Quantum Gowdy T^3 model: Schrödinger representation with unitary dynamics

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The linearly polarized Gowdy T^3 model is paradigmatic for studying technical and conceptual issues in the quest for a quantum theory of gravity since, after a suitable and almost complete gauge fixing, it becomes an exactly soluble midisuperspace model. Recently, a new quantization of the model, possessing desired features such as a unitary implementation of the gauge group and of the time evolution, has been put forward and proven to be essentially unique. An appropriate setting for making contact with other approaches to canonical quantum gravity is provided by the Schrödinger representation, where states are functionals on the configuration space of the theory. Here we construct this functional description, analyze the time evolution in this context and show that it is also unitary when restricted to physical states, i.e. states which are solutions to the remaining constraint of the theory.

I. INTRODUCTION

In the quest for a quantum theory of gravity, the use of simple models has proven to be very effective. The simplest possible models, where the most symmetries are imposed from the outset [1], have become important for the study of Planck scale modifications to the Big Bang scenario (see, e.g., Ref. [2]). However, these models suffer from an oversimplification since all inhomogeneous degrees of freedom are neglected. A natural question is how the inclusion of these inhomogeneous modes affects the qualitative picture near the singularity that the homogeneous models possess. In this regard, the linearly polarized Gowdy T^3 model is a natural candidate for a detailed study. It is the simplest inhomogeneous, spatially closed, cosmological model *in vacuo* [3]. One important reason for the appeal of such a model is that, after a convenient almost complete gauge fixing and the introduction of a geometrically motivated internal time, the model becomes soluble. Any solution of the full set of Einstein equations can be obtained from the solutions of an auxiliary scalar field in a fixed fiducial background. However, this auxiliary scalar field system is not unique. Different "field parametrizations" of the metric may give rise to different scalar field systems. Classically they are all equivalent, but in the quantum theory this may not be so. In addition, the quantization of field systems possesses an infinite degree of ambiguity, even if one restricts all considerations to standard quantizations, e.g., of the Fock type. As a consequence, there exist in principle infinitely many inequivalent quantizations of the Gowdy T^3 midisuperspace model.

Among the different possibilities available in this route to quantization, two field parametrizations have received special attention in past years. One of them can be considered a somewhat conventional field parametrization from the viewpoint of a dimensional reduction of the model [4]. However, this proposal for the choice of fundamental field has the undesirable property of not implementing the dynamics (generated by the internal notion of time) unitarily. Actually, although this lack of unitarity was first proven [5] for a "natural quantization" of the associated scalar field, introduced by Pierri [4], it has been recently shown that there exists no Fock quantization with a unitary dynamics, at least if one also demands an invariant unitary implementation of the gauge group that remains on the model after gauge fixing [6]. To solve this problem, a new field parametrization, together with an essentially unique quantum representation, was recently introduced. In this case, not only is the evolution unitary and the gauge group naturally implemented, but also it has been shown that any other Fock quantization of the new field with such properties is unitarily equivalent to the constructed one [6–9]. Furthermore, the adopted field parametrization turns out to be unique in a precise sense under the condition of the existence of a Fock representation (FR) with an invariant unitary action of the gauge group and a unitary dynamics [6]. These results were mainly formulated in the language of Fock space, which is natural from the perspective of a scalar field in a fixed background.

On the other hand, quantum gravity in its canonical formulation is commonly defined in the Schrödinger func-

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tional picture, where states are functionals on the configuration space of the theory. Therefore, it is important to have a Schrödinger functional description of any symmetry reduced model, such as the Gowdy T^3 model. The purpose of this paper is to present this description for the quantization which admits a unitary time evolution [7,8], and analyze the implementation of such a unitary evolution in this framework, both before and after imposing the remaining constraint of the theory.

We will adopt here the same viewpoint as in Refs. [7,8]: instead of working with a fixed quantum representation and considering the unitary implementability of the family of symplectic transformations defined by the evolution (together with the corresponding unitary evolution operator), we will construct the associated 1-parameter family of representations. Notice that this is precisely the family of representations which is obtained by "evolving in time" a fixed GNS state (Gelfand-Naimark-Segal state), and hence the complex structure defining the FR. The equivalence between the two viewpoints is then established by the fact that evolution between any two given times admits a unitary implementation if, and only if, the corresponding representations are unitarily equivalent.

Finally, we note that the 1-parameter family of complex structures that gives rise to the 1-parameter family of unitarily equivalent representations can be obtained both on the canonical phase space (the space of Cauchy data for the auxiliary scalar field) or on the covariant phase space (the space of solutions). Since we are interested in the canonical functional description, we will obtain the family of complex structures directly on the canonical phase space. As a particular consequence of unitarity, we will obtain a family of mutually equivalent Gaussian measures in the (quantum) configuration space.

The structure of the paper is the following. In Sec. II we recall the quantization of the linearly polarized Gowdy T^3 model constructed by Corichi, Cortez and Mena Marugán, in which the time evolution is implemented unitarily [7,8]. In Sec. III we construct the Schrödinger representation (SR) corresponding to this (unique) Fock quantization. In Sec. IV, we implement the canonical notion of time evolution within the Schrödinger description, showing explicitly the equivalence of the family of representations at different times. The conclusions are presented in Sec. V.

II. THE QUANTUM GOWDY MODEL

In this section we will review the quantization of the Gowdy T^3 cosmological model as performed in Refs. [7,8]. We will start with a description of the classical model and its dynamics.

A. The classical model

The linearly polarized Gowdy T^3 model describes globally hyperbolic four-dimensional vacuum spacetimes, with two commuting hypersurface orthogonal spacelike Killing

fields and compact spacelike hypersurfaces homeomorphic to a three-torus. In a coordinate system $\{(t, \theta, \nu, \delta), t \in \mathbb{R}^+; \theta, \nu, \delta \in S^1\}$ with $(\partial_\nu)^A$ and $(\partial_\delta)^A$ being the hypersurface orthogonal Killing fields, the line element can be expressed as

$$ds^2 = e^{\gamma - (\xi/\sqrt{p}t) - \xi^2/(4pt)}(-dt^2 + d\theta^2) + e^{-\xi/\sqrt{p}t}t^2 p^2 d\nu^2 + e^{\xi/\sqrt{p}t}d\delta^2 \tag{1}$$

after a gauge fixing procedure which removes all the gauge degrees of freedom except for a homogeneous one [8]. The spatially homogeneous variable p is a positive constant of motion. On the other hand, the fields ξ and γ depend only on the time coordinate t and the spatial coordinate θ. The field γ is completely determined by ξ, by $P := \ln p$ and by their respective momentum and configuration (canonically) conjugate variables, P_ξ and Q (see Ref. [6] for details). Therefore, all local degrees of freedom reside in the field ξ.

As we have mentioned, the model is just partially gauge fixed: there is still a global constraint,

$$\mathcal{C}_0 = \frac{1}{\sqrt{2\pi}} \oint d\theta P_\xi \xi' = 0, \tag{2}$$

which comes from the homogeneous part of the θ-momentum constraint. Here, the prime denotes the derivative with respect to θ.

After the reduction process, the Hamiltonian becomes[1]

$$H = \frac{1}{2} \oint d\theta \left(P_\xi^2 + (\xi')^2 + \frac{1}{4t^2}\xi^2 \right). \tag{3}$$

Note that, since the reduced Hamiltonian does not depend on the degrees of freedom Q and P, these are constants of motion, and will be obviated in our subsequent discussion.

Thus, the resulting system consists of a real scalar field ξ subject to the constraint (2). Its Hamiltonian (3) is that of a massless field with a quadratic time dependent potential $V(\xi) = \xi^2/(4t^2)$ propagating in a (fictitious) background $(\mathcal{M}^{(f)}, g_{AB})$, where $\mathcal{M}^{(f)} \simeq S^1 \times \mathbb{R}^+$ and $g_{AB} = -(dt)_A \times (dt)_B + (d\theta)_A(d\theta)_B$.

We will now describe the (linear) dynamics of this field system, starting with the covariant description. The reduced Hamiltonian (3) leads to the field equations

$$\dot{\xi} = P_\xi, \qquad \dot{P}_\xi = \xi'' - \frac{\xi}{4t^2}, \tag{4}$$

where the dot denotes the derivative with respect to t. Hence, the field ξ satisfies the second order differential equation

$$\ddot{\xi} - \xi'' + \frac{\xi}{4t^2} = 0. \tag{5}$$

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[1]We set $4G/\pi = c = 1$, G and c being Newton's constant and the speed of light, respectively.

Since the general solution is most conveniently expressed in Fourier series, let us introduce the notation

$$e_k := \frac{e^{-ik\theta}}{\sqrt{2\pi}} \quad \forall \ k \in \mathbb{Z}. \tag{6}$$

With respect to some reference ("initial") time $t = t_0$, all smooth solutions can then be written as [8]

$$\xi(t, \theta) = \sqrt{t}[q_0 + p_0 \ln(t)] + \sum_{k \in \mathbb{Z}-\{0\}} [b_k(t_0)G_k^{(t_0)}(t, \theta)$$
$$+ b_k^*(t_0)G_k^{(t_0)*}(t, \theta)], \tag{7}$$

where we have singled out the homogeneous mode $k = 0$ and used the symbol $*$ to represent complex conjugation. The constants $b_k(t_0)$ are complex coefficients, q_0 and p_0 are canonically conjugate variables and the mode solutions $G_k^{(t_0)}(t, \theta)$ are given by

$$G_k^{(t_0)}(t, \theta) = \sqrt{\frac{\pi t}{4}}[c^*(|k|t_0)H_0(|k|t) - d^*(|k|t_0)H_0^*(|k|t)]e_k^*, \tag{8}$$

where

$$d(x) = \sqrt{\frac{\pi x}{8}}\left[\left(1 + \frac{i}{2x}\right)H_0^*(x) - iH_1^*(x)\right],$$
$$c(x) = \sqrt{\frac{\pi x}{2}}H_0(x) - d^*(x), \tag{9}$$

and H_n ($n = 1, 2$) is the nth order Hankel function of the second kind [10]. Note that the mode solutions satisfy

$$G_k^{(t_0)}(t, \theta)|_{t=t_0} = \frac{e_k^*}{\sqrt{2|k|}}, \qquad \partial_t G_k^{(t_0)}(t, \theta)|_{t=t_0} = -i\sqrt{\frac{|k|}{2}}e_k^*. \tag{10}$$

We will refer to the linear space of solutions (7), equipped with the symplectic structure $\Omega(\xi_1, \xi_2) = \oint d\theta(\xi_2\partial_t\xi_1 - \xi_1\partial_t\xi_2)$, as the covariant phase space S.

Alternatively, instead of S, we can consider the canonical phase space. This is the linear space Γ coordinatized by the canonical pair which is formed by the configuration φ and the momentum P_φ of the field ξ on a given section of constant time. We take this time to be some fixed reference time t_0. As we have seen above, the section of constant time $t = t_0$ can be identified with the compact space S^1. In the following, we will refer to this section as the reference Cauchy surface (RCS). Let us also point out that, via Eq. (10), one can understand the way in which the solutions (7) are expressed as being specially adapted to the choice of RCS, or vice versa (given the RCS, such an adapted expression of the solutions obviously simplifies the explicit form of the map between Γ and S). On the other hand, the symplectic structure on Γ is, of course,

$$\sigma[(\varphi_1, P_{\varphi_1}), (\varphi_2, P_{\varphi_2})] = \oint d\theta(P_{\varphi_1}\varphi_2 - P_{\varphi_2}\varphi_1). \tag{11}$$

The evolution generated by the Hamiltonian (3) in the canonical phase space gives rise to a 1-parameter family of symplectic linear transformations $\tau_{(t_f, t_0)}: \Gamma \to \Gamma$ (with t_0 fixed) as follows. An initial state (φ, P_φ) at $t = t_0$ determines a solution $\xi \in S$, which in turn determines a canonical pair of fields $(\xi|_{t=t_f}, \partial_t\xi|_{t=t_f})$ for any value of t_f. This pair is then naturally interpreted as new initial data at $t = t_0$. More rigorously, we have a natural 1-parameter family of embeddings $E_t: S^1 \to \mathcal{M}^{(f)}$, together with a 1-parameter family of isomorphisms I_{E_t}, mapping Cauchy data at $E_t(S^1)$ into solutions. Then, the classical evolution operator is

$$\tau_{(t_f, t_0)} = (E_{t_0}^*)^{-1}E_{t_f}^*I_{E_{t_f}}^{-1}I_{E_{t_0}}, \tag{12}$$

with E_t^* denoting the pullback of the map E_t. In this work we mostly ignore the distinction between S^1 and our RCS, $E_{t_0}(S^1)$, so that E_{t_0} is trivialized. In addition, note that the canonical evolution maps provide the transformations $I_{E_{t_0}}\tau_{(t_f, t_0)}I_{E_{t_0}}^{-1}: S \to S$ in the covariant phase space (this notion of time evolution in the covariant description was employed in Ref. [11]).

In order to present the evolution maps in explicit form, it is convenient to use the Fourier components of the field φ and its momentum. We then define

$$\varphi_k := \oint d\theta\varphi e_k, \qquad P_\varphi^k := \oint d\theta P_\varphi e_k^*. \tag{13}$$

It is clear from the form of the Hamiltonian (3) that modes with different values of $|k|$ decouple. Furthermore, from now on we will concentrate ourselves on the infinite set of inhomogeneous modes $k \neq 0$, since no relevant aspect of our discussion depends on the single zero mode (being single and decoupled, the quantum treatment of this mode can be made independently by standard methods, and included in the final description by means of a tensor product).

Employing Eq. (10), one can check that the Fourier coefficients φ_k and P_φ^k are related to those appearing in expression (7) by

$$b_k(t_0) = \frac{1}{\sqrt{2|k|}}(|k|\varphi_k + iP_\varphi^{-k}),$$
$$b_{-k}^*(t_0) = \frac{1}{\sqrt{2|k|}}(|k|\varphi_k - iP_\varphi^{-k}). \tag{14}$$

We will adopt this convenient set of (complex) variables as alternative coordinates in Γ. In the following, to simplify the notation, we will let b_k and b_{-k}^* denote the variables $b_k(t_0)$ and $b_{-k}^*(t_0)$, respectively, and collect them in the set of pairs $\{(b_k, b_{-k}^*)\}$ with $k \in \mathbb{Z} - \{0\}$.[2] It is then straightforward to check that each of the considered pairs of

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[2]Note that the pairs with $k > 0$ and $k < 0$ are related by complex conjugation.

variables decouples in the evolution, so that the evolution transformations are 2×2 block-diagonal in these coordinates.

In more detail, time evolution $\tau_{(t_f, t_0)}$ maps (b_k, b_{-k}^*) to a new pair $(b_k(t_f), b_{-k}^*(t_f))$ [seen as new data at $t = t_0$, related to the new configuration and momentum of the field as in Eq. (14)], such that

$$
b_k(t_f) = \alpha_k(t_f, t_0)b_k + \beta_k(t_f, t_0)b_{-k}^*,
$$
$$
b_{-k}^*(t_f) = \beta_k^*(t_f, t_0)b_k + \alpha_k^*(t_f, t_0)b_{-k}^*,
$$
(15)

where

$$
\alpha_k(t_f, t_0) = c(|k|t_f)c^*(|k|t_0) - d(|k|t_f)d^*(|k|t_0),
$$
$$
\beta_k(t_f, t_0) = d(|k|t_f)c(|k|t_0) - c(|k|t_f)d(|k|t_0).
$$
(16)

Note that the functions c and d, given in Eqs. (9), satisfy $|c|^2 - |d|^2 = 1$, and we thus have that $|\alpha_k(t_f, t_0)|^2 - |\beta_k(t_f, t_0)|^2 = 1$ for all $t_f > 0$ (and any $t_0 > 0$). In addition,

$$
\alpha_{-k}(t_f, t_0) = \alpha_k(t_f, t_0), \qquad \beta_{-k}(t_f, t_0) = \beta_k(t_f, t_0),
$$
$$
\alpha_k(t_0, t_f) = \alpha_k^*(t_f, t_0), \qquad \beta_k(t_0, t_f) = -\beta_k(t_f, t_0).
$$
(17)

B. Fock quantization

Let us summarize now the Fock quantization of the model, i.e. the Fock quantization of the sector of nonzero modes of the associated scalar field system, as performed in Refs. [7,8]. We will call this sector of nonzero modes the inhomogeneous sector.

By construction, the set of mode solutions $\{G_k^{(t_0)}(t, \theta), G_k^{(t_0)*}(t, \theta)\}$ in Eq. (7) (with $k \in \mathbb{Z} - \{0\}$) is complete in the inhomogeneous sector of the space of solutions S, and "orthonormal" in the product $(G_k^{(t_0)}, G_m^{(t_0)}) = -i\Omega(G_k^{(t_0)*}, G_m^{(t_0)})$, in the sense that

$$
(G_k^{(t_0)}, G_m^{(t_0)}) = \delta_{km}, \qquad (G_k^{(t_0)*}, G_m^{(t_0)*}) = -\delta_{km},
$$
$$
(G_k^{(t_0)}, G_m^{(t_0)*}) = 0.
$$
(18)

Associated to the field decomposition (7), there is a natural Ω-compatible complex structure J_0:

$$
J_0[G_k^{(t_0)}(t, \theta)] = iG_k^{(t_0)}(t, \theta),
$$
$$
J_0[G_k^{(t_0)*}(t, \theta)] = -iG_k^{(t_0)*}(t, \theta).
$$
(19)

This complex structure defines (and is defined by) the annihilation and creationlike variables $b_k(t_0) = \Omega(J_0 G_k^{(t_0)*}, \xi)$ and $b_k^*(t_0) = \Omega(J_0 G_k^{(t_0)}, \xi)$. We notice that J_0 is invariant under the group of S^1 translations $T_\omega: \theta \mapsto \theta + \omega$ generated by the global constraint (2).

Starting with (S, J_0), we can construct the so-called "one particle" Hilbert space $\mathcal{H}_0$. It is the Cauchy completion of the space of "positive" frequency solutions

$$
S^+ := \left\{ \xi^+ = \frac{1}{2}(\xi - iJ_0\xi) \right\}
$$
(20)

with respect to the norm $\|\xi^+\| = \sqrt{\langle \xi^+, \xi^+ \rangle}$. Here, $\langle \cdot, \cdot \rangle$ denotes the inner product $\langle \xi_1^+, \xi_2^+ \rangle := -i\Omega(\xi_1^-, \xi_2^+)$ with $\xi^- = (\xi + iJ_0\xi)/2 \in \bar{\mathcal{H}}_0$ (the complex conjugate space of $\mathcal{H}_0$). The kinematical Hilbert space of the quantum theory is then the symmetric Fock space

$$
\mathcal{F}(\mathcal{H}_0) = \bigoplus_{n=0}^{\infty} \left(\bigotimes_{(s)}^{n} \mathcal{H}_0 \right),
$$
(21)

where $\otimes_{(s)}^n \mathcal{H}_0$ is the Hilbert space of all nth rank symmetric tensors over $\mathcal{H}_0$. Following this prescription, the formal field operator $\hat{\xi}$ yields

$$
\hat{\xi}(t; \theta) = \sum_{k \in \mathbb{Z} - \{0\}} [G_k^{(t_0)}(t, \theta)\hat{b}_k + G_k^{(t_0)*}(t, \theta)\hat{b}_k^\dagger].
$$
(22)

Here, $\hat{b}_k$ and $\hat{b}_k^\dagger$ are, respectively, the annihilation and creation operators corresponding to the positive and "negative" frequency decomposition defined by J_0, and represent the classical variables b_k and b_k^*.

A crucial aspect of this quantization is that the dynamics is unitarily implementable, i.e. for each symplectic transformation in the 1-parameter family $\tau_{(t_f, t_0)}$ (15) defined by time evolution $\forall t_f > 0$, there exists a unitary quantum evolution operator $\hat{U}(t_f, t_0)$ such that

$$
\begin{aligned}
\hat{b}_k(t_f) &= \alpha_k(t_f, t_0)\hat{b}_k + \beta_k(t_f, t_0)\hat{b}_{-k}^\dagger \\
&= \hat{U}^{-1}(t_f, t_0)\hat{b}_k \hat{U}(t_f, t_0), \\
\hat{b}_{-k}^\dagger(t_f) &= \beta_k^*(t_f, t_0)\hat{b}_k + \alpha_k^*(t_f, t_0)\hat{b}_{-k}^\dagger \\
&= \hat{U}^{-1}(t_f, t_0)\hat{b}_{-k}^\dagger \hat{U}(t_f, t_0).
\end{aligned}
$$
(23)

As shown in Refs. [7,8], this follows from the fact that the sequences $\{\beta_k(t_f, t_0)\}$ are square summable.[3]

In addition, since J_0 is invariant under the group of translations T_ω, we have an invariant unitary implementation of the gauge group on the (kinematical) Fock space $\mathcal{F}(\mathcal{H}_0)$.

The physical Hilbert space $\mathcal{F}_{\text{phys}}$ consists of all states in $\mathcal{F}(\mathcal{H}_0)$ that belong to the kernel of the quantum constraint

$$
\hat{\mathcal{C}}_0 = \sum_{k=1}^{\infty} k(\hat{b}_k^\dagger \hat{b}_k - \hat{b}_{-k}^\dagger \hat{b}_{-k}).
$$
(24)

Starting with the basis of "n-particle" states determined by the annihilation and creation operators $\{(\hat{b}_k, \hat{b}_k^\dagger)\}$, one can then construct physical states by restricting the elements of that basis to the subset of states which are physical,

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[3]Let us recall that a symplectic transformation is unitarily implementable with respect to a FR if, and only if, its antilinear part is Hilbert-Schmidt on the "one particle" Hilbert space [12]. In the present case this condition reduces to $\sum_k |\beta_k(t_f, t_0)|^2 < \infty$.

namely, the "n-particle" states with zero field momentum $\sum_{k=1}^{\infty} k(N_k - N_{-k}) = 0$, where N_k is the corresponding eigenvalue of the *partial kth number operator* $\hat{N}_k := \hat{b}_k^\dagger \hat{b}_k$. Furthermore, it is straightforward to check that $\hat{C}_0$ is invariant under the time evolution (23). This invariance ensures that the dynamics is unitarily implementable not just on $\mathcal{F}(\mathcal{H}_0)$, but also on the space of physical states $\mathcal{F}_{\text{phys}}$.

Let us conclude with a comment regarding an apparent ambiguity. Our fixed reference time t_0 certainly plays a role in the definition of J_0, and it is clear that, by changing t_0 and keeping the definition (19), one obtains new complex structures with the same properties of S^1 invariance and unitary dynamics, since the results of Refs. [7,8] do not depend on the value of t_0. However, since these different complex structures are, by construction, related by evolution transformations, they give rise to unitarily equivalent quantizations, precisely because the evolution is unitary [8] (see also Sec. IV B). Moreover, as we mentioned in the introduction, much stronger results have indeed been proven regarding the uniqueness of the quantization [6,9].

III. THE SCHRÖDINGER REPRESENTATION

We will now obtain the Schrödinger functional description of the quantum representation of the canonical commutation relations (CCRs) provided by the quantum fields of the system at a given fixed time. Let us stress again that, just because of the unitary implementation of the field dynamics, the choice of this fixed time is irrelevant, in the sense that different choices lead to unitarily equivalent representations of the CCRs. So, for convenience, we will take this fixed time to be our reference time t_0.

The SR that we are going to construct is that defined by the specific complex structure that is induced from J_0 on the canonical phase space Γ by means of the isomorphism $I_{E_{t_0}}$. Taking into account that the complex structure J_0 effectively declares that the classical variables $\{b_k\}$ and $\{b_k^*\}$ are to be quantized as the respective annihilation and creation operators of the representation, and recalling Eq. (14), which gives the relation between these variables and the field modes, it should not come as a surprise that the representation of the CCRs which we will obtain is essentially that associated with the free massless field in S^1. We will nevertheless present this construction in some detail, both for completeness and to clarify the relation that, for the quantization of the Gowdy model, exists between the covariant approach adopted in Refs. [7,8] and its canonical version.

A. General framework

Let us start by considering the canonical phase space Γ (more precisely, its inhomogeneous sector). The set of elementary observables $\mathcal{O}$ is taken to be the vector space of linear functionals

$$L_\lambda(Y) := \sigma(\lambda, Y) = \oint d\theta(f\varphi + gP_\varphi) \qquad (25)$$

and the unit functional **1**, namely $\mathcal{O} = \text{Span}\{\mathbf{1}, L_\lambda\}$. Here, Y is a vector in Γ of the form (φ, P_φ) and λ denotes a pair of smooth test functions $(-g, f)$ which have both a vanishing integral on S^1. The set $\mathcal{O}$ is closed under Poisson brackets, $\{L_\lambda(Y), L_\nu(Y)\} = L_\nu(\lambda)$, and is complete, in the sense that its elements separate points in (the inhomogeneous sector of) Γ.

The configuration and momentum observables are particular cases of functionals L_λ. Whereas $L_\lambda|_{\lambda=(0,f)}$ defines the configuration observable[4]

$$\bar{\varphi}(f) := \oint d\theta f\varphi = \sum_{k\in\mathbb{Z}-\{0\}} f^k \varphi_k, \qquad (26)$$

the momentum observable is defined by considering the label $\lambda = (-g, 0)$,

$$\bar{P}_\varphi(g) := \oint d\theta g P_\varphi = \sum_{k\in\mathbb{Z}-\{0\}} g_k P_\varphi^k. \qquad (27)$$

From the Poisson brackets between the configuration and momentum observables (and setting $\hbar = 1$), one obtains for their respective quantum operators $\hat{\bar{\varphi}}[f]$ and $\hat{\bar{P}}_\varphi[g]$ the CCRs:

$$[\hat{\bar{\varphi}}[f], \hat{\bar{P}}_\varphi[g]] = i\hat{\mathbf{1}} \sum_{k\in\mathbb{Z}-\{0\}} f^k g_k. \qquad (28)$$

At this point of the discussion and in order to make the analysis self-contained, it is convenient to succinctly review how a Schrödinger functional representation of the CCRs is determined by a complex structure on the canonical phase space. We will start by describing the most general form of a complex structure on Γ. This discussion can then be easily applied to the general setting of a scalar field in a globally hyperbolic spacetime (see Refs. [13,14]) and, in particular, to the case of the Gowdy model.

A (σ-compatible) complex structure j on Γ has the generic form

$$j(\varphi, P_\varphi) = (A\varphi + BP_\varphi, CP_\varphi + D\varphi), \qquad (29)$$

where A, B, C and D are linear operators that satisfy

$$A^2 + BD = -\mathbf{1}, \qquad AB + BC = 0,$$
$$C^2 + DB = -\mathbf{1}, \qquad DA + CD = 0 \qquad (30)$$

(so that $j^2 = -\mathbf{1}$), and

$$(f, Bf') = (Bf, f'), \qquad (g, Dg') = (Dg, g'),$$
$$(f, Ag) = -(Cf, g), \qquad (f, Bf) < 0, \qquad (g, Dg) > 0 \qquad (31)$$

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[4]The Fourier components of f and g in λ are, respectively, $f^k = \oint d\theta f e_k^*$ and $g_k = \oint d\theta g e_k$.

for all smooth test functions g, g', f and f' (so that j is σ compatible). Here, we have introduced the notation $(f, g) := \oint d\theta f g$. Notice that C and D can be obtained from A and B: indeed, from the two first relations in Eq. (30) one gets $C = -B^{-1}AB$ and $D = -B^{-1}(\mathbf{1} + A^2)$ (when B^{-1} exists). Thus, the set of all compatible complex structures on Γ can be parametrized by the operators A and B (assuming B is invertible); that is, this set can be identified with $\{j_{(A,B)}\}$ where (in matrix notation)

$$ j_{(A,B)} = \begin{pmatrix} A & B \\ -B^{-1}(\mathbf{1} + A^2) & -B^{-1}AB \end{pmatrix}. \quad (32) $$

Given a complex structure j on the canonical phase space Γ, a Schrödinger, or ''configuration'' wave functional representation—which we will call the j-SR—is determined as follows.[5] The j-SR consists of a representation of the basic operators of configuration and momentum on a space of complex-valued functionals Ψ on the ''quantum'' configuration space $\bar{\mathcal{C}}$ (generally an extension of the classical configuration space). These functionals are square integrable with respect to a Gaussian measure μ with covariance $-B/2$.[6] On the Hilbert space defined in this way, the basic operators of configuration and momentum are

$$ (\hat{\bar{\varphi}}[f]\Psi)[\bar{\varphi}] = \bar{\varphi}(f)\Psi[\bar{\varphi}], \quad (33) $$

$$ (\hat{\bar{P}}_\varphi[g]\Psi)[\bar{\varphi}] = -i\frac{\delta\Psi}{\delta\bar{\varphi}}[g] - i\bar{\varphi}(B^{-1}(\mathbf{1} - iA)g)\Psi[\bar{\varphi}], \quad (34) $$

where $\bar{\varphi} \in \bar{\mathcal{C}}$.

It is worth noticing that, while the measure is determined just by B, there is an extra freedom in the momentum operator, given by the operator A (see Ref. [15] for discussion). Finally, let us also recall that two complex structures j and j' on Γ lead to unitarily equivalent representations of the CCRs if, and only if, $j - j'$ defines a Hilbert-Schmidt operator on the ''one particle'' Hilbert space determined by j (or equivalently by j').

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[5] We are only presenting the outcome, obtained under suitable regularity conditions. The full process involves the construction of an inner product from j and σ, which is used to determine a state of the Weyl algebra associated with the CCRs. The GNS representation defined by this state can be realized as an SR, since the restriction of the state to the Weyl configuration observables defines a measure.

[6] We define the covariance of a Gaussian measure as twice the positive bilinear form appearing in the exponential of the Fourier transform of the measure. We follow the standard practice of using the term ''covariance'' to refer not only to this bilinear form, but also to the operator which defines it with respect to a fiducial integration in the space of test functions, which in our case is given by $d\theta$.

B. The canonical complex structure

As we explained in Sec. II, given our RCS, which is determined by the chosen reference time t_0, there is a preferred isomorphism between the canonical phase space and the space of solutions to the field Eq. (5). In order to simplify the notation, we will denote this isomorphism by I_{E_0} instead of $I_{E_{t_0}}$. Then, $I_{E_0}: \Gamma \to S$ is such that

$$ S \ni \xi \mapsto I_{E_0}^{-1}(\xi) = (\varphi, P_\varphi) = (\xi|_{t=t_0}, \partial_t\xi|_{t=t_0}). \quad (35) $$

Therefore, a complex structure J on the covariant phase space S determines (and is determined by) a corresponding complex structure $j = I_{E_0}^{-1}JI_{E_0}$ on the canonical phase space. In particular, the complex structure J_0 of Sec. II has the canonical counterpart $j_0 = I_{E_0}^{-1}J_0I_{E_0}: \Gamma \to \Gamma$. The SR we are looking for is thus specified by j_0, following the prescription of the previous subsection. We will now obtain the explicit form of j_0.

Recalling the field decomposition (7) (for the inhomogeneous sector) and employing Eq. (10), we get the explicit relation between (φ, P_φ) and the set of pairs of variables $\{(b_k, b_{-k}^*)\}$:

$$ \varphi = \sum_{k\in\mathbb{Z}-\{0\}} \frac{1}{\sqrt{2|k|}}[b_k e_k^* + b_k^* e_k], $$

$$ P_\varphi = -i\sum_{k\in\mathbb{Z}-\{0\}} \sqrt{\frac{|k|}{2}}[b_k e_k^* - b_k^* e_k]. \quad (36) $$

For a given $(\varphi, P_\varphi) \in \Gamma$ and the corresponding solution $\xi = I_{E_0}(\varphi, P_\varphi) \in S$, we obtain the new canonical fields $j_0(\varphi, P_\varphi) = I_{E_0}^{-1}J_0(\xi) \in \Gamma$, which we will call $(\tilde{\varphi}, \tilde{P}_\varphi)$. Taking into account that $J_0(\xi) = i\xi^+ - i\xi^-$, with ξ^+ (ξ^-) being the positive (negative) frequency part spanned by $\{G_k^{(t_0)}\}$ ($\{G_k^{(t_0)*}\}$), with $k \in \mathbb{Z} - \{0\}$, we get

$$ \tilde{\varphi} = J_0(\xi)|_{t_0} = i\xi^+|_{t_0} - i\xi^-|_{t_0}, $$
$$ \tilde{P}_\varphi = \partial_t J_0(\xi)|_{t_0} = i\partial_t\xi^+|_{t_0} - i\partial_t\xi^-|_{t_0}. \quad (37) $$

Hence, it is easy to check that

$$ \tilde{\varphi} = \sum_{k\in\mathbb{Z}-\{0\}} \frac{i}{\sqrt{2|k|}}[b_k e_k^* - b_k^* e_k], $$

$$ \tilde{P}_\varphi = \sum_{k\in\mathbb{Z}-\{0\}} \sqrt{\frac{|k|}{2}}[b_k e_k^* + b_k^* e_k]. \quad (38) $$

From Eqs. (36) and (38), one obtains that $\tilde{\varphi} = -(-\Delta)^{-1/2}P_\varphi$ and $\tilde{P}_\varphi = (-\Delta)^{1/2}\varphi$, where Δ is the second order differential operator $d^2/d\theta^2$. The explicit expression for the canonical counterpart of J_0 is then

$$ j_0 = \begin{pmatrix} 0 & -(-\Delta)^{-1/2} \\ (-\Delta)^{1/2} & 0 \end{pmatrix}. \quad (39) $$

A comparison with Eq. (32) shows that, in this case, $A = 0$

and $B = -(-\Delta)^{-1/2}$. Therefore, the momentum operators are completely determined by the covariance $(-\Delta)^{-1/2}/2$ of the Gaussian measure.

In terms of the Fourier coefficients $\{(\varphi_k, P_\varphi^{-k})\}$ with $k \in \mathbb{Z} - \{0\}$, the complex structure (39) yields

$$(j_0)_k = \begin{pmatrix} 0 & -\frac{1}{|k|} \\ |k| & 0 \end{pmatrix}. \qquad (40)$$

So, in this alternative description of Γ provided by the Fourier components of φ and P_φ, the counterparts of A and B are given by $A_k = 0$ and $B_k = -\frac{1}{|k|}$, respectively, (recall that $k \neq 0$).

C. The functional representation of the Gowdy cosmologies

Let us now complete the construction of the j_0-SR. We will call $\mathcal{T}$ our space of test functions, i.e. the space of smooth real functions on S^1 with vanishing integral. By standard arguments in the theory of measures in infinite dimensional spaces (see e.g. Ref. [16]), the space $\mathcal{T}$ can be equipped with a so-called nuclear topology, and the covariance $(-\Delta)^{-1/2}/2$ defines a Gaussian measure μ on the topological dual of $\mathcal{T}$, namely, the real vector space $\mathcal{T}^\star$ of continuous linear functionals on $\mathcal{T}$. This will be the quantum configuration space $\bar{\mathcal{C}}$.

Designating a generic element of $\mathcal{T}^\star$ as $\bar{\varphi}$ and its action on elements of $\mathcal{T}$ as $f \mapsto \bar{\varphi}(f)$, the measure μ is defined by its Fourier transform

$$\int_{\mathcal{T}^\star} e^{i\bar{\varphi}(f)} d\mu = \exp\left[-\frac{1}{4}(f, (-\Delta)^{-1/2}f) \right]. \qquad (41)$$

The configuration wave functional representation of $\hat{\bar{\varphi}}$ and $\hat{\bar{P}}_\varphi$ on $\mathcal{H}_s := L^2(\mathcal{T}^\star, d\mu)$ is then

$$(\hat{\bar{\varphi}}[f]\Psi)[\bar{\varphi}] = \bar{\varphi}(f)\Psi[\bar{\varphi}], \qquad (42)$$

$$(\hat{\bar{P}}_\varphi[g]\Psi)[\bar{\varphi}] = -i\frac{\delta\Psi}{\delta\bar{\varphi}}[g] + i\bar{\varphi}((-\Delta)^{1/2}g)\Psi[\bar{\varphi}]. \qquad (43)$$

An alternative description is obtained in Fourier space as follows. By means of the Fourier correspondence $f \mapsto \{f^k\} = \{\oint d\theta f e_k^*\}$, one can identify $\mathcal{T}$ with the space of rapidly decreasing complex sequences $\{f^k\}$ with $k \in \mathbb{Z} - \{0\}$, i.e. sequences such that $k^r f^k$ goes to zero as $|k| \to \infty$, for all $r > 0$ (and which, moreover, satisfy $f^{k*} = f^{-k}$, so that the corresponding functions f are real). Likewise, the dual space $\mathcal{T}^\star$ can be identified with a subspace (of sequences of appropriate behavior) of the space of all complex sequences $\{\varphi_k\}$ with $k \in \mathbb{Z} - \{0\}$ and $\varphi_{-k} = \varphi_k^*$. This correspondence is given by $\mathcal{T}^\star \ni \bar{\varphi} \leftrightarrow \{\varphi_k\} := \{\bar{\varphi}(e_k)\}$, so that

$$\bar{\varphi}(f) = \sum_{k\neq 0} f^k \varphi_k = \sum_{k>0} f^k \varphi_k + \sum_{k>0}(f^k \varphi_k)^*. \qquad (44)$$

In order to present the measure without unnecessary complications, we note that, since the sequences $\{f^k\} \in \mathcal{T}$ and $\{\varphi_k\} \in \mathcal{T}^\star$ are both determined by their values for $k > 0$, one can simply work with sequences whose index is defined in $\mathbb{N}$, rather than in $\mathbb{Z} - \{0\}$. Actually, one can view μ as a measure on the space of all complex sequences $\{\varphi_k\}$ with $k \in \mathbb{N}$ that happens to be supported on the subspace $\mathcal{T}^\star$.[7]

In this description, μ is a product measure on (a subset of) the product space $\mathbb{C}^{\mathbb{N}}$ of complex sequences $\{\varphi_k\}$ with $k \in \mathbb{N}$:

$$d\mu = \prod_{k\in\mathbb{N}} \frac{2|k|}{\pi} \exp(-2|k||\varphi_k|^2) d\mu_k^0, \qquad (45)$$

where $d\mu_k^0$ is the Lebesgue measure on the plane coordinatized by (φ_k, φ_k^*). It is easily seen that this measure corresponds to that appearing in Eq. (41).

Note that we are using here complex canonical variables. This accounts for the factors 2 in Eq. (45), which no longer appear when the quantization is recast in terms of real canonical variables, namely, the coefficients in the Fourier decompositions of φ and P_φ in terms of normalized sine and cosine functions.

It is worth pointing out that one can reinterpret the measure μ described above as a measure on the original space of sequences $\{\varphi_k\}$ with integer index ($k \in \mathbb{Z} - \{0\}$) and such that $\varphi_{-k} = \varphi_k^*$. Using the one-to-one correspondence between these sequences and their restrictions to $k \in \mathbb{N}$, one can define both the measurable sets and the measure.

The operators which present the simplest expressions correspond to the Fourier components of the field operators, $\hat{\bar{\varphi}}[e_k]$ and $\hat{\bar{P}}_\varphi[e_k^*]$, i.e. to the quantization of the classical variables φ_k and P_φ^k:

$$\hat{\varphi}_k \Psi = \varphi_k \Psi, \qquad (46)$$

$$\hat{P}_\varphi^k \Psi = -i\frac{\partial\Psi}{\partial\varphi_k} + i|k|\varphi_{-k}\Psi, \qquad (47)$$

where Ψ is a functional of the Fourier components φ_k.

The CCRs (28) are clearly satisfied. Moreover, the same happens with the reality conditions $\hat{\varphi}_k^\dagger = \hat{\varphi}_{-k}$ and $\hat{P}_\varphi^{k\dagger} = \hat{P}_\varphi^{-k}$ with respect to the $L^2(\mathcal{T}^\star, d\mu)$-inner product. Equivalently, the operators $\hat{\bar{\varphi}}[f]$ and $\hat{\bar{P}}_\varphi[g]$ are symmetric, leading to self-adjoint operators on an appropriate domain of definition.

In addition, from Eq. (14) the variables b_k and b_k^* are quantized as

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[7]On the other hand, one can certainly find proper subsets of $\mathcal{T}^\star$ which support the measure. See e.g. Ref. [17] for reviews of results and for techniques concerning support properties of field measures.

$$\hat{b}_k = \frac{1}{\sqrt{2|k|}} \frac{\partial}{\partial \varphi_{-k}}, \qquad \hat{b}_k^\dagger = -\frac{1}{\sqrt{2|k|}} \frac{\partial}{\partial \varphi_k} + \sqrt{2|k|} \varphi_{-k}.$$
(48)

These are precisely the annihilation and creation operators of the j_0-SR. By construction, the "zero particle" state of the j_0-SR, which we will call the vacuum, is the unit constant functional $\Psi_0[\bar{\varphi}] = 1$ (up to a constant phase).

As we have already mentioned, the invariance of J_0—and therefore of j_0—under the group of S^1 translations $T_\omega: \theta \to \theta + \omega$, $\omega \in S^1$, provides us with corresponding unitary operators $\hat{T}_\omega$ which leave the vacuum invariant, and whose explicit action, in the Fourier description, is given by

$$(\hat{T}_\omega \Psi)[\varphi_k] = \Psi[e^{-i\omega k} \varphi_k], \qquad \Psi \in \mathcal{H}_s.$$
(49)

The generator of the unitary group $\hat{T}_\omega$,

$$\hat{\mathcal{C}}_0 = \sum_{k=1}^\infty |k| \left(\varphi_{-k} \frac{\partial}{\partial \varphi_{-k}} - \varphi_k \frac{\partial}{\partial \varphi_k} \right),$$
(50)

is the quantum constraint operator in the functional approach.

The space of physical states consists of all states in $\mathcal{H}_s$ which are invariant under the action of $\hat{T}_\omega$ for every $\omega \in S^1$. That is, physical states are invariant under the group of phase transformations $\varphi_k \to e^{-i\omega k} \varphi_k$ $\forall \omega \in S^1$. This property allows a characterization of physical states alternative to that presented at the end of Sec. II. One can obtain the Hilbert space of physical states $\mathcal{H}_{\mathrm{phys}}$ as the quotient of the kinematical Hilbert space $\mathcal{H}_s$ by the action of the considered gauge group. Since this group is compact, the projection of any kinematical state Ψ onto the space of physical states can then be easily determined by a group averaging procedure (see e.g. Ref. [18]):

$$\Psi_{\mathrm{phys}}[\varphi_k] = \oint \frac{d\omega}{2\pi} (\hat{T}_\omega \Psi)[\varphi_k].$$
(51)

It is important to emphasize that, because the gauge group is unitary and compact, the physical state Ψ_{phys} has a finite norm for any $\Psi \in \mathcal{H}_s$. Therefore, the space of physical states is just a Hilbert subspace of the kinematical Hilbert space.

In summary, the j_0-SR consists of a (kinematical) Hilbert space $\mathcal{H}_s$ defined by a Gaussian measure of covariance $(-\Delta)^{-1/2}/2$, on which the CCRs are implemented by the operators $\hat{\bar{\varphi}}$ and $\hat{\bar{P}}_\varphi$ (42) and (43) [or equivalently, by $\hat{\varphi}_k$ and $\hat{P}_\varphi^k$ (46) and (47)]. The physical Hilbert space consists of the invariant subspace under S^1 translations. It follows from the results of Refs. [6–9] that the j_0-SR is the (essentially) unique S^1-invariant configuration wave functional representation with a unitary dynamics. Finally, it is worth emphasizing that the SR here presented is not equivalent to the Schrödinger representa-

tions (SRs) constructed in Ref. [19], where the considered basic field was $\bar{\xi} = \xi/\sqrt{t}$ instead of ξ [6–8].

IV. TIME EVOLUTION

In this section we will address the issue of how time evolution is implemented in our model in the context of the functional representation.

A. Creation and annihilation operators and the vacuum

In the J_0-Fock quantization, classical dynamics is implemented in the Heisenberg picture by a unitary operator $\hat{U}(t_f, t_0)$ relating annihilation and creation operators at different times as in Eq. (23). Recalling that $\hat{U}^{-1}(t_f, t_0) = \hat{U}(t_0, t_f)$ and the last two relations in Eq. (17), we can now introduce the annihilation and creation operators corresponding to evolution "backwards in time,"

$$\begin{aligned}
\hat{\bar{b}}_k(t_f) &= \hat{U}(t_f, t_0) \hat{b}_k \hat{U}^{-1}(t_f, t_0) \\
&= \alpha_k^*(t_f, t_0) \hat{b}_k - \beta_k(t_f, t_0) \hat{b}_{-k}^\dagger, \\
\hat{\bar{b}}_k^\dagger(t_f) &= \hat{U}(t_f, t_0) \hat{b}_k^\dagger \hat{U}^{-1}(t_f, t_0) \\
&= \alpha_k(t_f, t_0) \hat{b}_k^\dagger - \beta_k^*(t_f, t_0) \hat{b}_{-k}.
\end{aligned}$$
(52)

Obviously, $\hat{\bar{b}}_k(t_0)$ and $\hat{\bar{b}}_k^\dagger(t_0)$ coincide with $\hat{b}_k$ and $\hat{b}_k^\dagger$, respectively.

Because of the mixing of annihilation and creation operators, the Heisenberg vacuum state $|0\rangle_{\mathrm{H}}$ which is annihilated by all the operators $\hat{b}_k$ fails to be in the kernel of all the time-evolved operators $\hat{\bar{b}}_k(t_f)$ for any $t_f \neq t_0$. Instead, these operators annihilate the state

$$|0, t_f\rangle := \hat{U}(t_f, t_0)|0\rangle_{\mathrm{H}},$$
(53)

which is just the time-evolved vacuum, i.e. the counterpart of the state $|0\rangle_{\mathrm{H}}$ in the Schrödinger picture. Of course, $|0, t_0\rangle = |0\rangle_{\mathrm{H}}$.

We will refer to $|0, t_f\rangle$ and to states of the form

$$|n, t_f\rangle = \hat{\bar{b}}_{k_1}^\dagger(t_f) \hat{\bar{b}}_{k_2}^\dagger(t_f) \dots \hat{\bar{b}}_{k_n}^\dagger(t_f)|0, t_f\rangle$$
(54)

as the t_f-vacuum and the t_f "n-particle" states, respectively. From Eqs. (52) and (54) one concludes that the t_f "n-particle" states are related with the Heisenberg "n-particle" states $|n\rangle_{\mathrm{H}} := |n, t_0\rangle$ as follows

$$\begin{aligned}
|n, t_f\rangle &= \hat{U} \hat{b}_{k_1}^\dagger \hat{b}_{k_2}^\dagger \dots \hat{b}_{k_n}^\dagger \hat{U}^{-1}|0, t_f\rangle = \hat{U} \hat{b}_{k_1}^\dagger \hat{b}_{k_2}^\dagger \dots \hat{b}_{k_n}^\dagger|0\rangle_{\mathrm{H}} \\
&= \hat{U}|n\rangle_{\mathrm{H}},
\end{aligned}$$
(55)

where we have used $\hat{U}$ as an abbreviation for $\hat{U}(t_f, t_0)$. The t_f "n-particle" states are thus the result of evolving the states $|n\rangle_{\mathrm{H}}$ from t_0 to t_f. Therefore, in order to specify the evolution to time t_f of all Heisenberg "n-particle" states—and hence determine the time evolution operator—we only need to supply the operators (52). In this respect, we note

that an equivalent condition for unitarity of the evolution to time t_f is the existence of a vector which is annihilated by all the operators $\hat{\bar{b}}_k(t_f)$. If this vector exists, then it is unique (up to a constant phase), so that the considered annihilation operators contain indeed all the necessary information to fix the evolved vacuum (53).

Turning back to the functional description, let us now write the operators $\hat{\bar{b}}_k(t_f)$ and determine the explicit form of the state $|0, t_f\rangle$ in the j_0-SR. From Eqs. (48) and (52) one obtains[8]

$$\hat{\bar{b}}_k(t_f) = \frac{\alpha_k^* + \beta_k}{\sqrt{2|k|}} \frac{\partial}{\partial \varphi_{-k}} - \sqrt{2|k|}\beta_k \varphi_k. \quad (56)$$

Here, α_k and β_k denote $\alpha_k(t_f, t_0)$ and $\beta_k(t_f, t_0)$, respectively, a simplified notation that we will use in the following. It is straightforward to see that, formally, the solution of the set of conditions $\hat{\bar{b}}_k(t_f)\Psi = 0$ ($\forall k \in \mathbb{Z} - \{0\}$) is given by

$$\Psi_0^{(t_f)} := \prod_{k \in \mathbb{N}} \frac{1}{|\alpha_k^* + \beta_k|} \exp\left(2|k| \frac{\beta_k}{\alpha_k^* + \beta_k} |\varphi_k|^2\right), \quad (57)$$

where we have already normalized each of the factors in the infinite product. Actually, owing to the summability of the sequences $\{|\beta_k|^2\}$ (i.e. thanks to unitarity), one can check that the normalized sequence formed by the finite number of factors $1 \le k \le K$ with $K \in \mathbb{N}$ is a Cauchy sequence in the $L^2(\mathcal{T}^\star, d\mu)$ norm. Hence, the t_f vacuum $|0, t_f\rangle$ in the j_0-SR is (up to a constant phase) the state $\Psi_0^{(t_f)}$, rigorously defined as the L^2 limit of the sequence of products with a finite number of factors.

B. Complex structures induced by time evolution

Regardless of its unitary implementability in the quantum theory, the classical evolution, being defined by a family of symplectic transformations, generates a family of representations of the CCRs starting from a given one. In the present case, this family of representations is associated with the family of complex structures

$$j_{t_f} := \tau_{(t_f, t_0)} j_0 \tau_{(t_f, t_0)}^{-1}, \qquad j_{t_f} : \Gamma \to \Gamma, \quad (58)$$

obtained by evolving the complex structure j_0. Here, $\tau_{(t_f, t_0)}$ is the classical evolution operator for an arbitrary time $t_f > 0$. Clearly, the condition of unitary implementability of time evolution in the j_0 representation translates into the condition of unitary equivalence between that representation and the representations defined by the complex structures j_{t_f}, $\forall t_f > 0$. Thus, one can address the question of time evolution by considering the representations constructed from the 1-parameter family of complex structures j_{t_f}. The relationship between the members of this family of

representations provides us with an alternative, equivalent description of the time evolution. In the present case, given the unitary implementability of the evolution, established in Refs. [7,8], we obtain a family of unitarily equivalent representations. In particular, the family of SRs defined by the complex structures j_{t_f}, which we will refer to as the family of j_{t_f}-SRs, is associated with a family of mutually absolutely continuous Gaussian measures.

Before determining explicitly the complex structures j_{t_f} and the corresponding j_{t_f}-SRs, we will give an equivalent characterization of them which is related to the discussion in the previous subsection. Let us consider the set of (pairs of) coefficients $\{(\bar{b}_k(t_f), \bar{b}_k^*(t_f))\}$ which is obtained from $\{(b_k, b_k^*)\}$ by applying $\tau_{(t_f, t_0)}^{-1}$ [i.e. the relation between the two sets is the direct classical counterpart of Eq. (52)]. It is clear that, when expressed in terms of the pairs $\{(\bar{b}_k(t_f), \bar{b}_k^*(t_f))\}$, the complex structure j_{t_f} adopts the same form as j_0 in terms of the pairs $\{(b_k, b_k^*)\}$ [namely, it is given by a block-diagonal matrix with the 2×2 blocks $(j_{t_f})_k = \text{diag}(i, -i)$]. Therefore, the j_{t_f} representation is such that the classical variables which are quantized as the creation and annihilation operators are $\{\bar{b}_k^*(t_f)\}$ and $\{\bar{b}_k(t_f)\}$, respectively, rather than $\{b_k^*\}$ and $\{b_k\}$.[9]

Returning to the covariant description for a moment, the family $\{j_{t_f}\}$ determines a family of complex structures on the covariant phase space via the isomorphism I_{E_0} (35). These are given by $J_{t_f} = \bar{\tau}_{(t_f, t_0)} J_0 \bar{\tau}_{(t_f, t_0)}^{-1} = I_{E_0} j_{t_f} I_{E_0}^{-1}$, where $\bar{\tau}_{(t_f, t_0)} = I_{E_0} \tau_{(t_f, t_0)} I_{E_0}^{-1}$ is the classical evolution map in covariant phase space. Just as J_0 is associated with the field decomposition (7), J_{t_f} can be understood as being associated with the decomposition

$$\xi(t, \theta) = \sum_{k \neq 0} [\bar{b}_k(t_f) G_k^{(t_f)}(t, \theta) + \bar{b}_k^*(t_f) G_k^{(t_f)*}(t, \theta)], \quad (59)$$

where $G_k^{(t_f)} = \bar{\tau}_{(t_f, t_0)} G_k^{(t_0)}$ are the time-evolved modes. One can thus see that, as commented above, changing the time used to define our fiducial complex structure on the covariant phase space corresponds in fact to evolution.

C. The family of unitarily equivalent functional representations

Explicit expressions for the complex structures j_{t_f} (58) are obtained quite straightforwardly. Taking into account expression (40) for j_0, relations (14) and the evolution (15), one concludes that j_{t_f}, given in terms of the Fourier coefficients $\{(\varphi_k, P_\varphi^{-k})\}$, is defined by the following 2×2

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[8]One may also obtain the operators $\hat{\bar{b}}_k^\dagger(t_f)$ in the same way.

[9]Let us point out that a different but equivalent way to recast time evolution is with the family of representations arising from the set of complex structures $\{\tilde{j}_{t_f} = \tau^{-1} j_0 \tau\}$. In that case, the annihilation and creationlike variables defined by $\tilde{j}_{t_f}$ are $\{b_k(t_f)\}$ and $\{b_k^*(t_f)\}$, introduced in Eq. (15).

matrices:

$$
(j_{t_f})_k = \begin{pmatrix} 2\,\mathrm{Im}(\alpha_k\beta_k) & -\frac{|\alpha_k^* + \beta_k|^2}{|k|} \\ |k||\alpha_k^* - \beta_k|^2 & -2\,\mathrm{Im}(\alpha_k\beta_k) \end{pmatrix}.
\tag{60}
$$

One can now easily determine the corresponding family of j_{t_f}-SRs. Comparing with the case (40) for j_0, and referring to the general form (32), we find a change in the terms B_k, which now become $B_k = -|\alpha_k^* + \beta_k|^2/|k|$ and correspond to a new Gaussian measure. In addition, we note the appearance of the term $A_k = 2\,\mathrm{Im}(\alpha_k\beta_k)$ (owing to the mixing between positive and negative frequency parts during evolution). The respective contribution $\mathbf{1} - iA_k$ in the general expression for the momentum operators (34) can be written in this case as $(\alpha_k + \beta_k^*)(\alpha_k^* - \beta_k)$. Thus,

$$
B_k^{-1}(\mathbf{1} - iA_k) = -|k|\frac{\alpha_k^* - \beta_k}{\alpha_k^* + \beta_k}.
\tag{61}
$$

Adopting the same Fourier space description as in Sec. III C, the j_{t_f}-SR is then realized in the Hilbert space $L^2(\mathcal{T}^\star, \mathrm{d}\mu_{t_f})$ defined by the Gaussian product measure

$$
\mathrm{d}\mu_{t_f} = \prod_{k\in\mathbb{N}} \frac{2|k|}{\pi|\alpha_k^* + \beta_k|^2} \exp\left(-\frac{2|k|}{|\alpha_k^* + \beta_k|^2}|\varphi_k|^2\right)\mathrm{d}\mu_k^0,
\tag{62}
$$

where $\mathrm{d}\mu_k^0$ is again the Lebesgue measure in $\mathbb{C}$.

The (Fourier components of the) basic field operators are now represented by

$$
\hat{\varphi}_k\Psi = \varphi_k\Psi,
\tag{63}
$$

$$
\hat{P}_\varphi^k\Psi = -i\frac{\partial\Psi}{\partial\varphi_k} + i|k|\frac{\alpha_k^* - \beta_k}{\alpha_k^* + \beta_k}\varphi_{-k}\Psi.
\tag{64}
$$

Notice that, in order to avoid an excessively complicated notation, we have used the same symbols as in Eqs. (46) and (47) to denote quantum operators and states in the j_{t_f}-SR. For completeness, let us also present the form of the annihilation and creation operators of the j_{t_f}-SR, which are given by

$$
\hat{\bar{b}}_k(t_f) = \frac{\alpha_k^* + \beta_k}{\sqrt{2|k|}}\frac{\partial}{\partial\varphi_{-k}},
$$

$$
\hat{\bar{b}}_k^\dagger(t_f) = -\frac{\alpha_k + \beta_k^*}{\sqrt{2|k|}}\frac{\partial}{\partial\varphi_k} + \frac{\sqrt{2|k|}}{\alpha_k^* + \beta_k}\varphi_{-k}.
\tag{65}
$$

As we have discussed above, they represent the classical variables $\{(\bar{b}_k(t_f), \bar{b}_k^*(t_f))\}$. The quantization of the variables $\{(b_k, b_k^*)\}$ in this representation can be obtained from (the inverse of) relations (52), or from Eqs. (63) and (64), using relation (14).

Let us now analyze the issue of unitarity in this context, namely, the unitary equivalence between the j_0-SR and the j_{t_f}-SRs. We first remark that, since unitarity is granted for any finite number of degrees of freedom, unitary equivalence (for a case of compact spatial topology such as the present one) rests just on the behavior of the high frequency modes. In our case, the asymptotic limit for large k of the sequences $\beta_k(t, t_0)$ and $\alpha_k(t, t_0)$ is zero and one, respectively. Therefore, the factors in the measure (62) and the momentum operators (64) approach the corresponding expressions for the j_0-SR. Actually, this is a necessary condition for unitarity, but not sufficient. Unitary equivalence between the j_{t_f} and the j_0 representations amounts to requiring that $j_{t_f} - j_0$ be a Hilbert-Schmidt operator. In turn, this is equivalent to the summability of the sequences $\{|\beta_k|^2\}$, a condition which is indeed satisfied, as shown in Refs. [7,8]. So, all the representations in the 1-parameter family of j_{t_f}-SRs are equivalent to the j_0-SR, and hence any two members of the family are equivalent to each other.

Consider now in more detail the momentum operators (64), and, in particular, the extra multiplicative term (that cannot be obtained from the measure)

$$
-B_k^{-1}A_k = 2|k|\frac{\mathrm{Im}(\alpha_k\beta_k)}{|\alpha_k^* + \beta_k|^2},
\tag{66}
$$

coming from the diagonal component $2\,\mathrm{Im}(\alpha_k\beta_k)$ in $(j_{t_f})_k$. The presence of this term means that the unitary group generated by the momentum operators is not simply the natural unitary implementation in $L^2(\mathcal{T}^\star, \mathrm{d}\mu_{t_f})$ of translations (by elements of $\mathcal{T}$) in $\mathcal{T}^\star$. In addition to the contribution coming from the transformation under translations of the quasi-invariant measure μ_{t_f} [which corresponds to the term $-iB_k^{-1}$ in Eq. (64)], the elements of that unitary group carry additional (nonconstant) phases. Such phases, responsible for the extra term in Eq. (64), can be viewed in our case as generated by the unitary transformation $T: L^2(\mathcal{T}^\star, \mathrm{d}\mu_{t_f}) \rightarrow L^2(\mathcal{T}^\star, \mathrm{d}\mu_{t_f})$, with

$$
(T\Psi)[\varphi_k] = \exp\left(i\sum_{k=1}^\infty B_k^{-1}A_k|\varphi_k|^2\right)\Psi[\varphi_k]
$$

$$
= \exp\left(-i\sum_{k=1}^\infty \frac{2|k|\,\mathrm{Im}(\alpha_k\beta_k)}{|\alpha_k^* + \beta_k|^2}|\varphi_k|^2\right)\Psi[\varphi_k].
\tag{67}
$$

In fact, one can check that T^{-1} maps the j_{t_f}-SR to the representation defined by the complex structure $\bar{j}_{t_f}$, with

$$
(\bar{j}_{t_f})_k = \begin{pmatrix} 0 & -\frac{|\alpha_k^* + \beta_k|^2}{|k|} \\ \frac{|k|}{|\alpha_k^* + \beta_k|^2} & 0 \end{pmatrix}.
\tag{68}
$$

It is also worth noting that the summability of $\{|\beta_k|^2\}$

guarantees that the unitary transformation T is well defined.[10]

Adopting the above perspective, the unitary transformation mapping the j_{t_f}-SR to the j_0-SR can be obtained as the composition of T^{-1} with the natural unitary transformation between the $\bar{j}_{t_f}$-SR and the j_0-SR, namely $\Psi \mapsto (d\mu_{t_f}/d\mu)^{1/2}\Psi$. We also notice that the existence of both derivatives $d\mu_{t_f}/d\mu$ and $d\mu/d\mu_{t_f}$, i.e. the mutual absolute

continuity of the Gaussian measures, depends on whether the operator $C_{t_f}C^{-1} - \mathbf{1}$ is Hilbert-Schmidt, where C and C_{t_f} denote the covariances of μ and μ_{t_f}, respectively. In the present case this leads to the condition that $\{|\alpha_k^* + \beta_k|^2 - 1\}$ be a square summable sequence. Again, this is ensured by the summability of $\{|\beta_k|^2\}$.

Summarizing, the unitary transformation mapping the j_{t_f}-SR to the j_0-SR is the multiplicative transformation

$$L^2(\mathcal{T}^\star, d\mu_{t_f}) \ni \Psi \mapsto \left(\frac{d\mu_{t_f}}{d\mu}\right)^{1/2} \exp\left(i \sum_{k=1}^{\infty} \frac{2|k|\mathrm{Im}(\alpha_k\beta_k)}{|\alpha_k^* + \beta_k|^2} |\varphi_k|^2\right)\Psi \in L^2(\mathcal{T}^\star, d\mu). \tag{69}$$

Of course, the multiplicative factor in this expression is simply the image of the unit functional $\Psi_0[\varphi_k] = 1$ of the j_{t_f}-SR, and therefore supplies the state $|0, t_f\rangle$ (53) in the j_0-SR, namely, it coincides with $\Psi_0^{(t_f)}$ given in Eq. (57) [one can check this by introducing the explicit form of $d\mu_{t_f}/d\mu$ obtained from Eqs. (45) and (62)].

Finally, we want to comment that any unitary transformation between two SRs admits a form like that displayed in Eq. (69). In fact, given two normalized measures μ_1 and μ_2 (not necessarily Gaussian), if a unitary transformation U exists such that it maps one SR to the other, then it is necessarily of the multiplicative form $\Psi \mapsto \Psi_0\Psi$, where Ψ_0 is the image under U of the unit functional.[11] Moreover, the identity $\int |\Psi|^2 d\mu_1 = \int |\Psi_0|^2 |\Psi|^2 d\mu_2$, valid $\forall \Psi$, implies that μ_1 is continuous with respect to μ_2, with $d\mu_1/d\mu_2 = |\Psi_0|^2$. By interchanging the roles of μ_1 and μ_2, one concludes that the measures are mutually continuous. Thus, the equivalence of the measures is a necessary condition for the unitary equivalence between two SRs, and any possible unitary equivalence is of the form $\Psi \mapsto (d\mu_1/d\mu_2)^{1/2}e^{iF}\Psi$, where F is a real functional. As one can easily realize from the discussion of Ref. [15], in the case of two representations defined by

equivalent complex structures, the functional F is a bilinear form of the type appearing in Eq. (67) and its introduction results in a modification of the action of the momentum operators by linear terms.

V. CONCLUSION

In full canonical quantum gravity formulated on a compact spatial section Σ, there is no fundamental notion of time. There is no Hamiltonian, and therefore no time with respect to which one might define evolution (this is one of the manifestations of the notorious problem of time). The Gowdy model that we have considered here is somewhat special in this respect since, through a partial gauge fixing, a particular notion of internal time is introduced in order to "deparametrize" the theory. Even when this parameter has no physical meaning in the final description, it is used as an intermediate step in order to construct the corresponding physical operators that define the true quantum geometry. This is the strategy that has also been followed in the quantization of homogeneous cosmologies [1]. Therefore, within the model, it is important to implement this notion of time evolution in a unitary way. Furthermore, the strategy that we have followed of implementing the internal notion of time at the quantum level, together with the remaining gauge group, receives support from the fact that a quantization with such properties exists [7,8] and is essentially unique [6,9]. This consistent quantization has to be contrasted to a previous proposal [4] that does not admit a unitary time evolution [5].

The purpose of this paper was to bridge the gap between the formalism of Refs. [7,8] and the standard formulation of canonical quantum gravity, and thus to recast the quantization of the Gowdy model into the Schrödinger functional representation, where the states of the theory are functionals $\Psi(\bar{\varphi})$ on the quantum configuration space. Let us now summarize the results found here. First, we have constructed the Schrödinger functional version of this quantum Gowdy model, and analyzed the (unitary) time evolution in this context. Second, we have solved the remaining constraint that is present in the model. In this way, we have been able to define the space of physical

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[10]In general, the presence of phases in the unitary representation of the group of (appropriate) translations in the quantum configuration space is a source of unitary inequivalence in quantum field theory, in addition to the existence of nonequivalent quasi-invariant measures in infinite dimensions (see, e.g., Refs. [20,21]). From the viewpoint of the momentum operators, rather than from that of the corresponding unitary group, this issue was addressed more recently in Ref. [15], where the possible lack of unitary equivalence between representations with and without an extra linear term in the momentum operators was discussed, and related to the possibility or impossibility of defining unitarity transformations of the type (67).

[11]This can be seen using the fact that, by construction, configuration operators such as the unitary groups generated by the basic field operators generate a dense set when applied to the unit functional. On the other hand, the action of the configuration operators is the same in both representations. Thus, for $\Psi = e^{i\bar{\varphi}(f)}$, $U\Psi = Ue^{i\hat{\bar{\varphi}}[f]}1 = Ue^{i\hat{\bar{\varphi}}[f]}U^{-1}\Psi_0 = e^{i\bar{\varphi}(f)}\Psi_0$. The general expression follows from linearity and continuity.

states in the Schödinger picture, where unitary evolution is again well defined.

As a general strategy, we have approached the problem from a functional perspective. In this fashion, we have constructed explicitly the 1-parameter family of representations that gives rise to the quantum description at any time. These different representations are unitarily equivalent precisely because time evolution is unitarily implementable. We have discussed in some detail the unitary transformations between such representations, confirming that, in the Schrödinger representation, they are associated with a corresponding 1-parameter family of mutually continuous measures in the quantum configuration space. This has to be contrasted with the functional description [19] of the quantization proposed in Ref. [4], which does not admit unitary evolution. In that case, the fact that the dynamics fails to be unitarily implementable implies that any two representations at different times correspond to inequiva-lent measures. In fact, a 1-parameter family of mutually singular measures is obtained in that case [19].

To conclude, our functional representation leads to a consistent framework where the standard probabilistic interpretation of quantum physics is applicable. In particular, the Heisenberg and Schrödinger pictures are well defined and conciliated. The present description can thus be taken as a starting point for a detailed study of the quantum geometric aspects of linearly polarized Gowdy models.

[1] C. W. Misner, in *Magic Without Magic: John Archibald Wheeler*, edited by J. Klauder (Freeman, San Francisco, 1972).

[2] M. Bojowald, Living Rev. Relativity **8**, 11 (2005); A. Ashtekar, M. Bojowald, and J. Lewandowski, Adv. Theor. Math. Phys. **7**, 233 (2003); A. Ashtekar, T. Pawlowski, and P. Singh, Phys. Rev. D **74**, 084003 (2006).

[3] R. H. Gowdy, Ann. Phys. (N.Y.) **83**, 203 (1974); C. G. Torre, Int. J. Theor. Phys. **38**, 1081 (1999).

[4] M. Pierri, Int. J. Mod. Phys. D **11**, 135 (2002).

[5] A. Corichi, J. Cortez, and H. Quevedo, Int. J. Mod. Phys. D **11**, 1451 (2002); C. G. Torre, Phys. Rev. D **66**, 084017 (2002); J. Cortez and G. A. Mena Marugán, Phys. Rev. D **72**, 064020 (2005).

[6] J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Phys. Rev. D **75**, 084027 (2007).

[7] A. Corichi, J. Cortez, and G. A. Mena Marugán, Phys. Rev. D **73**, 041502 (2006).

[8] A. Corichi, J. Cortez, and G. A. Mena Marugán, Phys. Rev. D **73**, 084020 (2006).

[9] A. Corichi, J. Cortez, G. A. Mena Marugán, and J. M. Velhinho, Classical Quantum Gravity **23**, 6301 (2006).

[10] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, NBS Appl. Math. Ser. Vol. LV (U.S. GPO, Washington, DC, 1970), 9th ed.

[11] C. G. Torre and M. Varadarajan, Classical Quantum Gravity **16**, 2651 (1999).

[12] D. Shale, Trans. Am. Math. Soc. **103**, 149 (1962); R. Honegger and A. Rieckers, J. Math. Phys. (N.Y.) **37**, 4292 (1996).

[13] A. Corichi, J. Cortez, and H. Quevedo, Phys. Rev. D **66**, 085025 (2002); Ann. Phys. (N.Y.) **313**, 446 (2004).

[14] A. Ashtekar and A. Magnon, Proc. R. Soc. A **346**, 375 (1975).

[15] A. Corichi, J. Cortez, and H. Quevedo, Classical Quantum Gravity **20**, L83 (2003).

[16] J. Glimm and A. M. Jaffe, *Quantum Physics: A Functional Integral Point of View* (Springer, New York, 1987).

[17] J. M. Mourão, T. Thiemann, and J. M. Velhinho, J. Math. Phys. (N.Y.) **40**, 2337 (1999).

[18] A. Ashtekar, J. Lewandowski, D. Marolf, J. M. Mourão, and T. Thiemann, J. Math. Phys. (N.Y.) **36**, 6456 (1995); D. Marolf, arXiv:gr-qc/9508015.

[19] C. G. Torre, Classical Quantum Gravity **24**, 1 (2007).

[20] I. M. Gel'fand and N. Vilenkin, *Generalized Functions* (Academic Press, New York, 1964), Vol. IV.

[21] J. Baez, I. Segal, and Z. Zhou, *Introduction to Algebraic and Constructive Quantum Field Theory* (Princeton University Press, Princeton, NJ, 1992).