirac operator on S^3, the operator D^2 was decomposed into a direct sum of operators of the form

$$H_n^\pm = -\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{(n+\frac{3}{2})^2}{a^2} \pm \frac{(n+\frac{3}{2})a'}{a^2}\right),$$

which are then used to compute the spectral action expansion via a Feynman–Kac formula. We will discuss this setting more in detail in Sect. 3, where we present a different method of computing these coefficients, also in terms of the Feynman–Kac formula and the Brownian bridge integrals, but with a computationally simpler choice of coordinates.

The heat kernel semi-group of a Dirac Laplacian is trace class when the underlying manifold is compact. In the case of a non-compact manifold $\mathbb{R} \times S^3$ with a Robertson–Walker metric, this trace class property may hold or not depending on the specific form of the scaling factor $a(t)$. For certain cosmological models (such as an expanding universe) where the scaling factor introduces divergences, a regularization method (such as a cutoff) would need to be introduced. Our main results in the present paper focus on the coefficients of the heat kernel expansion prior to integration in the time variable, hence as functions $a_{2m}(t)$ (as explained more explicitly in the next subsection) that apply to arbitrary choices of the scaling factor $a(t)$.

2.2. Pseudodifferential Calculus

We let D be the Dirac operator of the Robertson–Walker metric with a general cosmic factor $a(t)$:

$$\mathrm{d}s^2 = \mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2. \tag{2.4}$$

We can express the heat kernel as

$$\mathrm{e}^{-\tau D^2} = \frac{1}{2\pi i} \int_\gamma \mathrm{e}^{-\tau\lambda} (D^2 - \lambda)^{-1} \, \mathrm{d}\lambda, \tag{2.5}$$

for γ a contour in the complex plane traveling clockwise around the nonnegative reals. Since D^2 is an elliptic operator of order 2, we can approximate $(D^2 - \lambda)^{-1}$ by a right parametrix with symbol

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x, \xi, \lambda), \tag{2.6}$$

where each of the $r_j(x, \xi, \lambda)$ is a pseudodifferential symbol of order $-2-j$ so that $r_j(x, \tau\xi, \tau^2\lambda) = \tau^{-2-j} r_j(x, \xi, \lambda)$. It is then possible to determine recursively the homogeneous pseudodifferential symbols r_j in the expansion of the parametrix, with $r_0(x, \xi, \lambda) = (p_2(x, \xi) - \lambda)^{-1}$, and for any $n > 1$

$$r_n(x, \xi, \lambda) = -\sum \frac{1}{\alpha!} \partial_\xi^\alpha r_j(x, \xi, \lambda) \partial_x^\alpha p_k(x, \xi) r_0(x, \xi, \lambda), \tag{2.7}$$

where the summation runs over all $\alpha \in \mathbb{Z}_+^4$, $j \in 0, 1, \ldots, n-1$, $k \in \{0, 1, 2\}$ such that $|\alpha| + j + 2 - k = n$, and the p_k are the homogeneous components of the symbol of D^2, see [19,23].

The small time asymptotic expansion of the heat kernel

$$\mathrm{Tr}(\mathrm{e}^{-\tau D^2}) \sim_{\tau \to 0^+} \sum_{n=0}^{\infty} \frac{\tau^{(n-4)/2}}{16\pi^4} \int \mathrm{tr}(e_n(x)) \, \mathrm{d}vol_g \tag{2.8}$$

and (2.5) with the parametrix expansion (2.6) give

$$e_n(x) \cdot \sqrt{det(g)} = -\frac{1}{2\pi i} \int_\gamma \mathrm{e}^{-\lambda} r_n(x, \xi, \lambda) \, \mathrm{d}\lambda \, \mathrm{d}\xi, \tag{2.9}$$

and the coefficients a_n of the heat kernel expansion can be written as in [23] in the form:

$$\begin{aligned} a_n &= \frac{1}{16\pi^4} \int_{S^3_\alpha} \mathrm{tr}(e_n)\, \mathrm{d}vol_g \\ &= \frac{1}{16\pi^4} \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \mathrm{tr}(e_n) a^3(t) \sin(\eta) \cos(\eta)\, \mathrm{d}\eta\, \mathrm{d}\phi_1\, \mathrm{d}\phi_2. \end{aligned} \quad (2.10)$$

In fact, only the even coefficients a_{2m} are non-trivial. The coefficients a_{2m} can then be determined in terms of the recursive formula (2.7) for the parametrix. Using this method, it is proved in [23] that the coefficients a_n satisfy a rationality phenomenon conjectured in [7]. Namely, if we denote by $a_{2m}(t)$ the coefficient a_{2m} prior to time integration, that is, $a_{2m} = \int a_{2m}(t)\, \mathrm{d}t$, then the result proved in [23] shows that each $a_{2m}(t)$ is described by a polynomial in several variables whose coefficients are rational numbers,

$$a_{2m}(t) = \frac{Q_{2m}\left(a(t), a'(t), \ldots, a^{(2m)}(t)\right)}{a(t)^{2m-3}}, \quad (2.11)$$

where $Q_{2m} \in \mathbb{Q}[x_0, x_1, \ldots, x_{2m}]$. Moreover, the degree of each monomial appearing in Q_{2m} is either $2m-2$ or $2m$. More concretely,

$$Q_{2m}\left(x_0, x_1, \ldots, x_{2m}\right) = \sum_k c_{2m,k}\, x_0^{k_0} x_1^{k_1} \cdots x_{2m}^{k_{2m}}, \quad (2.12)$$

where $c_{2m,k} \in \mathbb{Q}$, and for each multi-index $k = (k_0, k_1, \ldots, k_{2m})$ in the summation we have:

$$\text{either} \qquad \sum_{j=0}^{2m} k_j = \sum_{j=0}^{2m} j k_j = 2m-2 \qquad \text{or} \qquad \sum_{j=0}^{2m} k_j = \sum_{j=0}^{2m} j k_j = 2m. \quad (2.13)$$

As we will see later, structure (2.13) of the summation in the polynomials (2.12) is reminiscent of the structure of a combinatorially very interesting family of polynomials, the Bell polynomials, that describe the combinatorial structure of derivatives of composite functions. Indeed, we will prove in the next sections that the coefficients $a_{2m}(t)$ can be computed explicitly in terms of Bell polynomials.

2.3. Physical Examples: Expansion Models

The first few coefficients a_0, a_2, a_4, a_6 in the expansion of the spectral action for the Robertson–Walker metric, as computed in [7], give the following expressions (written without time integration)

$$
\begin{aligned}
a_0(t) &= \frac{1}{2}a^3(t),\\
a_2(t) &= \frac{a^3(t)}{4}\left(\frac{a''(t)}{a(t)} + \frac{(a'(t))^2-1}{a^2(t)}\right),\\
a_4(t) &= \frac{1}{120}\Big(3a^2(t)a^{(4)}(t) + 9a(t)a'(t)a^{(3)}(t) + 3a(t)(a'')^2(t)\\
&\quad -4(a')^2(t)a''(t) - 5a''(t)\Big),\\
a_6(t) &= -\frac{a'(t)^2a''(t)}{240a^2(t)} - \frac{a'(t)^4a''(t)}{84a^2(t)} + \frac{a''(t)^2}{120a(t)} + \frac{a'(t)^2a''(t)^2}{21a(t)} - \frac{a''(t)^3}{90}\\
&\quad +\frac{a'(t)a^{(3)}(t)}{240a(t)} + \frac{a'(t)a^{(3)}(t)}{84a(t)} - \frac{a'(t)a''(t)a^{(3)}(t)}{20} - \frac{a(t)a^{(3)}(t)^2}{1680}\\
&\quad -\frac{a^{(4)}(t)}{240} - \frac{a'(t)^2a^{(4)}(t)}{120} + \frac{a(t)a''(t)a^{(4)}(t)}{840} + \frac{a(t)a'(t)a^{(5)}(t)}{140}\\
&\quad +\frac{a(t)^2a^{(6)}(t)}{560}.
\end{aligned}
$$

While the spectral action is computed for a compact 4-dimensional Riemannian manifold (for example, the sphere S^4 for which $a(t) = \sin(t)$), the expressions obtained above for the coefficients as functions of the scaling factor $a(t)$ of the Robertson–Walker metric continue to make sense for more realistic universe models where the scaling factor describes different phases of the expansion of the universe. In the case of an expanding universe (as opposed to the expansion and contraction case of the sphere S^4), the time integration of the expressions above may introduce divergences that requires cutoff regularization.

Throughout this paper, we will not assume any fixed form for the scaling factor $a(t)$ and we will work in complete generality for an arbitrary smooth function. We list here some cosmologically relevant examples of scaling factors of an expanding universe and the corresponding form of the first few coefficients of the spectral action expansion, computed using the pseudodifferential calculus discussed above.

2.3.1. Inflation Dominated Universe.

For an inflation dominated universe model, the scaling factor derived from the Friedmann equations and the Robertson–Walker metric produces an exponentially expanding universe, [15], with scaling factor $a(t) = \mathrm{e}^{Ht}$. This gives

$$a_0(t) = \frac{1}{2}\mathrm{e}^{2Ht},$$
$$a_2(t) = \frac{2H^2\mathrm{e}^{3Ht} - \mathrm{e}^{Ht}}{4},$$
$$a_4(t) = 11H^4\mathrm{e}^{3Ht} - 5H^2\mathrm{e}^{Ht},$$
$$a_6(t) = \frac{-31}{2510}H^6\mathrm{e}^{3Ht} + \frac{1}{240}H^4\mathrm{e}^{Ht}.$$

2.3.2. Radiation-Dominated Universe. In the radiation-dominated phase of the universe expansion, the scaling factor grows like $a(t) = (2Ht)^{1/2}$. This gives terms of the form

$$a_0(t) = \sqrt{2}(Ht)^{3/2},$$
$$a_2(t) = -\frac{\sqrt{2Ht}}{4},$$
$$a_4(t) = -\frac{\sqrt{2Ht}(-11H + 30t)}{1440t^2},$$
$$a_6(t) = \frac{-919 \cdot 2^{1/6}H^2t + 189 \cdot 2^{1/6}Ht^2 + 30 \cdot 6^{1/3}H(Ht)^{5/6}}{20160 \cdot 2^{2/3}t^5\sqrt{Ht}} + \frac{21 \cdot 6^{1/3}t(Ht)^{5/6} + 126 \cdot 3^{2/3}H(Ht)^{7/6}}{20160 \cdot 2^{2/3}t^5\sqrt{Ht}}.$$

2.3.3. Matter-Dominated Universe. In a matter-dominated universe model [15], the scale factor has an expansion rate of the form $a(t) = (\frac{3}{2}Ht)^{2/3}$ and gives terms

$$a_0(t) = \frac{9}{8}H^2t^2,$$
$$a_2(t) = \frac{H^2}{8} - \frac{1}{4}\left(\frac{3}{2}\right)^{2/3}(Ht)^{2/3},$$
$$a_4(t) = \frac{1}{216}\frac{H^2}{t^2} + \frac{1}{72}\left(\frac{2}{3}\right)^{1/3}\frac{H^{2/3}}{t^{4/3}},$$
$$a_6(t) = \frac{5}{2916}\frac{H^2}{t^4} + \frac{11}{810 \cdot 2^{2/3} \cdot 3^{1/3}}\frac{H^{2/3}}{t^{10/3}}.$$

2.3.4. Empty Universe. In an empty universe with scaling factor $a(t) = Ht$, the spectral action coefficients take the form

$$a_0(t) = \frac{1}{2}(Ht)^3,$$
$$a_2(t) = \frac{H^3t - Ht}{4},$$

with vanishing higher terms.

3. Combinatorial Structures in the Spectral Action Expansion

We work here and in the rest of the paper with a Robertson–Walker metric with an arbitrary choice of the scaling factor $a(t)$.

In [7], a method for computing the spectral action expansion on Robertson–Walker metrics based on the Feynman–Kac formula and Brownian bridge integrals was developed. The main steps of their argument are summarized as follows. Let D^2 be the square of the Dirac operator on a Euclidean Robertson–Walker metric. The spectral action, for a test function of the form $f(u) = \mathrm{e}^{-su}$, is written as

$$\mathrm{Tr}(f(D^2)) \sim \sum_{n\geq 0} \mu(n)\, \mathrm{Tr}(f(H_n)),$$

with multiplicities $\mu(n) = 4(n+1)(n+2)$ and with the operator H_n of the form

$$H_n = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V_n(t),$$

$$V_n(t) = \frac{(n+\frac{3}{2})}{a(t)^2}\left(\left(n+\frac{3}{2}\right) - a'(t)\right). \tag{3.1}$$

In order to evaluate the trace $\mathrm{Tr}(\mathrm{e}^{-sH_n})$, one then uses the Feynman–Kac formula of [41], Theorem 6.6,

$$\mathrm{e}^{-sH_n}(t,t) = \frac{1}{2\sqrt{\pi s}} \int \exp\left(-s\int_0^1 V_n(t+\sqrt{2s}\alpha(u))\mathrm{d}u\right) D[\alpha]. \tag{3.2}$$

Here $D[\alpha]$ denotes the Brownian bridge integrals, [41], where the Brownian bridge is the Gaussian process characterized by the covariance

$$\mathbb{E}(\alpha(v_1)\alpha(v_2)) = v_1(1-v_2), \qquad 0 \leq v_1 \leq v_2 \leq 1. \tag{3.3}$$

One then uses the Euler–Maclaurin formula to replace the summation

$$\sum_n \mu(n)\mathrm{e}^{-sH_n}(t,t)$$

by a continuous integration over $x \geq 3/2$, which gives

$$\int_{3/2}^{\infty} k_s(x)\,\mathrm{d}x + \frac{1}{2}k_s(3/2) - \frac{k_s'(3/2)}{12} + \cdots,$$

with the functions

$$\begin{aligned} k_s(x) &= (4x^2-1)\frac{1}{2\sqrt{\pi s}}\int \mathrm{e}^{u(b-x)x}\, D[\alpha], \\ u &= s\int_0^1 a^{-2}(t+\sqrt{2s}\alpha(v))\,\mathrm{d}v, \\ ub &= s\int_0^1 a'a^{-2}(t+\sqrt{2s}\alpha(v))\,\mathrm{d}v. \end{aligned} \tag{3.4}$$

The asymptotic expansion is then obtained in [7] using Taylor expansions of the form

$$\int_0^1 F(t+\sqrt{2s}\alpha(v))\,\mathrm{d}v = F(t) + \sum_k \frac{F^{(k)}(t)}{k!}(\sqrt{2s})^k x_k(\alpha),$$

where

$$x_k(\alpha) = \int_0^1 \alpha(v)^k\,\mathrm{d}v. \tag{3.5}$$

In computational terms, the Laurent series expansion for b, which is given as the quotient

$$b = \frac{\int_0^1 a^{-2}(t+\sqrt{2s}\alpha(v))\,\mathrm{d}v}{\int_0^1 a' a^{-2}(t+\sqrt{2s}\alpha(v))\,\mathrm{d}v}, \tag{3.6}$$

introduces a considerable amount of complication that slows down the computation. We argue here that a simpler choice of variables significantly simplifies the computational complexity of the terms of this expansion and provides a more transparent description of the resulting terms of the asymptotic expansion, which reveals the presence of a richer combinatorial structure.

3.1. A Convenient Choice of Variables for Brownian Bridge Integrals

We set

$$A(t) = 1/a(t), \qquad B(t) = A(t)^2. \tag{3.7}$$

We can then write the potential $V_n(t)$ of (3.1) in the form

$$V_n(t) = x^2 A(t)^2 + xA'(t) = x^2 B(t) + xA'(t), \qquad \text{with } x = n + 3/2. \tag{3.8}$$

We then write the integral in the Feynman–Kac formula (3.2) as

$$-s\int_0^1 V_n(t+\sqrt{2s}\,\alpha(v))\,\mathrm{d}v = -x^2 U - xV, \tag{3.9}$$

in terms of the expressions

$$U = s\int_0^1 A^2\left(t+\sqrt{2s}\,\alpha(v)\right)\,\mathrm{d}v = s\int_0^1 B\left(t+\sqrt{2s}\,\alpha(v)\right)\,\mathrm{d}v, \tag{3.10}$$

$$V = s\int_0^1 A'\left(t+\sqrt{2s}\,\alpha(v)\right)\,\mathrm{d}v. \tag{3.11}$$

In order to do the summation over n using the Poisson summation formula (cf. [6,7]), we set

$$f_s(x) := \left(x^2 - \frac{1}{4}\right)\mathrm{e}^{-x^2 U - xV}, \tag{3.12}$$

and we obtain

$$\int_{-\infty}^{\infty} f_s(x)\,\mathrm{d}x = \frac{\sqrt{\pi}\,\mathrm{e}^{\frac{V^2}{4U}}\left(-U^2+2U+V^2\right)}{4U^{5/2}}. \tag{3.13}$$

Considering the variables U and V given in definitions (3.10) and (3.11), the task of computing the terms of the asymptotic expansion becomes significantly easier (even for the computer) compared to the series for the specific functions (3.4) and (3.6) of the u and b defined in [7]. It is in particular the Laurent series (3.6) of b in [7] which creates a great amount of unnecessary computational difficulties.

Using (3.12), we obtain the function that generates the full expansion in the form

$$\frac{1}{\sqrt{\pi s}} \frac{\sqrt{\pi}\, \mathrm{e}^{\frac{V^2}{4U}} \left(-U^2+2U+V^2\right)}{4U^{5/2}} = \frac{1}{\sqrt{s}} \frac{\mathrm{e}^{\frac{V^2}{4U}} \left(-U^2+2U+V^2\right)}{4U^{5/2}}. \tag{3.14}$$

We consider then the Laurent series expansion of the function given by (3.14) in the variable s.

3.2. Laurent Series Expansion

In order to keep the notation concise and more efficient, we let $\tau = s^{1/2}$ and write:

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!}\, \tau^n, \qquad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!}\, \tau^n, \tag{3.15}$$

where

$$\begin{aligned} u_n &= B^{(n)}(t)\, 2^{n/2} x_n(\alpha) = \left(\sum_{k=0}^{n} \binom{n}{k} A^{(k)}(t) A^{(n-k)}(t) \right) 2^{n/2}\, x_n(\alpha), \\ v_n &= A^{(n+1)}(t) 2^{n/2}\, x_n(\alpha)\,, \end{aligned}$$

with A and B as in (3.7) and with $x_k(\alpha)$ as in (3.5).

Lemma 3.1. *For $r \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathrm{e}^{\frac{V^2}{4U}}\, U^r\, V^m = \tau^{2(r+m)} \sum_{M=0}^{\infty} C_M^{(r,m)} \tau^M, \tag{3.16}$$

where

$$C_M^{(r,m)} = \sum_{\substack{0\leq k,p,N \leq M \\ 0 \leq n \leq M/2 \\ N+2n=M \\ 1 \leq \ell_1,\dots,\ell_k,q_1,\dots,q_p \leq N \\ \ell_1+\cdots+\ell_k+q_1+\cdots+q_p = N}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p} \frac{u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p}}{\ell_1!\cdots \ell_k!\, q_1!\cdots q_p!} \tag{3.17}$$

with the convention that when $k=0$ we have $u_{\ell_1}\cdots u_{\ell_k} = 1$, and when $p=0$ we have $v_{q_1}\cdots v_{q_p} = 1$. Note that it follows that when $k=p=0$ one considers $u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p} = 1$.

Proof. By direct computation, we find that

$$\begin{aligned}
\mathrm{e}^{\frac{V^2}{4U}}\, U^r\, V^m &= \sum_{n=0}^{\infty} \frac{1}{4^n n!}\, U^{-n+r}\, V^{2n+m} \\
&= \sum_{n=0}^{\infty} \frac{1}{4^n n!}\, u_0^{-n+r}\, \tau^{-2n+2r} \sum_{k=0}^{\infty} \binom{-n+r}{k} \left(\sum_{\ell=1}^{\infty} \frac{u_\ell}{\ell!\, u_0}\tau^\ell\right)^k \tau^{4n+2m} v_0^{2n+m} \\
&\quad \times \sum_{p=0}^{2n+m} \binom{2n+m}{p} \left(\sum_{q=1}^{\infty} \frac{v_q}{q!\, v_0}\tau^q\right)^p \\
&= \tau^{2(r+m)} \sum_{n,k,p\geq 0} \frac{u_0^{-n+r} v_0^{2n+m}}{4^n n!} \binom{-n+r}{k}\binom{2n+m}{p} \tau^{2n} \\
&\quad \times \left(\sum_{\ell=1}^{\infty} \frac{u_\ell}{\ell!\, u_0}\tau^\ell\right)^k \left(\sum_{q=1}^{\infty} \frac{v_q}{q!\, v_0}\tau^q\right)^p \\
&= \tau^{2(r+m)} \sum_{n,k,p\geq 0} \frac{u_0^{-n+r} v_0^{2n+m}}{4^n n!} \binom{-n+r}{k}\binom{2n+m}{p} \tau^{2n} \\
&\quad \times \sum_{N=0}^{\infty} \left(\sum_{|\ell|+|q|=N} \frac{u_{\ell_1}\cdots u_{\ell_k}}{\ell_1!\cdots \ell_k!\, u_0^k} \frac{v_{q_1}\cdots v_{q_p}}{q_1!\cdots q_p!\, v_0^p} \right) \tau^N,
\end{aligned}$$

where the inner summation is over all $\ell_1, \ldots, \ell_k \geq 1$ and $q_1, \ldots, q_p \geq 1$ such that $\ell_1 + \cdots + \ell_k + q_1 + \cdots + q_p = N$ with the convention that when $p = k = 0$ the sum is equal to 1. For simplicity we denoted this condition by $|\ell| + |q| = N$. □

We then obtain an explicit formula for the general term in the full expansion of the spectral action.

Theorem 3.2. *As* $\tau = s^{1/2} \to 0^+$, *we have:*

$$\mathrm{Tr}(\exp(-\tau^2 D^2)) \sim \sum_{M=0}^{\infty} \tau^{2M-4} \int a_{2M}(t)\, \mathrm{d}t,$$

where

$$a_0(t) = \frac{1}{2} C_0^{(-3/2,0)}, \tag{3.18}$$

and

$$a_{2M}(t) = \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4}\left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)} \right)\right) D[\alpha], \qquad M \in \mathbb{Z}_{\geq 1}. \tag{3.19}$$

Proof. Using Lemma 3.1, we write the desired expansion for the function given by (3.14) as

$$\begin{aligned}
&\frac{1}{\tau} \frac{\mathrm{e}^{\frac{V^2}{4U}} \left(-U^2 + 2U + V^2\right)}{4U^{5/2}} \\
&\quad = \frac{1}{4} \sum_{M=0}^{\infty} \left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)} \right) \tau^{M-2} + \frac{1}{2} \sum_{M=0}^{\infty} C_M^{(-3/2,0)} \tau^{M-4}.
\end{aligned} \tag{3.20}$$

The statement of the theorem then follows directly from this expression for the expansion of (3.14). □

Form (2.12) with relations (2.13) of the terms in the expansion of the spectral action for Robertson–Walker metrics, obtained in [23], suggests that the explicit terms obtained above should be expressible in terms of Bell polynomials. We show in the next subsections that this is indeed the case.

3.3. Bell Polynomials

Bell polynomials arise naturally in the Faà di Bruno formula that expresses the derivatives of composite functions, [26,40]

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n} f(g(t)) = \sum_{m=1}^{n} f^{(m)}(g(t))\, B_{n,m}(g'(t), g''(t), \dots, g^{(n-m+1)}(t)). \quad (3.21)$$

More precisely, the multivariable Bell polynomials are defined as

$$B_{\beta,k}(x_1,\dots,x_{\beta-k+1}) = \sum_{\lambda} \frac{\beta!}{\lambda_1!\lambda_2!\cdots\lambda_{\beta-k+1}!} \left(\frac{x_1}{1!}\right)^{\lambda_1} \left(\frac{x_2}{2!}\right)^{\lambda_2} \cdots \left(\frac{x_{\beta-k+1}}{(\beta-k+1)!}\right)^{\lambda_{\beta-k+1}}, \quad (3.22)$$

where the summation is over all sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers such that

$$|\lambda|' := \sum_{i=1}^{\infty} i\,\lambda_i = \beta, \qquad |\lambda| := \sum_{i=1}^{\infty} \lambda_i = k.$$

Note that these conditions imply that $\lambda_{\beta-k+2} = \lambda_{\beta-k+3} = \cdots = 0$. We shall use the following conventions:

$$B_{0,0}(x_1) = 1,$$
$$B_{\beta,0}(x_1,\dots,x_{\beta+1}) = 0, \qquad \beta > 0,$$
$$B_{\beta,k} = 0, \qquad 0 \le \beta < k.$$

The structure of polynomials (2.12) that arise in the spectral action expansion of the Robertson–Walker metric [23] suggests that the combinatorial structure of the asymptotic expansion may be describable in terms of Bell polynomial, arising from the time-derivatives of expressions depending on the scaling factor $a(t)$ in the recursive formula (2.7). Although it may be possible to see this directly at the level of the recursive formula obtained by the pseudodifferential calculus, it seems difficult to control the terms explicitly by that method. We show instead that the explicit form of the terms of the asymptotic expansion obtained in Theorem 3.2 can indeed be expressed directly in terms of Bell polynomials.

Proposition 3.3. *For $r \in \mathbb{R}$ and $m, M \in \mathbb{Z}_{\geq 0}$, we have:*

$$C_{2M}^{(r,m)} = \sum_{\substack{0\leq k,p\leq 2M\\ 0\leq n\leq M\\ 0\leq \beta\leq 2M-2n}} \left(\frac{\binom{-n+r}{k}\binom{2n+m}{p}\binom{2M-2n}{\beta}\,k!\,p!}{4^n\,n!\,(2M-2n)!} u_0^{-n+r-k} v_0^{2n+m-p} \right.$$
$$\left. \times B_{\beta,k}\left(u_1,\ldots,u_{\beta-k+1}\right) B_{2M-2n-\beta,\,p}\left(v_1,\ldots,v_{2M-2n-\beta-p+1}\right) \right).$$

Proof. We have

$$\begin{aligned}
&C_{2M}^{(r,m)}\\
&= \sum_{\substack{0\leq k,p,N\leq 2M\\ 0\leq n\leq M\\ N+2n=2M\\ 1\leq \ell_1,\ldots,\ell_k,q_1,\ldots,q_p\leq N\\ \ell_1+\cdots+\ell_k+q_1+\cdots+q_p=N}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p} \frac{u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p}}{\ell_1!\cdots\ell_k!\,q_1!\cdots q_p!}\\
&= \sum_{\substack{0\leq k,p\leq 2M\\ 0\leq n\leq M\\ 1\leq \ell_1,\ldots,\ell_k,q_1,\ldots,q_p\leq 2M-2n\\ \ell_1+\cdots+\ell_k+q_1+\cdots+q_p=2M-2n}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p} \frac{u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p}}{\ell_1!\cdots\ell_k!\,q_1!\cdots q_p!}\\
&= \sum_{\substack{0\leq k,p\leq 2M\\ 0\leq n\leq M}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p}\\
&\quad \times \sum_{\beta=0}^{2M-2n} \sum_{\substack{1\leq \ell_1,\ldots,\ell_k,q_1,\ldots,q_p\leq 2M-2n\\ \ell_1+\cdots+\ell_k=\beta\\ q_1+\cdots+q_p=2M-2n-\beta}} \frac{u_{\ell_1}\cdots u_{\ell_k} v_{q_1}\cdots v_{q_p}}{\ell_1!\cdots\ell_k!\,q_1!\cdots q_p!}.
\end{aligned}$$

In the inner summation, assume the integers $\ell_1,\ldots,\ell_k$ consist of λ_1 copies of 1, …, λ_β copies of β, and the integers $q_1,\ldots,q_p$ consist of μ_1 copies of 1, …, $\mu_{2M-2n-\beta}$ copies of $2M-2n-\beta$. We then obtain

$$\begin{aligned}
&C_{2M}^{(r,m)}\\
&= \sum_{\substack{0\leq k,p\leq 2M\\ 0\leq n\leq M}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p}\\
&\quad \times \sum_{\beta=0}^{2M-2n} \sum_{\lambda,\mu} \binom{k}{\lambda_1,\ldots,\lambda_\beta}\binom{p}{\mu_1,\ldots,\mu_{2M-2n-\beta}}\\
&\quad \times \frac{u_1^{\lambda_1}\cdots u_\beta^{\lambda_\beta} v_1^{\mu_1}\cdots v_{2M-2n-\beta}^{\mu_{2M-2n-\beta}}}{(1!)^{\lambda_1}\cdots(\beta!)^{\lambda_\beta}(1!)^{\mu_1}\cdots((2M-2n-\beta)!)^{\mu_{2M-2n-\beta}}}\\
&= \sum_{\substack{0\leq k,p\leq 2M\\ 0\leq n\leq M}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p} \left(\sum_{\beta=0}^{2M-2n} \frac{k!\,p!}{\beta!\,(2M-2n-\beta)!} \right.
\end{aligned}$$

$$\times \sum_{\lambda,\mu} \frac{\beta!}{\lambda_1! \cdots \lambda_\beta!} \frac{(2M-2n-\beta)!}{\mu_1! \cdots (\mu_{2M-2n-\beta})!}$$

$$\times \frac{u_1^{\lambda_1} \cdots u_\beta^{\lambda_\beta} v_1^{\mu_1} \cdots v_{2M-2n-\beta}^{\mu_{2M-2n-\beta}}}{(1!)^{\lambda_1} \cdots (\beta!)^{\lambda_\beta} (1!)^{\mu_1} \cdots ((2M-2n-\beta)!)^{\mu_{2M-2n-\beta}}} \Bigg),$$

where the last summation is over all sequences of nonnegative integers λ and μ such that

$$\lambda_1 + 2\lambda_2 + \cdots + \beta\lambda_\beta = \beta, \qquad \lambda_1 + \lambda_2 + \cdots + \lambda_\beta = k,$$
$$\mu_1 + 2\mu_2 + \cdots + (2M-2n-\beta)\mu_{2M-2n-\beta} = 2M-2n-\beta,$$
$$\mu_1 + \mu_2 + \cdots + \mu_{2M-2n-\beta} = p.$$

Now we can use the Bell polynomials to write:

$$C_{2M}^{(r,m)} = \sum_{\substack{0\le k,p \le 2M \\ 0 \le n \le M}} \frac{\binom{-n+r}{k}\binom{2n+m}{p} k!\, p!}{4^n\, n!\, (2M-2n)!} u_0^{-n+r-k} v_0^{2n+m-p}$$
$$\times \left(\sum_{\beta=0}^{2M-2n} \binom{2M-2n}{\beta} B_{\beta,k}\left(u_1, \ldots, u_{\beta-k+1}\right) \right.$$
$$\left. \times\, B_{2M-2n-\beta,\, p}\left(v_1, \ldots, v_{2M-2n-\beta-p+1}\right) \right).$$

This gives the stated result. □

4. Brownian Bridge and Combinatorial Structure

In this section, we compute explicitly the Brownian bridge integrals and obtain the full combinatorial structure of the spectral action expansion for Robertson–Walker metrics.

4.1. Brownian Bridge Integrals

We provide combinatorial formulas for the Brownian bridge integrals that we need in order to write combinatorial expressions for the integrals in Theorem 3.2 describing the coefficients $a_{2M}(t)$ in the full expansion of the spectral action for the Robertson–Walker metric.

We need a preliminary result about integrals of monomials on the standard simplex.

Lemma 4.1. *Let Δ^n denote the simplex*

$$\Delta^n = \{(v_1, v_2, \ldots, v_n) \in \mathbb{R}^n : 0 \le v_1 \le v_2 \le \cdots \le v_n \le 1\}.$$

Then, the integral of a monomial $v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}$ *is given by the expression*

$$\int_{\Delta^n} v_1^{k_1}v_2^{k_2}\cdots v_n^{k_n}\,\mathrm{d}v_1\,\mathrm{d}v_2\cdots\mathrm{d}v_n = \frac{1}{(k_1+1)(k_1+k_2+2)\cdots(k_1+k_2+\cdots+k_n+n)}. \tag{4.1}$$

In particular, we have the following case.

Corollary 4.2. *If* $1\leq j_1<j_2<\cdots<j_k\leq n$*, then*

$$\int_{\Delta^n} v_{j_1}v_{j_2}\cdots v_{j_k}\,\mathrm{d}v_1\,\mathrm{d}v_2\cdots\mathrm{d}v_n = \frac{j_1(j_2+1)(j_3+2)\cdots(j_k+k-1)}{(n+k)!}. \tag{4.2}$$

Proof. The case of (4.2) is a special case of Lemma 4.1. The right-hand side is directly obtained from the corresponding expression in (4.1). □

We then consider the Brownian bridge integrals. The defining property (3.3) of the Brownian bridge implies the following.

Lemma 4.3. *For* $(v_1,v_2,\ldots,v_{2n})\in\Delta^{2n}$*,* $n\in\mathbb{Z}_{\geq 0}$*, the Brownian bridge integrals can be computed as*

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\,D[\alpha] = \sum v_{i_1}(1-v_{j_1})v_{i_2}(1-v_{j_2})\cdots v_{i_n}(1-v_{j_n}), \tag{4.3}$$

where the summation is over all indices such that $i_1<j_1$*,* $i_2<j_2$*, …,* $i_n<j_n$*, and* $\{i_1,j_1,i_2,j_2,\ldots,i_n,j_n\}=\{1,2,\ldots,2n\}$.

It is convenient to reformulate expression (4.3) in a slightly different notation as follows.

Corollary 4.4. *For* $(v_1,v_2,\ldots,v_{2n})\in\Delta^{2n}$*,* $n\in\mathbb{Z}_{\geq 0}$*, we have*

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\,D[\alpha] = \sum_{\sigma\in S_{2n}^*} v_{\sigma(1)}(1-v_{\sigma(2)})v_{\sigma(3)}(1-v_{\sigma(4)})\cdots v_{\sigma(2n-1)}(1-v_{\sigma(2n)}), \tag{4.4}$$

where S_{2n}^* *is the set of all permutations* σ *in the symmetric group* S_{2n} *such that* $\sigma(1)<\sigma(2)$*,* $\sigma(3)<\sigma(4)$*, …,* $\sigma(2n-1)<\sigma(2n)$.

This then gives a reformulation that will be useful in the following.

Lemma 4.5. *For* $(v_1,v_2,\ldots,v_{2n})\in\Delta^{2n}$*,* $n\in\mathbb{Z}_{\geq 0}$*, we have*

$$\int \alpha(v_1)\alpha(v_2)\cdots\alpha(v_{2n})\,D[\alpha] = \sum_{\sigma\in S_{2n}^*} v_{\sigma(1)}v_{\sigma(3)}\cdots v_{\sigma(2n-1)} \times\left(1+\sum_{k=1}^{n}(-1)^k\sum_{1\leq j_1<j_2<\cdots<j_k\leq n} v_{\sigma(2j_1)}v_{\sigma(2j_2)}\cdots v_{\sigma(2j_k)}\right).$$

Proof. This follows directly from Lemma 4.4. □

Definition 4.6. For any nonnegative integer n, we let $\mathcal{J}_{0,n}$ be the set consisting of the 0-tuple. For any $k = 1, \ldots, n$, we let $\mathcal{J}_{k,n}$ be the set of all k-tuples of integers $J = (j_1, j_2, \ldots, j_k)$ such that $1 \le j_1 < j_2 < \cdots < j_k \le n$. For any $J \in \mathcal{J}_{k,n}$, and $\sigma \in S^*_{2n}$, we define $\sigma_J(1)$, $\sigma_J(2)$, $\ldots$, $\sigma_J(n+k)$ by the property that

$$\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k),$$

and that the set of such σ_J's is given by

$$\begin{aligned}&\{\sigma_J(1) < \sigma_J(2) < \cdots < \sigma_J(n+k)\}\\ &\quad = \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1), \sigma(2j_1), \ldots, \sigma(2j_k)\}.\end{aligned}$$

Using notation (3.5) as in [7],

$$x_k(\alpha) = \int_0^1 \alpha(v)^k \, \mathrm{d}v\,,$$

we can then write the Brownian bridge integrals of the $x_k(\alpha)$ in the following form.

Lemma 4.7. *We have*

$$\begin{aligned}&\int x_1(\alpha)^{2n}\, D[\alpha] = \int \Big(\int_0^1 \alpha(v)\,\mathrm{d}v\Big)^{2n} D[\alpha]\\ &\quad = (2n)! \sum_{\sigma \in S^*_{2n}} \sum_{k=0}^{n} \sum_{J \in \mathcal{J}_{k,n}} (-1)^k \frac{\sigma_J(1)\,(\sigma_J(2)+1)\cdots(\sigma_J(n+k)+n+k-1)}{(3n+k)!}.\end{aligned}$$

Proof. This is obtained using the expression of Lemma 4.5 and the notation as in Definition 4.6. □

We can then formulate the monomial Brownian bridge integrals as follows:

Proposition 4.8. *For $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and for $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\ge 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\ge 0}$, we have*

$$\begin{aligned}\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] &= \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!}\\ &\quad\times \Bigg(\sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1}\sum_{r_2=0}^{K_2}\cdots\sum_{r_n=0}^{K_n}\prod_{p=1}^{n}(-1)^{r_p} v_p^{i_p - r_p}\Bigg),\end{aligned} \tag{4.5}$$

where $I = (i_1, i_2, \ldots, i_n)$, the first summation is over all nonnegative integers $k_{j,m}$, $j, m = 1, 2, \ldots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = \frac{|I|}{2}, \qquad \sum_{m=1}^{n} (k_{j,m} + k_{m,j}) = i_j \ \text{ for all } j = 1, 2, \ldots, n,$$

and we set for each $m = 1, 2, \dots, n$,

$$K_m := k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j}).$$

Proof. First observe that, for Brownian bridge integrals in exponential form, we have the identity

$$\int \exp\left(\sqrt{-1}\sum_{j=1}^{n} u_j \alpha(v_j)\right) D[\alpha] = \exp\left(-\frac{1}{2}\sum_{j,m=1}^{n} c_{j,m} u_j u_m\right),$$

where the terms $c_{j,m}$ are given by

$$c_{j,m} = v_j(1 - v_m) \quad \text{if} \quad j \leq m, \qquad \text{and} \qquad c_{j,m} = v_m(1 - v_j) \quad \text{if} \quad m \leq j.$$

This implies that we have

$$\begin{aligned}
&\frac{(\sqrt{-1})^{i_1+i_2+\cdots+i_n}}{(i_1+i_2+\cdots+i_n)!}\binom{i_1+i_2+\cdots+i_n}{i_1, i_2, \dots, i_n}\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] \\
&\quad = \frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!} \\
&\qquad \times \left(\text{Coefficient of } u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n} \text{ in } \left(\sum_{j,m=1}^{n} c_{j,m}u_j u_m\right)^{(i_1+i_2+\cdots+i_n)/2}\right) \\
&\quad = \frac{(-1/2)^{(i_1+i_2+\cdots+i_n)/2}}{((i_1+i_2+\cdots+i_n)/2)!}\sum \binom{(i_1+i_2+\cdots+i_n)/2}{k_{1,1}, k_{1,2}, \dots, k_{1,n}, k_{2,1}, \dots, k_{n,n}} \prod_{j,m=1}^{n} c_{j,m}^{k_{j,m}},
\end{aligned}$$

where the summation is over all nonnegative integers $k_{j,m}$, $j, m = 1, 2, \dots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = (i_1 + i_2 + \cdots + i_n)/2$$

and, for any $j = 1, 2, \dots, n$,

$$2k_{j,j} + \sum_{1 \leq m \leq n, m \neq j} (k_{j,m} + k_{m,j}) = i_j.$$

This then gives

$$\begin{aligned}
&\int \alpha(v_1)^{i_1}\alpha(v_2)^{i_2}\cdots\alpha(v_n)^{i_n}\, D[\alpha] \\
&\quad = \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!}\left(\sum \binom{|I|/2}{k_{m,j}} \prod_{j,m=1}^{n} c_{j,m}^{k_{j,m}}\right) \\
&\quad = \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!}\left(\sum \binom{|I|/2}{k_{m,j}} \prod_{m=1}^{n} v_m^{i_m - K_m}(1 - v_m)^{K_m}\right),
\end{aligned}$$

where $I = (i_1, i_2, \ldots, i_n)$ and for each $m = 1, 2, \ldots, n$,

$$K_m := k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j})$$

Therefore, we obtain

$$\begin{aligned}
&\int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n}\, D[\alpha] \\
&= \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \left(\sum \binom{|I|/2}{k_{m,j}} \prod_{m=1}^{n} \sum_{r_m=0}^{K_m} (-1)^{r_m} \binom{K_m}{r_m} v_m^{i_m - r_m} \right) \\
&= \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \\
&\quad \times \left(\sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1} \sum_{r_2=0}^{K_2} \cdots \sum_{r_n=0}^{K_n} \prod_{p=1}^{n} (-1)^{r_p} \binom{K_p}{r_p} v_p^{i_p - r_p} \right).
\end{aligned}$$

□

We then obtain the following expression for integration over a simplex of Brownian bridge monomial integrals.

Lemma 4.9. *For $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$, we have*

$$\begin{aligned}
&\int_{\Delta^n} \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n}\, D[\alpha]\, \mathrm{d}v_1\, \mathrm{d}v_2 \cdots \mathrm{d}v_n \\
&\quad = \binom{|I|}{I}^{-1} \frac{|I|!}{(\sqrt{-1})^{|I|}} \frac{(-1/2)^{|I|/2}}{(|I|/2)!} \\
&\qquad \times \left(\sum \binom{|I|/2}{k_{m,j}} \sum_{r_1=0}^{K_1} \sum_{r_2=0}^{K_2} \cdots \sum_{r_n=0}^{K_n} \prod_{p=1}^{n} \frac{(-1)^{r_p} \binom{K_p}{r_p}}{p + \sum_{\ell=1}^{p} (i_\ell - r_\ell)} \right),
\end{aligned}$$

where $I = (i_1, i_2, \ldots, i_n)$ and the first summation is over all nonnegative integers $k_{j,m}$, $j, m = 1, 2, \ldots, n$ such that

$$\sum_{j,m=1}^{n} k_{j,m} = |I|/2, \qquad \sum_{m=1}^{n} (k_{j,m} + k_{m,j}) = i_j \ \text{ for all } j = 1, 2, \ldots, n,$$

and where, for each $m = 1, 2, \ldots, n$, we set

$$K_m = k_{m,m} + \sum_{j=1}^{m-1} (k_{j,m} + k_{m,j}).$$

Proof. This follows directly from Lemma 4.1 and Proposition 4.8. □

4.2. Shuffle Product

We introduce the following notation for the integrals described combinatorially in Proposition 4.8.

Definition 4.10. For $(v_1, v_2, \ldots, v_n) \in \Delta^n$ and for $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ such that $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$, we set

$$V^b(i_1, i_2, \ldots, i_n) := \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha]\,. \tag{4.6}$$

We view $(i_1, i_2, \ldots, i_n)$ as a word constructed with the letters $i_1, i_2, \ldots, i_n$, and we extend the definition of V^b linearly to the vector space generated by all such words.

Definition 4.11. The *shuffle product* of two words $(i_1, i_2, \ldots, i_p)$ and $(j_1, j_2, \ldots, j_q)$ is defined to be the sum of the $\binom{p+q}{p}$ words obtained by interlacing the letters of the two words in such a way that in each term the order of the letters of each word is preserved. The shuffle product is denoted by $\sqcup\!\sqcup$.

Lemma 4.12. *Assume that $2n = m_1 i_1 + m_2 i_2 + \cdots + m_r i_r$ is an even positive integer (where $i_1, i_2, \ldots, i_r$ are distinct positive integers and $m_1, m_2, \ldots, m_r$ are positive integers). Then, with the $x_k(\alpha)$ as in (3.5),*

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha]$$
$$= m! \int_{\Delta^{|m|}} V^b\Big(\underbrace{(i_1, \ldots, i_1)}_{m_1} \sqcup\!\sqcup \underbrace{(i_2, \ldots, i_2)}_{m_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \underbrace{(i_r, \ldots, i_r)}_{m_r} \Big) \mathrm{d}v_1 \mathrm{d}v_2 \cdots \mathrm{d}v_{|m|},$$

where

$$m! = (m_1!)(m_2!) \cdots (m_r!), \qquad |m| = m_1 + m_2 + \cdots + m_r\,.$$

Proof. It follows directly from writing

$$\int x_{i_1}(\alpha)^{m_1} x_{i_2}(\alpha)^{m_2} \cdots x_{i_r}(\alpha)^{m_r} \, D[\alpha]$$
$$= \int \Big(\int_0^1 \alpha(v_1)^{i_1} \, \mathrm{d}v_1 \Big)^{m_1} \Big(\int_0^1 \alpha(v_2)^{i_2} \, \mathrm{d}v_2 \Big)^{m_2} \cdots \Big(\int_0^1 \alpha(v_r)^{i_r} \, \mathrm{d}v_r \Big)^{m_r} D[\alpha],$$

and considering Definitions 4.10 and 4.11. □

Note that Lemma 4.9 gives a formula for $\int_{\Delta^n} V^b(i_1, \ldots, i_n) \mathrm{d}v_1 \cdots \mathrm{d}v_n$. Thus, using Lemmas 4.12 and 4.9 we have achieved a combinatorial description of all the Brownian bridge integrals involved in the calculation of the spectral action expansion.

4.3. The Integrals in Terms of the Dawson Function

In Lemma 4.9, we gave a combinatorial formula for

$$\int_{\Delta^n} \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha] \, \mathrm{d}v_1 \, \mathrm{d}v_2 \, \cdots \mathrm{d}v_n.$$

A crucial fact that we used for deriving the combinatorial formula is that since the Brownian bridge is a Gaussian process, for $(v_1, \ldots, v_n) \in \Delta^n$ we have:

$$\int \exp\left(\sqrt{-1} \sum_{j=1}^n u_j \alpha(v_j) \right) D[\alpha] = \exp\left(-\frac{1}{2} \sum_{j,m=1}^n c_{j,m} u_j u_m \right),$$

where

$$c_{j,m} = v_j(1 - v_m) \quad \text{if} \quad j \leq m, \qquad \text{and} \qquad c_{j,m} = v_m(1 - v_j) \quad \text{if} \quad m \leq j.$$

Therefore, if $i_1, i_2, \ldots, i_n \in \mathbb{Z}_{\geq 0}$ and $i_1 + i_2 + \cdots + i_n \in 2\mathbb{Z}_{\geq 0}$, then, setting $I = (i_1, \ldots, i_n)$,

$$\frac{\left(\sqrt{-1}\right)^{|I|} \binom{|I|}{I}}{|I|!} \int_{\Delta^n} \int \alpha(v_1)^{i_1} \alpha(v_2)^{i_2} \cdots \alpha(v_n)^{i_n} \, D[\alpha] \, \mathrm{d}v_1 \, \mathrm{d}v_2 \cdots \mathrm{d}v_n$$

is equal to the coefficient of $u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n}$ in the Maclaurin series of

$$\int_{\Delta^n} \exp\left(-\frac{1}{2} \sum_{j,m=1}^{n} c_{j,m} u_j u_m \right) \mathrm{d}v_1 \, \mathrm{d}v_2 \cdots \mathrm{d}v_n. \tag{4.7}$$

By writing the expansion of the integrand in the latter, we derived the combinatorial formula presented in Lemma 4.9.

It is natural to ask whether there is a closed formula for the result of the integral given by (4.7). It turns out that it is possible to obtain such closed expressions in terms of the Dawson function

$$F(x) := \exp\left(-x^2\right) \int_0^x \exp\left(y^2\right) \, dy. \tag{4.8}$$

We show the first few cases of integral (4.7) and their explicit form in terms of function (4.8).

Example 4.13. When $n = 1, 2, 3$, we find the following explicit expressions:

$$\int_{\Delta^1} \exp\left(-\frac{1}{2} c_{1,1} u_1^2 \right) \mathrm{d}v_1 = \frac{2\sqrt{2} F\left(\frac{u_1}{2\sqrt{2}}\right)}{u_1},$$

$$\int_{\Delta^2} \exp\left(-\frac{1}{2} \sum_{j,m=1}^{2} c_{j,m} u_j u_m \right) \mathrm{d}v_1 \, \mathrm{d}v_2$$
$$= \frac{4\sqrt{2} \left(F\left(\frac{u_1}{2\sqrt{2}}\right) + F\left(\frac{u_2}{2\sqrt{2}}\right) - F\left(\frac{u_1+u_2}{2\sqrt{2}}\right) \right)}{u_1 u_2 \left(u_1 + u_2\right)},$$

$$\int_{\Delta^3} \exp\left(-\frac{1}{2} \sum_{j,m=1}^{3} c_{j,m} u_j u_m \right) \mathrm{d}v_1 \, \mathrm{d}v_2 \, \mathrm{d}v_3$$
$$= \frac{8\sqrt{2} F\left(\frac{u_1}{2\sqrt{2}}\right)}{u_1 u_2 \left(u_1 + u_2\right) \left(u_2 + u_3\right) \left(u_1 + u_2 + u_3\right)}$$
$$+ \frac{8\sqrt{2} F\left(\frac{u_2}{2\sqrt{2}}\right)}{u_1 u_2 \left(u_1 + u_2\right) \left(u_2 + u_3\right) \left(u_1 + u_2 + u_3\right)}$$
$$- \frac{8\sqrt{2} F\left(\frac{u_1+u_2}{2\sqrt{2}}\right)}{u_1 u_2 \left(u_1 + u_2\right) \left(u_2 + u_3\right) \left(u_1 + u_2 + u_3\right)}$$

$$
\begin{aligned}
&+\frac{8\sqrt{2}F\left(\frac{u_2}{2\sqrt{2}}\right)}{u_1\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&-\frac{8\sqrt{2}F\left(\frac{u_1+u_2}{2\sqrt{2}}\right)}{u_1\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&-\frac{8\sqrt{2}F\left(\frac{u_2+u_3}{2\sqrt{2}}\right)}{u_1\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&+\frac{8\sqrt{2}F\left(\frac{u_1+u_2+u_3}{2\sqrt{2}}\right)}{u_1\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&+\frac{8\sqrt{2}F\left(\frac{u_2}{2\sqrt{2}}\right)}{u_2\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&+\frac{8\sqrt{2}F\left(\frac{u_3}{2\sqrt{2}}\right)}{u_2\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}\\
&-\frac{8\sqrt{2}F\left(\frac{u_2+u_3}{2\sqrt{2}}\right)}{u_2\left(u_1+u_2\right)u_3\left(u_2+u_3\right)\left(u_1+u_2+u_3\right)}.
\end{aligned}
$$

An explicit expression for

$$\int_{\Delta^4} \exp\left(-\frac{1}{2}\sum_{j,m=1}^{4} c_{j,m}u_j u_m\right) \mathrm{d}v_1\,\mathrm{d}v_2\,\mathrm{d}v_3\,\mathrm{d}v_4$$

in terms of the Dawson function (4.8) is included in "Appendix A" of the arXiv version of this paper, namely https://arxiv.org/abs/1811.02972.

4.4. Combinatorial Description of the Full Spectral Action Expansion

In Proposition 3.3, we showed that

$$
\begin{aligned}
&C_{2M}^{(r,m)}\\
&\quad=\sum_{\substack{0\le k,p\le 2M\\ 0\le n\le M\\ 0\le\beta\le 2M-2n}} \frac{\binom{-n+r}{k}\binom{2n+m}{p}}{4^n n!} u_0^{-n+r-k} v_0^{2n+m-p} k!\,p! \left(\sum_{\lambda,\mu}\prod_{i=1}^{\infty}\frac{\left(\frac{u_i}{i!}\right)^{\lambda_i}\left(\frac{v_i}{i!}\right)^{\mu_i}}{\lambda_i!\,\mu_i!}\right),
\end{aligned}
$$

where for each fixed k, p, n, β, the inner summation is over all sequences $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ of nonnegative integers such that

$$|\lambda|' := \sum_{i=1}^{\infty} i\,\lambda_i = \beta, \qquad |\lambda| := \sum_{i=1}^{\infty}\lambda_i = k, \qquad |\mu|' = 2M-2n-\beta, \qquad |\mu| = p.$$

Note that these conditions imply that only finitely many λ_i and μ_i can be nonzero, namely: $\lambda_{\beta-k+2} = \lambda_{\beta-k+3} = \cdots = 0$ and $\mu_{2M-2n-\beta-p+2} = \mu_{2M-2n-\beta-p+3} = \cdots = 0$.

Lemma 4.14. *For $r \in \mathbb{R}$ and $m, M \in \mathbb{Z}_{\geq 0}$, the Brownian bridge integral of the expressions $C_{2M}^{(r,m)}$ above gives*

$$\int C_{2M}^{(r,m)}\, D[\alpha]$$

$$= \sum \left(\frac{\binom{-n+r}{k}\binom{2n+m}{p} k!\, p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b \left(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \right) \mathrm{d}v_1 \cdots \mathrm{d}v_{k+p} \right.$$

$$\left. \times B(t)^{-n+r-k}\ (A'(t))^{2n+m-p} \prod_{i=1}^{\infty} \binom{\lambda_i + \mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \right),$$

where the summation is over all integers $0 \leq k, p \leq 2M, 0 \leq n \leq M$, $0 \leq \beta \leq 2M - 2n$, and over all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of nonnegative integers for each choice of k, p, n, β, such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M - 2n - \beta, |\mu| = p$.

Proof. This follows directly from the above fact from Proposition 3.3 using Lemma 4.12. □

In Theorem 3.2, we showed that the coefficients appearing in the asymptotic expansion

$$\mathrm{Tr}(\exp(-\tau^2 D^2)) \sim \sum_{M=0}^{\infty} \tau^{2M-4} \int a_{2M}(t)\, \mathrm{d}t \qquad (\text{as } \tau \to 0^+)$$

are given by

$$a_0(t) = \frac{1}{2} C_0^{(-3/2,0)},$$

and

$$a_{2M}(t) = \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4}\left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)} \right) \right) D[\alpha], \qquad M \in \mathbb{Z}_{\geq 1}.$$

The result of Lemma 4.14 gives a combinatorial description of the integral $\int C_{2M}^{(r,m)}\, D[\alpha]$; hence, we can write a combinatorial formula for an arbitrary coefficient in the full expansion of the spectral action for the Robertson–Walker metric with the expansion factor $a(t)$. We use again the notation $A(t) = 1/a(t), B(t) = A(t)^2$ as in (3.7) and the expressions

$$V^b\left((1,\ldots,1) \sqcup (2,\ldots,2) \sqcup \cdots\right)$$

as in Definition 4.10.

Theorem 4.15. *For any $M \in \mathbb{Z}_{\geq 1}$, the coefficients of the expansion of the spectral action of a Robertson–Walker metric are given by*

$$
\begin{aligned}
&a_{2M}(t)\\
&= \frac{1}{2}\sum{}' \Bigg(\frac{\binom{-n-3/2}{k}\binom{2n}{p} k!\,p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b \Bigg(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \Bigg)\, \mathrm{d}v_1 \cdots \mathrm{d}v_{k+p}\\
&\quad \times B(t)^{-n-(3/2)-k}\, (A'(t))^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg)\\
&\quad + \frac{1}{4}\sum{}'' \Bigg(\Bigg(\binom{-n-5/2}{k} \binom{2n+2}{p} B(t)^{-5/2}\, (A'(t))^2\\
&\quad - \binom{-n-1/2}{k}\binom{2n}{p} B(t)^{-1/2} \Bigg)\\
&\quad \times \frac{k!\,p!}{4^n\, 2^{n-M}\, n!} \int_{\Delta^{k+p}} V^b \Bigg(\underbrace{(1,\ldots,1)}_{\lambda_1+\mu_1} \sqcup \underbrace{(2,\ldots,2)}_{\lambda_2+\mu_2} \sqcup \cdots \Bigg)\, \mathrm{d}v_1 \cdots \mathrm{d}v_{k+p}\\
&\quad \times B(t)^{-n-k}\, (A'(t))^{2n-p} \prod_{i=1}^{\infty} \binom{\lambda_i+\mu_i}{\lambda_i} \left(\frac{B^{(i)}(t)}{i!}\right)^{\lambda_i} \left(\frac{A^{(i+1)}(t)}{i!}\right)^{\mu_i} \Bigg),
\end{aligned}
$$

where the structure of the summations is as follows. The first summation $\sum{}'$ is over all integers $0 \leq k, p \leq 2M, 0 \leq n \leq M$, $0 \leq \beta \leq 2M - 2n$, and over all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ of nonnegative integers (for each choice of k, p, n, β) such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M-2n-\beta, |\mu| = p$. Similarly, the second summation $\sum{}''$ is over all integers $0 \leq k, p \leq 2M-2, 0 \leq n \leq M-1$, $0 \leq \beta \leq 2M-2-2n$, and over all sequences $\lambda = (\lambda_1, \lambda_2, \ldots)$, $\mu = (\mu_1, \mu_2, \ldots)$ of nonnegative integers such that $|\lambda|' = \beta, |\lambda| = k$, $|\mu|' = 2M-2-2n-\beta, |\mu| = p$.

Proof. The result follows directly from Theorem 3.2 and Lemma 4.14. □

4.5. Explicit Form of the Coefficients

Using the combinatorial formula obtained in Theorem 4.15, we can compute explicitly the coefficients $a_{2M}(t)$ in terms of the expressions $A(t) = 1/a(t), B(t) = A(t)^2$. For the first few coefficients, this gives the following expressions:

$$
\begin{aligned}
a_0(t) &= \frac{1}{2B(t)^{3/2}},\\
a_2(t) &= \frac{3A'(t)^2}{8B(t)^{5/2}} - \frac{B''(t)}{8B(t)^{5/2}} + \frac{5B'(t)^2}{32B(t)^{7/2}} - \frac{1}{4B(t)^{1/2}},\\
a_4(t) &= \frac{A''(t)^2}{16B(t)^{5/2}} - \frac{5A'(t)^2 B''(t)}{32B(t)^{7/2}} + \frac{35A'(t)^2 B'(t)^2}{128B(t)^{9/2}}\\
&\quad + \frac{5A'(t)^4}{64B(t)^{7/2}} - \frac{A'(t)^2}{16B(t)^{3/2}}
\end{aligned}
$$

$$+\frac{A^{(3)}(t)A'(t)}{8B(t)^{5/2}}-\frac{5A'(t)A''(t)B'(t)}{16B(t)^{7/2}}-\frac{B^{(4)}(t)}{80B(t)^{5/2}}+\frac{3B''(t)^2}{64B(t)^{7/2}}$$
$$+\frac{B''(t)}{48B(t)^{3/2}}+\frac{105B'(t)^4}{1024B(t)^{11/2}}$$
$$-\frac{B'(t)^2}{64B(t)^{5/2}}+\frac{B^{(3)}(t)B'(t)}{16B(t)^{7/2}}-\frac{77B'(t)^2B''(t)}{384B(t)^{9/2}}.$$

Similar explicit expressions for the coefficients $a_6(t)$ and $a_8(t)$ are reported in "Appendix B" of the arXiv version of this paper, namely https://arxiv.org/abs/1811.02972. When written in terms of the scaling factor $a(t)$ through relation (3.7), these expressions agree with those computed in [7,23].

4.6. The Faà di Bruno Hopf Algebra

The Bell polynomials and the Faà di Bruno formula have a Hopf algebra interpretation, where one considers the group $G^{\text{diff}}(A)$ of formal diffeomorphisms tangent to the identity,

$$f(t)=t+\sum_{n\geq 2} f_n\, t^n \in tA[[t]],$$

with A a unital commutative algebra over a field $\mathbb{K}$, with the product given by composition. Viewed as an affine group scheme, G^{diff} is dual to the Faà di Bruno Hopf algebra $\mathcal{H}_{\text{FdB}}$,

$$G^{\text{diff}}(A)=\text{Hom}(\mathcal{H}_{\text{FdB}},A). \tag{4.9}$$

As an algebra $\mathcal{H}_{\text{FdB}}$ is a polynomial algebra $\mathbb{K}[x_1,x_2,x_3,\ldots,x_n,\ldots]$ in countably many variables x_i, with coproduct (see [25])

$$\Delta(x_n)=\sum_{m=0}^{n}\frac{(m+1)!}{(n+1)!}\,B_{n+1,m+1}(1,2!x_1,3!x_2,\ldots,(n-m+1)!x_{n-m})\otimes x_m$$
$$=\sum_{m=0}^{n}\left(\sum_{\substack{k_0+k_1+k_2+\cdots+k_n=m+1\\ k_1+2k_2+\cdots+nk_n=n-m}}\frac{(m+1)!}{k_0!k_1!\cdots k_n!}\prod_{i=1}^{n}x_i^{k_i}\right)\otimes x_m,$$

with $x_0=1$. The Hopf algebra is graded by $\deg(x_n)=n$ and connected $\mathcal{H}_{\deg=0}=\mathbb{K}$. The counit is given by $\epsilon(x_n)=\delta_{n,0}$, and the antipode is determined inductively for graded connected Hopf algebras.

It is known that the Faà di Bruno Hopf algebra embeds in the Connes–Kreimer Hopf algebra of planar rooted trees, [25], $\mathcal{H}_{\text{FdB}}\hookrightarrow\mathcal{H}_{CK}$ and dually the affine group schemes map surjectively $G_{CK}\twoheadrightarrow G^{\text{diff}}$.

While we will not consider this question in the present paper, it is worth pointing out that the structure of the asymptotic expansion of the spectral action of the Robertson–Walker metrics that we obtained here in terms of Bell polynomials suggests the presence of an interesting Hopf algebra action, similar to the one regulating the renormalization of quantum field theories, see [14]. Understanding the structure and meaning of the role of the Faà di Bruno Hopf algebra in the spectral action expansion appears to be an especially interesting question in view of a better understanding of the spectral action as a gravity model. Indeed, one usually considers the spectral action functional as an effective field theory (at energies around or below unification, as indicated by the resulting models of gravity coupled to matter) and treats it semiclassically using the leading terms of the asymptotic expansion as a classical action functional for (modified) gravity. The full spectral action expansion provides a series of higher-derivatives correction terms, which are known to improve renormalizability. In particular, the role of the full spectral action expansion in renormalizability in the case of Yang–Mills models was studied in [43] and for general almost-commutative geometries in [44,45]. The use of the full expansion of the spectral action functional is crucial in these renormalizability arguments, see [45]. The description of the coefficients of the spectral action in terms of Brownian bridge integrals appears especially suitable for analyzing the spectral action as a quantum theory and their expression in terms of Bell polynomials, with the Faà di Bruno Hopf algebra action, suggests what the underlying Hopf-algebraic renormalization structure should be. We hope to return to this question in future work.

5. Multifractal Robertson–Walker Cosmologies

In this second part of the paper, we turn to consider the multifractal cases of Robertson–Walker cosmologies, where the spatial sections are obtained as an arrangement of 3-spheres such as an Apollonian packing, generalizing the static cases considered in [3].

5.1. Packed Swiss Cheese Cosmologies

The hypothesis of multifractal structures in cosmology was proposed to justify the observable distribution of clusters of galaxies, see for instance [31]. A particularly interesting model exhibiting fractality and multifractality is known as the "Packed Swiss Cheese Cosmology", [38]. These are constructed on the model of the fractality of an Apollonian packing of 3-spheres inside a 4-dimensional spacetime. Equivalently, one can obtain these spacetimes as an Apollonian arrangement of 4-dimensional Robertson–Walker metrics. This does not require an embedding into higher dimensions, as the individual scaled 4-dimensional building blocks and their incidence relations completely specify the resulting spacetime, which is topologically 4-dimensional, but with an associated non-integer Hausdorff dimension due to the fractality of the arrangement.

In [3], it was shown that a spectral action model of gravity can be applied to these fractal cosmologies. Under some regularity assumptions on the structure of the fractal, the spectral action can be computed, using a general method of [10,11] for constructing spectral triples (the non-commutative analogs of spin geometry) on fractals. Unlike the case of an ordinary smooth manifold, in the presence of a fractal structure the heat kernel of the Dirac operator acquires some log-periodic terms. These correspond to the presence of poles off the real line in the zeta function of the Dirac operators. In turn, these poles determine additional terms in the asymptotic expansion of the spectral action. These are corrections to the action functional of gravity that detect the presence of fractality. In particular, the shape of a slow–roll potential for inflation derived from the spectral action model was shown in [3] to be also affected by these terms, so that corrections due to the presence of fractality also appear in the slow–roll coefficients, which in principle are detectable via observational data. The regularity assumptions on the fractal geometry used in [3] were aimed at obtaining sufficiently good analytic properties of the corresponding zeta functions, in the sense of [32].

However, the model of spectral action on multifractal Swiss Cheese type cosmologies considered in [3] is not entirely realistic, because the relevant spacetime is assumed to be a product of a fractal packing of spheres (or of spherical manifolds) times a compactified Euclidean time dimension S^1, so that, in particular, the scaling factor of the spatial sections remains constant. This corresponds to a static, rather than a more realistic expanding universe.

In order to make the model more realistic and physically interesting, our goal in the present paper is to reformulate the multifractal spectral action model in terms of Robertson–Walker metrics.

Although this may at first look like a simple modification, in fact it requires a completely different set of analytical tools to derive the spectral action computation from heat kernel and zeta function information. In particular, the analytic techniques involved are based on the derivation of the terms of the asymptotic expansion of the spectral action for Robertson–Walker metrics discussed in the previous sections, based on the Feynman–Kac formula and Brownian bridge integrals as in [7]. In the case of Robertson–Walker metrics that are round 4-spheres, the result we obtain can also be obtained using the technique of [3], based on the results of [33] counting the contributions of the different levels in the fractal structure, and on results on the heat kernel on fractals, [16–18].

5.2. Dirac Operator Decomposition

We consider here a Robertson–Walker geometry on a spacetime of the form $\mathbb{R} \times \mathcal{P}$, where the spatial sections, instead of being a single 3-sphere, form an Apollonian packing of 3-spheres, as in a Packed Swiss Cheese Cosmology.

More precisely, here $\mathcal{P}$ is a packing of 3-dimensional spheres with radii $\{a_{n,k} : n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}\}$, where, in an iterative construction of the packing, at each stage $n \in \mathbb{N}$, a number of spheres equal to $6 \cdot 5^{n-1}$ are added to the packing. We denote these spheres by $S_{a_{n,k}}$, see [3,27] for a detailed description of the Apollonian packings of higher-dimensional spheres and their iterative construction.

In a Robertson–Walker metric on a spacetime with an Apollonian packing of spheres in the spatial sections, we can assume that each sphere in the packing inflates at time t with the same rate $a(t)$. We consider two possible rescalings of the Robertson–Walker metric by effect of the scaling radii $a_{n,k}$ of the packing.

1. The first choice is to rescale the whole 4-dimensional spacetime, that is, to consider a metric of the form

$$\mathrm{d}s_{n,k}^2 = a_{n,k}^2 \,(\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\sigma^2), \qquad n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}. \tag{5.1}$$

2. The second choice is to rescale only the spatial sections, that is, to consider a metric of the form

$$\mathrm{d}s_{n,k}^2 = \mathrm{d}t^2 + a(t)^2\, a_{n,k}^2\,\mathrm{d}\sigma^2, \qquad n \in \mathbb{N}, k = 1, \ldots, 6 \cdot 5^{n-1}. \tag{5.2}$$

In both cases, we write $D_{n,k}$ for the resulting Dirac operators on $\mathbb{R} \times S_{a_{n,k}}$ for metric (5.1) or (5.2).

We then encode the geometry of the inflating sphere packing $\mathcal{P}$ in the Dirac operator of a spectral triple associated with the entire (fractal) space $\mathbb{R} \times \mathcal{P}$. The main advantage of the spectral triples formalism of non-commutative geometry, [12], is the fact that it makes it possible to adapt fundamental properties of Riemannian geometry to spaces that are not smooth manifolds, including fractals.

In our case, we follow the construction of spectral triples on fractals obtained in [10,11]. For a space of the form $\mathbb{R} \times \mathcal{P}$, with $\mathcal{P}$ an Apollonian packing of 3-spheres, the spectral triple we consider is as in [3], with $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ an involutive subalgebra of $C_0(\mathbb{R} \times \mathcal{P})$ of functions f with bounded commutator $[D, \pi(f)]$, where π is the representation of the algebra as multiplication operators on the Hilbert space $\mathcal{H} = \oplus_{n,k} \mathcal{H}_{n,k}$ with $\mathcal{H}_{n,k} = L^2(S_{a_{n,k}}, \mathbb{S})$ the spinor spaces of the individual spheres in the packing and with Dirac operator $D = D_{\mathbb{R} \times \mathcal{P}}$ of the form

$$D_{\mathbb{R} \times \mathcal{P}} := \bigoplus_{n \in \mathbb{N}} \bigoplus_{k=1}^{6 \cdot 5^{n-1}} D_{n,k}, \tag{5.3}$$

with the $D_{a_{n,k}}$ the Dirac operators on the individual spaces $\mathbb{R} \times S_{a_{n,k}}$ with the Robertson–Walker metric (5.2).

We will discuss both choices (5.1) and (5.2) in Sect. 6.

Note that, depending on the form of the scaling factor $a(t)$ of the Robertson–Walker metric, the heat kernel and spectral action of the Dirac operator $D_{\mathbb{R} \times \mathcal{P}}$ may need regularization. Regularization is not always necessary: cases when $a(t)$ consists of an expansion followed by a contraction can give rise to a compact spacetime (the round metric on S^4 is an example of such a Robertson–Walker metric). In the following, we will be computing the coefficients of the heat kernel expansion as functions $a_{2m}(t)$ of the time variable, that is, prior to integration in the $\mathbb{R}$ direction in the $\mathbb{R} \times \mathcal{P}$ spacetime. The cases of scaling factors causing divergences can be treated by regularizing the spectral action through a cutoff regularization of the $\mathbb{R}$ integration of the coefficients. Appropriate boundary conditions for a self-adjoint Dirac operator, in the case where such a cutoff is introduced, can be obtained following the procedure of [8,29]. A more detailed discussion of this type of regularization is left to future work.

5.3. Mellin Transform and Zeta Functions

We first recall the relation between the terms in asymptotic expansion of a function and the poles of its Mellin transform, [24]. Given a meromorphic function $\phi(z)$ with set of poles $\mathcal{S} \subset \mathbb{C}$ and its Laurent series expansion at a pole $z_0 \in \mathcal{S}$,

$$\phi(z) = \sum_{-N \leq k} c_k (z - z_0)^k,$$

the singular element $S(\phi, z_0)$ of ϕ at z_0 is the projection onto the polar part of the Laurent expansion at z_0,

$$S(\phi, z_0) := \sum_{-N \leq k \leq 0} c_k (z - z_0)^k.$$

The singular expansion of ϕ is the formal sum of all the singular elements of ϕ at all poles in $\mathcal{S}$,

$$S_\phi(z) := \sum_{z \in \mathcal{S}} S(\phi, z).$$

We write $\phi(z) \asymp S_\phi(z)$ to denote the singular expansion. For example, the singular expansion of the Gamma function is

$$\Gamma(z) \asymp \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{z + k}.$$

Then, the relation between the asymptotic expansion at $u \to 0$ of a function $f(u)$ and the singular expansion of its Mellin transform $\phi(z) = \mathcal{M}(f)(z)$ is as follows. The small time asymptotic expansion is of the form

$$f(u) \sim_{u \to 0^+} \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha, k_\alpha} u^\alpha \log(u)^{k_\alpha},$$

where the coefficients c_{α, k_α} are determined by the singular expansion of the Mellin transform,

$$\mathcal{M}(f)(z) \asymp S_{\mathcal{M}(f)}(z) = \sum_{\alpha \in \mathcal{S}, k_\alpha} c_{\alpha, k_\alpha} \frac{(-1)^{k_\alpha} k_\alpha!}{(s + \alpha)^{k_\alpha + 1}},$$

where the index k_α ranges over the terms in the singular element of $\phi(z) = \mathcal{M}(f)(z)$ at $z = \alpha$, up to the order of pole at α. A similar expression holds for the asymptotic expansion at $u \to \infty$, see [24] and the appendix to [47]. In the case where there are no logarithmic terms in the asymptotic expansion,

$$f(u) \sim_{u \to 0^+} \sum_{\alpha \in \mathcal{S}} c_\alpha u^\alpha,$$

the Mellin transform $\mathcal{M}(f)(z)$ has analytic continuation to a meromorphic function on $\mathbb{C}$ with simple poles at $z = -\alpha$ with residue c_α.

5.4. Packings of 4-Spheres

Before discussing the general structure of the spectral action on Packed Swiss Cheese Cosmologies based on Robertson–Walker metrics, which we will be discussing in Sect. 6, we begin by discussing explicitly a special case: choice (5.1) of the scaled metrics in the special setting where the underlying Robertson–Walker metric is a round 4-sphere. Thus, we are considering a packing of 4-spheres, where one scales the entire 4-sphere over each 3-sphere in the packing and not just the spatial directions. The resulting spacetimes then have a different scaling of the time coordinate in each sphere of the Apollonian packing. This case is much simpler than the general case, because one can directly see the corrections to the spectral action due to fractality simply by looking at the zeta function of the Dirac operator as in the cases considered in [3]. Thus, it serves as a good model case to identify the expected correction terms we will encounter in the general case.

Lemma 5.1. *In the case of a 4-dimensional sphere S^4_r of radius $r > 0$, the zeta function of the Dirac operator is given by*

$$\zeta_D(s) = \mathrm{Tr}(|D_{S^4_r}|^{-s}) = \sum_{\ell,\pm} \mathrm{m}_{\ell,\pm} |\lambda_{\ell,\pm}|^{-s} = \frac{4}{3} r^s \left(\zeta(s-3) - \zeta(s-1)\right),$$

with $\zeta(s)$ the Riemann zeta function.

Proof. The spectrum of the Dirac operator on a round $(D-1)$-dimensional sphere S^{D-1}_r of radius r is

$$\mathrm{Spec}(D_{S^{D-1}_r}) = \left\{ \lambda_{\ell,\pm} = \pm r^{-1}\left(\frac{D-1}{2} + \ell\right) \mid l \in \mathbb{Z}_+ \right\}, \tag{5.4}$$

with multiplicities

$$\mathrm{m}_{\ell,\pm} = 2^{[\frac{D-1}{2}]} \binom{\ell + D}{\ell}. \tag{5.5}$$

As shown in [7], this can also be obtained by regarding the unit 4-dimensional sphere as a Robertson–Walker metric with $a(t) = \sin(t)$. In the case of a 4-sphere of radius r, this gives

$$\zeta_D(s) = \mathrm{Tr}(|D_{S^4_r}|^{-s}) = \sum_{n\geq 0} \mu(n) \mathrm{Tr}(f(H_n))$$

with $f(x) = x^{-s/2}$ and where $H_n = H_n^+ \oplus H_n^-$ with

$$H_n^{\pm} = -r^{-2}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{(n+\frac{3}{2})^2}{a^2} \pm \frac{(n+\frac{3}{2})a'}{a^2}\right),$$

with multiplicity $\mu(n) = 2(n+1)(n+2)$ and with $a(t) = \sin(t)$ with $t \in [0,\pi]$. Here we are scaling both the factor $a(t)$ and the t direction by the same factor r, since the whole S^4 is rescaled. Indeed, one then has

$$\sum_{n\geq 0} \mu(n) \mathrm{Tr}(f(H_n)) = \sum_{n\geq 0} \mu(n) \sum_{k\geq n+2} (r^{-2}k^2)^{-s/2}$$

$$= 4r^s \sum_{k\geq n+2} \frac{k^2-k}{3} k^{-s} = \frac{4}{3} r^s (\zeta(s-3) - \zeta(s-1)).$$

□

The fractal string zeta function [32] of an Apollonian packing $\mathcal{P}$ of 3-spheres with sequence of radii $\mathcal{L} = \{a_{n,k}\}$ is given by the generating series

$$\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s\,. \tag{5.6}$$

The exponent of convergence $\sigma_{\mathcal{P}}$ of this series (the packing constant of the Apollonian packing) is an upper bound on the Hausdorff dimension measuring its fractality. In general, these fractal zeta functions do not necessarily have analytic continuation to meromorphic functions on $\mathbb{C}$. However, as shown in [32], there are a screen and a window, that is, a curve of the form $S(t) + it$ for a continuous function $S : \mathbb{R} \to (-\infty, \sigma_{\mathcal{P}}]$ and a region $\mathcal{W}$ to the right of this curve, within which analytic continuation exists.

Corollary 5.2. *For a packing $\mathcal{P}$ of 3-spheres with sequence of radii $\mathcal{L} = \{a_{n,k}\}$ and the collection of 4-spheres obtained from it by making each 3-sphere the equator inside a fixed hyperplane of a corresponding 4-sphere, we obtain a Dirac operator $\mathcal{D}_{\mathcal{P}}$ (of the general form discussed in Sect. 5.2) with zeta function $\zeta_{\mathcal{D}_{\mathcal{P}}}(s) = \zeta_{\mathcal{L}}(s)\zeta_{D_{S^4}}(s)$, given by the product of the fractal string zeta function of the sphere packing and the zeta function of the Dirac operator on the unit 4-sphere.*

Proof. This follows directly from the previous lemma with form (5.3) of the Dirac operator, which gives

$$\zeta_{\mathcal{D}_{\mathcal{P}}}(s) = \mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n,k} \frac{4}{3} a_{n,k}^s (\zeta(s-3) - \zeta(s-1)) = \zeta_{\mathcal{L}}(s)\zeta_{D_{S^4}}(s).$$

□

This shows that, for a spacetime geometry constructed in this way the leading terms in the spectral action expansion have the following form.

Lemma 5.3. *The leading terms in the expansion of the spectral action for the Dirac operator $\mathcal{D}_{\mathcal{P}}$ of the 4-sphere packings described above have the form*

$$\begin{aligned} \mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim\, & f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0) + f_2\Lambda^2 \frac{\zeta_{\mathcal{L}}(2)}{2} + f_4\Lambda^4 \frac{\zeta_{\mathcal{L}}(4)}{2} \\ & + \sum_{\sigma \in \mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma, \end{aligned} \tag{5.7}$$

where $\mathcal{S}(\mathcal{L})$ is the set of poles of the fractal string zeta function $\zeta_{\mathcal{L}}(s)$ of the sequence of radii of the packing and $\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$ are the corresponding residues, and the coefficients f_α are the momenta of the test function f,

$$f_\alpha = \int_0^\infty f(\nu)\nu^{\alpha-1}\mathrm{d}\nu.$$

Proof. The result follows using the Mellin transform relation between the zeta function of the Dirac operator and the heat kernel of the Dirac Laplacian

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}(\mathrm{e}^{-t\mathcal{D}_{\mathcal{P}}^2})\, t^{s/2-1} \mathrm{d}t \tag{5.8}$$

give the terms in the expansion of the spectral action of form (5.7), through the relation between heat kernel coefficients and residues of the zeta function via Mellin transform, as in Sect. 5.3. □

The terms $\zeta_{\mathcal{L}}(2)$ and $\zeta_{\mathcal{L}}(4)$ replace the radii r^2 and r^4 of the corresponding terms in the spectral action on a single sphere S^4_r of radius r of Lemma 5.1 with the zeta regularizations of the sums $\sum_{n,k} a^2_{n,k}$ and $\sum_{n,k} a^4_{n,k}$. Since the packing dimension of a 3-dimensional sphere packing is smaller than four but larger than three, by the estimate in [3], the sum $\zeta_{\mathcal{L}}(4) = \sum_{n,k} a^4_{n,k}$ is an actual convergent sum while $\zeta_{\mathcal{L}}(2)$ is a zeta-regularized value. However, for more general packings of 4-spheres, not obtained from a packing of 3-spheres, the packing dimension may be larger than 4; hence, $\zeta_{\mathcal{L}}(4)$ would also be a regularized value.

The multifractal nature of the sphere packing is reflected in the fact that the zeta function $\zeta_{\mathcal{L}}(s)$ has poles off the real line.

In [3], the packing $\mathcal{P}$ of 3-spheres was assumed to satisfy certain strong *analyticity* assumptions (listed in Sect. 3.3 of [3]), requiring that the fractal string zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a^s_{n,k}$ of the sequence of radii of the packing would have analytic continuation to a meromorphic function on a region of the complex plane that contains the nonnegative real axis; the analytic continuation would have only one pole on the nonnegative real axis, located at the packing dimension of $\mathcal{P}$; all the poles of $\zeta_{\mathcal{L}}(s)$ would be simple; and $\mathcal{L}$ would have a good approximation by a family $\mathcal{L}_n$ of self-similar fractal strings with the lattice property (see [32]) so that the complex poles of $\zeta_{\mathcal{L}}(s)$ are approximated by the poles of $\zeta_{\mathcal{L}_n}(s)$. More precisely, this approximation property means that, for all $\epsilon > 0$ there is an $n \in \mathbb{N}$ and $R = R(\epsilon, n) > 0$ such that within a vertical region of size at most R the complex poles of $\zeta_{\mathcal{L}}(s)$ are within distance at most ϵ from the poles of $\zeta_{\mathcal{L}_n}(s)$. Under these assumptions, the following result is obtained by applying directly the results of [3] to the packing of 4-spheres.

Proposition 5.4. *Under the analyticity assumption of* [3] *on the sphere packing, the term*

$$\sum_{\sigma \in \mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma$$

consists of a leading real term

$$\Lambda^\sigma \frac{4 f_\sigma}{3} (\zeta(\sigma - 3) - \zeta(\sigma - 1)) \mathcal{R}_\sigma$$

with $\sigma = \sigma(\mathcal{P}) \in \mathbb{R}_+$ *the packing dimension of* $\mathcal{P}$*, and an oscillatory term* $\mathcal{S}^{\mathrm{osc}}_{\mathcal{P}}(\Lambda)$ *involving the contribution of the poles of* $\zeta_{\mathcal{L}}(s)$ *off the real line. Its* *truncation* $\mathcal{S}^{\rm osc}_{\mathcal{P}}(\Lambda)_{\leq R}$ *given by counting only the poles in a strip of vertical width* R *satisfies*

$$\mathcal{S}^{\rm osc}_{\mathcal{P}}(\Lambda)_{\leq R} \sim \sum_{j=0}^{N_r} \Lambda^{\sigma_{n,j}} \phi_{\sigma_{n,j}}(\theta_n(\Lambda)),$$

where $s_{n,j} = \sigma_{n,j} + i(\alpha_{n,j} + \frac{2\pi m}{\log b_n})$, *for* $j = 0, \dots, N_n$ *are the poles of the approximating* $\zeta_{\mathcal{L}_n}(s)$ *in the same strip,* $\theta_n = \frac{\log \Lambda}{\log b_n}$ *and* $\phi_{\sigma_{n,j}}(\theta_n) = \sum_m f_{s_{n,j}} \mathrm{e}^{2\pi i m \theta_n}$ *with*

$$f_{s_{n,j}} = \frac{4 f_\sigma}{3} (\zeta(s_{n,j} - 3) - \zeta(s_{n,j} - 1)) \mathcal{R}_{s_{n,j}} \int_0^\infty f(u) u^{s_{n,j}-1} \mathrm{d}u.$$

In the following subsection, we look at a simpler and more explicit lower-dimensional example based on a special case of Apollonian circle packings, the Ford circles. This provides an example where the correction terms to the spectral action due to fractality can be computed completely explicitly, although the packing in this case does not satisfy the analyticity assumption since it does not have a good approximation by self-similar fractal strings with the lattice property.

5.5. Lower-Dimensional Example: Circle Packings

It is useful to consider first a simpler lower-dimensional example where, instead of a 4-dimensional spacetime $\mathbb{R} \times \mathcal{P}$, with $\mathcal{P}$ an Apollonian packing of 3-spheres, one considers the case of a 2-dimensional spacetime $\mathbb{R} \times \mathcal{C}$ where $\mathcal{C}$ is an Apollonian packing of circles. The reasons for considering this example, although it is not directly of physical relevance, are that it simplifies two important features of the more general case we will be analyzing in the following: the sequence of the radii of the packing can be described more explicitly, especially in the more interesting cases with number theoretic structure, [28]; moreover, the Dirac spectrum for the Dirac operator on the circle is simpler than the Dirac spectrum on higher-dimensional spheres.

5.5.1. Ford Circles. We consider here in particular the lower-dimensional example of a 2-dimensional spacetime $\mathbb{R} \times \mathcal{C}$ where the Apollonian circle packing $\mathcal{C}$ is given by the Ford circles. These are circles tangent to the real line at points $(k/n, 0)$ with centers at the points $(k/n, 1/(2n^2))$. The advantage of this case is that the sequence of radii is known and given by a simple explicit expression. This example will also be helpful in showing that the condition mentioned above on the existence of a good approximation of $\mathcal{L}$ and the poles of $\zeta_{\mathcal{L}}(s)$ by a family $\mathcal{L}_n$ of self-similar fractal strings and the poles of their zeta functions $\zeta_{\mathcal{L}_n}(s)$ is in fact a very delicate property and even very simple and apparently very regular examples of Apollonian packings need not satisfy it.

Lemma 5.5. *The fractal string zeta function of the Apollonian packing of Ford circles is given by*

$$\zeta_{\mathcal{L}}(s) = 2^{-s} \, \frac{\zeta(2s-1)}{\zeta(2s)}. \tag{5.9}$$

Proof. In the case of Ford circles, the number of circles of radius $r_n = (2n^2)^{-1}$ is equal to the number of integers $1 \leq k \leq n$ that are coprime to n, $\gcd\{k, n\} = 1$. This means that the multiplicity $m(r_n)$ is given by the value of the Euler totient function

$$m(r_n) = \varphi(n),$$

where the Euler totient function is equivalently given by

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

with product over the distinct prime numbers dividing n. Thus, the fractal string zeta function in the sense of [32] of the Apollonian packing of Ford circles is essentially the Dirichlet series generating function of the Euler totient function,

$$\zeta_{\mathcal{L}}(s) = \sum_{n\geq 1} \varphi(n)\,(2n^2)^{-s} = 2^{-s} \sum_{n\geq 1} \varphi(n)\, n^{-2s} = 2^{-s} \mathcal{D}_{\varphi}(2s). \tag{5.10}$$

The Dirichlet series generating function of the Euler totient function

$$\mathcal{D}_{\varphi}(s) = \sum_{n\geq 1} \frac{\varphi(n)}{n^s} \tag{5.11}$$

can be computed using the fact that for a prime power p^k the totient function satisfies $\varphi(p^k) = p^k - p^{k-1}$ so that

$$1 + \sum_{k} \varphi(p^k) p^{-sk} = \frac{1 - p^{-s}}{1 - p^{1-s}}$$

which then gives, using the Euler product formula,

$$\mathcal{D}_{\varphi}(s) = \frac{\zeta(s-1)}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta function. Thus, the zeta function of the Ford circles packing is given by (5.9).

Lemma 5.6. *The Ford circles packing does not satisfy the approximation condition by a family $\mathcal{L}_n$ of self-similar fractal strings with the lattice property.*

Proof. The set $\mathcal{S}(\mathcal{L})$ of poles of $\zeta_{\mathcal{L}}(s)$ consists of three subsets $\mathcal{S}(\mathcal{L}) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ where $\mathcal{S}_1 = \{s = 1\}$, the point where the function $\zeta(2s-1)$ has a pole, $\mathcal{S}_2 = \{s = -k \,:\, k \in \mathbb{N}\}$, the points that are trivial zeros of $\zeta(2s)$, and $\mathcal{S}_3 = \{\rho \in \mathbb{C} \smallsetminus \mathbb{R}_- \,:\, \zeta(2\rho) = 0\}$ consisting of all the non-trivial zeros of the Riemann zeta function $\zeta(2s)$. Assuming that the Riemann hypothesis holds, the poles of $\zeta_{\mathcal{L}}(s)$ in $\mathcal{S}_3$ are all on a single vertical line of real part $1/4$. Thus, in the case of the Apollonian packing given by Ford circles, the question of whether poles of $\zeta_{\mathcal{L}}(s)$ that lie off the real line have a good approximation by self-similar fractal strings, is in fact the question of whether the non-trivial Riemann zeros admit such an approximation. It is known (Theorem 11.1 of [32]) that the non-trivial Riemann zeros do not contain any infinite arithmetic progression. The possibility of a finite arithmetic progression within a certain vertical strip is limited (Theorem 11.5 of [32]) by an estimate of the following form: if $\zeta(a+inb)=0$ for some $a\in(0,1)$ and $b>0$ and for all integers with $0<|n|<\Lambda$ then

$$\Lambda < 60\log b\left(\frac{b}{2\pi}\right)^{\frac{1}{a}-1}$$

and $\Lambda<13b$ when $a=1/2$, see Theorem 11.5 of [32]. Also, the possibility of having an infinite sequence of non-trivial Riemann zeros approximated by an arithmetic progression, namely having $\zeta(a+inb)\to 0$ as $|n|\to\infty$ for some $a\in(0,1)$ and $b>0$, is ruled out (Theorem 11.16 of [32]). Thus, the explicit example of Ford circles provides a simple case where one can see that the approximation condition by self-similar fractal strings is very difficult to satisfy even for very regular packings. □

Nonetheless, in the case of the Ford circles, one can explicitly see the corrections to the spectral action due to the fractality of the Apollonian packing.

Since our main focus here is on 4-dimensional spacetime geometries, we can also construct a 4-dimensional example using the 1-dimensional Apollonian circle packing by Ford circles, by increasing dimension to a collection of 2-spheres with the Ford circles as equators in a given hyperplane and then considering these 2-spheres as equators of a collection of 3-spheres and similarly pass to a collection of 4-spheres.

Proposition 5.7. *For a packing of* 4*-spheres obtained from the Ford packing of circles as above, the leading terms of the spectral action expansion are of the form*

$$\begin{aligned}\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim\; & \frac{11}{140}\,f(0)+\frac{f_1}{\pi^2}\Lambda+\frac{45\,\zeta(3)}{4\pi^4}\,f_2\Lambda^2+\frac{4725\,\zeta(7)}{16\pi^8}\,f_4\Lambda^4\\ &+\sum_{k\in\mathbb{N}}\frac{2^{k+1}f_{-k}}{3}\,\frac{\zeta(-k-3)-\zeta(-k-1)}{\zeta(-2k-1)}\Lambda^{-k}\\ &+\sum_{\sigma=a+ib}\frac{2^{-a}\cos(b\log 2)}{3}\Re(Z_\sigma)\,r(f)_\sigma\,\cos(b\log\Lambda)\Lambda^{a},\end{aligned}$$

where σ ranges over the non-trivial zeros of $\zeta(2s)$, with

$$r(f)_\sigma=\int_0^\infty f(u)u^{a-1}\cos(bu)\,\mathrm{d}u$$

and

$$Z_\sigma=(\zeta(\sigma-3)-\zeta(\sigma-1))\zeta(2\sigma-1).$$

Proof. The packing is given by a collection of 4-spheres of radii the $r_n=(2n^2)^{-1}$ as the Ford circles and with the same multiplicities given by the Euler totient function $m(r_n)=\varphi(n)$. Thus, we can apply the expansion obtained previously for an arrangement of 4-spheres with a given sequence of radii and multiplicities, and we obtain

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda))\sim f(0)\zeta_{\mathcal{D}_{\mathcal{P}}}(0)+f_2\Lambda^2\frac{\zeta_{\mathcal{L}}(2)}{2}+f_4\Lambda^4\frac{\zeta_{\mathcal{L}}(4)}{2}$$

$$+ \sum_{\sigma\in\mathcal{S}(\mathcal{L})} f_\sigma \Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2} \mathcal{R}_\sigma,$$

where here we have

$$\begin{aligned} f(0)\zeta_{\mathcal{D}_\mathcal{P}}(0) &= f(0)\zeta_{D_{S^4}}(0)\,\zeta_\mathcal{L}(0) \\ &= \frac{4f(0)}{3}\left(\zeta(-3)-\zeta(-1)\right)\frac{\zeta(-1)}{\zeta(0)} = \frac{11}{140}\, f(0), \end{aligned}$$

with $\zeta(0) = -1/2$, $\zeta(-1) = -1/12$, and $\zeta(-3) = 1/120$, and

$$\begin{aligned} f_2\Lambda^2 \frac{\zeta_\mathcal{L}(2)}{2} &= \frac{45\,\zeta(3)}{4\pi^4}\, f_2\Lambda^2, \\ f_4\Lambda^4 \frac{\zeta_\mathcal{L}(4)}{2} &= \frac{4725\,\zeta(7)}{16\pi^8}\, f_4\Lambda^4. \end{aligned}$$

The sequence of additional terms corresponding to poles of the zeta function $\zeta_\mathcal{L}(s)$ can be subdivided into the contributions of the three sets $\mathcal{S}_i$ described above so that we have a term coming from $\mathcal{S}_1 = \{\sigma = 1\}$ with

$$f_1\Lambda \frac{\zeta_{D_{S^4}}(1)}{2}\mathcal{R}_1 = \frac{1}{\pi^2}\, f_1\Lambda,$$

then a series of contributions from the poles in $\mathcal{S}_2$, at the even negative integers,

$$f_{-k}\Lambda^{-k}\frac{\zeta_{D_{S^4}}(-k)}{2}\mathcal{R}_{-k} = \frac{2^{k+1}}{3}\frac{\zeta(-k-3)-\zeta(-k-1)}{\zeta(-2k-1)} f_{-k}\Lambda^{-k},$$

and a series of contributions from the set $\mathcal{S}_3$ that involves the non-trivial zeros of the Riemann zeta function, of the form

$$f_\sigma\Lambda^\sigma \frac{\zeta_{D_{S^4}}(\sigma)}{2}\mathcal{R}_\sigma = f_\sigma\Lambda^\sigma\frac{2^{-\sigma+1}}{3}(\zeta(\sigma-3)-\zeta(\sigma-1))\zeta(2\sigma-1),$$

at points $\sigma\in\mathbb{C}\smallsetminus\mathbb{R}_-$ with $\zeta(2\sigma) = 0$. We can write these equivalently as a series of terms

$$\frac{2^{-a}\cos(b\log 2)}{3}\Re(Z_\sigma)\, r(f)_\sigma\, \Lambda^a\cos(b\log\Lambda),$$

where $\sigma = a + ib$ (with $a = 1/4$ under the assumption that the Riemann hypothesis holds) and with $r(f)_\sigma = \int_0^\infty f(u)u^{a-1}\cos(bu)\,\mathrm{d}u$ and with $Z_\sigma = (\zeta(\sigma-3)-\zeta(\sigma-1))\zeta(2\sigma-1)$. □

Thus, assuming the Riemann hypothesis holds, the correction terms due to the presence of fractality introduce a term of order Λ in the energy scale and a term of order $\Lambda^{1/4}$, where the latter occurs together with a series of log-periodic terms $\cos(b\log\Lambda)$ with b the imaginary parts of the non-trivial Riemann zeros.

Remark 5.8. Observe also that the usual cosmological term and Einstein–Hilbert term now no longer have rational coefficients as in the case of the ordinary Robertson–Walker metrics with a single S^3 as spatial section. An effect of the fractality introduced by the sphere packing is the zeta regularization of the sphere radius powers in these terms, which introduce non-rational coefficients like $\zeta(3)$, $\zeta(7)$, and powers of π. In relation to the results of [20,22], observe that the coefficients in this example are no longer rational numbers but are still periods of mixed Tate motives. One can ask the question, for more general sphere packings with associated zeta function $\zeta_{\mathcal{L}}(z)$, of whether the argument given in [22] can be modified to obtain a motivic description of the coefficients of the spectral action expansion and what conditions on the fractal string $\mathcal{L} = \{a_{n,k}\}$ of the packing radii will give rise to mixed Tate periods.

6. Feynman–Kac Formula on Sphere Packings

In this section, we consider the general case of a Packed Swiss Cheese Cosmology on $\mathbb{R} \times \mathcal{P}$, where $\mathcal{P}$ is an Apollonian packing of 3-spheres S^3, with radii sequence $\mathcal{L} = \{a_{n,k}\}$, where each $\mathbb{R} \times S^3_{n,k}$ is endowed with a scaled Robertson–Walker metric of either form (5.1) or (5.2), for a given underlying Robertson–Walker metric $\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2$. We use the full expansion of the heat kernel for the underlying Robertson–Walker metric, obtained in the previous sections using the Brownian bridge and the Feynman–Kac formula, and an analysis of the effect of the scaling by the radii $a_{n,k}$ to derive via a Mellin transform argument the full heat kernel expansion for the Dirac operator on the Packed Swiss Cheese Cosmology $\mathbb{R} \times \mathcal{P}$.

We first consider the case of $\mathbb{R} \times \mathcal{P}$ with the scaled Robertson–Walker metrics of the form $a^2_{n,k}(\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2)$, as in (5.1), which we refer to as the "round scaling" case. We compute the Feynman–Kac formula for the entire sphere packing using a Mellin transform with respect to the s variables of the heat kernel, together with the results on the asymptotic expansion for the underlying Robertson–Walker metric to obtain the full heat kernel expansion for the Packed Swiss Cheese Cosmology.

As we have seen in the simpler examples of the previous section, one finds two series of terms, one that corresponds to the expansion of the underlying Robertson–Walker metric, with zeta-regularized coefficients, and one additional series that corresponds to the poles of the zeta function $\zeta_{\mathcal{L}}(z)$ of the fractal string of the packing.

We then consider the case of $\mathbb{R} \times \mathcal{P}$ with the scaled Robertson–Walker metrics of the form $\mathrm{d}t^2 + a^2_{n,k} \cdot a(t)^2\,\mathrm{d}\sigma^2$ as in (5.2), or the "non-round scaling" case. We illustrate in this case a different argument based on the Mellin transform of the function $f_s(x)$ with respect to the "multiplicity" variable x, and we interpret the integral $\int_{\mathbb{R}} f_s(x)\,\mathrm{d}x$ as a special value of a combination of Mellin transforms. This shows the occurrence of zeta-regularized sums over the radii in the resulting Feynman–Kac formula. We also explain how one obtains the contributions of the poles of the zeta function $\zeta_{\mathcal{L}}(z)$ to the asymptotic expansion of the spectral action in this case.

6.1. Zeta-Regularized Series and Mellin Transform

We consider the Packed Swiss Cheese Cosmology $\mathbb{R} \times \mathcal{P}$ with radii $\mathcal{L} = \{a_{n,k}\}$ and with the Robertson–Walker metrics of the form $a^2_{n,k}(\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2)$, as in (5.1). We present a method for computing the asymptotic expansion of the heat kernel based on the Mellin transform with respect to the variable τ of the heat kernel expansion $\exp(-\tau^2\mathcal{D}^2)$.

We consider a slightly more general form of the series considered in Sect. 4 of [47]. In particular, we consider the case of a function $f(\tau)$ with small time asymptotic expansion

$$f(\tau) \sim \sum_N c_N \tau^N \tag{6.1}$$

and we consider an associated series of the form

$$g_R(\tau) = \sum_n f(r_n \tau), \tag{6.2}$$

where $R = \{r_n\}$ is an assigned sequence of $r_n \in \mathbb{R}^*_+$ with the property that the zeta function $\zeta_R(z) = \sum_n r_n^{-z}$ converges for $\Re(z) > C$ for some $C > 0$ and has an analytic continuation to a meromorphic function in $\mathbb{C}$ for which $z = -N$ for $N \in \mathbb{N}$ are regular points. We also assume that $\zeta_R(z)$ has only simple poles and that the poles of ζ_R are regular values of the Mellin transform $\mathcal{M}(f)(z)$.

Proposition 6.1. *Let $R = \{r_n\}$ be a sequence as above with $f(\tau)$ and $g_R(\tau)$ as in (6.1) and (6.2). Then, the small time asymptotic expansion of $g_R(\tau)$ is given by*

$$g_R(\tau) \sim_{\tau\to 0^+} \sum_N c_N\, \zeta_R(-N)\, \tau^N + \sum_{\sigma\in\mathcal{S}(\zeta_R)} \mathcal{R}_{R,\sigma}\, \mathcal{M}(f)(\sigma)\, \tau^{-\sigma}, \tag{6.3}$$

where $\mathcal{S}(\zeta_R)$ is the set of poles of $\zeta_R(z)$ with residues $\mathcal{R}_{R,\sigma} := \mathrm{Res}_{z=\sigma}\zeta_R(z)$.

Proof. We can formally write for the series g_R the expansion

$$g_R(\tau) \sim \sum_{N,n} c_N r_n^N \tau^N = \sum_N \zeta_R(-N)\tau^N. \tag{6.4}$$

One observes then, as in [47], that if $\mathcal{M}(f)(z)$ is the Mellin transform

$$\mathcal{M}(f)(z) = \int_0^\infty f(\tau)\tau^{z-1}\mathrm{d}\tau$$

then the Mellin transform $\mathcal{M}(g)(z)$ is given by

$$\mathcal{M}(g)(z) = \zeta_R(z)\cdot\mathcal{M}(f)(z). \tag{6.5}$$

This means that we can obtain the asymptotic expansion of $g_R(\tau)$ when $\tau \to 0^+$ by analyzing the singular expansion of the Mellin transform $\mathcal{M}(g_R)(z)$, as recalled in Sect. 5.3. In this case, we obtain the singular expansion

$$S_{\mathcal{M}(g_R)}(z) = \sum_{\sigma\in\mathcal{S}(\zeta_R)} \frac{\mathcal{R}_{R,\sigma}\, \mathcal{M}(f)(\sigma)}{z-\sigma} + \sum_{\sigma\in\mathcal{S}(\mathcal{M}(f))} \frac{\zeta_R(\sigma)c_\sigma}{z-\sigma},$$

where $\mathcal{S}(\zeta_R)$ is the set of poles of $\zeta_R(z)$, which are assumed to be simple, with residue $\mathcal{R}_{R,\sigma}$, and where $\mathcal{S}(\mathcal{M}(f))$ is the set of poles of $\mathcal{M}(f)$ with singular expansion

$$S_{\mathcal{M}(f)}(z) = \sum_{\sigma\in\mathcal{S}(\mathcal{M}(f))} \frac{c_\sigma}{z-\sigma}.$$

Since we are assuming that $f(\tau)$ has small time asymptotic expansion (6.1), the relation to the singular expansion of the Mellin transform gives

$$S_{\mathcal{M}(f)}(z) = \sum_N \frac{c_N}{z+N},$$

hence we obtain the result. □

6.2. The Feynman–Kac Formula and Asymptotic Expansion

We use the method described in Proposition 6.1 together with the Feynman–Kac formula, to obtain the asymptotic expansion of the trace of the heat kernel for the square of the Dirac operator on the sphere packing $\mathcal{P}$ endowed with the Robertson–Walker metric as in (5.1).

Theorem 6.2. *Let $\mathcal{D} := \mathcal{D}_{\mathbb{R}\times\mathcal{P}} = \oplus_{n,k} D_{n,k}$ be the Dirac operator on a sphere packing $\mathcal{P}$, with sequence of radii $\mathcal{L} = \{a_{n,k}\}$, with the Robertson–Walker metrics (5.1). Assume that the zeta function $\zeta_{\mathcal{L}}(z)$ of the fractal string $\mathcal{L}$ has analytic continuation to a meromorphic function on $\mathbb{C}$ that is regular at the points $z \in \{M \in \mathbb{Z} : M \leq 4\}$ and only has simple poles. Then, the heat kernel expansion is given by*

$$\begin{aligned}
&\mathrm{Tr}(\exp(-\tau^2\mathcal{D}^2)) \\
&\quad\sim \sum_{M=0}^{\infty} \tau^{2M-4}\zeta_{\mathcal{L}}(-2M+4) \int \left(\frac{1}{2}C_{2M}^{(-3/2,0)} + \frac{1}{4}\left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}\right)\right) D[\alpha] \\
&\qquad + \sum_{\sigma\in\mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma)\cdot \mathrm{Res}_{z=\sigma}\zeta_{\mathcal{L}} \cdot \tau^{-\sigma},
\end{aligned} \tag{6.6}$$

where $\mathcal{S}_{\mathcal{L}}$ is the set of poles of $\zeta_{\mathcal{L}}$ and $\tilde{f}(z) = \mathcal{M}(f)(z)$ is the Mellin transform of the function $f(\tau) = \mathrm{Tr}(\exp(-\tau^2 D^2))$ with D the Dirac operator on $\mathbb{R}\times S^3$ with the Robertson–Walker metric $\mathrm{d}t^2 + a(t)^2\mathrm{d}\sigma^2$.

Proof. We consider again the Feynman–Kac formula, focusing first on a single sphere in the packing $\mathcal{P}$, with radius $a_{n,k}$, endowed with a Robertson–Walker metric of form (5.1). This means that we consider an operator of the form

$$H_{m,n,k} = -a_{n,k}^{-2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V_{m,n,k}(t) \tag{6.7}$$

where the potential $V_{m,n,k}$ is as in (6.20). The Feynman–Kac formula then reads as

$$\begin{aligned}
\mathrm{e}^{-\tau^2 H_{m,n,k}}(t,t) &= \mathrm{e}^{-\frac{\tau^2}{a_{n,k}^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_{n,k}^2 V_{m,n,k}\right)}(t,t) \\
&= \frac{a_{n,k}}{2\sqrt{\pi}\tau}\int \exp\left(-\tau^2\int_0^1 V_{m,n,k}\left(t+\sqrt{2}\frac{\tau}{a_{n,k}}\alpha(u)\right)\mathrm{d}u\right) D[\alpha].
\end{aligned}$$

Using the same Taylor expansion method described earlier, after replacing as above the sum

$$\sum_m \mu(m)\mathrm{e}^{-\tau^2 H_{m,n,k}}(t,t)$$

with multiplicities $\mu(m)$ with the integral

$$\int_{-\infty}^{\infty} f_{\tau,\,n,k}(x)\,\mathrm{d}x$$

where

$$f_{\tau,\,n,k}(x) = \left(x^2 - \frac{1}{4}\right)\mathrm{e}^{-x^2 a_{n,k}^{-2} U - x a_{n,k}^{-1} V},$$

with U and V as in (3.10) and (3.11), we obtain

$$\begin{aligned}
&\sum_m \mu(m)\mathrm{e}^{-\tau^2 H_{m,n,k}}(t,t)\\
&\quad = \int \frac{a_{n,k}}{\tau}\left(\frac{\mathrm{e}^{\frac{V^2}{4U}}}{4}(-a_{n,k}U^{-1/2} + 2a_{n,k}^3 U^{-3/2} + a_{n,k}^3 V^2 U^{-5/2})\right) D[\alpha]\\
&\quad = \int \frac{1}{\tau}\left(\frac{\mathrm{e}^{\frac{V^2}{4U}}}{4}(-a_{n,k}^2 U^{-1/2} + a_{n,k}^4(2U^{-3/2} + V^2 U^{-5/2}))\right) D[\alpha]
\end{aligned}$$

with the Taylor expansion

$$\mathrm{e}^{\frac{V^2}{4U}} U^r V^\ell = \tau^{2(r+\ell)} \sum_{M=0}^{\infty} a_{n,k}^{-M-2(r+\ell)}\, C_M^{(r,\ell)} \tau^M$$

with $C_M^{(r,\ell)}$ as in (3.17) and the resulting expansion as in (3.20), which after scaling appropriately by the factors $a_{n,k}$ becomes

$$\begin{aligned}
&\frac{1}{\tau}\left(\frac{\mathrm{e}^{\frac{V^2}{4U}}}{4}(-a_{n,k}^2 U^{-1/2} + a_{n,k}^4(2U^{-3/2} + V^2 U^{-5/2}))\right)\\
&\quad = \frac{1}{4}\sum_{M=0}^{\infty}\left(C_M^{(-5/2,2)} - C_M^{(-1/2,0)}\right)\zeta_{\mathcal{L}}(-M+2)\tau^{M-2}\\
&\qquad + \frac{1}{2}\sum_{M=0}^{\infty} C_M^{(-3/2,0)}\zeta_{\mathcal{L}}(-M+4)\tau^{M-4}.
\end{aligned}$$

Thus, we can write the Feynman–Kac formula for the whole $\mathcal{P}$

$$\begin{aligned}
&\sum_{n,k}\sum_m \mu(m)\mathrm{e}^{-\tau^2 H_{m,n,k}}(t,t)\\
&\quad = \sum_{M=0}^{\infty}\tau^{2M-4}\zeta_{\mathcal{L}}(-2M+4)\int\left(\frac{1}{2}C_{2M}^{(-3/2,0)} + \frac{1}{4}(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)})\right) D[\alpha],
\end{aligned}$$

with only the term $\frac{1}{2}C_0^{(-3/2,0)}$ when $M=0$, as in Theorem 3.2. This series should be interpreted in the sense discussed in Sect. 6.1, as a series

$$\begin{aligned}
&g_{\mathcal{L}}(\tau) = \sum_{n,k} f(a_{n,k}^{-1}\tau),\\
&\quad f(\tau) \sim \sum_M \tau^{2M-4}\int\left(\frac{1}{2}C_{2M}^{(-3/2,0)} + \frac{1}{4}(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)})\right) D[\alpha].
\end{aligned}$$

Thus, applying the method of Proposition 6.1, we find that $g_{\mathcal{L}}(\tau)$ has an asymptotic expansion for $\tau \to 0^+$ of form (6.6). □

Theorem 6.2 determines the full expansion of the spectral action on a multifractal Packed Swiss Cheese Cosmology $\mathbb{R}\times\mathcal{P}$ with the Robertson–Walker metrics (5.1).

Corollary 6.3. *Under the same hypotheses as Theorem 6.2, the full expansion of the spectral action on $\mathbb{R}\times\mathcal{P}$ with the Robertson–Walker metrics (5.1) is of the form*

$$\begin{aligned} &\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \\ &\quad\sim \sum_{M=0}^{\infty} \Lambda^{4-2M} f_{4-2M}\, \zeta_{\mathcal{L}}(-2M+4) \\ &\qquad \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4}(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)})\right) D[\alpha] \\ &\qquad + \sum_{\sigma\in\mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma)\cdot f_\sigma \cdot \mathrm{Res}_{z=\sigma}\zeta_{\mathcal{L}} \cdot \Lambda^{\sigma}. \end{aligned} \tag{6.8}$$

Proof. As in [5,6,42], we relate the coefficients of the spectral action expansion to the coefficients of the heat kernel as in (2.2) and (2.3), by computing the spectral action $\mathrm{Tr}(f(\mathcal{D}/\Lambda))$ with respect to a test function of the form $f(x) = \int_0^\infty \mathrm{e}^{-\tau^2 x^2} \mathrm{d}\mu(\tau)$ for some measure μ, with $\int_0^\infty \mathrm{d}\mu(\tau) = f(0)$. Assuming that the full expansion of the heat kernel is of the form

$$\mathrm{Tr}(\mathrm{e}^{-\tau^2\mathcal{D}^2}) \sim \sum_{\alpha} \tau^{2\alpha} c_{2\alpha},$$

we obtain from (2.2) and (2.3)

$$\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \sim \sum_{\alpha<0} f_{2\alpha} c_{2\alpha} \Lambda^{-2\alpha} + a_0\, f(0) + \sum_{\alpha>0} f_{2\alpha} c_{2\alpha} \Lambda^{-2\alpha},$$

where for $\alpha < 0$

$$f_{2\alpha} = \int_0^\infty f(v)\, v^{-2\alpha-1}\, \mathrm{d}v,$$

and when $\alpha = M > 0$

$$f_{2M} = \int_0^\infty \tau^{2M} \mathrm{d}\mu(\tau) = (-1)^M f^{(2M)}(0).$$

Thus, we obtain a series of the form

$$\begin{aligned} &\mathrm{Tr}(f(\mathcal{D}/\Lambda)) \\ &\quad\sim \sum_{M=0}^{\infty} \Lambda^{-2M+4} f_{-2M+4}\, \zeta_{\mathcal{L}}(-2M+4) \\ &\qquad \int \left(\frac{1}{2} C_{2M}^{(-3/2,0)} + \frac{1}{4}\left(C_{2M-2}^{(-5/2,2)} - C_{2M-2}^{(-1/2,0)}\right)\right) D[\alpha] \\ &\quad + \sum_{\sigma\in\mathcal{S}_{\mathcal{L}}} \tilde{f}(\sigma)\cdot f_\sigma \cdot \mathrm{Res}_{z=\sigma}\zeta_{\mathcal{L}} \cdot \Lambda^{\sigma}, \end{aligned}$$

where the Mellin transform relation (5.8) between heat kernel and zeta function

$$\mathrm{Tr}(|\mathcal{D}|^{-z}) = \frac{2}{\Gamma(z/2)} \int_0^\infty \mathrm{e}^{-\tau^2 \mathcal{D}^2} \tau^{z-1} \mathrm{d}\tau$$

gives

$$\tilde{f}(z) = \mathcal{M}(\mathrm{Tr}(\mathrm{e}^{-\tau^2 \mathcal{D}^2}))(z) = \frac{\Gamma(z/2)}{2}\, \zeta_{\mathcal{D}}(z).$$

□

6.3. Scaling Properties for Non-round Scaling

We now consider the effect of rescaling the Robertson–Walker metric $\mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2$ to metrics of the form $\mathrm{d}t^2 + a_{n,k}^2 \cdot a(t)^2\, \mathrm{d}\sigma^2$, as in (5.2).

In this case, it is technically considerably more difficult than in the round scaling case to obtain exact results based on the spectral action expansion of a single Robertson–Walker metric. Here we present first a brief heuristic argument based on scaling properties of the terms U and V of (3.10), (3.11), aimed at identifying the kind of spectral action expansion that one expects in this case, see (6.11). We then justify this expression using a Mellin transform argument, where we now take a Mellin transform with respect to the real variable x whose integral provides the count of the multiplicities of the eigenspaces in the decomposition of the Dirac operator. This gives us an expression for this integration in terms of values of the Kummer confluent hypergeometric function. The effect of scaling is then considered in terms of this Mellin transform expression. This confirms the form of the zeta-regularized coefficients identified in (6.11). The part of the argument that is more delicate in this case, with respect to the round metric, is identifying the form of the log-periodic terms coming from the poles off the real line of the fractal string zeta function of the packing. Rather than providing a full explicit computation, we only present in Sect. 6.7 an argument based again on properties of the Mellin transform that justifies the presence of these log-periodic terms in the expansion.

Lemma 6.4. *Let U and V be as in (3.10), (3.11), for a given Robertson–Walker metric of the form $\mathrm{d}s^2 = \mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2$ on $\mathbb{R} \times S^3$. For a rescaled metric of the form $ds_a^2 = dt^2 + a^2 \cdot a(t)^2\, \mathrm{d}\sigma^2$, with a constant scaling factor $a > 0$, as in (5.2), we have*

$$U \mapsto a^{-2}\, U, \qquad V \mapsto a^{-1}\, V\,. \tag{6.9}$$

Proof. In the first case, expressions (3.7), (3.10), (3.11) show that the U and V functions defined by (3.10) and (3.11) change like $U \mapsto a^{-2}\, U$ and $V \mapsto a^{-1}\, V$, under the rescaling $a(t) \mapsto a \cdot a(t)$ of the scaling factor. □

Remark 6.5. Compare this with the case of a rescaled metric of the form $\mathrm{d}s_a^2 = a \cdot (\mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2)$ as in (5.1), that we discussed in Sect. 6.1, where the operator $\mathrm{d}^2/\mathrm{d}t^2$ also scales by a^{-2}; hence, in the Feynman–Kac formula, one modifies the term

$$\exp\left(-s \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right) \mapsto \exp\left(-\frac{s}{a^2}\, \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right).$$

This has the effect of scaling the variable $s \mapsto a^{-2}s$, so that the variables U and V are replaced by new variables U', V' with

$$U'=a^{-4}\, s \int_0^1 A^2\left(t+\frac{\sqrt{2s}}{a}\,\alpha(v)\right) \mathrm{d}v \quad \text{and} \quad V' = a^{-3}\, s\int_0^1 A'\left(t+\frac{\sqrt{2s}}{a}\,\alpha(v)\right)\mathrm{d}v.$$

with the effect of the rescaling $\sqrt{2s}/a$ on the expansion and on the resulting heat kernel asymptotics shown in Sect. 6.1 .

Lemma 6.6. *The rescaling $U \mapsto a^{-2}U$ and $V \mapsto a^{-1}V$ gives the rescaled expression*

$$\frac{1}{4}\sum_{M=0}^{\infty}\left(a^3\, C_M^{(-5/2,2)} - a\, C_M^{(-1/2,0)}\right)\tau^{M-2} + \frac{1}{2}\sum_{M=0}^{\infty} a^3\, C_M^{(-3/2,0)}\,\tau^{M-4}. \tag{6.10}$$

Proof. By Lemma 6.4, in this case the scaling takes the form

$$\mathrm{e}^{V^2/4U}U^rV^m \mapsto a^{-2r-m}\,\mathrm{e}^{V^2/4U}U^rV^m,$$

which means that the coefficients $C_M^{r,m}$ are rescaled by a factor a^{-2r-m} and we obtain the rescaled expression (6.10). □

Clearly, expression (6.10) suggests that, in this case, we should expect an asymptotic expansion with zeta-regularized terms of the form

$$\frac{1}{4}\sum_{M=0}^{\infty}(\zeta_{\mathcal{L}}(3)\, C_M^{(-5/2,2)} - \zeta_{\mathcal{L}}(1)\, C_M^{(-1/2,0)})\,\tau^{M-2}+\frac{1}{2}\sum_{M=0}^{\infty}\zeta_{\mathcal{L}}(3)\, C_M^{(-3/2,0)}\,\tau^{M-4}, \tag{6.11}$$

with the zeta function $\zeta_{\mathcal{L}}(s) = \sum_{n,k} a_{n,k}^s$ of the sphere packing radii. This will be justified more precisely in the following subsections.

6.4. Mellin Transform and Hypergeometric Function

We provide an argument for the presence of the zeta-regularized terms (6.11) based on taking a Mellin transform with respect to the "multiplicity variable" x in the function $f_s(x)$ of (3.12).

With the notation $a^{(n)} = a(a+1)\cdots(a+n-1)$ and $a^{(0)} = 1$, the Kummer confluent hypergeometric function is defined by the series

$${}_1F_1(a,b,t) = \sum_{n=0}^{\infty}\frac{a^{(n)}t^n}{b^{(n)}n!}$$

and is a solution of the Kummer equation

$$t\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + (b-t)\frac{\mathrm{d}f}{\mathrm{d}t} - af = 0.$$

The function $f_s(x)$ of (3.12) has a Mellin transform that can be computed explicitly in terms of the Kummer confluent hypergeometric function.

Lemma 6.7. *The Mellin transform in the x-variable of the function*

$$f_{s,-}(x) := f_s(x) = \left(x^2 - \frac{1}{4}\right) \mathrm{e}^{-x^2U - xV}$$

is given in terms of the Kummer confluent hypergeometric function ${}_1F_1$ by the expression

$$\mathcal{M}\left(\left(x^2-\frac{1}{4}\right)\mathrm{e}^{-x^2U-xV}\right)(z) = \frac{1}{8}U^{-(z+3)/2}$$
$$\times\Bigg(U^{1/2}\,\Gamma\left(\frac{z}{2}\right)\left(-U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)+2z\,{}_1F_1\left(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right)$$
$$+V\,\Gamma\left(\frac{z+1}{2}\right)\left(U\,{}_1F_1\left(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right)-2(z+1)\,{}_1F_1\left(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}\right)\Bigg).$$

Similarly, the Mellin transform in the x-variable of the function

$$f_{s,+}(x) := \left(x^2 - \frac{1}{4}\right) \mathrm{e}^{-x^2U + xV}$$

is also given in terms of the Kummer confluent hypergeometric function as

$$\mathcal{M}\left(\left(x^2-\frac{1}{4}\right)\mathrm{e}^{-x^2U+xV}\right)(z) = \frac{1}{8}U^{-(z+3)/2}$$
$$\times\Bigg(U^{1/2}\,\Gamma(z/2)\left(-U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)+2z\,{}_1F_1\left(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right)$$
$$+V\,\Gamma\left(\frac{z+1}{2}\right)\left(-U\,{}_1F_1\left(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right)+2(z+1)\,{}_1F_1\left(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}\right)\Bigg).$$

As discussed earlier, the real variable x is a continuous variable replacing the discrete $m + \frac{3}{2}$ in the expression for the potential $V_m(t)$ of (3.1).

Lemma 6.8. *The multiplicity integral is a special value at $z = 1$*

$$\int_{-\infty}^{\infty} f_s(x)\,\mathrm{d}x$$
$$= \left(-\frac{1}{4}\,U^{-\left(1+\frac{z}{2}\right)}\,\Gamma\left(\frac{z}{2}\right)\left(U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)-2z\,{}_1F_1\left(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right)\right)|_{z=1}$$
$$= \mathrm{e}^{\frac{V^2}{4U}}\,\frac{\sqrt{\pi}}{4}(-U^{-1/2}+2U^{-3/2}+V^2U^{-5/2}). \tag{6.12}$$

Proof. We consider again integral (3.13), which we write in the form

$$\int_{-\infty}^{\infty} f_s(x)\,\mathrm{d}x = \int_0^{\infty} f_{s,-}(x)\,\mathrm{d}x + \int_0^{\infty} f_{s,+}(x)\,\mathrm{d}x,$$

where

$$f_{s,\pm}(x) = \left(x^2 - \frac{1}{4}\right)\mathrm{e}^{-x^2U \pm xV}.$$

In turn, we can write these integrals in terms of Mellin transforms as

$$\int_{-\infty}^{\infty} f_s(x)\,\mathrm{d}x = \mathcal{M}(f_{s,-})(z)|_{z=1} + \mathcal{M}(f_{s,+})(z)|_{z=1} \tag{6.13}$$

The Mellin transform on the right-hand side is given by

$$\mathcal{M}(f_{s,-})(z)+\mathcal{M}(f_{s,+})(z)=-\frac{1}{4}U^{-1-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right)\left(U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right.$$
$$\left.-2z\,{}_1F_1\left(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right), \tag{6.14}$$

and the evaluation at $z=1$ of this expression gives back the expression we used before

$$\left(-\frac{1}{4}U^{-(1+\frac{z}{2})}\,\Gamma\left(\frac{z}{2}\right)\left(U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)-2z\,{}_1F_1\left(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right)\right)|_{z=1}$$
$$=\mathrm{e}^{\frac{V^2}{4U}}\,\frac{\sqrt{\pi}}{4}(-U^{-1/2}+2U^{-3/2}+V^2U^{-5/2}). \tag{6.15}$$

□

6.5. Scaling and Hypergeometric Functions

To simplify some of the following expressions, we introduce the notation

$$H_\lambda(\tau,z):=U^{-z/2}\,\Gamma(z/2)\,{}_1F_1\left(\frac{z}{2},\lambda,\frac{V^2}{4U}\right),$$
$$H(\tau,z):=H_{1/2}(\tau,z)=U^{-z/2}\,\Gamma(z/2)\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right), \tag{6.16}$$

where as above, the variables s and τ are related by $s=\tau^2$. We also introduce, for later use, the notation

$$H_{\mathcal{L}}(\tau,z):=U^{-z/2}\,\zeta_{\mathcal{L}}(z)\,\Gamma(z/2)\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)=\zeta_{\mathcal{L}}(z)\,H(\tau,z). \tag{6.17}$$

Corollary 6.9. *Consider the scaling of the S^3 spatial sections by a factor $a_{n,k}$ taken from the series $\mathcal{L}=\{a_{n,k}\}$ of radii of a given sphere packing, as in* (5.2)*.*

$$\mathcal{M}(f_{s,n,k})(z)=\frac{1}{8}a_{n,k}^{z}\left(H_{\frac{3}{2}}(\tau,z+1)\,V-H_{\frac{1}{2}}(\tau,z)\right)$$
$$-\frac{1}{2}a_{n,k}^{z+2}\left(H_{\frac{3}{2}}(\tau,z+3)\,V-H_{\frac{1}{2}}(\tau,z+2)\right). \tag{6.18}$$

Proof. Scaling the Robertson–Walker metric by $a_{n,k}$ as in (5.2), we find that the Mellin transform of $f_s(x)$ of (3.12) satisfies

$$\mathcal{M}(f_s)(z)\mapsto\mathcal{M}(f_{s,n,k})(z):=\frac{1}{8}a_{n,k}^{(z+3)}\,U^{-(z+3)/2}$$
$$\times\left(a_{n,k}^{-1}\,U^{1/2}\,\Gamma\left(\frac{z}{2}\right)\left(-a_{n,k}^{-2}\,U\,{}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)+2z\,{}_1F_1\left(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right)\right.$$
$$+a_{n,k}^{-1}\,V\,\Gamma\left(\frac{z+1}{2}\right)\left(a_{n,k}^{-2}\,U\,{}_1F_1\left(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right.$$
$$\left.\left.-2(z+1)\,{}_1F_1\left(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right)\right)$$
$$=\frac{1}{8}a_{n,k}^{z}\left(U^{-(z+1)/2}\,V\,\Gamma\left(\frac{z+1}{2}\right)\,{}_1F_1\left(\frac{z+1}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right)$$

$$
\begin{aligned}
&-U^{-z/2}\Gamma\left(\frac{z}{2}\right){}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\\
&+\frac{1}{4}a_{n,k}^{z+2}\left(U^{-(\frac{z}{2}+1)}\Gamma\left(\frac{z}{2}\right)\, z\,{}_1F_1\left(\frac{z+2}{2},\frac{1}{2},\frac{V^2}{4U}\right)\right.\\
&\left.-U^{-\frac{z+3}{2}}\,V\,\Gamma\left(\frac{z+1}{2}\right)(z+1){}_1F_1\left(\frac{z+3}{2},\frac{3}{2},\frac{V^2}{4U}\right)\right).
\end{aligned}
$$

We then write the above as (6.18), where we used $\Gamma(z/2)z = 2\Gamma((z+2)/2)$ and $\Gamma((z+1)/2)(z+1) = 2\Gamma(z+3)/2)$. □

Similarly, we obtain the scaling of the integral $\int_{-\infty}^{\infty} f_s(x)\,\mathrm{d}x$, viewed in terms of sums of Mellin transforms as above.

Corollary 6.10. *For a scaled metric of the form* $\boldsymbol{d}t^2 + a_{n,k}^2 a(t)^2 \mathrm{d}\sigma^2$ *as in* (5.2), *we have*

$$
\begin{aligned}
&\mathcal{M}(f_{s,n,k,-})(z) + \mathcal{M}(f_{s,n,k,+})(z)\\
&\quad= -\frac{1}{4}\,a_{n,k}^{z}\,U^{-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right){}_1F_1\left(\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\\
&\qquad+a_{n,k}^{z+2}U^{1-\frac{z}{2}}\,\Gamma\left(1+\frac{z}{2}\right){}_1F_1\left(1+\frac{z}{2},\frac{1}{2},\frac{V^2}{4U}\right)\\
&\quad-\frac{1}{4}a_{n,k}^{z}\,H(\tau,z) + a_{n,k}^{z+2}\,H(\tau,z+2).
\end{aligned}
\tag{6.19}
$$

6.6. Sphere Packing and Mellin Transform

We consider then the full sphere packing $\mathcal{P}$ with sequence of radii $\mathcal{L} = \{a_{n,k}\}$ and with the Robertson–Walker metrics of the form as in (5.2). The potential $V_m(t)$, $m \in \mathbb{N}$, of (3.1) is replaced by a sequence

$$
V_{m,n,k} = \frac{(m+\frac{3}{2})}{a_{n,k}^2\cdot a(t)^2}\left(\left(m+\frac{3}{2}\right) - a_{n,k}\cdot a'(t)\right).
\tag{6.20}
$$

Each potential in this sequence corresponds to a scaled choice $a_{n,k}^{-1}A(t)$ and $a_{n,k}^{-2}B(t)$ of the variables of (3.7), and corresponding scaled variables $a_{n,k}^{-2}\,U$ and $a_{n,k}^{-1}\,V$ in (3.10) and (3.11), as discussed above. We then consider a function $f_{\mathcal{P},s}$ associated to the full sphere packing $\mathcal{P}$ of the form

$$
f_{\mathcal{P},s}(x) = \left(x^2 - \frac{1}{4}\right)\sum_{n,k} \mathrm{e}^{-x^2 a_{n,k}^{-2}\,U - x a_{n,k}^{-1}\,V}.
\tag{6.21}
$$

As above, we write

$$
\int_{-\infty}^{\infty} f_{\mathcal{P},s}(x)\,\mathrm{d}x = \mathcal{M}(f_{\mathcal{P},s,-})(z)|_{z=1} + \mathcal{M}(f_{\mathcal{P},s,+})(z)|_{z=1},
$$

with $f_{\mathcal{P},s,\pm} = (x^2 - 1/4)\sum_{n,k}\exp(-x^2 a_{n,k}^{-2}\,U \pm x a_{n,k}^{-1}\,V)$.

Lemma 6.11. *For* $\mathbb{R}\times\mathcal{P}$ *with sequence of radii* $\mathcal{L} = \{a_{n,k}\}$ *and with the Robertson–Walker metrics of the form as in* (5.2), *the Mellin transform of*

$f_{\mathcal{P},s,-} + f_{\mathcal{P},s,+}$ *satisfies, with* $s = \tau^2$,

$$\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) = -\frac{1}{4} H_{\mathcal{L}}(\tau, z) + H_{\mathcal{L}}(\tau, z+2). \tag{6.22}$$

Proof. This follows directly from (6.12) since we have

$$\begin{aligned}\mathcal{M}(f_{\mathcal{P},s})(z) = \frac{1}{8} &\left(H_{\frac{3}{2}}(\tau, z+1)\, V - H_{\frac{1}{2}}(\tau, z)\right) \sum_{n,k} a_{n,k}^{z} \\ &-\frac{1}{2} \left(H_{\frac{3}{2}}(\tau, z+3)\, V - H_{\frac{1}{2}}(\tau, z+2)\right) \sum_{n,k} a_{n,k}^{z+2}. \\ = \frac{1}{8} &\left(H_{\frac{3}{2}}(\tau, z+1)\, V - H_{\frac{1}{2}}(\tau, z)\right) \zeta_{\mathcal{L}}(z) \\ &-\frac{1}{2} \left(H_{\frac{3}{2}}(\tau, z+3)\, V - H_{\frac{1}{2}}(\tau, z+2)\right) \zeta_{\mathcal{L}}(z+2),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{M}(f_{\mathcal{P},s,-})(z) + \mathcal{M}(f_{\mathcal{P},s,+})(z) \\ &\quad = -\frac{1}{4} \left(\sum_{n,k} a_{n,k}^{z}\right) H(\tau, z) + \left(\sum_{n,k} a_{n,k}^{z+2}\right) H(\tau, z+2),\end{aligned}$$

which gives (6.22). □

This shows that, indeed, we obtain the zeta-regularized coefficients $\zeta_{\mathcal{L}}(3)$ and $\zeta_{\mathcal{L}}(1)$ as in (6.11), when we evaluate at $z = 1$ expression (6.22). This argument, however, does not suffice to identify all the modified terms in the asymptotic expansion of the spectral action due to the fractality of the sphere packing, as we expect also in this case to see contributions from the poles of the zeta function $\zeta_{\mathcal{L}}(z)$. We cannot apply the same argument used for the round scaling here, since the fact that the τ variable is not rescaled prevents us from applying the same argument of Sect. 6.1 based on Sect. 4 of [47]. However, we show in the next subsection that one can still extract the information on the contribution of the poles of $\zeta_{\mathcal{L}}(z)$ from a further analysis of these Mellin transforms.

6.7. Pole Contributions

The discussion above shows why one obtains zeta-regularized coefficients in the spectral action expansion in the case of the non-round scaling (5.2) of the Robertson–Walker metrics. However, it does not explain why in the asymptotic expansion for $\tau \to 0^+$ of the heat kernel one should also find contributions associated with the poles of the zeta function $\zeta_{\mathcal{L}}(z)$, as in the case of the round scaling discussed before. In fact, one can see that such terms will occur in this case too, when one applies the τ expansion (3.15)

$$U = \tau^2 \sum_{n=0}^{\infty} \frac{u_n}{n!}\, \tau^n, \quad V = \tau^2 \sum_{n=0}^{\infty} \frac{v_n}{n!}\, \tau^n$$

to the function $H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z)\Gamma(z/2)U^{-z/2}\,{}_1F_1(z/2, 1/2, V^2/4U)$ as in (6.17), equivalently written as

$$H_{\mathcal{L}}(\tau, z) = \zeta_{\mathcal{L}}(z)\Gamma(z/2)U^{-z/2} \sum_{n=0}^{\infty} \frac{(z/2)_n}{4^n\, n!\,(1/2)_n}\, V^{2n}U^{-n},$$

by expanding the confluent hypergeometric function, with $(a)_n = a(a+1)\cdots(a+n-1)$ denoting the rising factorial. One sees from (3.15) that the term $U^{-z/2}$ will contribute a term with τ^z times a power series in τ, while the confluent hypergeometric function will contribute power series in τ.

Rather than giving a complete computation, we simply explain here why the presence of the term with τ^z will generate the pole contributions, by illustrating the same phenomenon in a simplified case.

The product of the Mellin transforms $\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z)$ corresponds to the transform of the Mellin convolution

$$\mathcal{M}(f_1)(z) \cdot \mathcal{M}(f_2)(z) = \mathcal{M}(f_1 \star f_2)(z),$$
$$(f_1 \star f_2)(x) = \int_0^{\infty} f_1\left(\frac{x}{u}\right) f_2(u)\, \frac{\mathrm{d}u}{u}.$$

Moreover, Mellin transform can be applied to distributions [30,37], by considering the space of test functions $\mathcal{D}(\mathbb{R}_+)$ and the space $\mathcal{Q} = \mathcal{M}(\mathcal{D}(\mathbb{R}_+))$ of their Mellin transforms. Let $\mathcal{D}'_+$ and $\mathcal{Q}'$ denote the dual spaces. Denoting by $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ the duality pairing between $\mathcal{Q}'$ and $\mathcal{Q}$ and by $\langle\cdot,\cdot\rangle$ the duality pairing between $\mathcal{D}'_+$ and $\mathcal{D}(\mathbb{R}_+)$, for a distribution $\Lambda \in \mathcal{D}'_+$ one defines $\mathcal{M}(\Lambda)$ by the property that $\langle \mathcal{M}(\Lambda), \mathcal{M}(\phi)\rangle = \langle \Lambda, \phi\rangle$. Then, the Mellin transform of a distribution in $\mathcal{D}'_+$ belongs to the space $\mathcal{Q}'$, which is a space of analytic functions. In particular, the Mellin transform of a delta distribution is given by

$$\tau^{z-1} = \mathcal{M}(\delta(x-\tau)). \tag{6.23}$$

Similarly, we can write as Mellin transform of a distribution

$$\tau^z\, \zeta_{\mathcal{L}}(z) = \mathcal{M}\left(\sum_{n,k} \tau\, a_{n,k}\; \delta\left(x - \tau \cdot a_{n,k}\right)\right). \tag{6.24}$$

This is the distribution acting as

$$\left\langle \sum_{n,k} \tau\, a_{n,k}\; \delta(x - \tau \cdot a_{n,k}), \phi(x) \right\rangle = \sum_{n,k} \tau\, a_{n,k}\; \phi(\tau\, a_{n,k}).$$

We write this distribution as

$$\Lambda_{\mathcal{P},\tau} := \sum_{n,k} \tau\, a_{n,k}\; \delta(x - \tau \cdot a_{n,k}).$$

Consider then a given function $g(x)$. In particular, for our application we should think of the function $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2)\,{}_1F_1(z/2, 1/2, \gamma))$. The product of Mellin transforms is then

$$\mathcal{M}(\Lambda_{\mathcal{P},\tau})(z) \cdot \mathcal{M}(g)(z) = \mathcal{M}(\Lambda_{\mathcal{P},\tau} \star g)(z)$$

$$= \mathcal{M}\left(\sum_{n,k} \tau a_{n,k} \int_0^\infty \delta(u - \tau a_{n,k})\, g\left(\frac{x}{u}\right) \frac{\mathrm{d}u}{u}\right) = \sum_{n,k} \mathcal{M}\left(g\left(\frac{x}{\tau \cdot a_{n,k}}\right)\right).$$

We then let $h_z(\tau) := \mathcal{M}(g(\frac{x}{\tau})$, and we write the above as

$$L_z(\tau) := \sum_{n,k} h_z(\tau \cdot a_{n,k}).$$

One can then obtain the asymptotic expansion for this function by using the same technique that we used in the case of the round scaling, by considering now the variable z fixed (it will be evaluated at $z = 1$ in the end) and taking a Mellin transform with respect to the variable τ. To avoid confusing notation, we write $\mathcal{M}_\tau$ for the Mellin transform taken with respect to the variable τ, and we write this Mellin transform as a function of a complex variable β. Arguing as in the round scaling case, we have

$$\mathcal{M}_\tau(L_z(\tau))(\beta) = \zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta).$$

It then follows that the terms in the asymptotic expansion for $\tau \to 0$ of $L_z(\tau)$ are determined by the terms in the singular expansion of the Mellin transform $\zeta_{\mathcal{L}}(\beta) \cdot \mathcal{M}(h_z(\tau))(\beta)$. These contain a series of terms that correspond to the poles $\sigma \in \mathcal{S}_{\mathcal{L}}$ of the zeta function $\zeta_{\mathcal{L}}(\beta)$ with coefficient given by the product of the residue $\mathcal{R}_\sigma = \mathrm{Res}_{\beta=\sigma}\zeta_{\mathcal{L}}(\beta)$ and the value $\mathcal{M}(h_z(\tau))(\sigma)$ with a power $\tau^{-\sigma}$, as well as terms that correspond to the poles of $\mathcal{M}(h_z(\tau))(\beta)$ (which are terms of the asymptotic expansion of $h_z(\tau)$. When we apply this argument to $g_\gamma(x) := \mathcal{M}^{-1}(\Gamma(z/2)\, {}_1F_1(z/2, 1/2, \gamma))$, the resulting asymptotic expansion then needs to be modified by replacing $\gamma = V^2/4U$ and expanding U and V in powers of τ according to (3.15). This makes writing out in full the explicit computation lengthy, but it does not change the fundamental structure of the expansion, which will still have a series of terms arising from the poles of $\zeta_{\mathcal{L}}(\beta)$. Thus, even without carrying out a full explicit computation of all these terms, we then see that the structure of the asymptotic expansion of the heat kernel (hence of the spectral action) is similar to the case of the round scaling, with a series of terms generated by the poles of $\zeta_{\mathcal{L}}(\beta)$ and a series of terms coming from the asymptotic expansion of the underlying (unscaled) Robertson–Walker metric, appearing with zeta-regularized coefficients as in (6.11).

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