Black Hole State Counting in Loop Quantum Gravity: A Number-Theoretical Approach

Iván Agulló,[1,*] J. Fernando Barbero G.,[2,†] Jacobo Díaz-Polo,[3,‡]
Enrique Fernández-Borja,[1,3,§] and Eduardo J. S. Villaseñor[4,2,||]

[1]*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC, Universidad de Valencia, Burjassot-46100, Valencia, Spain*
[2]*Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain*
[3]*Departamento de Astronomía y Astrofísica, Universidad de Valencia, Burjassot-46100, Valencia, Spain*
[4]*Instituto Gregorio Millán, Grupo de Modelización y Simulación Numérica, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain*

(Received 12 March 2008; published 27 May 2008)

We give an efficient method, combining number-theoretic and combinatorial ideas, to *exactly* compute black hole entropy in the framework of loop quantum gravity. Along the way we provide a complete characterization of the relevant sector of the spectrum of the area operator, including degeneracies, and *explicitly* determine the number of solutions to the projection constraint. We use a computer implementation of the proposed algorithm to confirm and extend previous results on the detailed structure of the black hole degeneracy spectrum.

DOI: 10.1103/PhysRevLett.100.211301 PACS numbers: 04.70.Dy, 04.60.Pp

Any proposed quantum theory of gravity must account for the states responsible for black hole entropy. Within loop quantum gravity (LQG), entropy can be studied by using the isolated horizon framework [1]. The counting of states is reduced in this setting to a well-defined combinatorial problem. It gives rise, in the asymptotic limit, to the semiclassical Bekenstein-Hawking formula [2,3] corrected to the next relevant order by a term logarithmic in the area. A computer assisted study has been carried out for small black holes up to two hundred Planck areas ℓ_P^2 [4]. This has unearthed a very interesting behavior in their degeneracy spectrum, namely, an equidistant "band structure" with important physical consequences. The most relevant of them is the effective quantization of black hole entropy [4]. A qualitative understanding of the origin of this behavior has been obtained in [5]. However, no detailed theoretical description of this phenomenon has been available to date, owing to the incomplete characterization of the area spectrum on one hand, and the lack of exact manageable solutions for some combinatorial problems (involving the so called projection constraint) on the other.

In this Letter we present a satisfactory solution to both types of difficulties, giving a precise characterization of the area spectrum by relying on number-theoretic methods, and addressing the combinatorial problems related to the projection constraint. We do it for the original counting of states proposed in [1] and carried out in [2], and also for the one described in [3]. The method that we discuss in the following will allow us to have a full understanding of the different factors that come into play to reproduce the features previously observed in the black hole degeneracy spectrum. In addition, it can be efficiently used to perform *exact* entropy computations—extensible to large areas—that improve and confirm the results obtained by brute force methods in [4].

We start by characterizing the area eigenvalues and their degeneracies. In LQG the black hole area is given by an eigenvalue A of the area operator

$$A = 8\pi\gamma\ell_P^2 \sum_{I=1}^{N} \sqrt{j_I(j_I+1)}, \tag{1}$$

where γ denotes the Immirzi parameter. Notice that these do not give the full area spectrum, but for the case of isolated horizons relevant here we only need Eq. (1). The labels j_I are half-integers, $j_I \in \mathbb{N}/2$, associated to the edges of a given spin network state. They pierce the horizon at a finite set of N distinguishable points called punctures [1]. Horizon quantum states are further characterized by an additional label m_I. In the case where we have spherical symmetry, a *projection constraint*,

$$\sum_{I=1}^{N} m_I = 0, \tag{2}$$

must be satisfied by the m_I. There are two inequivalent proposals in the literature to account for the relevant microscopic configurations [2,3]. When taken as a purely combinatorial problem, they differ in the range of the label m_I. In the standard (DLM) counting performed in [2] one takes $m_I \in \{-j_I, j_I\}$, whereas the counting proposed in [3] (that we will refer to as the GM counting) assumes that m_I can take all the allowed values for a spin component $m_I \in \{-j_I, -j_I+1, \ldots, j_I-1, j_I\}$.

The first problem we address is the characterization of the numbers belonging to the spectrum of the area operator restricted to the vector subspace spanned by spin network states having no vertices nor edges lying on the black hole horizon. In the following when we talk about the area spectrum we in fact refer to this restriction. The first question that we want to consider is the following: Given

$A \in \mathbb{R}$, when does it belong to the spectrum of the area? In order to simplify the algebra and work with integer numbers we will write $j_I = k_I/2$ in the following, so that the area eigenvalues become

$$A = \sum_{I=1}^{N} \sqrt{(k_I + 1)^2 - 1} = \sum_{k=1}^{k_{\max}} n_k \sqrt{(k+1)^2 - 1}.$$

Here we have chosen units such that $4\pi\gamma\ell_P^2 = 1$, and the n_k (satisfying $n_1 + \cdots n_{k_{\max}} = N$) denote the number of punctures corresponding to edges carrying spin $k/2$. An elementary but useful comment is that we can always write $\sqrt{(k+1)^2 - 1}$ as the product of an integer and the square root of a square-free positive integer number (SRSFN) by using its prime factor decomposition. Hence, with our choice of units, only integer linear combinations of SRSFNs can appear in the area spectrum. The questions now are the following: First, given such a linear combination, when does it correspond to an eigenvalue of the area operator? If the answer is in the affirmative, then what are the permissible choices of k and n_k compatible with this value for the area?

In the following we will take advantage of the fact that SRSFNs are linearly independent over the rational numbers (and, hence, over the integers), i.e., $q_1\sqrt{p_1} + \cdots + q_r\sqrt{p_r} = 0$, with $q_i \in \mathbb{Q}$ and p_i different square-free integers, implies that $q_i = 0$ for every $i = 1, \ldots, r$. This can be easily checked for concrete choices of the p_i and can be proved in general (see, for instance, [6]). We can answer the two questions previously posed in the following way. Given an *integer* linear combination of SRSFNs $\sum_{i=1}^{r} q_i\sqrt{p_i}$, where $q_i \in \mathbb{N}$, we need to determine the values of the k and n_k, if any, that solve the equation

$$\sum_{k=1}^{k_{\max}} n_k \sqrt{(k+1)^2 - 1} = \sum_{i=1}^{r} q_i \sqrt{p_i}. \tag{3}$$

Each $\sqrt{(k+1)^2 - 1}$ can be written as an integer times a SRSFN so the left-hand side of (3) will also be a linear combination of a SRSFN with coefficients given by integer linear combinations of the unknowns n_k. As a preliminary step, let us find out—for a given square-free positive integer p_i—the values of k satisfying

$$\sqrt{(k+1)^2 - 1} = y\sqrt{p_i}, \tag{4}$$

for some positive integer y. This is equivalent to solving the Pell equation $x^2 - p_i y^2 = 1$ where the unknowns are $x := k + 1$ and y. Equation (4) admits an infinite number of solutions (k_m^i, y_m^i), where $m \in \mathbb{N}$ (see, for instance, [7]). These can be obtained from the fundamental one (k_1^i, y_1^i) corresponding to the minimum, nontrivial, value of both k_m^i and y_m^i. They are given by the formula

$$k_m^i + 1 + y_m^i\sqrt{p_i} = (k_1^i + 1 + y_1^i\sqrt{p_i})^m.$$

The fundamental solution can be obtained by using continued fractions [7]. Tables of the fundamental solution for the smallest p_i can be found in standard references on number theory. As we can see, both k_m^i and y_m^i grow exponentially in m. By solving the Pell equation for all the different p_i we can rewrite (3) as

$$\sum_{i=1}^{r} \sum_{m=1}^{\infty} n_{k_m^i} y_m^i \sqrt{p_i} = \sum_{i=1}^{r} q_i \sqrt{p_i}.$$

Using the linear independence of the $\sqrt{p_i}$, the previous equation can be split into r different equations of the type

$$\sum_{m=1}^{\infty} y_m^i n_{k_m^i} = q_i, \qquad i = 1, \ldots, r. \tag{5}$$

Several comments are in order now. First, these are diophantine linear equations in the unknowns $n_{k_m^i}$ with the solutions restricted to take non-negative values. They can be solved by standard algorithms (for example, the Fröbenius method or techniques based on the use of Smith canonical forms). These are implemented in commercial symbolic computing packages. Second, although we have extended the sum in (5) to infinity it is actually finite because the y_m^i grow with m without bound. Third, for different values of i the equations in (5) are written in terms of disjoint sets of unknowns. This means that they can be solved independently of each other—a very convenient fact when performing actual computations. Indeed, if $(k_{m_1}^{i_1}, y_{m_1}^{i_1})$ and $(k_{m_2}^{i_2}, y_{m_2}^{i_2})$ are solutions to the Pell equations associated to different square-free integers p_{i_1} and p_{i_2}, then $k_{m_1}^{i_1}$ and $k_{m_2}^{i_2}$ must be different. This can be easily proved by *reductio ad absurdum*.

It may happen that some of the equations in (5) admit no solutions. In this case $\sum_{i=1}^{r} q_i\sqrt{p_i}$ does not belong to the relevant part of the area spectrum. On the other hand, if these equations do admit solutions, the $\sum_{i=1}^{r} q_i\sqrt{p_i}$ belong to the spectrum of the area operator, the numbers k_m^i tell us the spins involved, and the $n_{k_m^i}$ count the number of times that the edges labeled by the spin $k_m^i/2$ pierce the horizon. A set of pairs $\{(k_m^i, n_{k_m^i})\}$ obtained from the solutions to Eqs. (3)–(5) will define what we call a *spin configuration*. The number of different quantum states associated to each of these is given by two degeneracy factors, namely, the one coming from reorderings of the k_I-labels over the distinguishable punctures (r degeneracy) and the other originating in all the different choices of m_I labels satisfying (2) (m degeneracy). The combinatorial factors associated to the r degeneracy are straightforward to obtain and appear in the relevant literature.

Let us consider then the m degeneracy. The problem that we have to solve is this: Given a set of (possibly equal) spin labels j_I, $I = 1, \ldots, N$, what are the different choices for the allowed m_I such that (2) is satisfied? Notice that an obvious necessary condition for the existence of solutions is that $\sum_{I=1}^{N} j_I \in \mathbb{N}$.

In the standard DLM approach the number of different solutions for the projection constraint can be found by solving the following combinatorial problem (closely related to the so called *partition problem*): Given a set $\mathcal{K} = \{k_1, \ldots, k_N\}$ of N—possibly equal—natural numbers, how many different partitions of $\mathcal{K}$ into two disjoint sets $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\sum_{k\in\mathcal{K}_1} k = \sum_{k\in\mathcal{K}_2} k$ do exist? The answer to this question can be found in the literature (see, for example, [8], and references therein) and is

$$\frac{2^N}{M}\sum_{s=0}^{M-1}\prod_{I=1}^{N}\cos(2\pi s k_I/M), \tag{6}$$

where $M = 1 + \sum_{I=1}^{N} k_I$. This expression can be seen to be zero if there are no solutions to the projection constraint.

Let us consider now the GM proposal. The problem is equivalent in this case to counting the number of irreducible representations, taking into account multiplicities, that appear in the tensor product $\bigotimes_{I=1}^{N}[j_I]$, where $[j_I] = [k_I/2]$ denotes the irreducible representation of $SU(2)$ corresponding to spin j_I. In order to solve this problem, we rely on techniques developed in the context of conformal field theories [9] (see also [10]) and in the spectral theory of Toeplitz matrices [11]. The starting point is to write the tensor product of two $SU(2)$ representations in the form

$$\left[\frac{k_1}{2}\right]\otimes\left[\frac{k_2}{2}\right] = \bigoplus_{k_3=0}^{\infty}\mathcal{N}_{k_1k_2}^{k_3}\left[\frac{k_3}{2}\right],$$

where the integers $\mathcal{N}_{k_1k_2}^{k_3}$, called *fusion numbers* [9], tell us the number of times that the representation labeled by $k_3/2$ appears in the tensor product of $[k_1/2]$ and $[k_2/2]$. For each $k\in\mathbb{N}\cup\{0\}$, we introduce now the infinite *fusion matrices* $(C_k)_{k_1k_2} := \mathcal{N}_{k_1k}^{k_2}$, where $k_1, k_2 \in \mathbb{N}\cup\{0\}$. These can be shown to satisfy the recursion relation

$$C_{k+2} = XC_{k+1} - C_k, \qquad k = 0, 1, \ldots, \tag{7}$$

where we have introduced the notation $X := C_1$. Explicitly $X_{k_1k_2} = \delta_{k_1,k_2-1} + \delta_{k_1,k_2+1}$, which shows that X is a Toeplitz matrix [11]. The solution to (7), with initial conditions $C_0 = I$ and $C_1 = X$, can be written as

$$C_k = U_k(X/2), \qquad k = 0, 1, \ldots$$

in terms of the Chebyshev polynomials of the second kind U_k. The tensor product of an arbitrary number of representations can be decomposed as a direct sum of irreducible representations by multiplying the fusion matrices introduced in the previous equations. By proceeding in this way we get

$$\left[\frac{k_1}{2}\right]\otimes\left[\frac{k_2}{2}\right]\otimes\cdots\otimes\left[\frac{k_N}{2}\right] = \bigoplus_{k=0}^{\infty}(C_{k_2}C_{k_3}\cdots C_{k_N})_{k_1k}\left[\frac{k}{2}\right].$$

Notice that the product of matrices appearing in the previous formula is, in fact, a polynomial in X. The total number of representations, which gives the solution to the combinatorial problem at hand, is simply given by

$$\sum_{k=0}^{\infty}(C_{k_2}C_{k_3}\cdots C_{k_N})_{k_1k}. \tag{8}$$

This is just the sum of the (finite number of non zero) elements in the k_1 row of the matrix $C_{k_2}C_{k_3}\cdots C_{k_N}$. A useful integral representation for this sum can be obtained by introducing a resolution of the identity for X as in [11] and the well-known identity $U_n(\cos\theta) = \sin[(n+1)\theta]/\sin\theta$. In fact, Eq. (8) can be equivalently written as

$$\frac{2}{\pi}\int_0^{\pi}d\theta\cos\frac{\theta}{2}\left[\cos\frac{\theta}{2} - \cos\left(K+\frac{3}{2}\right)\theta\right]\prod_{I=1}^{N}\frac{\sin(k_I+1)\theta}{\sin\theta}, \tag{9}$$

where $K = k_1 + \cdots + k_N$. This is related to the well-known Verlinde formula for $SU(2)$ [9].

The procedure to calculate the black hole spectrum described in the first part of this Letter can be efficiently implemented in a computer, for instance using MATHEMATICA. This allows us to analyze in detail the different factors that shape the degeneracy spectrum. First of all, the fact that the diophantine equations are decoupled allows us to obtain the configurations compatible with a given value of area $A = \sum_{i=1}^{r} q_i\sqrt{p_i}$ as the cartesian product of the sets of solutions to the diophantine equations for each p_i. Let us then begin by analyzing the results for area values of the form $A = q\sqrt{p}$, with $q\in\mathbb{N}$ and $\sqrt{p}$ a fixed SRSFN. What we see in this case is that the r degeneracy—coming from the reordering of puncture labels—will be maximized by those configurations having both a large number of different values of k and a large number of punctures. For a fixed area value these two factors compete with each other because higher values of k imply a lower number of punctures. On the other hand, the m degeneracy shows an exponential growth with area (both in the DLM and GM countings). When the two sources of degeneracy are taken into account—in the present case involving a single SRSFN—the total degeneracy can be seen to be dominated by the m degeneracy. The reason for this dominance of the m degeneracy is that the number of different (small) values of k available within the set of solutions to the Pell equation for a given p is limited, and hence only a few possibilities of reordering exist.

This situation is expected to change drastically when we consider areas $A = \sum_{i=1}^{r} q_i\sqrt{p_i}$, with $r > 1$, built as linear combinations of different SRSFNs. In this case it is possible to obtain configurations with a large number of different small values of k (associated to different SRSFNs). The effect of considering linear combinations involving several SRSFNs produces a very distinctive feature when the r degeneracy is plotted as a function of area; namely, it creates a band structure where high values of degeneracy alternate with much lower ones. Furthermore, maxima and

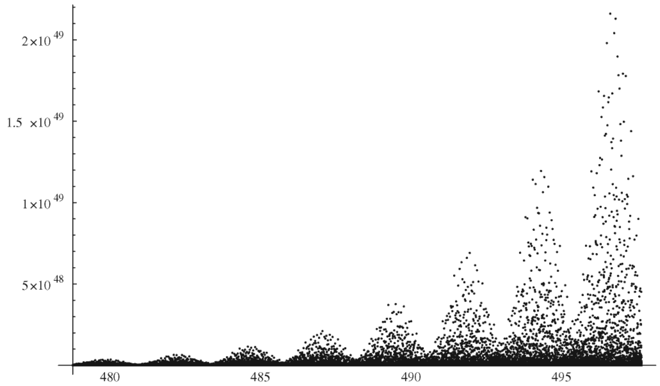

FIG. 1. Plot of the black hole degeneracy as a function of the area (expressed in units of ℓ_P^2 to facilitate the comparison with the results obtained in [4]).

minima are evenly spaced. When this behavior is considered together with the m degeneracy, we obtain the regular pattern shown in Fig. 1.

Several remarks are in order now. First, we want to point out that the result obtained from the explicit computational analysis carried out in [4] (by using the GM counting) is exactly recovered with the new approach. The fact that the same result is obtained from two completely independent procedures (a brute force approach and the algorithm proposed here) provides strong evidence for the reliability of both computations. Second, the structure of the degeneracy spectrum obtained by using the DLM and GM countings is basically equal. They differ only in the absolute values of the degeneracy, whereas the band structure (including the position and spacing of the bands) is the same. This can be understood in our framework because the terms accounting for the r degeneracy, responsible for this effect, coincide for both counting procedures. This justifies the appearance of the constant χ obtained in [4,5]. Third, once we understand how the r degeneracy works, we see that the area values for which the degeneracy is large are those that can be written as linear combinations of the SRSFNs originating from small solutions k to the corresponding Pell equation. Thus, considering these linear combinations will suffice to account for the band structure. The remaining area values give rise only to very low degeneracies.

Summarizing, we have been able to find a number-theoretic or combinatorial way to tackle the problem of calculating the degeneracy spectrum of spherical black holes in LQG. Our procedure has several advantages over previous approaches. First, we have been able to characterize the area spectrum in a proper way, giving an algorithm to explicitly find every single spin configuration contributing to each value of the area spectrum. In particular, the degeneracies of the area eigenvalues can be obtained. This has allowed us to reproduce and understand the band structure already observed in [4] for the black hole degeneracy spectrum in a much more efficient way. We not only recover previous results obtained by using a *brute force* algorithm, but easily extend them to area values significantly larger than those reached in [4] (see Fig. 1). Moreover, with our methods it is possible to compute the configurations and degeneracy even for much larger values of area. As a token we give the degeneracy for an area of $8320\sqrt{2} + 14\,400\sqrt{3} + 2240\sqrt{6} + 4640\sqrt{15} + 1120\sqrt{35}$, which is $3.464\,372\,965\,079\,75 \cdots \times 10^{24\,420}$. Finally, the concrete procedures and explicit formulas given in the letter offer a good starting point to study the asymptotic behavior of the entropy as a function of the area of a black hole. This could help us investigate whether the effective entropy quantization discussed here is present in macroscopic black holes.

We want to thank J. Baez, I. Garay, H. Sahlmann, and J. Salas for their comments. This work is supported by the Spanish MEC Grants No. FIS2005-05736-C03-02, No. FIS2005-05736-C03-03, No. ESP2005-07714-C03-01, and No. FIS2005-02761. I. A. and J. D.-P. acknowledge financial support provided by the Spanish Ministry of Science and Education under the FPU program.

*Ivan.Agullo@uv.es
†Corresponding author.
fbarbero@iem.cfmac.csic.es
‡Jacobo.Diaz@uv.es
§Enrique.Fernandez@uv.es
‖ejsanche@math.uc3m.es

[1] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998); A. Ashtekar, J. Baez, and K. Krasnov, Adv. Theor. Math. Phys. **4**, 1 (2000).
[2] M. Domagala and J. Lewandowski, Classical Quantum Gravity **21**, 5233 (2004); K. Meissner, Classical Quantum Gravity **21**, 5245 (2004).
[3] A. Ghosh and P. Mitra, Phys. Lett. B **616**, 114 (2005).
[4] A. Corichi, J. Diaz-Polo, and E. Fernandez-Borja, Phys. Rev. Lett. **98**, 181301 (2007); A. Corichi, J. Diaz-Polo, and E. Fernandez-Borja, Classical Quantum Gravity **24**, 243 (2007).
[5] I. Agullo, J. Diaz-Polo, and E. Fernandez-Borja, arXiv:0802.3188.
[6] D. J. Newman, W. F. Furr, and L. K. Willians, Am. Math. Mon. **66**, 321 (1959).
[7] D. M. Burton, *Elementary Number Theory* (McGraw-Hill, New York, 2002).
[8] H. De Raedt, K. Michielsen, K. De Raedt, and S. Miyashita, Phys. Lett. A **290**, 227 (2001).
[9] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory* (Springer, New York, 1997).
[10] R. Kaul and P. Majumdar, Phys. Rev. Lett. **84**, 5255 (2000).
[11] A. Böttcher and S. M. Grudsky, *Spectral Properties of Banded Toeplitz Matrices* (SIAM, Philadelphia, 2005).