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Asymptotics of regulated field commutators for Einstein-Rosen waves

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We discuss the asymptotic behavior of regulated field commutators for linearly polarized, cylindrically symmetric gravitational waves and the mathematical techniques needed for this analysis. We concentrate our attention on the effects brought about by the introduction of a physical cutoff in the study of the microcausality of the model and describe how the different physically relevant regimes are affected by its presence. Specifically we discuss how genuine quantum gravity effects can be disentangled from those originating in the introduction of a regulator. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1864251]

I. INTRODUCTION

Linearly polarized cylindrical waves, also known as Einstein-Rosen waves,[1,2] provide a symmetry reduction of general relativity that can be used as a test bed for the quantization of the theory. This system displays several interesting features that contribute to its relevance. On one hand, it has an infinite number of local degrees of freedom and, hence, it is a genuine quantum field theory (in contradistinction to other symmetry reductions, such as Bianchi models, that have a finite number of global degrees of freedom). On the other, the system is tractable both classically and quantum mechanically, thus allowing us to derive exact consequences independent of any approximation scheme.[3–8] The main reason behind this success and tractability is the fact that the gravitational degrees of freedom of the model are encoded in a free, massless, axially symmetric, scalar field that evolves in an auxiliary Minkowskian background.

In previous papers we have analyzed the issue of microcausality in this system; in particular, we have studied in detail the smearing of light cones owing to the quantization of the gravitational field.[7,8] The main tool for this type of analysis is the study *in vacuo* of the field commutator evaluated at different space–time points. As is well known, the commutator of quantum fields reflects the causal structure of space–time (Minkowskian space–time in ordinary perturbative quantum field theory) in the sense that the quantum fields in spatially separated space–time points commute. This is true for all standard types of quantum fields, i.e., scalar, fermion, or vector fields, though issues related to gauge invariance must be carefully considered in this last case. In the specific model that we are interested in, gauge invariance has been discussed in Ref. 9. The authors of that paper conclude that it is correct to use the Ashtekar-Pierri gauge fixed action, written in terms of the axially symmetric scalar field, to derive gauge invariant information about the model.

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In a recent work[8] we discussed the situation when no cutoff is introduced in the system, studying the unregulated commutator. The main results reached were the following. First, one can clearly see that light cones are smeared by quantum gravity effects; in fact it is possible to obtain a quantitative measure of this smearing and show how sharp light cones are recovered in the limit of large distances as compared to the natural length scale of the model, the Planck length. It is also interesting to point out that the asymptotic behavior of the commutator in the different physically relevant regimes strongly depends on the causal relationship between the different space–time points involved. Second, one finds a singularity structure in the commutator that differs from that of the free theory; in particular, the field commutator for equal values of the radial coordinate R is singular. Finally, one observes that, in the case when one of the space–time points that appear in the commutator corresponds to the symmetry axis, there are quantum effects that persist for large values of the difference of the time coordinates. This effect is reminiscent of the large quantum gravity effects first discussed by Ashtekar.[4,6,10,11]

The purpose of this paper is to study how the conclusions of Ref. 8 are changed by the introduction of a cutoff. As is well known, regulators are generally necessary in order to have well-defined quantum field theories. One can justify its use, for example, by noticing that the action of the field operator on the vacuum in a Fock space is not a vector in the Hilbert space because it has infinite norm. In order to have a well-defined action of the field operator one regulates it by introducing smearing functions that render the norms of these states finite. The problem then consists in removing these regulators (or rather showing that the physical results are independent of them).

In principle, it is possible to argue that the results derived in the absence of regulators somehow approximate those derived after their introduction; this is straightforward to see in the case of cutoffs. In the presence of a cutoff Λ, the improper integrals that define the field commutator become proper because the integration region is a closed interval $[0,\Lambda]$. For a given value of the parameters that appear in the integral (involving the values of the space–time coordinates of the quantum fields and the gravitational constant) it is always possible to choose a value for Λ such that the integral with the cutoff is well approximated by the integral extended to $[0,\infty)$.

Of course it is conceivable that the cutoff is not just a mathematical device but rather a physical scale defining a fundamental limit for the resolution of our measurements. If space–time becomes discrete at short distances (such as the Planck length), the continuum space–time picture breaks down and, certainly, it would be difficult to justify the extension of the integrals involved in the definition of field commutators (or S matrix elements, for that matter) to infinite intervals in momenta (inverse length). Our point of view here is that the introduction of a cutoff can mimic some of the effects appearing after a successful quantization of gravity (for example, in the loop quantum gravity approach), and hence we plan to study its effect within the consistent framework provided by the Einstein-Rosen waves. It is also interesting to point out here that the cutoff by itself can produce some of the effects expected from quantum gravity. In particular, it is possible to show that light cones are also smeared by cutoffs.[12] In our opinion this makes it necessary to study in detail how the effects of the cutoff and quantum gravity can be disentangled.

The paper is organized as follows. After this introduction, we briefly review the main results about microcausality in quantum cylindrical gravitational waves and introduce the commutators that we will discuss in the rest of the paper. We will then study the field commutators in the presence of a cutoff with the help of the asymptotic techniques already employed in Ref. 8. Here the situation is simpler because we will only have to consider integrals over closed intervals. We will discuss one by one the asymptotic behaviors in the different parameters involved. In Sec. VI we will derive a power series expansion in the gravitational constant for the commutator in the presence of a cutoff in the spirit of ordinary perturbative quantum field theory, and discuss the uniform convergence of this series under appropriate conditions on the cutoff in Sec. VII. We end the paper with a discussion of our results and our conclusions.

II. THE FIELD COMMUTATOR

Einstein-Rosen waves describe topologically trivial space–times with two linearly independent, commuting, spacelike, and hypersurface orthogonal Killing vector fields[1,2,13] endowed with a metric that can be written as

$$ds^2 = e^{\gamma-\psi}(-dT^2 + dR^2) + e^{-\psi}R^2 d\theta^2 + e^{\psi}dZ^2. \tag{1}$$

Here we use the coordinates (T,R,θ,Z), $T\in\mathbb{R}$, $R\in[0,\infty)$, $\theta\in[0,2\pi)$, $Z\in\mathbb{R}$, and ψ and γ are functions only of R and T. The Einstein field equations are very simple. The scalar field ψ satisfies the wave equation for a massless, axially symmetric scalar field in three dimensions,

$$\partial_T^2\psi - \partial_R^2\psi - \frac{1}{R}\partial_R\psi = 0,$$

and the function γ can be expressed in terms of this field[3,6] as

$$\gamma(R) = \frac{1}{2}\int_0^R d\overline{RR}[(\partial_T\psi)^2 + (\partial_{\overline{R}}\psi)^2].$$

We will use in the following a system of units such that $c=\hbar=1$ and define $G\equiv\hbar G_3$, where G_3 denotes the gravitational constant per unit length in the direction of the symmetry axis.[14] The function $\gamma(R)$ (apart from a factor of $8G$) has a simple physical interpretation: it is the energy of the scalar field in a ball of radius R whereas γ_∞ denotes $\lim_{R\to\infty}\gamma(R)$ (the energy of the whole two-dimensional flat space). It is also possible to show[3,7] that $\gamma_\infty/(8G)$ coincides with the Hamiltonian H_0 of the system obtained by a linearization of the metric (1).

In order to have a unit asymptotic timelike Killing vector and a physical notion of energy (per unit length) we introduce the coordinates (t,R,θ,Z) defined by $T=e^{-\gamma_\infty/2}t$. In these coordinates the metric takes the form[2,13]

$$ds^2 = e^{\gamma-\psi}(-e^{-\gamma_\infty}dt^2 + dR^2) + e^{-\psi}R^2 d\theta^2 + e^{\psi}dZ^2.$$

By taking a sufficiently fast fall-off for ψ as $R\to\infty$, this metric describes asymptotically flat cylindrical space–times such that ∂_t is a unit timelike Killing vector in the asymptotic region. In the $2+1$-dimensional framework these space–times are asymptotically flat at spacelike and null infinities[15,16] (the appropriate introduction of null infinity will be needed in order to study the S matrix of the model). It is also worthwhile noting that these space–times have a nonzero deficit angle.

The Einstein field equations can be obtained from a Hamiltonian action principle.[13,17,18] A remarkable (and useful) feature of the physical Hamiltonian H (associated with the physical time t) is the fact that it is a function of the Hamiltonian corresponding to the free scalar field, H_0:

$$H = E(H_0) = \frac{1}{4G}(1 - e^{-4GH_0}).$$

In terms of the T-time and imposing regularity at the axis $R=0$[3], the classical solutions for the field ψ can be written as

$$\psi(R,T) = \sqrt{4G}\int_0^\infty dk J_0(Rk)[A(k)e^{-ikT} + A^\dagger(k)e^{ikT}],$$

where $A(k)$ and its complex conjugate $A^\dagger(k)$ are fixed by the initial conditions. The free Hamiltonian H_0 can be written now as

$$\frac{\gamma_\infty}{8G} = H_0 = \int_0^\infty dk k A^\dagger(k) A(k).$$

Using this expression, we obtain the t-evolution of the field

$$\psi_E(R,t) = \sqrt{4G}\int_0^\infty dk J_0(Rk)[A(k)e^{-ikte^{-\gamma_\infty/2}} + A^\dagger(k)e^{ikte^{-\gamma_\infty/2}}].$$

The quantization can be carried out in the usual way by introducing a Fock space where $\hat{\psi}(R,0)$, the quantum counterpart of $\psi(R,0)$, is the operator-valued distribution[19] given by[20]

$$\hat{\psi}(R,0;\Lambda=\infty) = \hat{\psi}_E(R,0;\Lambda=\infty) = \sqrt{4G}\int_0^\infty dk J_0(Rk)[\hat{A}(k) + \hat{A}^\dagger(k)]. \tag{2}$$

Its action on Fock space is determined by those of $\hat{A}(k)$ and $\hat{A}^\dagger(k)$, the annihilation and creation operators, with nonvanishing commutators given by $[\hat{A}(k_1),\hat{A}^\dagger(k_2)] = \delta(k_1,k_2)$.

We can regulate the field by introducing suitable functions $g(k)$ that render finite the norms of the states obtained by acting with the quantum field on Fock space vectors. In the following we will make the simplest choice $g(k) = \chi_{[0,\Lambda_k]}(k)$ (here $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$). By doing this the integration region in (2) becomes compact and we have

$$\hat{\psi}(R,0) = \hat{\psi}_E(R,0) = \sqrt{4G}\int_0^{\Lambda_k} dk J_0(Rk)[\hat{A}(k) + \hat{A}^\dagger(k)]. \tag{3}$$

Evolution in T is given by the unitary operator $\hat{U}_0(T) = \exp(-iT\hat{H}_0)$, where

$$\hat{H}_0 = \int_0^\infty dk k \hat{A}^\dagger(k)\hat{A}(k)$$

is the quantum Hamiltonian operator of a three-dimensional, axially symmetric scalar field. The cutoff-regulated quantum scalar field in the Heisenberg picture is hence given by

$$\hat{\psi}(R,T) = \hat{U}_0^\dagger(T)\hat{\psi}(R,0)\hat{U}_0(T) = \sqrt{4G}\int_0^{\Lambda_k} dk J_0(Rk)[\hat{A}(k)e^{-ikT} + \hat{A}^\dagger(k)e^{ikT}].$$

If we describe the evolution in our model in terms of the physical time t, the quantum Hamiltonian is $\hat{H} = E(\hat{H}_0) = (1-e^{-4G\hat{H}_0})/(4G)$ and unitary evolution is given by $\hat{U}(t) = \exp(-it\hat{H})$. With this time evolution the annihilation and creation operators in the Heisenberg picture are

$$\hat{A}_E(k,t) \equiv \hat{U}^\dagger(t)\hat{A}(k)\hat{U}(t) = \exp[-itE(k)e^{-4G\hat{H}_0}]\hat{A}(k),$$

$$\hat{A}_E^\dagger(k,t) = \hat{A}^\dagger(k)\exp[itE(k)e^{-4G\hat{H}_0}],$$

where $E(k) = (1-e^{-4Gk})/(4G)$, and the regulated field operator evolved with the physical Hamiltonian [that we denote as $\hat{\psi}_E(R,t)$] is given by

$$\hat{\psi}_E(R,t) = \sqrt{4G}\int_0^{\Lambda_k} dk J_0(Rk)[\hat{A}_E(k,t) + \hat{A}_E^\dagger(k,t)].$$

The field commutator $[\hat{\psi}_E(R_1,t_1),\hat{\psi}_E(R_2,t_2)]$ can be computed from these expressions.[7] Since we are dealing with an effectively interacting theory this operator is not proportional to the identity in the Fock space basis that we are using and, hence, we have to consider its matrix elements. As in

previous work we will concentrate on the vacuum expectation value. If a cutoff Λ_k is introduced, this is given by

$$\frac{1}{8iG}\langle 0|[\hat{\psi}_E(R_1,t_1),\hat{\psi}_E(R_2,t_2)]|0\rangle = \int_0^{\Lambda_k} dk J_0(R_1k)J_0(R_2k)\sin\left[\frac{t_2-t_1}{4G}(1-e^{-4Gk})\right], \tag{4}$$

which can be seen to depend on the time coordinates only through their difference t_2-t_1, which we will assume in the following to be positive. Notice that it depends symmetrically on R_1 and R_2. The functional dependence in G is less trivial, a fact that requires special attention when studying the limit in which the relevant lengths and time differences are much larger than the Planck length.[8]

It is convenient to refer the dimensional parameters of these integrals to another length scale, which we choose as R_1. We hence introduce $R_2=\rho R_1$, $t_2-t_1=R_1\tau$, and $\lambda=R_1/4G$ and rewrite (4) as

$$\frac{1}{8iG}\langle 0|[\hat{\psi}_E(R_1,t_1),\hat{\psi}_E(R_2,t_2)]|0\rangle = \frac{\lambda}{R_1}\Im\left\{\int_0^{\Lambda_q} dq J_0(\lambda q)J_0(\rho\lambda q)e^{i\tau\lambda(1-e^{-q})}\right\} \tag{5}$$

after introducing the new variable $k=q/(4G)$. Here $\Im$ denotes the imaginary part and $\Lambda_q=4G\Lambda_k$.

Physically, the cutoff Λ_k (which in principle has the dimensionality of an inverse length) could be interpreted as having its origin in the existence of a minimum length. This comes out naturally in loop quantum gravity where space is discrete and the area and volume operators have minimum eigenvalues of the order of the Planck area and volume, respectively. In fact, the existence of a minimum length (of the size of the Planck scale) can be considered a generic feature of essentially every quantum theory.[21] The interpretation of the adimensional cutoff Λ_q would follow from that of Λ_k, so it may be reasonable to expect that it be a number of order unity; nevertheless we will treat it as a free parameter in the following.

III. ASYMPTOTIC BEHAVIOR IN ρ

Let us start by considering the behavior of (5) when the parameter ρ grows to infinite or approaches $\rho=0$. This integral can be written as a standard h-transform[22] by the change of variables $t=q\lambda$. The most convenient way to get its asymptotic behavior in $\rho\to\infty$ is by rewriting it in the form

$$\frac{1}{R_1}\Im\left[\int_0^{\infty} dt J_0(\rho t)J_0(t)e^{i\tau\lambda(1-e^{-t/\lambda})} - \int_{R_1\Lambda_k}^{\infty} dt J_0(\rho t)J_0(t)e^{i\tau\lambda(1-e^{-t/\lambda})}\right]. \tag{6}$$

One can then use the asymptotic behavior obtained for the first integral in Ref. 8, and find the asymptotics of the second integral by standard integration by parts [employing the fact that[23] $J_0(k)=-J_0'(k)/k-J_0''(k)$]. By doing this one gets the following two contributions:

$$\frac{1}{R_1}\left[\frac{\tau}{2\lambda\rho^3}+\frac{1}{\rho^5}\left(\frac{9\tau}{8\lambda}-\frac{3\tau}{8\lambda^3}+\frac{9\tau^2}{2\lambda}\right)\right]+O\left(\frac{1}{\rho^7}\right),$$

$$\frac{1}{\rho R_1}\Im[J_1(\Lambda_k R_1\rho)J_0(\Lambda_k R_1)e^{i\tau\lambda(1-e^{-\Lambda_k R_1/\lambda})}]+O\left(\frac{1}{\rho^{5/2}}\right).$$

The first one is cutoff independent but subdominant with respect to the second, hence we see that the presence of a cutoff changes the asymptotic behavior in ρ. This is the kind of behavior that one would expect even in a Lorentz covariant theory after the introduction of a cutoff because of the breaking of the Lorentz symmetry. The novel feature here is the presence of cutoff independent terms. Although the cutoff-dependent one dominates in the asymptotic limit, there may be a transient regime, whose onset will be controlled by the value of Λ_k, in which the asymptotic behavior is given by the first term. This will be most evident when $\Lambda_k\to\infty$.

In the $\rho \to 0$ limit we get

$$\frac{1}{R_1}\Im \int_0^{R_1\Lambda_k} dt J_0(t) e^{i\tau\lambda(1-e^{-t/\lambda})} + O(\rho) = \frac{\lambda}{R_1}\Im \int_0^{\Lambda_q} dq J_0(\lambda q) e^{i\tau\lambda(1-e^{-q})} + O(\rho), \tag{7}$$

as a result of the continuity at $\rho=0$ of the integral defining the commutator (5).

IV. ASYMPTOTIC BEHAVIOR IN τ

The integral in (5) has the convenient form of an *h*-transform and, hence, it can be studied by standard Mellin transform methods[22] if the asymptotic parameter is chosen to be ρ; however, this is no longer true if the asymptotic parameter is taken to be τ (which corresponds to considering large separations in the time coordinates). This fact introduces some mathematical difficulties in the asymptotic analysis. In this case one has to consider the cases $\rho=0$ and $\rho\neq 0$ separately.

If $\rho=0$, one finds that the asymptotic behavior when $\tau\to\infty$ without the cutoff is given by[8]

$$\frac{1}{R_1}\sqrt{\frac{\lambda}{2\pi\log\tau}}\Im\{e^{i[(\pi/4)+\tau\lambda-\lambda\log(\tau\lambda)]}e^{(\pi/2)\lambda}\Gamma(i\lambda) + e^{-i[(\pi/4)-\tau\lambda-\lambda\log(\tau\lambda)]}e^{-(\pi/2)\lambda}\Gamma(-i\lambda)\} + O\left(\frac{1}{\log\tau}\right), \tag{8}$$

whereas for $\rho\neq 0$ we get

$$\begin{aligned}
&\frac{1}{2\pi R_1\sqrt{\rho}\log\tau}\Im\{e^{i[(\pi/2)+\tau\lambda-\lambda(1+\rho)\log(\tau\lambda)]}e^{(\pi/2)\lambda(1+\rho)}\Gamma[i\lambda(1+\rho)] \\
&+ e^{-i[(\pi/2)-\tau\lambda-\lambda(1+\rho)\log(\tau\lambda)]}e^{-(\pi/2)\lambda(1+\rho)}\Gamma[-i\lambda(1+\rho)] \\
&+ e^{i[\tau\lambda-\lambda(1-\rho)\log(\tau\lambda)]}e^{(\pi/2)\lambda(1-\rho)}\Gamma[i\lambda(1-\rho)] \\
&+ e^{i[\tau\lambda-\lambda(\rho-1)\log(\tau\lambda)]}e^{(\pi/2)\lambda(\rho-1)}\Gamma[i\lambda(\rho-1)]\} + O\left(\frac{1}{(\log\tau)^2}\right).
\end{aligned} \tag{9}$$

The most interesting feature of these expressions is their unusual dependence on the asymptotic parameter τ; in fact, the dependence on inverse powers of logarithms (especially on the inverse square root of $\log\tau$) cannot be obtained by direct application of the usual asymptotic expressions derived by Mellin transform techniques.[8] It is also remarkable how slowly the commutator decays in τ, in particular in the axis $\rho=0$, a fact that is suggestive of the large quantum gravity effects discussed by Ashtekar.[4] Outside the axis the decay is faster but still quite slow. A consequence of the different asymptotic behaviors in τ for $\rho=0$ and $\rho\neq 0$ is the impossibility to recover (8) as the limit when $\rho\to 0$ of (9). As we can see, the frequency of the oscillations of the commutator in τ is controlled by λ (proportional to the inverse of G) in such a way that although the amplitude of the oscillations decays very slowly, they will average to zero on scales larger than the Planck length.

When we introduce a cutoff Λ_k, the above asymptotic expressions change to

$$\frac{1}{R_1}\Im\left\{\frac{i}{\tau}[1-e^{i\tau\lambda(1-e^{-\Lambda_k R_1/\lambda})}e^{\Lambda_k R_1/\lambda}J_0(\Lambda_k R_1)J_0(\rho\Lambda_k R_1)]\right\} + O\left(\frac{1}{\tau^2}\right), \tag{10}$$

valid both for $\rho=0$ and $\rho\neq 0$. This can be obtained by straightforward integration by parts. An interesting situation develops at this point because the asymptotic behavior of the integral in τ behaves in a discontinuous way in the cutoff. In the analysis carried out to study the asymptotic behavior in ρ we found out that the cutoff-dependent term, in spite of being dominant, goes to zero in the limit $\Lambda_k\to\infty$. Here the situation is different: taking now $\Lambda_k\to\infty$ in (10) does not lead to the asymptotic expressions corresponding to $\Lambda_k=\infty$. That is, the asymptotic behavior of the improper integral in (4) is not the limit when $\Lambda_k\to\infty$ of (5). As in the case of the asymptotics in ρ, one expects that there must be a transient regime in which the behavior in τ of (5) is given by (8) and

(9). We will not consider here a precise characterization of this transient behavior for arbitrary values of the relevant parameters because its main properties can be conveniently discussed, at least for large λ, by looking at the $\lambda\to\infty$ asymptotics of the commutator in the (ρ,τ) plane.

The $\tau\to 0$ limit is easy to analyze. In fact what we find, both with and without the cutoff, is that the series obtained by expanding $e^{i\pi\lambda(1-e^{-t/\lambda})}$ in powers of $e^{-t/\lambda}$, exchanging integration and infinite sum, and computing the resulting integrals gives a series that converges to the value of the commutator.

V. ASYMPTOTIC BEHAVIOR IN λ

The asymptotic behavior in λ is studied by following the procedure described in Ref. 8. It is worth remarking that the limit $\lambda\to\infty$ of the regulated field commutator cannot be identified with that in which the gravitational constant G vanishes if one admits that the dimensionful cutoff $\Lambda_k<\infty$ is kept constant in principle. On the contrary, the two limits could be considered equivalent only under the assumption that Λ_k increases as the inverse of G for small gravitational constant, so that its dimensionless counterpart $\Lambda_q=4G\Lambda_k$ may remain fixed.

The analysis of the asymptotics in λ when the cutoff is present is simultaneously simpler in some respects and more complicated in others compared with the case when no cutoff is introduced. It is simpler because the lengthy analysis needed to discuss the asymptotics of the improper integral is not necessary now. It is more complicated in the sense that the final asymptotic expressions contain additional terms and also because the number of regions with different λ-asymptotic regimes in the (ρ,τ) plane increases.

We have to consider now the cases $\rho=0$ and $\rho\neq 0$ separately. Let us consider first $\rho=0$ and write the rhs of (5) as[8]

$$\Im\left\{-\frac{i\lambda e^{i\tau\lambda}}{2\pi R_1}\int_0^{\Lambda_q}dq\oint_\gamma dt\frac{1}{t}e^{\lambda[(q/2)(t-1/t)-i\tau e^{-q}]}\right\},\tag{11}$$

after using the usual integral representation for the Bessel functions J_n $(n=0,1,\ldots)$,

$$J_n(z)=\frac{1}{2\pi i}\oint_\gamma\frac{dt}{t^{n+1}}e^{(z/2)(t-1/t)},$$

where γ is a closed, positively oriented, simple path in the complex plane surrounding the origin. Notice that we are integrating an integrable function in a compact region, so we can write the integrals in any order we want. The asymptotic analysis of (11) can be carried out by following the same steps as in Ref. 8. As we did there, it is useful to introduce neutralizers to split the integral in three pieces I_j, $j=1,2,3$, and choose appropriate contours for each of them. These integrals are

$$I_j\equiv\Im\left\{-\frac{i\lambda e^{i\tau\lambda}}{2\pi R_1}\int_0^{\Lambda_q}dq\oint_\gamma dt\nu_j(q)\frac{1}{t}e^{\lambda[(q/2)(t-1/t)-i\tau e^{-q}]}\right\},$$

where we have introduced the neutralizer functions $\nu_j(q)$, $j=1,2,3$, satisfying $\nu_1+\nu_2+\nu_3=1$ in $[0,\Lambda_q]$ and

$$\nu_1(q)=1 \text{ if } q\in[0,\alpha_1],$$

$$\nu_1(q)=0 \text{ if } q\in[\alpha_2,\Lambda_q],$$

$$\nu_2(q)=0 \text{ if } q\in[0,\alpha_1]\cup[\beta_2,\Lambda_q],$$

$$\nu_2(q)=1 \text{ if } q\in[\alpha_2,\beta_1],$$

$$\nu_3(q) = 0 \text{ if } q \in [0, \beta_1],$$

$$\nu_3(q) = 1 \text{ if } q \in [\beta_2, \Lambda_q],$$

with $0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2 < \Lambda_q$ (these parameters are chosen as in Ref. 8). By doing this the effective integration regions in q are $[0, \alpha_2]$, $[\alpha_1, \beta_2]$, and $[\beta_1, \Lambda_q]$ and the boundary $q=0$ appears only in the first.

The asymptotics in λ of the integral I_1 is best obtained by choosing an integration contour satisfying $\Re(t-1/t) \leqslant 0$ (that passes necessarily through $t=i$ and $t=-i$). By using the same method of Ref. 8 we see that the first two relevant terms are given by the contour integrals

$$\frac{1}{\pi R_1} \Im \left\{ i \oint_\gamma \frac{dt}{t^2 + 2i\pi t - 1} \right\},$$

$$- \frac{1}{2\pi R_1} \Im \left\{ \frac{i}{\lambda} \oint_\gamma dt \frac{8i\pi t^2}{(t^2 + 2i\pi t - 1)^3} \right\},$$

whose sum gives

$$\frac{1}{R_1 \sqrt{\tau^2 - 1}} \text{ for } \tau > 1$$

and

$$\frac{\tau(1 + 2\tau^2)}{2R_1 \lambda (1 - \tau^2)^{5/2}} \text{ for } \tau < 1.$$

Although the second term will be subdominant with respect to some of the contributions coming from I_2 and I_3, it improves the approximation of the full commutator obtained from the asymptotics in λ in the region $\tau < 1$.

The contribution of I_3 to the asymptotics in λ is obtained from the contour integral (corresponding to the boundary at $q=\Lambda_q$)

$$\Im \left\{ \frac{ie^{i\tau\lambda}}{2\pi R_1} \oint_\gamma dt \frac{2}{t} \frac{(t - 1/t + 2i\tau e^{-\Lambda_q}) e^{(\lambda\Lambda_q/2)(t-1/t) - i\tau e^{-\Lambda_q}}}{\Lambda_q^2 (t + 1/t)^2 - (t - 1/t + 2i\tau e^{-\Lambda_q})^2} \right\}.$$

The asymptotics in λ of this integral can be easily studied by using the method of steepest descents. This gives

$$\Im \left\{ \frac{2ie^{i\tau\lambda(1-e^{-\Lambda_q})}}{R_1 (\tau^2 e^{-2\Lambda_q} - 1) \sqrt{2\pi\lambda\Lambda_q}} \left[i\tau e^{-\Lambda_q} \sin\left(\lambda\Lambda_q - \frac{\pi}{4}\right) - \cos\left(\lambda\Lambda_q - \frac{\pi}{4}\right) \right] \right\}.$$

Finally, the integral I_2 (for which we choose for γ the curve $|t|=1$) only contributes when the stationary points of the exponent are in the integration region. This happens only when $1 < \tau < e^{\Lambda_q}$. The contribution to the first relevant order in λ is[8]

$$\Im \left\{ \frac{1}{R_1} \frac{e^{i\lambda(\tau - \log\tau - 1)}}{\sqrt{\log \tau}} \right\}.$$

Adding up the different terms we get

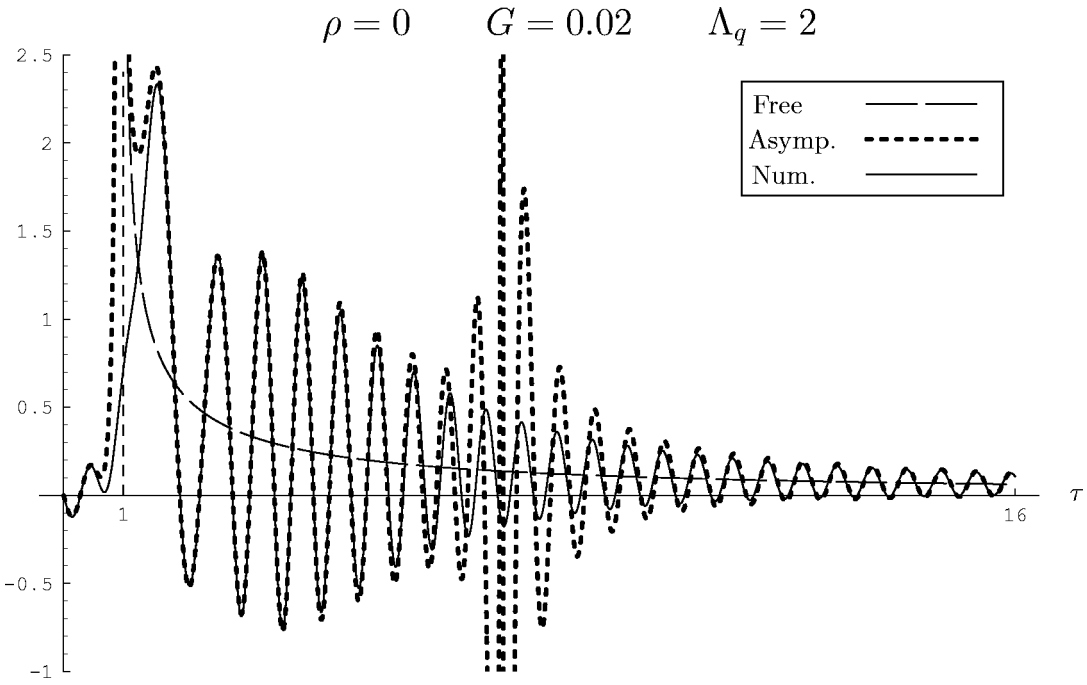

FIG. 1. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $\rho=0$, $G=0.02$, and $\Lambda_q=2$. We compare it both with a numerical computation of the integral that defines it and with the unregulated free commutator. As we can see, the approximation is good except at the points where the asymptotic behavior changes ($\tau=1$ and $\tau=e^{\Lambda_q}$). Notice the difference in the amplitude of the oscillations for $\tau<e^{\Lambda_q}$ and $\tau>e^{\Lambda_q}$.

$$\theta(1-\tau)\frac{\tau(1+2\tau^2)}{2R_1\lambda(1-\tau^2)^{5/2}}+\theta(\tau-1)\frac{1}{R_1\sqrt{\tau^2-1}}+\theta(\tau-1)\,\theta(e^{\Lambda_q}-\tau)\Im\left\{\frac{1}{R_1}\frac{e^{i\lambda(\tau-\log\,\tau-1)}}{\sqrt{\log\,\tau}}\right\}$$
$$+\Im\left\{\frac{2ie^{i\tau\lambda(1-e^{-\Lambda_q})}}{R_1(\tau^2e^{-2\Lambda_q}-1)\sqrt{2\pi\lambda\Lambda_q}}\left[i\tau e^{-\Lambda_q}\sin\left(\lambda\Lambda_q-\frac{\pi}{4}\right)-\cos\left(\lambda\Lambda_q-\frac{\pi}{4}\right)\right]\right\}, \tag{12}$$

where θ denotes the step function.

We see that the final result consists of several contributions: the free commutator for an infinite cutoff, a $1/\lambda$ correction for $\tau<1$, the term with the $1/\sqrt{\log\,\tau}$ dependence for $1<\tau<e^{\Lambda_q}$, and a cutoff-dependent contribution for all values of τ that fall off to zero when $\Lambda_q\to\infty$. If the cutoff goes to infinity, the commutator can be approximated by the one obtained in Ref. 8; however, if it is of order one (as would be the case if it is defined by the Planck length), then that approximation is no longer valid. Notice that the values of $\tau\in(1,e^{\Lambda_q})$ are those for which the asymptotics provided by the unregulated commutator are a correct approximation. This is roughly the transient region in the τ parameter mentioned in the previous subsection.

Figures 1–3 show the behavior of the field commutator (over $8iG$) when $\rho=0$ as a function of τ for several values of Λ_q. As we can see, the asymptotic approximation becomes singular between regions with different asymptotic regimes, but approximates well the exact value of the commutator (obtained by numerical methods) for the remaining values of τ. Notice that the singularity at $\tau=e^{\Lambda_q}$ of the asymptotic expansion lies outside the plotted region in Figs. 2 and 3.

In order to study the $\rho\neq0$ case we start by writing the rhs of (5) as

$$\Im\left\{-\frac{\lambda e^{i\tau\lambda}}{4\pi^2R_1}\int_0^{\Lambda_q}dq\oint_{\gamma_1}dt_1\oint_{\gamma_2}dt_2\frac{1}{t_1t_2}e^{\lambda[(q)(t_1-1/t_1)/2+\rho q(t_2-1/t_2)/2-i\tau e^{-q}]}\right\} \tag{13}$$

after employing the usual integral representation for the Bessel functions J_n ($n=0,1,\ldots$). Again it is helpful to introduce the same neutralizers as above to split the integral in three pieces I_j, $j=1,2,3$.

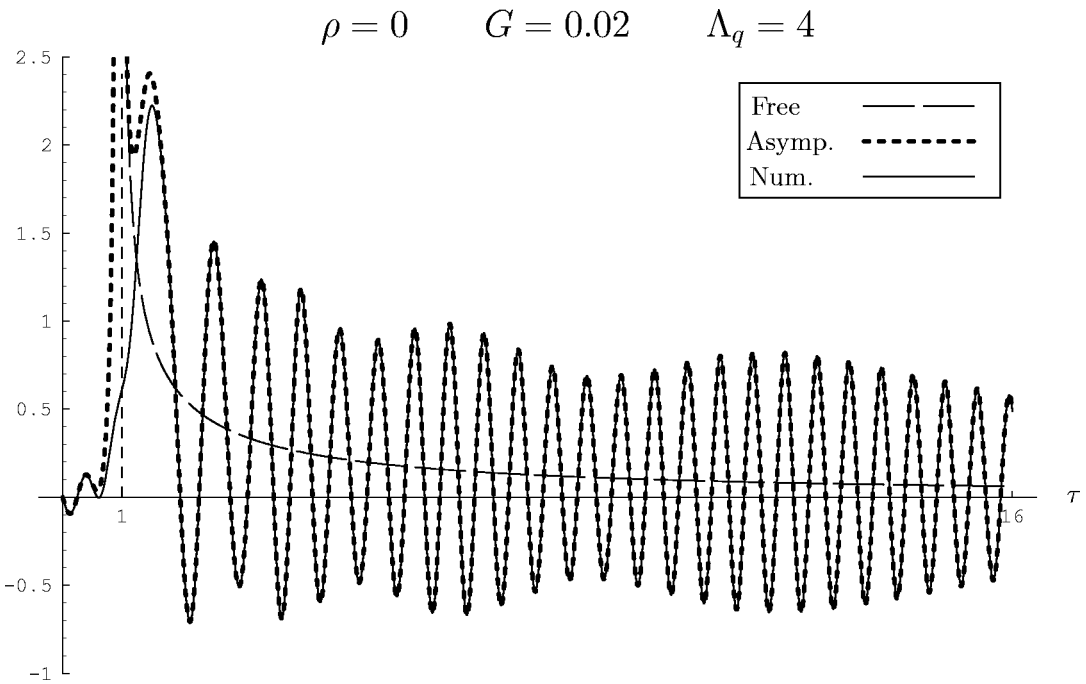

FIG. 2. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $\rho=0$, $G=0.02$, and $\Lambda_q=4$ (compared both with a numerical computation of the integral that defines it and with the unregulated free commutator). As we can see, the asymptotic approximation is good except at $\tau=1$. Notice that the cutoff introduces a modulation of the amplitude in the region $1<\tau<e^{\Lambda_q}$. The singularity of the asymptotic approximation at $\tau=e^{\Lambda_q}$ lies outside the plotted region.

The integral I_1 gives the following two contributions:

$$\Im\left\{\frac{1}{2\pi^2 R_1}\oint_{\gamma_1} dt_1 \oint_{\gamma_2} dt_2 \frac{1}{\rho t_1(t_2^2-1)+t_2(t_1^2+2i\pi t_1-1)}\right\} \tag{14}$$

and

$$\Im\left\{-\frac{2i\tau}{\pi^2 R_1 \lambda}\oint_{\gamma_1} dt_1 \oint_{\gamma_2} dt_2 \frac{t_1^2 t_2^2}{[\rho t_1(t_2^2-1)+t_2(t_1^2+2i\pi t_1-1)]^3}\right\}. \tag{15}$$

Both integrals can be computed exactly[8] in terms of complete elliptic integrals of first and second kinds. The first one gives the contribution of the unregulated free commutator (i.e., with infinite cutoff).

In order to describe it we define regions I, II, and III by $0<\tau<|\rho-1|$, $|\rho-1|<\tau<\rho+1$, and $\rho+1<\tau$, respectively. They are shown in Fig. 4. In region I the free commutator is zero, whereas in regions II and III it is given by

region II,

$$\frac{1}{\pi R_1\sqrt{\rho}} K\left(\sqrt{\frac{\tau^2-(\rho-1)^2}{4\rho}}\right),$$

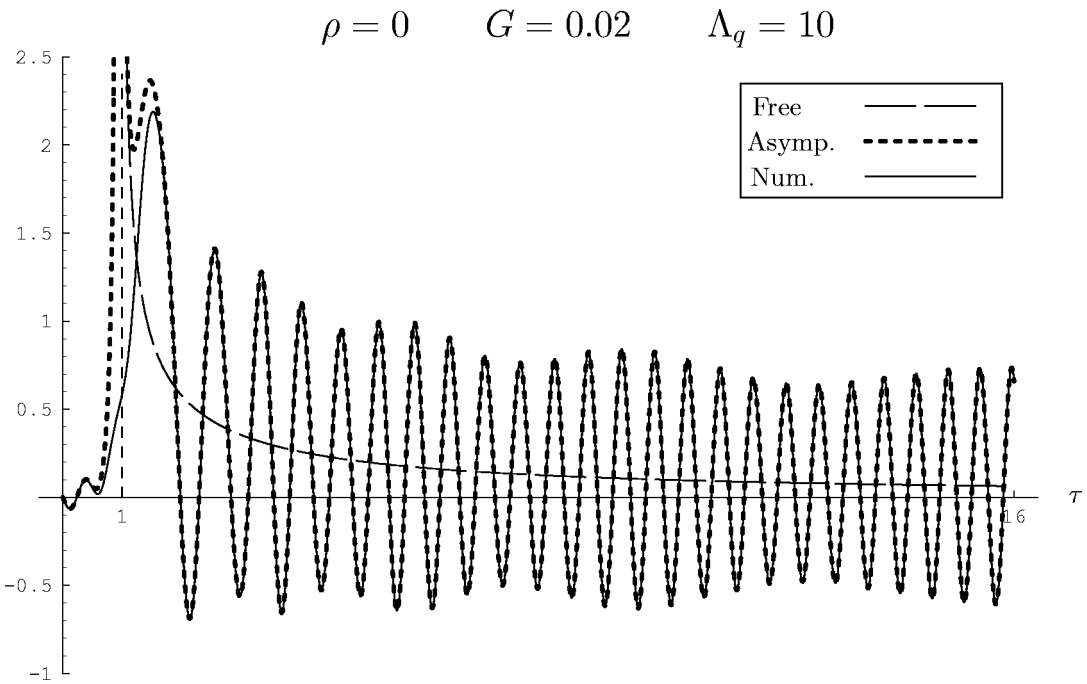

FIG. 3. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $\rho=0$, $G=0.02$, and $\Lambda_q=10$ (compared both with a numerical computation of the integral that defines it and with the unregulated free commutator). As we can see, the asymptotic approximation is good except at $\tau=1$. The asymptotic approximation obtained in Ref. 8 is good in a large region in the τ axis. The singularity of the asymptotic approximation at $\tau=e^{\Lambda_q}$ lies outside the plotted region.

region III,

$$\frac{2}{\pi R_1}\frac{1}{\sqrt{\tau^2-(1-\rho)^2}}K\left(\sqrt{\frac{4\rho}{\tau^2-(1-\rho)^2}}\right).$$

The second contribution (15) can be computed by the method outlined in Appendix IV of Ref. 8, obtaining

region I,

$$\frac{\tau}{2\pi R_1\lambda}\left\{\frac{2[1+\rho^4+2\tau^2-3\tau^4+2\rho^2(\tau^2-1)]\sqrt{(1+\rho)^2-\tau^2}}{(1+\rho-\tau)^2(1-\rho+\tau)^2(-1+\rho+\tau)^2(1+\rho+\tau)^2}E\left(\sqrt{\frac{4\rho}{(1+\rho)^2-\tau^2}}\right)\right.$$
$$\left.-\frac{2\tau^2}{[\rho^4+(\tau^2-1)^2-2(1+\tau^2)\rho^2]\sqrt{(1+\rho)^2-\tau^2}}K\left(\sqrt{\frac{4\rho}{(1+\rho)^2-\tau^2}}\right)\right\}, \tag{16}$$

region II,

$$\frac{\tau}{2\pi R_1\lambda}\left\{\frac{1-2\rho^2+\rho^4+\tau^2-2\rho\tau^2+\rho^2\tau^2-2\tau^4}{\sqrt{\rho}[(1-\rho)^2-\tau^2][(1+\rho)^2-\tau^2]^2}K\left(\sqrt{\frac{(1+\rho)^2-\tau^2}{4\rho}}\right)\right.$$
$$\left.+\frac{4\sqrt{\rho}[1-2\rho^2+\rho^4+2\tau^2-3\tau^4+2\rho^2\tau^2]}{[\rho^4+(\tau^2-1)^2-2(1+\tau^2)\rho^2]^2}E\left(\sqrt{\frac{(1+\rho)^2-\tau^2}{4\rho}}\right)\right\}. \tag{17}$$

The value of (15) in region III is zero.

The integral I_3, on the other hand, can also be studied by the methods described in Ref. 8. The first relevant term to its asymptotic expansion in inverse powers of λ is derived from the double contour integral

$$\Im\left\{\frac{e^{i\tau\lambda}}{4\pi^2 R_1}\oint_{\gamma_1} dt_1 \oint_{\gamma_2} dt_2 \frac{2}{t_1 t_2}\frac{[t_1 - 1/t_1 + \rho(t_2 - 1/t_2) + 2i\tau e^{-\Lambda_q}]e^{\lambda[(\Lambda_q/2)(t_1-1/t_1)+\rho(\Lambda_q/2)(t_2-1/t_2)-i\tau e^{-\Lambda_q}]}}{\Lambda_q^2[(t_1+1/t_1)^2+\rho^2(t_2+1/t_2)^2]-[t_1-1/t_1+\rho(t_2-1/t_2)+2i\tau e^{-\Lambda_q}]^2}\right\}, \tag{18}$$

corresponding to $q=\Lambda_q$ and whose asymptotic behavior can be determined by employing standard techniques for multiple integrals.[22] In this way we obtain the following contribution:

$$\frac{1}{2\pi R_1 \Lambda_q \lambda\sqrt{\rho}}\left\{\frac{\sin[\lambda\Lambda_q(1+\rho)-\tau\lambda(1-e^{-\Lambda_q})]}{1+\rho-\tau e^{-\Lambda_q}} - \frac{\sin[\lambda\Lambda_q(1+\rho)+\tau\lambda(1-e^{-\Lambda_q})]}{1+\rho+\tau e^{-\Lambda_q}}\right.$$
$$\left. + \frac{\cos[\lambda\Lambda_q(1-\rho)-\tau\lambda(1-e^{-\Lambda_q})]}{1-\rho-\tau e^{-\Lambda_q}} - \frac{\cos[\lambda\Lambda_q(1-\rho)+\tau\lambda(1-e^{-\Lambda_q})]}{1-\rho+\tau e^{-\Lambda_q}}\right\}. \tag{19}$$

Finally, the integral I_2 is written in terms of a neutralizer that vanishes at $q=0$ and $q=\Lambda_q$. This integral is best studied by choosing the unit circumference centered in the origin of the complex plane as the integration contour γ. The contributions of this integral to the asymptotics of (13) come from the stationary points of the exponent in the integrand whenever they are within the integration region. This fact is controlled by the value of the cutoff Λ_q. The result is

$$\{\theta(\tau-\rho+1)\theta[(\rho-1)e^{\Lambda_q}-\tau]\theta(\rho-1)$$
$$+\theta(\tau+\rho-1)\theta[(1-\rho)e^{\Lambda_q}-\tau]\theta(1-\rho)\}\Im\left\{\frac{e^{-i\pi/4}e^{i\lambda[\tau+|\rho-1|(1+\log\tau/|\rho-1|)]}}{R_1\sqrt{2\pi\lambda\rho|1-\rho|}\log\tau/|1-\rho|}\right\}$$
$$+\theta(\tau-\rho-1)\theta[(\rho+1)e^{\Lambda_q}-\tau]\Im\left\{\frac{e^{i\pi/4}e^{i\lambda[\tau+(\rho+1)(\log 1+\rho/\tau-1)]}}{R_1\sqrt{2\pi\lambda\rho(1+\rho)}\log\tau/1+\rho}\right\},$$

where the step functions define the regions where the different stationary points contribute. As we can see and it is explained in Fig. 5, there are two contributions in some parts of the (ρ,τ) plane,

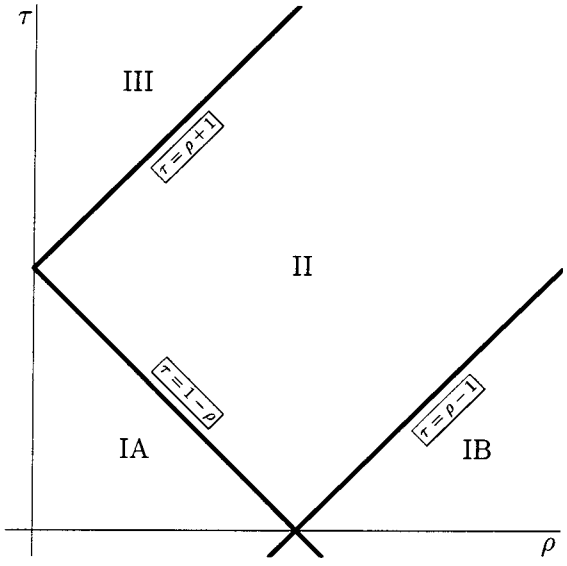

FIG. 4. Regions in the (ρ,τ) plane used in the discussion of the λ asymptotics and the free commutator. Region I is defined by $0<\tau<|\rho-1|$, region II by $|\rho-1|<\tau<\rho+1$, and region III by $\rho+1<\tau$.

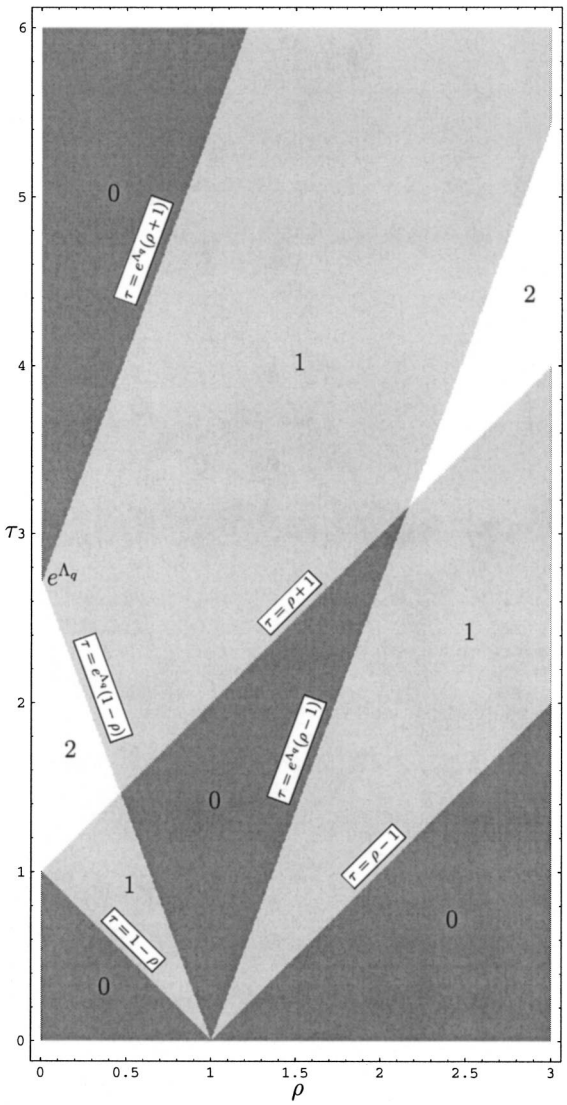

FIG. 5. Regions in the (ρ, τ) plane used in the discussion of the λ asymptotics in the presence of a cutoff $\Lambda_q = 1$. The label of each region indicates how many critical points contribute to the asymptotic expansion in λ.

only one in some other parts, and no contribution in the remaining ones. Notice that, whenever they differ from zero, these contributions are dominant with respect to those coming from the other integrals.

Several points are now in order. First it is interesting to realize that the singularity at $\rho = 1$ that exists when the cutoff is taken to be infinite (and is obviously absent now) shows up as the region defined by the lines $\tau = e^{\Lambda_q}(1-\rho)$, $\tau = e^{\Lambda_q}(\rho-1)$, and $\tau = e^{\Lambda_q}(\rho+1)$ shrinks with growing Λ_q. Another interesting feature of the commutator when the cutoff is present is the appearance of some regions where the leading asymptotic behavior is not given by the expressions obtained in Ref. 8 for infinite cutoff, namely the regions with $\tau > |1-\rho|$ labeled 0 in Fig. 5 and the region labeled 1 that connects them. On the contrary, there are two regions where two stationary points contribute to the asymptotics just as in the $\Lambda_q \to \infty$ case, showing the characteristic slow decay in the τ direction.[24] Of these two regions, the one closer to the axis is bounded, whereas the second one

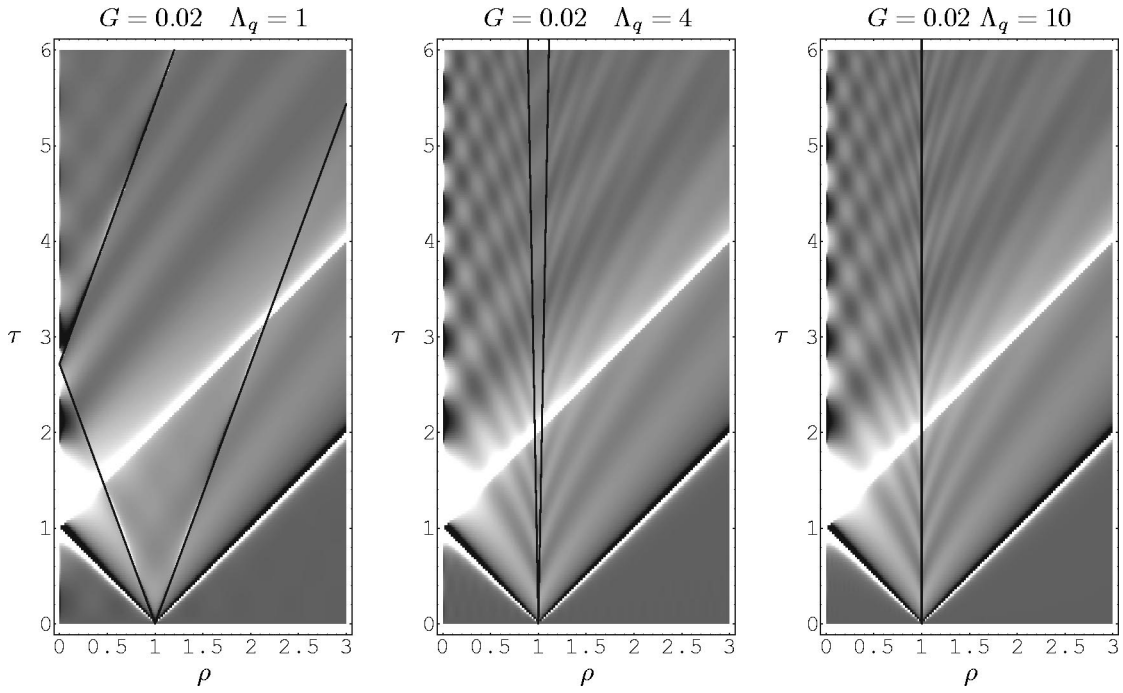

FIG. 6. Density plots of the commutator for $G=0.02$ and different values of Λ_q. Comparing the results with those of Ref. 8, we can see that the commutator in the regions labeled 2 in Fig. 4 is essentially equal to the one corresponding to an infinite cutoff. Notice also the process by which the singularity at R_1 appears.

[defined by the lines $\tau=e^{\Lambda_q}(\rho-1)$ and $\tau=\rho+1$] is not. The effect of the symmetry axis is evident in the sense that it is precisely there where one of the lines that limits the boundary of this region starts, namely $\tau=\rho+1$.

As we can see, the influence of the cutoff is important in some parts of the (ρ,τ) plane, but there are others where the asymptotic behavior is described at leading order(s) by the unregulated $\lambda\to\infty$ limit. The consideration of these different regions helps in describing the intermediate regimes where the infinite cutoff approximation is expected to work, at least for large values of λ. Finally, we want to point out that the most dramatic quantum effect observed when the cutoff is infinite, the very slow falloff of the commutator at the axis in the τ direction, is no longer present after introducing a regulator. This casts some doubts about the "observability" of large quantum gravitational fluctuations at the axis. These behaviors can be visually appreciated in Figs. 6–9.

VI. POWER EXPANSION IN *G*

In the above sections we have discussed the asymptotics of the regulated field commutator as a function of ρ, λ, and τ. These are dimensionless parameters obtained by using R_1 as a length scale. We want to discuss now the possibility of expanding this commutator as a power series in G. The main motivation to consider this issue is that one would expect to arrive at an expansion of this kind when adopting a standard perturbative approach for the treatment of the problem. As we will see, this can be done in a rather straightforward way if a cutoff is introduced in the system. However, our description breaks down when the cutoff is removed.

Let us analyze then the expansion of the vacuum expectation value of the commutator in powers of the quantum gravitational constant[25] $G=G_3\hbar$. To this end we rewrite (4) as

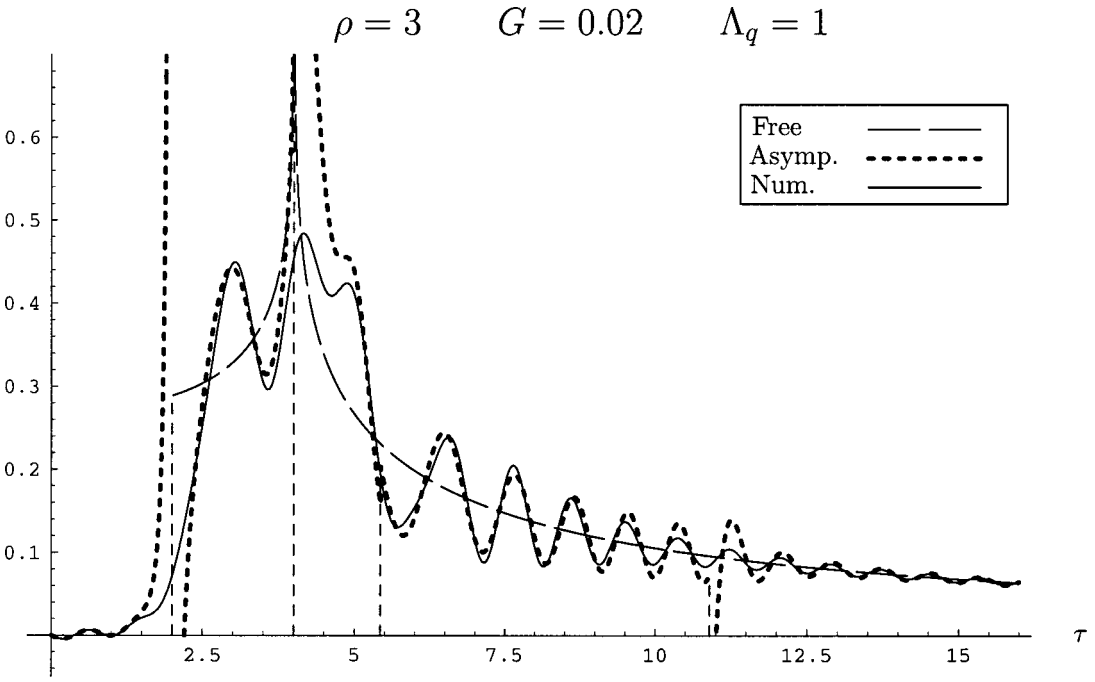

FIG. 7. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $G=0.02$, $\rho=3$, and $\Lambda_q=1$ compared with a numerical approximation. The regions with different asymptotic regimes are shown. The asymptotic approximation is good except at the boundaries between these regions. The different types of behavior are also evident.

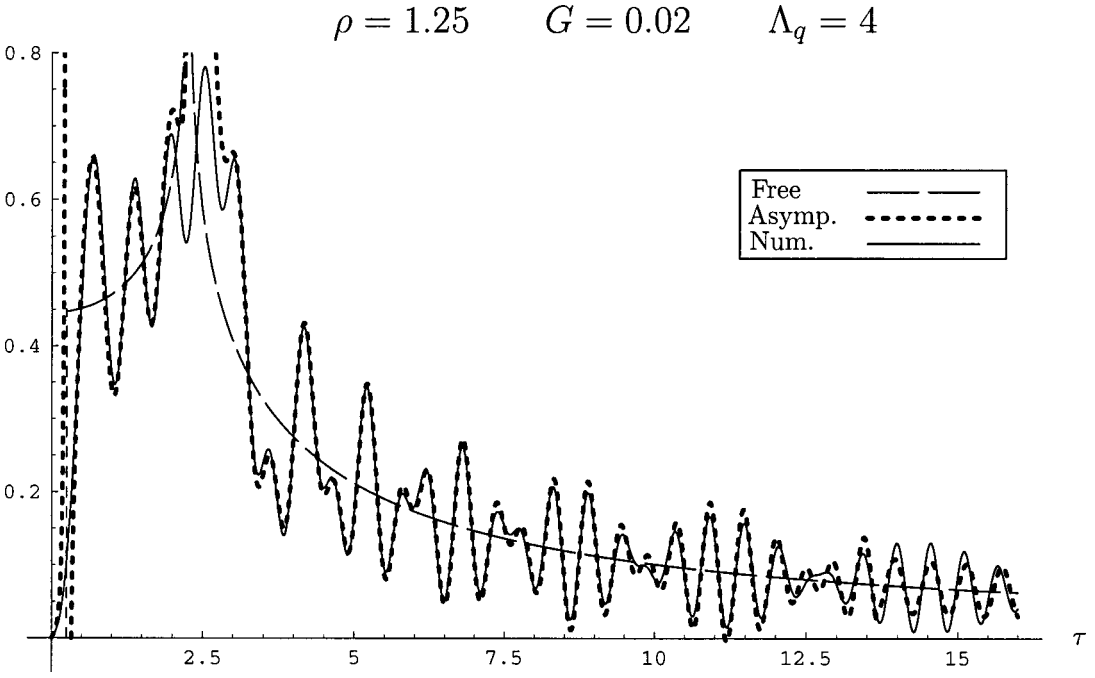

FIG. 8. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $G=0.02$, $\rho=1.25$, and $\Lambda_q=4$ compared with a numerical approximation.

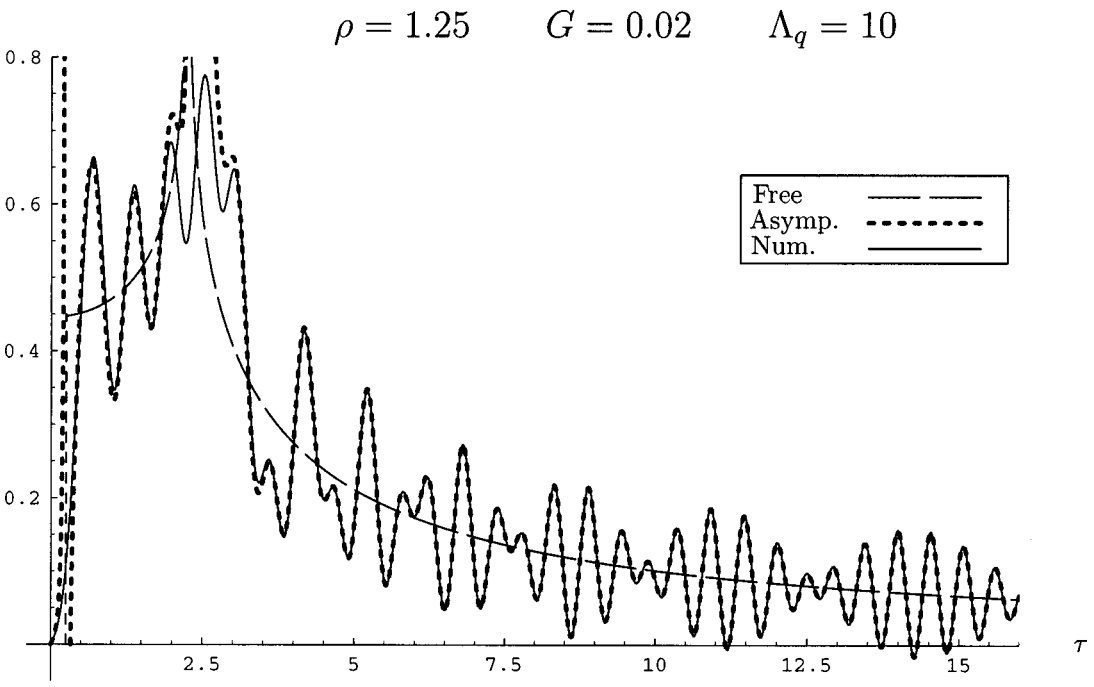

FIG. 9. Asymptotic approximation in λ for the field commutator over $8iG$ as a function of τ for $G=0.02$, $\rho=1.25$, and $\Lambda_q=10$ compared with a numerical approximation.

$$\frac{1}{8iG}\langle 0|[\hat{\psi}_E(R_1,t_1),\hat{\psi}_E(R_2,t_2)]|0\rangle = \int_0^{\Lambda_k} dk J_0(R_1k)J_0(R_2k)\sin\left[kR_1\tau\frac{(1-e^{-q})}{q}\right]. \tag{20}$$

With our conventions, both $kR_1\tau$ and $q=Gk$ are dimensionless, whereas k can be regarded to have dimensions of an inverse length. Note that all the dependence on G is contained in q, accepting that the cutoff Λ_k is fixed. Thus, in order to arrive at the desired series, we will expand the integrand in powers of the variable q. At this point, it is worth remarking that, had we described the regulated commutator by means of the dimensionless cutoff $\Lambda_q=4G\Lambda_k$ as in previous sections,[26] it would not have been possible to single out the dependence on the gravitational constant via that on q.

We will use the following formulas for the Taylor expansion of the functions involved in our expression (20) and the composition of the resulting series, assuming for the moment their convergence:

$$g(q) := \frac{1-q-e^{-q}}{q} = \sum_{n=1}^{\infty}\frac{(-q)^n}{(n+1)!}, \tag{21}$$

$$\sin(kR_1\tau+y) = \sin(kR_1\tau)\sum_{m=0}^{\infty}(-1)^m\frac{y^{2m}}{(2m)!} + \cos(kR_1\tau)\sum_{m=0}^{\infty}(-1)^m\frac{y^{2m+1}}{(2m+1)!}, \tag{22}$$

$$[g(q)]^m = \left[\sum_{n=1}^{\infty}\frac{(-q)^n}{(n+1)!}\right]^m = \sum_{p=m}^{\infty}a_p[m](-q)^p, \tag{23}$$

$$a_p[m] := \sum_{\sigma(p|m)} \prod_{i=1}^{m} \frac{1}{(n_i+1)!}, \tag{24}$$

where the range of the last sum extends to the sets of m integers n_i given by

$$\sigma(p|m) := \left\{ n_i \geqslant 1; \sum_{i=1}^{m} n_i = p \right\}. \tag{25}$$

Interchanging the sum and integration orders, one then obtains the formal series

$$\frac{1}{8iG}\langle 0|[\hat{\psi}_E(R_1,t_1), \hat{\psi}_E(R_2,t_2)]|0\rangle = \sum_{p=0}^{\infty} \int_0^{\Lambda_k} dk J_0(R_1 k) J_0(R_2 k)(-4Gk)^p F_p(kR_1\tau), \tag{26}$$

with

$$F_p(kR_1\tau) := \sin(kR_1\tau) \sum_{m=1}^{\mathrm{int}[p/2]} \frac{(-1)^m}{(2m)!} a_p[2m](kR_1\tau)^{2m} + \cos(kR_1\tau) \sum_{m=0}^{\mathrm{int}[(p-1)/2]} \frac{(-1)^m}{(2m+1)!} a_p[2m+1](kR_1\tau)^{2m+1}, \quad p \geqslant 1,$$

$$F_0(kR_1\tau) := \sin(kR_1\tau). \tag{27}$$

Here, the function int$[x]$ is the integer part of x, and the sum over m that multiplies the function $\sin(kR_1\tau)$ is understood to vanish when $p=1$. Note that, in the case of infinite cutoff, the first ($p=0$) term reproduces the commutator of the free-field theory. Moreover, then all the $p \geqslant 1$ additions to the free field contribution are integrals over $[0,\infty)$ of oscillating, nonbounded functions and, hence, at best conditionally convergent. So, in the unregulated theory ($\Lambda_k = \infty$), the above expansion should be taken only as a formal expression, and therefore we expect that the corresponding vacuum expectation value of the field commutator is not analytic in G.

Of course these problems disappear when we admit the existence of a finite cutoff $0 < \Lambda_k < \infty$. Taking into account that all the functions $F_p(kR_1\tau)$ are analytic in k around the positive real axis, because F_p is a finite combination of products of analytic functions, and that so are the zeroth-order Bessel functions that appear in the integrals of (26), it is easy to conclude that all those integrals are well defined when they are restricted to a compact interval $[0,\Lambda_k]$. Thus, each term in the power series (26) is finite for any finite positive value of Λ_k.

In the rest of this section, we will discuss the formal manipulations that we have carried out with infinite sums in order to deduce the above expansion. First, notice that the Taylor series in (21), which is obtained from that of the exponential function, has an infinite convergence radius. When this series is substituted in (20), one obtains a trigonometric function similar to that on the lhs of (22), but with $y = kR_1\tau g(q)$ expanded in powers of q. On the other hand, relation (22) is just the formula for the sine of the sum of two angles, with the resulting functions $\sin y$ and $\cos y$ replaced with their Taylor expansion. The series compositions $\sin[kR_1\tau g(q)]$ and $\cos[kR_1\tau g(q)]$ can then be rearranged without problems employing for $[g(q)]^m$ the value given in (23) because $g(q)$ (which we recall that converges for all $q \in \mathbb{R}^+$) is always smaller than the convergence radii of the sine and cosine series, which are in fact infinite.

In this way, one arrives at an expectation value of the regulated commutator that is equal to an integral over the interval $k \in [0,\Lambda_k]$ of the series of functions $\Sigma_p f_p(k|R_1,R_2,\tau)$, with

$$f_p(k|R_1,R_2,\tau) := J_0(R_1 k) J_0(R_2 k)(-4Gk)^p F_p(kR_1\tau). \tag{28}$$

Since the functions $f_p(k)$ are clearly continuous in $k \in [0,\Lambda_k]$ (for all allowed values of R_1, R_2, and τ) and this interval is compact, they are all integrable in that region. As a consequence, it is

sufficient that the considered series of functions converges uniformly in $k \in [0, \Lambda_k]$ to guarantee that the integration can be interchanged with the infinite sum. We will postpone to the next section the proof of this uniform convergence, at least for a convenient choice of the cutoff.

In conclusion, we have seen that the field commutator *in vacuo*, regulated with a (dimensionful) fixed cutoff, can be expanded as a power series in the gravitational constant G, each term in the series being finite. Besides, all the manipulations performed to deduce this series are rigorously justified provided that the cutoff is chosen so that the series $\Sigma_p f_p(k|R_1, R_2, \tau)$ converges uniformly in $k \in [0, \Lambda_k]$. Furthermore, in fact, this requirement of uniform convergence automatically ensures that the corresponding integrated power series (26) converges, and that it does so to the actual value of the expectation value of the regulated commutator.

VII. UNIFORM CONVERGENCE

We want to demonstrate that there exists a nonzero value of the cutoff for which the series $\Sigma_p f_p(k|R_1, R_2, \tau)$ converges uniformly in $k \in [0, \Lambda_k]$ for any fixed non-negative value of R_1, R_2, and τ. Let us start by finding a convenient upper bound for the coefficients $a_p[m]$, with $m \geqslant 1$, defined in (24). First, note that $a_p[m]=0$ unless $p \geqslant m$, because no set of the form $\sigma(p|m)$ exists with $n_i \geqslant 1$ if $\Sigma_{i=1}^m n_i = p < m$. In addition, since $(n_i+1)! \geqslant 2$ for $n_i \geqslant 1$, we have that

$$a_p[m] \leqslant \frac{1}{2^m} \sum_{\sigma(p|m)} 1. \tag{29}$$

From our definition (25), the last sum equals the different ways to arrange $p-m$ nondistinguishable elements [namely, the excess about its minimum of the sum of m elements $n_i \geqslant 1$, which equals $p-m$ for $\sigma(p|m)$] between m different sets (which correspond to the m integers n_i). The result is given by the permutations of $(p-m)+m-1$ elements (the latter $m-1$ elements representing movable delimiters between the m sets) with possible repetition in $p-m$ (the genuine, nondistinguishable elements) and in $m-1$ (the imaginary delimiters). Thus,

$$a_p[m] \leqslant \frac{1}{2^m} \frac{(p-1)!}{(p-m)!(m-1)!} \leqslant \frac{1}{2^m} \frac{p!}{(p-m)!}. \tag{30}$$

In the last inequality we have employed that $(m-1)!p \geqslant 1$ for all $p \geqslant m \geqslant 1$.

Using that the absolute value of the sine and the cosine is never greater than the unity, it is not difficult then to deduce from (27) the following bound for $F_p(kR_1\tau)$, with $p \geqslant 1$:

$$|F_p(kR_1\tau)| \leqslant \sum_{m=1}^{p} \frac{k^m R_1 \tau^m}{m!} a_p[m] \leqslant \sum_{m=1}^{p} \left(\frac{kR_1\tau}{2}\right)^m \binom{p}{m} = \left(1 + \frac{kR_1\tau}{2}\right)^p - 1. \tag{31}$$

In the last step, we have employed the formula of the binomial expansion. Likewise, since the zeroth-order Bessel function is bounded by the unity in the positive real axis, we get that, for all non-negative values of R_1 and R_2,

$$|f_p(k|R_1, R_2, \tau)| \leqslant (4Gk)^p \left[\left(1 + \frac{kR_1\tau}{2}\right)^p - 1\right] \leqslant \left[4Gk\left(1 + \frac{kR_1\tau}{2}\right)\right]^p. \tag{32}$$

The last inequality is trivial, given that $4Gk \geqslant 0$. Note also that the bound on the rhs is valid even in the case $p=0$, taking into account (28).

Finally, since $4Gk(1+kR_1\tau/2)$ is a strictly increasing function of k in $[0, \Lambda_k]$, we obtain a bound independent of the variable k in the interval considered:

$$|f_p(k|R_1, R_2, \tau)| \leqslant \left[4G\Lambda_k\left(1 + \frac{\Lambda_k R_1\tau}{2}\right)\right]^p. \tag{33}$$

To obtain the desired convergence properties, it will suffice to require that

$$4G\Lambda_k\left(1+\frac{\Lambda_k R_1\tau}{2}\right)<1. \tag{34}$$

We then get that both $|f_p(k|R_1,R_2,\tau)|$ and its integral over $[0,\Lambda_k]$ are small corrections for large p, which tend to zero in the limit $p\to\infty$.

In order to prove the uniform convergence of the series $\Sigma_p f_p(k|R_1,R_2,\tau)$, we have to check that, for each $\epsilon>0$, there exists an integer P such that, for every $k\in[0,\Lambda_k]$,

$$\left|\sum_{p=P}^{\infty} f_p(k|R_1,R_2,\tau)\right|\leqslant\epsilon. \tag{35}$$

Taking into account inequality (34), it is clear that, given $\epsilon>0$, we can always find a sufficiently large integer P for which

$$\left[4G\Lambda_k\left(1+\frac{\Lambda_k R_1\tau}{2}\right)\right]^P<\epsilon\left[1-4G\Lambda_k\left(1+\frac{\Lambda_k R_1\tau}{2}\right)\right]. \tag{36}$$

Note that the choice of this P depends only on the values of ϵ, Λ_k, G, and $R_1\tau$. Using (33), we then have

$$\left|\sum_{p=P}^{\infty} f_p(k|R_1,R_2,\tau)\right|\leqslant\sum_{p=P}^{\infty}\left[4G\Lambda_k\left(1+\frac{\Lambda_k R_1\tau}{2}\right)\right]^p=\frac{[4G\Lambda_k(1+\Lambda_k R_1\tau/2)]^P}{1-4G\Lambda_k(1+\Lambda_k R_1\tau/2)}\leqslant\epsilon.$$

So, inequality (35) is valid for all k in the considered interval $[0,\Lambda_k]$, as we wanted to prove.

We have thus shown that, for a given τ, every choice of the cutoff $\Lambda_k>0$ that satisfies condition (34) leads to a convergent power series in the gravitational constant G for the expectation vacuum of the regulated commutator, regardless of the radial coordinates R_1 and R_2. Moreover, the power expansion converges indeed to the true value of this regulated commutator *in vacuo*.

VIII. CONCLUSIONS AND COMMENTS

We have studied in this paper the issue of microcausality for quantum Einstein-Rosen waves after a suitable cutoff is introduced to regulate the quantum fields. In more detail, we have considered the introduction of a momentum cutoff Λ_k (or its dimensionless counterpart Λ_q). We have discussed first the asymptotic expansions in terms of the dimensionless parameters ρ, τ, and λ along the lines of Ref. 8. Owing to the fact that these parameters are defined with the help of R_1, in principle one does not need to make explicit the dependence of the cutoff Λ_k on G in this case. On physical grounds, one could view this cutoff, for example, as the inverse of the Planck length.

We have seen that the introduction of a finite cutoff modifies some of the conclusions obtained in Ref. 8. In particular we have seen that some of the most dramatic effects present when the cutoff is infinite (in particular the behavior of the field commutators in the symmetry axis) are now somewhat mitigated. Nevertheless, we have been able to show that the approximation provided by the unregulated field commutator is a good one in some regions of the (ρ,τ) plane, and, in fact, there is an unbounded region where that approximation prevails. This indicates that, even though the influence of the cutoff is felt in some regions of the parameter space, it is irrelevant in others.

In Secs. VI and VII, on the other hand, we have considered the expansion of the field commutator in terms of the gravitational constant G. We notice, nonetheless, that condition (34) on the cutoff Λ_k, which guarantees the convergence of the series, depends on G. At this stage, one possibility would be to admit that the cutoff depends on the gravitational constant; however, the expansion obtained would then fail to provide a genuine power series in G, because this parameter would also enter the different terms in the series via the implicit dependence of Λ_k on it. Another possibility that indeed respects the interpretation of our expansion as a power series in G is the following. Employing that condition (34) is an inequality equation for Λ_k given in terms of a

function of G that is strictly increasing, it is easy to see that the inequality is satisfied for all values of G in a certain interval $[0,G_M]$ if and only if it is satisfied for G_M. Something similar happens with respect to the dependence on the value of $R_1\tau=|t_2-t_1|$, so that if we want to consider a whole time interval of the form $|t_2-t_1|\in[0,t_M]$, we only have to evaluate our condition at the maximum time lapse. In other words, to ensure the convergence of the series for $G\in[0,G_M]$ and any time difference in $[0,t_M]$, we only have to demand the requirement (34) at $G=G_M$ and $R_1\tau=t_M$, because then

$$4G\Lambda_k\left(1+\frac{\Lambda_k R_1\tau}{2}\right)<4G_M\Lambda_k\left(1+\frac{\Lambda_k t_M}{2}\right)<1. \tag{37}$$

In this way we arrive at a cutoff that is independent of the particular values considered for $|t_2-t_1|$ and the gravitational constant (in the commented intervals), and our expansion becomes a true power series in G. The above inequality leads to the following positive upper bound for Λ_k:

$$\Lambda_k\leqslant\frac{1}{t_M}\left(\sqrt{1+\frac{t_M}{2G_M}}-1\right). \tag{38}$$

Therefore, with a cutoff that satisfies this condition, the power series (26) converges in the interval $[0,G_M]$ for all radial positions R_1 and R_2 and $R_1\tau=|t_2-t_1|\in[0,t_M]$.

When t_M is small, the bound on Λ_k is approximately $1/(4G_M)$, whereas for large t_M it is nearly equal to $1/\sqrt{2G_Mt_M}$. In particular, with this bound the cutoff would have to be vanishingly small if we want a good convergent behavior in an infinitely large time interval ($t_M\to\infty$). An open question is whether it is possible or not to find a different, nonzero time-independent cutoff such that the expansion of the regulated commutator converges for any value of the time elapsed, i.e., for all $|t_2-t_1|\in\mathbb{R}^+$. We expect to encounter convergence problems when the time interval is unbounded; for instance, one can prove that the series (26) does not converge uniformly in $\tau\in\mathbb{R}^+$ with any choice of the cutoff Λ_k (for generic R_1 and R_2). Nonetheless, one can in fact consider a kind of semi-classical limit in which G_M tends to zero (and hence so does the value of the gravitational constant, which had been restricted to $[0,G_M]$), while the time interval where the convergence is granted reaches infinity.

In order to do this, one only needs to allow a dependence of t_M on G_M, so that the assumed maximum value of the time difference varies with that of the gravitational constant. Suppose, let us say, that $t_M(G_M)=G_M^{-\alpha}$ with $0<\alpha<1$. Then, the bound (38) on the cutoff becomes

$$\Lambda_k\leqslant G_M^{\alpha}\left(\sqrt{1+\frac{1}{2G_M^{(\alpha+1)}}}-1\right). \tag{39}$$

Thus, when G_M tends to zero, we get the asymptotic behavior $\Lambda_k\leqslant G_M^{(\alpha-1)/2}/\sqrt{2}$. Since $G_M^{(\alpha-1)/2}$ and $G_M^{-\alpha}$ diverge for vanishing G_M, because $0<\alpha<1$, we therefore conclude that the cutoff can be removed in the limit $G_M\to0$ while ensuring that the time interval $[0,t_M(G_M)]$, where the expansion is well defined, covers the positive real axis.

We finally discuss the physical interpretation of this type of cutoff. It turns out to be intimately related to the maximum resolution that can be reached for the physical time when a certain perturbative approach is adopted to describe the quantum dynamics.[27] In such an approach, one expands the evolution generator in powers of G and regards the free-field Hamiltonian as the dominant contribution, with the higher powers seen as corrections. The auxiliary time T, associated with the free-field Hamiltonian, then plays the role of evolution parameter in the quantum theory, whereas the physical time becomes an operator. It was shown in Ref. 27 that, under these circumstances, a resolution limit Δt emerges for the physical time,

$$[\Delta t]^2\geqslant 4G^2+4GT. \tag{40}$$

Employing the inequality $\sqrt{1+x}\leqslant x/(\sqrt{1+x}-1)$ for $x>0$, evaluated at $x=t_M/(2G)$, one can easily check from condition (38) that the inverse of the cutoff satisfies

$$\Lambda_k^{-1} \geqslant \sqrt{4G_M^2 + 2G_M t_M}. \tag{41}$$

Therefore, the bound on Λ_k^{-1} equals that on the time resolution Δt for a value $G=G_M$ of the gravitational constant and a time elapsed $T=2t_M$ (and thus of the same order as t_M). In this sense, one can assign to Λ_k^{-1} the interpretation of a genuine resolution limit in the physical time.

The future prospects for this line of work will focus on the issue of deriving and obtaining meaningful physical information from the S matrix of the model. We feel that the mathematical techniques employed here to study the asymptotics of field commutators, with and without a cutoff, will also be helpful in analyzing this issue. We plan to concentrate on this problem in the future.

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[1] A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).
[2] K. Kuchař, Phys. Rev. D **4**, 955 (1971).
[3] A. Ashtekar and M. Pierri, J. Math. Phys. **37**, 6250 (1996).
[4] A. Ashtekar, Phys. Rev. Lett. **77**, 4864 (1996).
[5] M. Varadarajan, Class. Quantum Grav. **17**, 189 (2000).
[6] M. E. Angulo and G. A. Mena Marugán, Int. J. Mod. Phys. D **9**, 669 (2000).
[7] J. F. Barbero G., G. A. Mena Marugán, and E. J. S. Villaseñor, Phys. Rev. D **67**, 124006 (2003).
[8] J. F. Barbero G., G. A. Mena Marugán, and E. J. S. Villaseñor, J. Math. Phys. **45**, 3498 (2004).
[9] I. Kouletsis, P. Hájíček, and J. Bičák, Phys. Rev. D **68**, 104013 (2003).
[10] R. Gambini and J. Pullin, Mod. Phys. Lett. A **12**, 2407 (1997).
[11] A. E. Domínguez and M. H. Tiglio, Phys. Rev. D **60**, 064001 (1999).
[12] This is easy to understand for ordinary field theories in Minkowski space–time because cutoffs break Lorentz invariance in these models.
[13] J. D. Romano and C. G. Torre, Phys. Rev. D **53**, 5634 (1996).
[14] Notice that G has dimensions of length in this system of units.
[15] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D **55**, 669 (1997).
[16] A. Ashtekar, J. Bičák, and B. G. Schmidt, Phys. Rev. D **55**, 687 (1997).
[17] A. Ashtekar and M. Varadarajan, Phys. Rev. D **50**, 4944 (1994).
[18] M. Varadarajan, Phys. Rev. D **52**, 2020 (1995).
[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* (Academic, New York, 1975).
[20] In the main body of the paper we will be employing regulated fields; we will refer to the unregulated ones only in this section, so we label them with a $\Lambda=\infty$ tag.
[21] L. J. Garay, Int. J. Mod. Phys. A **10**, 145 (1995).
[22] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals*, 2nd ed. (Dover, New York, 1986).
[23] The appearance of $1/k$ terms forces us to split the original integral in two as in (6) to avoid singularities at $k=0$.
[24] Notice that the other contributions are subdominant with respect to this one.
[25] In previous papers[7,8] we chose to expand $\sin[kt(1-e^{-q})/q]$ as a power series in e^{-q}. This yielded a convergent series representation for the commutator. Nonetheless, the complicated dependence of this expansion in G did not allow us to obtain a power series in the gravitational constant.
[26] Let us comment that one can always make dimensionless the cutoff Λ_k (as well as k and $1/G$) by multiplication with an arbitrary, fixed length scale independent of the gravitational constant, so that no spurious dependence on G is introduced in the process. In this sense, one may always view Λ_k either as a G-independent dimensionful cutoff or equivalently as a fixed dimensionless parameter.
[27] J. F. Barbero G., G. A. Mena Marugán, and E. J. S. Villaseñor, Phys. Rev. D **69**, 044017 (2004).