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Probing quantized Einstein-Rosen waves with massless scalar matter

J. Fernando Barbero G.* and Iñaki Garay†
Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain

Eduardo J. S. Villaseñor‡
Grupo de Modelización y Simulación Numérica, Universidad Carlos III de Madrid, Avda. de la Universidad 30, 28911 Leganés, Spain
Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain
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The purpose of this paper is to discuss in detail the use of scalar matter coupled to linearly polarized Einstein-Rosen waves as a probe to study quantum gravity in the restricted setting provided by this symmetry reduction of general relativity. We will obtain the relevant Hamiltonian and quantize it with the techniques already used for the purely gravitational case. Finally, we will discuss the use of particlelike modes of the quantized fields to operationally explore some of the features of quantum gravity within this framework. Specifically, we will study two-point functions, the Newton-Wigner propagator, and radial wave functions for one-particle states.

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I. INTRODUCTION

There are reasons to believe that general relativity is not the last word as far as gravitational physics is concerned; for one thing, we still do not know how to reconcile it with quantum mechanics, the other fundamental pillar of our description of physical reality. As of today we lack a complete and consistent description of quantum gravity, although some proposals have made steady progress towards our understanding of the problem if not of its solution. The questions to be addressed by quantum gravity must go beyond the mere finding of a consistent quantization of general relativity because it is possible that many of the physical concepts that work classically are inappropriate in the quantum setting. Let us consider, for example, the metric. Though this is the fundamental concept in classical general relativity, formalisms such as loop quantum gravity suggest that it may not be the basic variable at the quantum level. It is perfectly conceivable that the metric is some kind of semiclassical construct that emerges at scales much larger than the Planck scale. We are left, then, with the problem of understanding what the relevant object at the shortest scales is and how a metric description appears. Another concept that may not be appropriate is that of test particles. Their classical meaning is clear; they are pointlike objects of negligible mass and, hence, with no influence as sources for the gravitational field described by a metric whose geodesics are the trajectories followed by these test particles in their motion through space and time. They are, in a definite sense, the *tools* that we have available to extract the nontrivial geometric content from the metric and describe spacetime physics. In the context of general relativity it is very useful and illuminating to adopt an operational point of view to define geometric quantities such as time intervals, lengths, and other geometrical objects with an immediate physical interpretation. In this way it is possible to avoid resting too much on our classical (nonrelativistic) intuition and isolate the basic and relevant physical concepts. If a consistent quantization of gravity was available, it would seem necessary to replace test particles by *quantum* objects. One of the obvious choices—though possibly not the only one—is to introduce quantum particles moving in the nontrivial background defined by the metric. The fact that this will, in general, be curved would manifest itself in the way wave functions spread throughout spacetime. Another possibility would be to couple *test fields* and use their quanta as test particles, yet many others may exist as we cannot be sure that the correct description at quantum gravity scales will be that provided by particlelike objects.

The purpose of this paper is to explore these issues in the restricted setting provided by the linearly polarized two-Killing vector reductions of general relativity, the so-called Einstein-Rosen waves. The general problem of introducing quantum test fields consistently is a nontrivial one. We will see, however, that it is possible to use some fields for this purpose because we will be able to solve the theory exactly both at the classical and quantum levels even in their presence. Specifically we will expand here the results presented in [1] showing that Einstein-Rosen waves coupled to a massless scalar field can be exactly quantized. After this, we will use the particlelike quanta of this scalar field as a way to extract information about this quantum gravity toy model in an operational way, much in the same way as test particles are used to derive spacetime physics in general relativity. Our concern will be the recovery of classical trajectories for particles and an approximate metric description. The results obtained here complement in a

*Electronic address: fbarbero@iem.cfmac.csic.es
†Electronic address: igael@iem.cfmac.csic.es
‡Electronic address: ejsanche@math.uc3m.es

sense those already derived by considering the microcausality of this model [2–4] that clearly show how a long distance limit appears in which microcausality reduces to the familiar one defined by the Minkowski metric.

The structure of the paper is the following. After this introduction we describe in Sec. II the classical solution of a system of linearly polarized Einstein-Rosen waves coupled to a massless, cylindrically symmetric, scalar field. We will use the Geroch formalism [5] and take advantage of the two-Killing vector fields available in order to describe the model as a $1+1$-dimensional system of coupled fields. As we will see, the structure of the solutions to the field equations strongly suggests that the Hamiltonian of the system has a very simple form. We discuss this in Sec. III. One of the points that merits special attention here is the treatment of the boundary terms that give rise to the Hamiltonian. In fact, the most salient feature of the model is the appearance of a Hamiltonian that is a nontrivial bounded function of the sum of the free Hamiltonians corresponding to two free, massless, axially symmetric scalars in a $2+1$-dimensional Minkowskian background. This fact will eventually allow us to quantize the model even though it is nontrivial and interacting. In Sec. IV we apply well-known Fock space techniques, similar to the ones already used in [6], to study its quantization. After this, we will describe several applications. We start in Sec. V by studying the two-point functions. They can be interpreted as approximate probability amplitudes for a particle created at a certain radial distance from the cylindrical symmetry axis at a certain time to be detected somewhere else at a different instant of time. The comparison of these amplitudes with the ones corresponding to a cylindrically symmetric massless scalar in a Minkowski background shows some characteristic features due to the presence of gravity such as an enhancement of the probability of finding the field quanta close to the axis. Microcausality is easily reviewed in this framework because the field commutator can be obtained from these two-point functions. As the reduced field model has two different fields, the "gravitational" and the "matter" scalars, we can study their commutators. We will see that they are not zero as a consequence of the nontrivial interacting character of the model; the commutator does not show the microcausal behavior that appears when only one type of field is considered.

The main reason why the interpretation of the above two-point functions as probability amplitudes is only approximate is the fact that the states obtained by acting with the field operators on the vacuum are not orthonormal. The obtention of an orthonormal basis of position states in quantum field theory is a difficult problem that was essentially solved by Newton and Wigner for some special types of fields [7]. By using this basis it is possible to introduce relativistic wave functions whose modulus squared can be interpreted as probabilities to find a particle around a certain spacetime point.[1] We introduce analogous objects in Sec. VI. Because of the absence of Lorentz invariance in the present model, some of the unpleasant features of the ordinary Newton-Wigner states are absent here. We use these states in a double fashion. First, we study the propagator $\langle R_2|\hat{U}(t_2,t_1)|R_1\rangle$, which can be interpreted now as a proper probability amplitude, and compare it with the two-point function considered above. After this we define "radial wave functions" in terms of these Newton-Wigner states in Sec. VII and study in detail a certain family of them for which the evolution can be explicitly obtained in closed form as a one-dimensional integral. This provides us with the main tool that we have been searching for in the paper: a way to introduce quantized particles with controllable wave functions that we can use as quantum test particles to explore the spacetime physics of this quantum gravitational toy model. We will use them to study the propagation of particles in this purely quantum spacetime.

Throughout the last sections of the paper, we will make use of asymptotic techniques developed by the authors in [3] to extract relevant information from the closed integral expressions that describe two-point functions, propagators, and wave functions. This is useful for two different reasons. In some cases it allows us to consider the "macroscopic" (or "classical") limit in which we only consider the behavior of the relevant objects at scales much larger than the quantum gravitational one. In others, they provide us with useful tools to obtain approximate values for these objects that are not easily derived by numerical methods. We end the paper with our conclusions and comments in Sec. VIII, followed by the Appendix where we discuss a useful representation for several integrals appearing in the paper.

II. EINSTEIN-ROSEN WAVES COUPLED TO MASSLESS SCALAR MATTER

In this section we review in some detail the classical theory of whole cylindrically symmetric spacetimes minimally coupled to a cylindrically symmetric massless scalar field. Although this problem has been extensively considered in the literature (see for instance [9] and references therein), it is useful to gain some insight about the Hamiltonian formulation of the model (that we will discuss in the next section) through the analysis of the field equations and the Geroch formalism. As we will see, the solutions to the field equations strongly suggest that the Hamiltonian of the system, obtained by carefully taking into account surface terms in the action principle, is a

[1] Although these so-called Newton-Wigner states provide a rather complete solution to the problem of localization in some relativistic quantum field theories, they also display some disturbing features mainly related to their behavior under Lorentz transformations [8].

simple function of the Hamiltonians for two massless cylindrically symmetric scalars. We start by considering the Einstein equations for the system,

$$\square^{(4)}\phi = 0, \tag{1}$$

$$R^{(4)}_{ab} = 8\pi G_N (\mathrm{d}\phi)_a(\mathrm{d}\phi)_b. \tag{2}$$

Here $R^{(4)}_{ab}$ and $\square^{(4)}$ are, respectively, the Ricci tensor and the d'Alembertian operator associated with the Levi-Civita connection $D^{(4)}_a$ compatible with the spacetime metric $g^{(4)}_{ab}$. The exterior derivative of the matter scalar field ϕ is denoted by $(\mathrm{d}\phi)_a$, and G_N is the Newton constant.

Whole cylindrical symmetry [10] is characterized by the existence of a $\mathbb{R}\times U(1)$ group of isometries with two mutually orthogonal, hypersurface-orthogonal, commuting, spacelike, globally defined Killing vectors ξ^a and σ^a. It is assumed also that the topology of the spacetime is $\mathbb{R}^4$ and that the set of fixed points of the rotations generated by $\sigma^a = (\partial/\partial\sigma)^a$ defines a two-dimensional timelike surface: the axis of symmetry $\{x \in \mathbb{R}^4 | g^{(4)}_{ab}\sigma^a\sigma^b = 0\}$. On the other hand, the translations along the symmetry axis generated by $\xi^a = (\partial/\partial z)^a$ act freely and satisfy the elementary flatness condition [11]. This condition guarantees the 2π periodicity of the axial coordinate σ. With these assumptions the Killing coordinates $z \in \mathbb{R}$ and $\sigma \in [0, 2\pi)$ are unique up to the trivial transformations $z \mapsto \alpha z + z_0$ and $\sigma \mapsto \pm\sigma + \sigma_0$. In terms of these Killing fields the symmetry of the system reflects on the vanishing of the Lie derivatives,

$$L_\xi g^{(4)}_{ab} = L_\sigma g^{(4)}_{ab} = 0, \qquad L_\xi \phi = L_\sigma \phi = 0.$$

Finally, we will restrict our discussion to the class of spacetimes for which the derivative $(\mathrm{d}R)_a$ of the scalar field defined by

$$R^2 = (g^{(4)}_{a_1a_2}\xi^{a_1}\xi^{a_2})(g^{(4)}_{b_1b_2}\sigma^{b_1}\sigma^{b_2})$$

is everywhere spacelike. R is the area density of the isometry group orbits.

In order to solve the Einstein equations (1) and (2) we will make use of the Geroch reduction technique [5]. First, owing to the fact that translations have no fixed points, it is possible to rewrite the Einstein equations as equations for fields defined on the quotient manifold—topologically $\mathbb{R}^3$—comprised by the translational orbits. To do this we need to introduce a scalar field λ and the three-dimensional spacetime metric $g^{(3)}_{ab}$ given by[2]

$$\lambda = \xi_a\xi^a > 0, \qquad g^{(4)}_{ab} = g^{(3)}_{ab} + \lambda^{-1}\xi_a\xi_b.$$

These fields, as well as ϕ, are well defined in the translational orbit manifold. In terms of them, the Eqs. (1) and (2) are equivalent to

$$\begin{aligned}
\square^{(3)}\phi &= -\frac{1}{2}g^{(3)ab}(\mathrm{d}\phi)_a(\mathrm{d}\log\lambda)_b,\\
\square^{(3)}\lambda &= \frac{1}{2\lambda}g^{(3)ab}(\mathrm{d}\lambda)_a(\mathrm{d}\lambda)_b,\\
R^{(3)}_{ab} &= \frac{1}{2\lambda}D^{(3)}_a(\mathrm{d}\lambda)_b - \frac{1}{4\lambda^2}(\mathrm{d}\lambda)_a(\mathrm{d}\lambda)_b\\
&\quad + 8\pi G_N(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b,
\end{aligned} \tag{3}$$

where $R^{(3)}_{ab}$, $D^{(3)}_a$, and $\square^{(3)}$ refer to $g^{(3)}_{ab}$. It is easy to prove that the rotations generated by σ^a are still a symmetry of the reduced theory. In particular, σ^a is well defined in the quotient manifold and satisfies

$$L_\sigma g^{(3)}_{ab} = 0, \qquad L_\sigma\lambda = 0, \quad \text{and} \quad L_\sigma\phi = 0.$$

At this point, although the rotations do not act freely, we can still apply the Geroch reduction to the manifold obtained by deleting the symmetry axis $\rho = g^{(3)}_{ab}\sigma^a\sigma^b = 0$ from $\mathbb{R}^3$, and incorporating these points at the end of the process by imposing regularity at the axis. In particular, outside of the axis, we can define

$$\rho = g^{(4)}_{ab}\sigma^a\sigma^b = g^{(3)}_{ab}\sigma^a\sigma^b > 0$$
$$\text{and} \quad g^{(3)}_{ab} = g^{(2)}_{ab} + \rho^{-1}\sigma_a\sigma_b.$$

The equations (3) can then be written as

$$\begin{aligned}
\square^{(2)}\phi &= -\frac{1}{2}g^{(2)ab}(\mathrm{d}\phi)_a(\mathrm{d}\log(\lambda\rho))_b,\\
\square^{(2)}\lambda &= -\frac{\lambda}{2}g^{(2)ab}(\mathrm{d}\log\lambda)_a(\mathrm{d}\log(\lambda^{-1}\rho))_b,\\
\square^{(2)}\rho &= \frac{\rho}{2}g^{(2)ab}(\mathrm{d}\log\rho)_a(\mathrm{d}\log(\lambda^{-1}\rho))_b,\\
R^{(2)}_{ab} &= \frac{1}{2\lambda}D^{(2)}_a(\mathrm{d}\lambda)_b + \frac{1}{2\rho}D^{(2)}_a(\mathrm{d}\rho)_b\\
&\quad - \frac{1}{4}(\mathrm{d}\log\lambda)_a(\mathrm{d}\log\lambda)_b - \frac{1}{4}(\mathrm{d}\log\rho)_a(\mathrm{d}\log\rho)_b\\
&\quad + 8\pi G_N(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b,
\end{aligned}$$

where the notation in the previous expressions is the natural one. These two-dimensional field equations can be solved in two steps. First, it is convenient to replace the field ρ in terms of $R = \sqrt{\lambda\rho}$ and λ to get

[2]Here and in the following $\xi_a = g^{(4)}_{ab}\xi^b$ and $\sigma_a = g^{(4)}_{ab}\sigma^b$.

$$\Box^{(2)}\phi + g^{(2)ab}(\mathrm{d}\phi)_a(\mathrm{d}\log R)_b = 0, \qquad \Box^{(2)}\log\lambda + g^{(2)ab}(\mathrm{d}\log\lambda)_a(\mathrm{d}\log R)_b = 0, \qquad \Box^{(2)}R = 0,$$
$$R^{(2)}_{ab} = \frac{1}{R}D^{(2)}_a(\mathrm{d}R)_b - \frac{1}{2}[\mathrm{d}\log\lambda]_{(a}[\mathrm{d}\log(\lambda^{-1}R^2)]_{b)} + 8\pi G_N(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b. \tag{4}$$

Second, owing to the fact that we are dealing now with two-dimensional field equations, $\Box^{(2)}R = 0$ allows us to introduce a new scalar field T—the harmonic function conjugate to R—whose derivative $(\mathrm{d}T)_a$ is everywhere timelike by means of

$$(\mathrm{d}T)_a = \epsilon^{(2)}_{ab}g^{(2)bc}(\mathrm{d}R)_c,$$

where $\epsilon^{(2)}_{ab}$ is the volume element associated with $g^{(2)}_{ab}$. Notice that this definition is conformally invariant. It is now possible to use R and T as coordinates in the 1 + 1 reduced spacetime and introduce the flat metric

$$\eta^{(2)}_{ab} = -(\mathrm{d}T)_a(\mathrm{d}T)_b + (\mathrm{d}R)_a(\mathrm{d}R)_b.$$

The degrees of freedom of $g^{(2)}_{ab}$ are encoded in the conformal factor e^{γ} defined by

$$g^{(2)}_{ab} = \frac{e^{\gamma}}{\lambda}\eta^{(2)}_{ab}.$$

In terms of these new fields, the equations (4) become

$$\Box\phi + \eta^{(2)ab}(\mathrm{d}\phi)_a(\mathrm{d}\log R)_b = 0, \qquad \Box\log\lambda + \eta^{(2)ab}(\mathrm{d}\log\lambda)_a(\mathrm{d}\log R)_b = 0,$$
$$\eta^{(2)cd}\left[\frac{1}{2}(\mathrm{d}\log\lambda)_c(\mathrm{d}\log\lambda)_d + 8\pi G_N(\mathrm{d}\phi)_c(\mathrm{d}\phi)_d - (\mathrm{d}\gamma)_c(\mathrm{d}\log R)_d\right]\eta^{(2)}_{ab}$$
$$= \frac{2}{R}\partial_a(\mathrm{d}R)_b - 2(\mathrm{d}\gamma)_{(a}(\mathrm{d}\log R)_{b)} + (\mathrm{d}\log\lambda)_a(\mathrm{d}\log\lambda)_b + 8\pi G_N(\mathrm{d}\phi)_a(\mathrm{d}\phi)_b,$$

where $\Box$ is the d'Alembertian operator defined by the flat metric $\eta^{(2)}_{ab}$. The general solution to these equations can be written in a very convenient form in terms of the fields

$$\phi_g := \log\lambda, \qquad \phi_s := \sqrt{16\pi G_N}\phi.$$

By using R and T as coordinates, the Einstein equations are equivalent to two uncoupled cylindrically symmetric Klein-Gordon equations,

$$\left[-\partial_T^2 + \partial_R^2 + \frac{1}{R}\partial_R\right]\phi_g = 0, \qquad \left[-\partial_T^2 + \partial_R^2 + \frac{1}{R}\partial_R\right]\phi_s = 0, \qquad \partial_T\gamma = R[(\partial_R\phi_g)(\partial_T\phi_g) + (\partial_R\phi_s)(\partial_T\phi_s)],$$
$$\partial_R\gamma = \frac{R}{2}[(\partial_T\phi_g)^2 + (\partial_R\phi_g)^2 + (\partial_T\phi_s)^2 + (\partial_R\phi_s)^2].$$

The first two are equations for massless, axially symmetric, scalar fields in a Minkowskian background, and the equations for γ satisfy an integrability condition that allows us to immediately write their solution as

$$\gamma = \frac{1}{2}\int[(\partial_T\phi_g)^2 + (\partial_R\phi_g)^2 + (\partial_T\phi_s)^2 + (\partial_R\phi_s)^2]R\mathrm{d}R.$$

Finally, the four-dimensional spacetime metric satisfying the Einstein field equations can be written as

$$g^{(4)}_{ab} = e^{\gamma-\phi_g}[-(\mathrm{d}T)_a(\mathrm{d}T)_b + (\mathrm{d}R)_a(\mathrm{d}R)_b] + R^2e^{-\phi_g}(\mathrm{d}\sigma)_a(\mathrm{d}\sigma)_b + e^{\phi_g}(\mathrm{d}z)_a(\mathrm{d}z)_b.$$

The form for the "C-energy" γ strongly suggests that the Hamiltonian of the system can be obtained from the one corresponding to Einstein-Rosen waves by adding the contribution of the extra scalar field. We show this in the next section.

III. HAMILTONIAN FORMALISM

In order to develop the Hamiltonian formalism, we start from the Einstein-Hilbert action in four dimensions for gravity coupled to a massless cylindrically symmetric scalar ϕ_s:

$$S^{(4)} = \frac{1}{16\pi G_N}\int_{\mathcal{M}^3\times Z}|g^{(4)}|^{1/2}\left(R^{(4)} - \frac{1}{2}g^{(4)ab}(\mathrm{d}\phi_s)_a(\mathrm{d}\phi_s)_b\right) + \frac{1}{8\pi G_N}\int_{\partial(\mathcal{M}^3\times Z)}(|h^{(3)}|^{1/2}K - |h^{(3)0}|^{1/2}K^0).$$

We have included the boundary terms needed to have a well-defined variational principle. The fields are taken to be regular in the symmetry axis and the boundary conditions at infinity correspond to the definition of asymptotic flatness introduced by Ashtekar and Varadarajan in the 2 + 1-dimensional setting [12]. We use here the rescaled scalar ϕ_s introduced above with the same dimensions of ϕ_g. As we will see, they play symmetric roles in the final formulation of the model. The four-dimensional manifold where the previous action is defined has the form of a product $\mathcal{M}^3 \times Z$ where $\mathcal{M}^3$ is a three-dimensional manifold orthogonal to the translational Killing vector $\xi^a = (\partial/\partial z)^a$ and $Z = [z_1, z_2]$ is a closed interval in this direction (axis of symmetry). We have introduced also a fiducial metric $g_{ab}^{(4)0}$ that provides us with an origin for the energy, ensures that the action is finite, and fixes the asymptotic behavior of the fields in such a way that the Minkowski metric has zero energy. Finally, $h_{ab}^{(3)}$ and $h_{ab}^{(3)0}$ are the induced metrics on the boundary. Owing to the translation symmetry it is possible to rewrite the previous action as an equivalent one in $2+1$ dimensions that can be interpreted as the Einstein-Hilbert action with two massless scalars after the conformal transformation $g_{ab} = e^{\phi_g} g_{ab}^{(3)}$ is performed. Hence, our starting point to get the Hamiltonian will be

$$\begin{aligned} S^{(3)} = \frac{1}{16\pi G_3} \int_{\mathcal{M}^3} |g|^{1/2}\Big(R^{(3)} - \frac{1}{2} g^{ab}(\mathrm{d}\phi_g)_a(\mathrm{d}\phi_g)_b \\ - \frac{1}{2} g^{ab}(\mathrm{d}\phi_s)_a(\mathrm{d}\phi_s)_b\Big) \\ + \frac{1}{8\pi G_3}\int_{\partial\mathcal{M}^3}(|h|^{1/2}K - |h^0|^{1/2}K^0). \end{aligned} \tag{5}$$

Here all the geometrical objects refer to the metric g_{ab}. The coupling constant G_3 is the gravitational constant per unit length along the symmetry axis and in the following we choose units such that $c = 1$. In this three-dimensional expression of the action, we notice that the scalar field term plays exactly the same role as the gravitational scalar. It is important to point out that both fields are coupled through the metric, but not directly (there are no cross terms in the action).

To obtain the Hamiltonian, we follow the procedure developed in [6] for the vacuum case. First of all, we choose a foliation of $\mathcal{M}^3$ with timelike unit normal n^a, a radial unit vector $\hat{r}^a$, and denote as σ^a the azimuthal, hypersurface-orthogonal, Killing vector field (notice that this is not a unit vector). It is possible now to write the metric as $g_{ab} = -n_a n_b + \hat{r}_a\hat{r}_b + \frac{1}{R^2}\sigma_a\sigma_b$ (with $R^2 = g_{ab}\sigma^a\sigma^b$). We also introduce two additional vector fields t^a and r^a defined as $t^a = Nn^a + N^r\hat{r}^a$ and $r^a = e^{\gamma/2}\hat{r}^a$, where N is the lapse function, N^r the radial shift, and at this point γ is just an extra field (that will eventually coincide with the one introduced in the previous section). We impose the condition that the commutators of these new vector fields are zero, so we can define coordinates t, r, θ and get the following consistency conditions

$$\partial_\sigma N = \partial_\sigma N^r = \partial_\sigma \gamma = 0; \qquad [\sigma, \hat{r}]^a = [\sigma, n]^a = 0;$$
$$n^a\partial_r N + \hat{r}^a(\partial_r N^r - \partial_t e^{\gamma/2}) + Ne^{\gamma/2}[\hat{r}, n]^a = 0.$$

Finally, the metric takes the form

$$\begin{aligned} g_{ab} = (N^{r2} - N^2)(\mathrm{d}t)_a(\mathrm{d}t)_b + 2e^{\gamma/2}N^r(\mathrm{d}t)_{(a}(\mathrm{d}r)_{b)} \\ + e^{\gamma}(\mathrm{d}r)_a(\mathrm{d}r)_b + R^2(\mathrm{d}\sigma)_a(\mathrm{d}\sigma)_b. \end{aligned}$$

Now, if we take the boundary $\partial\mathcal{M}^3$ to be orthogonal to the vectors $\hat{r}^a$ and n^a, it is straightforward to give an expression for the action in three dimensions in terms of the fields N, N^r, γ, R, and $\phi_{g,s}$:

$$\begin{aligned} S^{(3)} = \frac{1}{8G_3}\int_{t_1}^{t_2}\int_0^{\tilde{r}}\Big(Ne^{-\gamma/2}(\gamma' R' - 2R'') \\ - \frac{1}{N}(e^{\gamma/2}\dot{\gamma} - 2N^{r\prime})(\dot{R} - e^{-\gamma/2}N^r R') \\ + \frac{R}{2N}[e^{\gamma/2}\dot{\phi}_g^2 - 2N^r\dot{\phi}_g\phi_g' + e^{-\gamma/2}(N^{r2} - N^2)\phi_g'^2] \\ + \frac{R}{2N}[e^{\gamma/2}\dot{\phi}_s^2 - 2N^r\dot{\phi}_s\phi_s' + e^{-\gamma/2}(N^{r2} - N^2)\phi_s'^2]\Big) \\ \times \mathrm{d}r\mathrm{d}t + \frac{1}{4G_3}\int_{t_1}^{t_2}(Ne^{-\gamma/2}R' - 1)\mathrm{d}t, \end{aligned}$$

where we have denoted ∂_t with a dot and ∂_r with a prime. We get now the Hamiltonian for the special case in which the boundary is taken to infinity ($\tilde{r} \to \infty$) taking into account that the metric g_{ab} reduces to the Minkowskian metric when $N = 1$, $N^r = 0$, $\gamma = 0$, and $R = r$, assuming regularity in the axis, and the $2+1$-dimensional asymptotic flatness conditions for the fields introduced in [6,12]. The Hamiltonian is then

$$\begin{aligned} H = \int_0^\infty\Big(N^r e^{-\gamma/2}[p_R R' - 2p_\gamma' + p_\gamma\gamma' + \phi_g' p_g + \phi_s' p_s] \\ + Ne^{-\gamma/2}\Big[\frac{1}{8G_3}(2R'' - \gamma' R') - 8G_3 p_R p_\gamma + \frac{4G_3}{R}p_g^2 \\ + \frac{R}{16G_3}\phi_g'^2 + \frac{4G_3}{R}p_s^2 + \frac{R}{16G_3}\phi_s'^2\Big]\Big)\mathrm{d}r \\ + \frac{1}{4G_3}(1 - e^{-\gamma_\infty/2}), \end{aligned}$$

where p_R, p_γ, p_g, and p_s are the momenta canonically conjugate to R, γ, ϕ_g, and ϕ_s, respectively, and $\gamma_\infty := \lim_{\tilde{r}\to\infty}\gamma(\tilde{r})$. It is easy to read both the constraints and the reduced Hamiltonian from the last expression. In order to proceed further we fix the gauge with the same conditions as in the absence of matter [6],

$$R(r) = r \quad \text{and} \quad p_\gamma(r) = 0;$$

it is straightforward to show that these gauge fixing conditions are admissible. We can now solve the constraints to get

$$\gamma(\tilde{R}) = \frac{1}{2}\int_0^{\tilde{R}}\Big(\phi_g'^2 + \frac{(8G_3 p_g)^2}{R^2} + \phi_s'^2 + \frac{(8G_3 p_s)^2}{R^2}\Big)R\mathrm{d}R,$$
$$p_R = -p_s\phi_s' - p_g\phi_g'.$$

Finally, the three-dimensional line element can be written as

$$ds^2 = e^{\gamma}[-e^{-\gamma_\infty}dt^2 + dR^2] + R^2 d\sigma^2. \tag{6}$$

The reduced phase space is coordinatized by $\phi_s(R)$, $p_s(R)$, $\phi_g(R)$, and $p_g(R)$, and the reduced Hamiltonian is

$$H = \frac{1}{4G_3}(1 - e^{-\gamma_\infty/2})$$

where

$$\gamma_\infty = \frac{1}{2}\int_0^\infty \left(\phi_g'^2 + \frac{(8G_3 p_g)^2}{R^2} + \phi_s'^2 + \frac{(8G_3 p_s)^2}{R^2}\right) R \mathrm{d}R.$$

As we can see, γ_∞ is the Hamiltonian for a system of two free, axially symmetric scalar fields in $2+1$ dimensions. The true Hamiltonian H is a nonlinear and bounded *function* of this free Hamiltonian, similar to the one appearing in the absence of matter [12]. The Hamilton equations are

$$\dot{\phi}_{s,g} = e^{-\gamma_\infty/2}\frac{p_{s,g}}{R}, \qquad \dot{p}_{s,g} = e^{-\gamma_\infty/2}(R\phi'_{s,g})'.$$

Though they are nonlinear integro-differential equations, they can be easily solved by realizing that γ_∞ is a constant of motion. Taking this fact into account, we can perform a change in the time coordinate $T = e^{-\gamma_\infty/2}t$ and rewrite them as

$$\left[-\partial_T^2 + \partial_R^2 + \frac{1}{R}\partial_R\right]\phi_{s,g} = 0, \tag{7}$$

describing two massless, axially symmetric scalar fields in $2+1$-dimensional Minkowskian background. As we see, this change in the time variable provides a one-to-one map from the solutions to a simple linear system to those of the nonlinear one. This mapping encodes the nontrivial interaction present in the model. If we compare the classical evolution of two different sets of initial data, we can see that they both correspond to the evolution defined by the free Hamiltonian γ_∞ with times elapsing at different rates (defined by the conserved values of γ_∞). Notice that once we have a particular solution for Eqs. (7) we have the freedom to "change coordinates" in the metric and write it in terms of t or in terms of T. This is not the case quantum mechanically, because the evolution of arbitrary states involves, in general, the *superposition* of Hilbert space vectors with energy (and time) dependent phases. This implies that quantum dynamics will be much more complicated than the classical one. In conclusion, the fact that the Hamiltonian is a function of a certain free Hamiltonian makes it both nontrivial (we are, indeed, dealing with a coupled system) and solvable.

IV. CANONICAL QUANTIZATION

Once we have characterized the reduced phase space and obtained the classical Hamiltonian, we proceed to quantize the model. To this end we will use the Fock Hilbert spaces $\mathcal{F}_{g,s}$ associated with two different free, massless, axially symmetric scalar fields propagating in a Minkowskian background. These spaces are endowed with the usual creation and annihilation operators $\hat{a}_{g,s}(k)$, $\hat{a}^\dagger_{g,s}(k)$ satisfying

$$[\hat{a}_g(k), \hat{a}_g^\dagger(q)] = \delta(k,q), \qquad [\hat{a}_s(k), \hat{a}_s^\dagger(q)] = \delta(k,q);$$

their corresponding vacua are denoted as $|0\rangle^{g,s}$. The Hilbert space of the interacting model is taken as the tensor product $\mathcal{H} = \mathcal{F}_g \otimes \mathcal{F}_s$ of the Fock spaces corresponding to both scalars. We define the annihilation operators[3] for modes of gravitational or matter types as $\hat{A}_g(k) := \hat{a}_g(k) \otimes \mathbb{I}_s$, $\hat{A}_s(k) := \mathbb{I}_g \otimes \hat{a}_s(k)$, and the distribution-valued, field and momentum operators $\hat{\phi}_{g,s}(R)$, $\hat{p}_{g,s}(R)$,

$$\hat{\phi}_{g,s}(R) = \sqrt{4G_3\hbar}\int_0^\infty J_0(Rk)[\hat{A}_{g,s}(k) + \hat{A}^\dagger_{g,s}(k)]\mathrm{d}k,$$

$$\hat{p}_{g,s}(R) = \frac{iR}{2}\sqrt{\frac{\hbar}{4G_3}}\int_0^\infty kJ_0(Rk)[\hat{A}^\dagger_{g,s}(k) - \hat{A}_{g,s}(k)]\mathrm{d}k,$$

satisfying the usual commutation relations $[\hat{\phi}_{g,s}(R_1), \hat{p}_{g,s}(R_2)] = i\hbar\delta(R_1, R_2)$. Notice that we can construct states with a fixed number of quanta of gravitational or "scalar" type by acting with the corresponding creation operators on the vacuum state $|\Omega\rangle = |0\rangle^g \otimes |0\rangle^s \in \mathcal{H}$ that is the minimum energy eigenstate of the quantum Hamiltonian[4]

$$\hat{H} = \frac{1}{4G_3}\left[1 - \exp\left(-4G_3\hbar\int_0^\infty k[\hat{A}_g^\dagger(k)\hat{A}_g(k) + \hat{A}_s^\dagger(k)\hat{A}_s(k)]\mathrm{d}k\right)\right]. \tag{8}$$

This quantum Hamiltonian is a nonlinear and bounded function of the sum of the free Hamiltonians

$$\hat{H}_0^{g,s} = \int_0^\infty k\hat{A}^\dagger_{g,s}(k)\hat{A}_{g,s}(k)\mathrm{d}k$$

for two massless, cylindrically symmetric scalar fields in $2+1$ dimensions evolving in a fictitious Minkowskian background. Their sum $\hat{H}_0^g + \hat{H}_0^s$ is an observable but it is not the generator of the time evolution of the system. The physical evolution, from t_0 to t, is generated by $\hat{H}$ and is given by the unitary evolution operator

$$\hat{U}(t,t_0) = \exp\left(-\frac{i(t-t_0)}{4G_3\hbar}[1 - e^{-4G_3\hbar(\hat{H}_0^g + \hat{H}_0^s)}]\right). \tag{9}$$

This operator defines the S matrix of the system when we take the appropriate time limits. Its matrix elements on n-particle states are straightforward to compute because these are eigenstates of the free Hamiltonian $\hat{H}_0 = \hat{H}_0^g + \hat{H}_0^s$. As we can see, the only matrix elements—involving

[3] Creation operators are defined in an analogous way.
[4] We have normal ordered the exponent because, otherwise, the Hamiltonian is trivial.

state vectors with a definite number of both types of quanta—that are nonzero are those connecting states with the same number of particles of each type; hence there is no conversion of quanta of one type into the other. At this point, it is important to reflect upon the interpretation of these elementary excitations of the fields. One must be careful, for example, when interpreting states such as $|0\rangle^g \otimes |\Phi\rangle^s$ because one should not be led to think of them as matter (represented by $|\Phi\rangle^s$) evolving in a certain background (given by $|0\rangle^g$). The system that we are considering is a coupled one, and hence one is not entitled to think of the metric and matter fields as independent objects. They are coupled in the equations and, in particular, the classical metric depends on both the gravitational and matter fields. Conversely, the evolution of the scalar field depends on the metric. If we want to approximate the Minkowski metric, we see that the state that most closely resembles it is the vacuum of our total Hilbert space $|\Omega\rangle$ [by the way, this is the only coherent state of the system that we know under the evolution (9)]. In the next sections we will make extensive use of states of the type $|0\rangle^g \otimes |k\rangle^s$ consisting of tensor products of the vacuum state of one of the Fock spaces and a one-particle state of the other. As we will show in the last sections of the paper, these single-particle states are the closest ones to Minkowski if one wants to incorporate an extra element that can be used to explore and describe the geometry of the quantized model in an operational way.

In the following it will be convenient to explicitly keep the length scale of the system $G = G_3\hbar$ in the mathematical expressions of the relevant objects. We will nevertheless use units such as $\hbar = 1$. With the time evolution (9) defined by the Hamiltonian (8) the annihilation and creation operators in the Heisenberg picture are

$$\begin{aligned}\hat{A}_{s,g}(k;t,t_0) &= \hat{U}^\dagger(t,t_0)\hat{A}_{s,g}(k)\hat{U}(t,t_0)\\ &= \exp[-i(t-t_0)E(k)e^{-4G\hat{H}_0}]\hat{A}_{s,g}(k),\\ \hat{A}^\dagger_{s,g}(k;t,t_0) &= \hat{U}^\dagger(t,t_0)\hat{A}_{s,g}(k)\hat{U}(t,t_0)\\ &= \hat{A}^\dagger_{s,g}(k)\exp[i(t-t_0)E(k)e^{-4G\hat{H}_0}],\end{aligned}$$

where

$$E(k) := \frac{1}{4G}(1-e^{-4Gk}) \quad \text{and} \quad \hat{H}_0 := \hat{H}_0^g + \hat{H}_0^s.$$

Then the scalar field operators that describe the gravitational and the massless scalar field degrees of freedom at time t are

$$\begin{aligned}\hat{\phi}_{s,g}(R;t,t_0) = \sqrt{4G}\int_0^\infty J_0(Rk)[\hat{A}_{s,g}(k;t,t_0)\\ + \hat{A}^\dagger_{s,g}(k;t,t_0)]\mathrm{d}k.\end{aligned}$$

V. TWO-POINT FUNCTIONS

As commented in the Introduction, one of the main goals of the paper is to find a way to recover a physical picture of spacetime in the quantized symmetry reduction of gravity coupled to matter discussed above. We want to find ways to describe a quantized spacetime geometry in an operational way much in the same way as one explores a classical spacetime geometry by using test particles. To this end it is useful to have objects playing the role of particle propagators or, even better, one-particle states that could allow us to define suitable wave functions with a straightforward interpretation as spatial probability amplitudes. Hopefully the time evolution of these objects may give us some idea about the physical effects of quantizing the gravitational field and also tell us something about how the classical macroscopic geometry emerges. As the reader may expect, this is not easy; in fact, the problem of finding suitable *position* eigenstates in the usual Minkowskian quantum field theory (QFT) is already nontrivial. Our strategy will be to use some of the objects introduced in the discussion of these issues in the traditional approaches to QFT (such as two-point functions or Newton-Wigner states), interpret them in our framework, and use them to obtain a physical picture of the quantized geometry and gravity. Also, in order to disentangle genuine quantum gravitational phenomena from artifacts introduced by the symmetry of the problem and the reduction process, we will compare the relevant objects to the corresponding ones in a model of a quantized, axially symmetric, massless free scalar field moving in a Minkowskian background. This comparison is conceptually simpler if one works both with gravity and the scalar field.

The vacuum expectation values of the product of two fields at different spacetime points can be interpreted, at least in an approximate sense, as propagation amplitudes for particles or field quanta created at a certain event to be found at another. This is so because the (Schrödinger picture) scalar field operators that describe the gravitational and scalar degrees of freedom are

$$\hat{\phi}_{s,g}(R) = \sqrt{4G}\int_0^\infty J_0(Rk)[\hat{A}_{s,g}(k) + \hat{A}^\dagger_{s,g}(k)]\mathrm{d}k$$

and their action on the vacuum $|\Omega\rangle = |0\rangle^g \otimes |0\rangle^s$ satisfies

$$\begin{aligned}\frac{1}{\sqrt{4G}}\hat{\phi}_{s,g}(R)|\Omega\rangle &= \int_0^\infty \mathrm{d}k J_0(Rk)\hat{A}^\dagger_{s,g}(k)|\Omega\rangle\\ &= \int_0^\infty \mathrm{d}k J_0(Rk)|k\rangle_{s,g}.\end{aligned}$$

These are linear superpositions of (orthonormal) states $|k\rangle_{s,g} := \hat{A}^\dagger_{g,s}(k)|\Omega\rangle$ with well-defined "radial momentum" k. Notice that

$$J_0(Rk) = \frac{1}{\sqrt{4G}}\,{}_{s,g}\langle k|\hat{\phi}_{s,g}(R)|\Omega\rangle \tag{10}$$

is a solution of the radial part of the Schrödinger equations for states with zero angular momentum in two dimensions,

$$\left[\partial_R^2 + \frac{1}{R}\partial_R + k^2\right]J_0(Rk) = 0.$$

If we consider a small volume element ΔV at a distance R from the symmetry axis, the value of $J_0^2(Rk)\Delta V$ is proportional to the probability of finding a particle of type s or g inside it. Notice that *this is not*, in general, the probability to find the particle in a thin cylindrical shell of radius R.

In order to consider the amplitude for propagation of quanta of the matter scalar, we could consider more general situations; for example, we might take states of the form $|c\rangle^g \otimes |k\rangle^s$ with $|c\rangle^g$ a suitable "coherent" gravitational state under the evolution defined by the dynamics of the system. Notice, however, that the physical interpretation of such a state is not completely clear and, in particular, one should not be led to think that the gravitational part of the state (say $|c\rangle^g$) fixes the geometry (or a suitable classical approximation thereof) and the matter part fixes the scalar field; in fact, *both* parts of the state vector contribute to fix the metric on one hand and the matter field on the other (classically this can be understood by realizing that the metric depends on *both* the gravitational and matter scalars).

In the following we will consider the case $t_2 > t_1$ and interpret the matrix element $\langle\Omega|\hat{\phi}_{s,g}(R_2; t_2, t_0)\times \hat{\phi}_{s,g}(R_1; t_1, t_0)|\Omega\rangle$ as the (approximate) probability amplitude of a particle created at a point at a distance R_1 from the axis in the instant of time t_1 to be detected at a another point at R_2 distance in the instant of time t_2. We can now obtain in a straightforward way[5]

$$\begin{aligned}
&\langle\Omega|\hat{\phi}_{s,g}(R_2; t_2, t_0)\hat{\phi}_{s,g}(R_1; t_1, t_0)|\Omega\rangle \\
&\quad = 4G\int_0^\infty J_0(R_1k)J_0(R_2k) \\
&\qquad \times \langle\Omega|\hat{A}_{s,g}(k; t_2, t_0)\hat{A}^\dagger_{s,g}(k; t_1, t_0)|\Omega\rangle \mathrm{d}k \\
&\quad = 4G\int_0^\infty J_0(R_1k)J_0(R_2k)\exp[-i(t_2 - t_1)E(k)]\mathrm{d}k.
\end{aligned} \tag{11}$$

We have introduced an initial time t_0 that will not appear in the final expressions of the matrix elements that we will consider here so that in the following we will write $\langle\Omega|\hat{\phi}_{s,g}(R_2, t_2)\hat{\phi}_{s,g}(R_1, t_1)|\Omega\rangle$.

Let us consider the integral (11). First of all, it must be said that it is not possible to compute it in closed form, although there are suitable ways to compute it numerically and approximate it by means of asymptotic expansions [3]. The relevant parameters in the integral are R_1, R_2, $t_2 - t_1$—the arguments of the two-point function— and $4G$ which sets the length scale.[6] In view of this, it appears to be appropriate to refer both length and time to this scale and introduce the adimensional variables $\rho_1 = \frac{R_1}{4G}$, $\rho_2 = \frac{R_2}{4G}$, and $\tau = \frac{t_2 - t_1}{4G}$ together with the change of variables $q = 4Gk$ which gives a dimensionless integration variable. In this way we can rewrite (11) as

$$\begin{aligned}
&\langle\Omega|\hat{\phi}_{s,g}(R_2, t_2)\hat{\phi}_{s,g}(R_1, t_1)|\Omega\rangle \\
&\quad = \int_0^\infty J_0(\rho_1 q)J_0(\rho_2 q)\exp[-i\tau(1 - e^{-q})]\mathrm{d}q,
\end{aligned} \tag{12}$$

where ρ_1 and ρ_2 are to be considered as functions of R_1 and R_2 as defined above. An interesting consequence of (12) is that we can obtain the vacuum expectation value of the commutator of Heisenberg picture field operators by taking its imaginary part (as we did in [3] to discuss the microcausality in this system).

One can consider, in principle, the numerical computation of this type of improper integral, but this is rather difficult due to the oscillating nature of the integrand. In spite of this, there are efficient ways to do it, as the one described in the Appendix, by rewriting it as an integral over a torus plus a rapidly convergent improper integral. On the other hand, the advantage of using asymptotic approximations in some relevant parameters lies in the fact that it gives the limiting behavior at large scales in some physically relevant regimes and also allows us to get numerical estimates in a rapid manner. It is possible to consider separate expansions for each of the parameters ρ_1, ρ_2, and τ which are valuable in the sense that it is possible to study the behavior of the two-point function when only one of them is taken to be large. It is also possible to write down an expansion in which all of them are simultaneously large while keeping their relative values. Here this approximation corresponds to large length and time intervals as compared to the scale set by G. Let us consider these separately.

A. Asymptotic expansions in ρ_1 or ρ_2

For large values of ρ_1 and ρ_2 the asymptotic behaviors of (12) are, respectively,

$$\frac{1}{\rho_1} + \frac{1}{\rho_1^3}\left[\frac{\rho_2^2}{4} + \tau^2 - \frac{i\tau}{2}\right] + O(\rho_1^{-6}) \quad \text{and}$$

$$\frac{1}{\rho_2} + \frac{1}{\rho_2^3}\left[\frac{\rho_1^2}{4} + \tau^2 - \frac{i\tau}{2}\right] + O(\rho_2^{-6}).$$

They can be obtained in a straightforward way by considering (12) as a standard h transform with asymptotic parameters ρ_1 or ρ_2 and using Mellin-transform tech-

[5]The creation and annihilation operators as written above are specially suitable for this computation and similar ones where the states are eigenstates of the Hamiltonian operator.

[6]In fact, $4G$ plays the role of the Planck length. Notice also that it sets the time and energy scales as we are taking units such that $\hbar = c = 1$.

niques [13]. The imaginary part of the previous expressions corresponds to the vacuum expectation value of the field commutator. It does not show the sharp discontinuity present when one considers axially symmetric massless scalar fields in a Minkowskian background and, hence, is a quantitative measure of the spreading of the light cones expected in a quantized theory of gravity as discussed at length in [3].

B. Asymptotic expansions in τ

In order to study the asymptotic behavior in τ for (12) it is convenient to consider two separate cases: Both ρ_1 and ρ_2 different from zero or only one of them equal to zero (the other must be different from zero because, otherwise, the integral is badly divergent). In the first case, the expansion can be obtained by first writing (12) as an h transform, using the Mellin-Parseval formula to obtain a representation for it as a complex contour integral, and splitting it (by displacing the integration path close to the branch cuts of the integrand) into several integrals that can be studied by standard Mellin-transform techniques. The details of this procedure—which allows us to obtain the asymptotic behavior to any order—can be found in [3]. The result in the present case is

$$\frac{1}{2\pi\sqrt{\rho_1\rho_2}\log\tau}\bigg\{\exp\bigg(\frac{\pi}{2}(\rho_1+\rho_2)-i\bigg[\frac{\pi}{2}+\tau-(\rho_1+\rho_2)\log\tau\bigg]\bigg)\Gamma[-i(\rho_1+\rho_2)]+\exp\bigg(-\frac{\pi}{2}(\rho_1+\rho_2)+i\bigg[\frac{\pi}{2}-\tau-(\rho_1+\rho_2)\log\tau\bigg]\bigg)\Gamma[i(\rho_1+\rho_2)]+\exp\bigg(\frac{\pi}{2}(\rho_1-\rho_2)-i[\tau+(\rho_2-\rho_1)\log\tau]\bigg)\Gamma[i(\rho_2-\rho_1)]+\exp\bigg(\frac{\pi}{2}(\rho_2-\rho_1)-i[\tau+(\rho_1-\rho_2)\log\tau]\bigg)\Gamma[i(\rho_1-\rho_2)]\bigg\}+O(1/\log^2\tau).$$

When either ρ_1 or ρ_2 are equal to zero we, respectively, obtain

$$\frac{1}{\sqrt{2\pi\rho_2\log\tau}}\bigg\{\exp\bigg[\frac{\pi}{2}\rho_2-i\bigg(\frac{\pi}{4}+\tau-\rho_2\log\tau\bigg)\bigg]\Gamma(-i\rho_2)+\exp\bigg[-\frac{\pi}{2}\rho_2+i\bigg(\frac{\pi}{4}-\tau-\rho_2\log\tau\bigg)\bigg]\Gamma(i\rho_2)\bigg\}+O(1/\log^{3/2}\tau)$$

and

$$\frac{1}{\sqrt{2\pi\rho_1\log\tau}}\bigg\{\exp\bigg[\frac{\pi}{2}\rho_1-i\bigg(\frac{\pi}{4}+\tau-\rho_1\log\tau\bigg)\bigg]\Gamma(-i\rho_1)+\exp\bigg[-\frac{\pi}{2}\rho_1+i\bigg(\frac{\pi}{4}-\tau-\rho_1\log\tau\bigg)\bigg]\Gamma(i\rho_1)\bigg\}+O(1/\log^{3/2}\tau).$$

As we can see, for fixed values of ρ_1 and ρ_2 the decay when one of them is zero is slower (an inverse power of $\sqrt{\log\tau}$) than the decay when both ρ_1 and ρ_2 are different from zero (an inverse power of $\log\tau$). We can compare this result with the one corresponding to a massless, axially symmetric free scalar field, [7]

$$\langle 0|\hat{\phi}(R_2,t_2)\hat{\phi}(R_1,t_1)|0\rangle=\int_0^\infty J_0(\rho_1 q)J_0(\rho_2 q)\times\exp(-i\tau q)\mathrm{d}q.$$

For values of τ satisfying $\tau>\rho_1+\rho_2$, this integral is equal to[8]

$$-\frac{2i}{\pi\sqrt{\tau^2-(\rho_1-\rho_2)^2}}\mathrm{K}\bigg(\sqrt{\frac{4\rho_1\rho_2}{\tau^2-(\rho_1-\rho_2)^2}}\bigg)$$

and for large values of τ it behaves asymptotically as $-i/\tau$, i.e. it falls off to zero much faster than (12). As a consequence of this we interpret the very slow decay of the two-point function for quantized Einstein-Rosen waves as an enhanced probability amplitude to find quanta (either of the gravitational or matter field) in the symmetry axis. This is a gravitational effect as it is not present for a quantized, axially symmetric, massless scalar field in a Minkowskian background.

C. Asymptotic expansions for ρ_1, ρ_2, and τ simultaneously large

We discuss now the obtention of an asymptotic approximation that is valid in a "gravitational classical limit" corresponding to taking ρ_1, ρ_2, and τ large while keeping their relative values. This is equivalent to considering values for R_1, R_2, and t_2-t_1 which are much larger than the "Planck scale" provided by $4G$; it is in this sense that we talk of a classical limit here. To this end, let us rewrite the integral in (12) as

$$\int_0^\infty J_0(\rho_1 q)J_0(\rho_2 q)\exp[-i\tau(1-e^{-q})]\mathrm{d}q=\int_0^\infty J_0(\lambda r_1 q)J_0(\lambda r_2 q)\exp[-i\lambda t(1-e^{-q})]\mathrm{d}q$$

where $\rho_1=\lambda r_1$, $\rho_2=\lambda r_2$, and $\tau=\lambda t$ with r_1, r_2, t fixed,

[7] Here we denote the vacuum state of the system as $|0\rangle$.

[8] Complete elliptic integrals of the first, second, and third kind are, respectively, defined as $\mathrm{K}(k)=\int_0^{\pi/2}(\mathrm{d}\theta/\sqrt{1-k^2\sin^2\theta})$, $\mathrm{E}(k)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta$, and $\Pi(n|k)=\int_0^{\pi/2}\{\mathrm{d}\theta/[(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}]\}$.

and λ taken as a new parameter that we will consider large (we will use it as an asymptotic parameter $\lambda \to \infty$). The last integral can be written as

$$-\frac{e^{-it\lambda}}{4\pi^2}\int_0^\infty \mathrm{d}q \oint_{\gamma_1}\mathrm{d}z_1\oint_{\gamma_2}\mathrm{d}z_2\frac{1}{z_1 z_2}\exp\Big(\lambda\Big[\frac{qr_1}{2}\Big(z_1-\frac{1}{z_1}\Big)+\frac{qr_2}{2}\Big(z_2-\frac{1}{z_2}\Big)+ite^{-q}\Big]\Big)$$

by using the well-known representation of the Bessel functions as contour integrals (over contours γ that enclose $z=0$)

$$J_n(x)=\frac{1}{2\pi i}\oint_\gamma\frac{\mathrm{d}z}{z^{n+1}}\exp\Big[\frac{x}{2}\Big(z-\frac{1}{z}\Big)\Big].$$

As discussed in [3] it is useful to choose the contours $\gamma_{1,2}$ in the complex plane region satisfying $\Re(z-1/z)\le 0$, $z\in\mathbb{C}$. A suitable asymptotic expansion in λ can then be obtained by following the procedure outlined in [3]. We only quote the result here after reabsorbing the parameter λ (and, hence, expressing the integral again in terms of ρ_1, ρ_2, and τ). To do this, we need to define three different regions that cover the (ρ_1,ρ_2,τ) space,[9] referred to in the following as I, II, and III, and defined by the conditions $\tau\le|\rho_2-\rho_1|$ for region I, $|\rho_2-\rho_1|<\tau<\rho_1+\rho_2$ for region II, and $\tau\ge\rho_1+\rho_2$ for region III. The asymptotic expansion in each of them is given by the sum of a leading contribution ($\sim 1/\lambda$) and a first order correction behaving asymptotically as $1/\lambda^{3/2}$ or $1/\lambda^2$. The leading contribution in the different regions is

$$\text{Region I:}\quad \frac{2}{\pi\sqrt{(\rho_1+\rho_2)^2-\tau^2}}\mathrm{K}\Big(\sqrt{\frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2-\tau^2}}\Big), \tag{13a}$$

$$\text{Region II:}\quad \frac{1}{\pi\sqrt{\rho_1\rho_2}}\Big[\mathrm{K}\Big(\sqrt{\frac{(\rho_1+\rho_2)^2-\tau^2}{4\rho_1\rho_2}}\Big)-i\mathrm{K}\Big(\sqrt{\frac{\tau^2-(\rho_2-\rho_1)^2}{4\rho_1\rho_2}}\Big)\Big], \tag{13b}$$

$$\text{Region III:}\quad \frac{-2i}{\pi\sqrt{\tau^2-(\rho_2-\rho_1)^2}}\mathrm{K}\Big(\sqrt{\frac{4\rho_1\rho_2}{\tau^2-(\rho_2-\rho_1)^2}}\Big), \tag{13c}$$

and the first asymptotic correction

$$\begin{aligned}\text{Region I:}\quad & -\frac{i\tau}{2\pi}\Big\{\frac{2\sqrt{(\rho_1+\rho_2)^2-\tau^2}[\rho_1^4+\rho_2^4+2\rho_1^2\tau^2-3\tau^4+2\rho_2^2\tau^2-2\rho_1^2\rho_2^2]}{(\rho_1+\rho_2-\tau)^2(\rho_1-\rho_2+\tau)^2(-\rho_1+\rho_2+\tau)^2(\rho_1+\rho_2+\tau)^2}\mathrm{E}\Big(\sqrt{\frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2-\tau^2}}\Big)\\ &-\frac{2\tau^2}{\sqrt{(\rho_1+\rho_2)^2-\tau^2}[\rho_2^4+(\tau^2-\rho_1^2)^2-2\rho_2^2(\rho_1^2+\tau^2)]}\mathrm{K}\Big(\sqrt{\frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2-\tau^2}}\Big)\Big\},\end{aligned} \tag{14a}$$

$$\text{Region II:}\quad \frac{e^{i[(\pi/4)-\tau+|\rho_2-\rho_1|(1+\log(\tau/|\rho_2-\rho_1|))]}}{\sqrt{2\pi\rho_1\rho_2|\rho_1-\rho_2|\log\frac{\tau}{|\rho_2-\rho_1|}}}, \tag{14b}$$

$$\text{Region III:}\quad \frac{1}{\sqrt{2\pi\rho_1\rho_2}}\Big\{\frac{e^{i[(\pi/4)-\tau+|\rho_2-\rho_1|(1+\log(\tau/|\rho_2-\rho_1|))]}}{\sqrt{|\rho_1-\rho_2|\log\frac{\tau}{|\rho_2-\rho_1|}}}+\frac{e^{-i[(\pi/4)+\tau-(\rho_1+\rho_2)(1+\log(\tau/(\rho_1+\rho_2)))]}}{\sqrt{\rho_1+\rho_2\log\frac{\tau}{\rho_1+\rho_2}}}\Big\}. \tag{14c}$$

The imaginary part of these expressions gives the vacuum-to-vacuum matrix elements of the field commutator already discussed in [3]. Also it is important to point out that (13) is, precisely, the two-point function for an axially symmetric massless scalar field evolving in a 2+1-dimensional Minkowskian background. In fact, this is the leading contribution to the two-point function—corresponding to an ordinary quantum field theory for a massless scalar field in a Minkowskian background—whereas Eqs. (14) provide the first asymptotic corrections.

It is also possible to obtain similar expansions when either ρ_1 or ρ_2 is zero. In this case we only have a single J_0 function in the integrand and the computations are greatly simplified. The result is

$$\theta(\rho-\tau)\Big[\frac{1}{\sqrt{\rho^2-\tau^2}}-\frac{i\tau(\rho^2+2\tau^2)}{(\rho^2-\tau^2)^{5/2}}\Big]+\theta(\tau-\rho)\times\Big[\frac{-i}{\sqrt{\tau^2-\rho^2}}+\frac{\exp(i[\rho\log\frac{\tau}{\rho}-\tau+\rho])}{\rho\sqrt{\log\frac{\tau}{\rho}}}\Big] \tag{15}$$

where ρ is the remaining nonzero radial parameter. Notice that this cannot be obtained by simply putting $\rho_1=0$ or $\rho_2=0$ in (13) and (14).

[9]We restrict ourselves to positive values of τ—the extension to negative values is straightforward—and avoid the boundaries between these regions.

Figures 1 and 2 show the behavior of the two-point function. One can compare the exact values obtained by numerical computation and the approximation given by the asymptotic expansions. We also compare its value with the one corresponding to an axially symmetric massless scalar field evolving in a Minkowskian background. We emphasize again here that the value of these asymptotic approximations relies on the fact that they give the exact behavior of the two-point functions in the relevant limits. The most important physical information that can be gleaned from these plots is a significant enhancement of the probability to find field quanta (either gravitational or matter) in the vicinity of the symmetry axis (defined by $\rho = 0$) as compared with the result for an axially symmetric massless scalar field in a Minkowskian background. This is especially remarkable because far from the axis (both ρ_1 and ρ_2 large) the dominant contribution to the two-point function is given by the one corresponding to the free massless field.

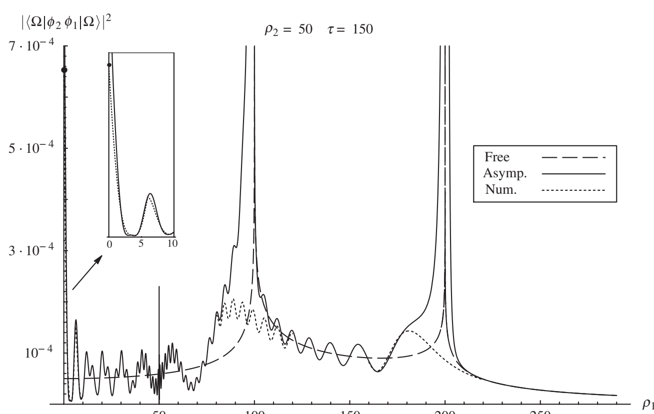

FIG. 1. This figure shows the absolute value squared for the two-point function for fixed values of ρ_2 and τ in terms of ρ_1. It shows the approximate probability of finding a field quantum at a small volume centered around ρ_1 after a certain time lapse τ if its position was ρ_2 at $\tau = 0$. Here radial distances and time are measured in units of $4G$. We also compare the exact values obtained by numerical computation (labeled as "Num."), the approximation given by the asymptotic expansions (labeled as "Asymp."), and the one corresponding to an axially symmetric massless scalar field evolving in a Minkowskian background (labeled as "Free"). The most salient feature is the significant enhancement of the probability for $\rho_1 = 0$. The dot on the vertical axis corresponds to the value of the two-point function on the axis $\rho_1 = 0$. The remarkable quality of the approximation provided by the asymptotic expansions given in the paper can also be seen, except in the boundary between regions in the ρ_1, ρ_2, and τ space, where the asymptotic expansions are divergent as expected on general grounds. The inset shows in detail the comparison of the exact values of the two-point function and the asymptotic approximation given by (13) and (14). It is worthwhile to point out that even though one does not expect the approximation to be valid for small values of ρ_1 or ρ_2 its general behavior is well described by it. Notice also the expected singularity at $\rho_1 = \rho_2$.

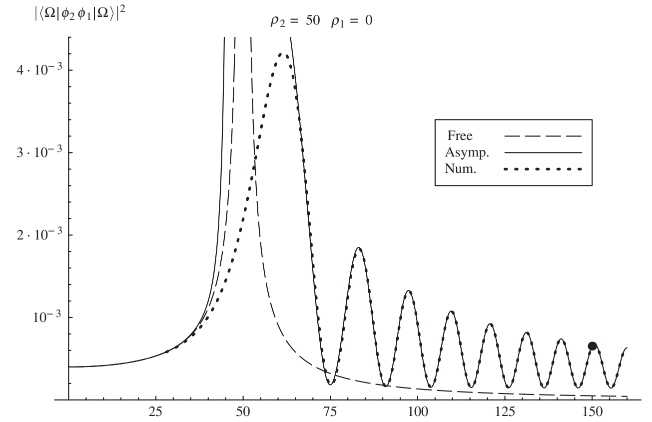

FIG. 2. This figure shows the absolute value squared for the two-point function for a fixed value of ρ_2 and $\rho_1 = 0$. The most interesting feature is the fact that the squared amplitude does not behave on average as the one corresponding to the "free part" as was apparent in the previous picture. The dot at $\tau = 150$ corresponds to the value of the two-point function at the symmetry axis shown in Fig. 1. Notice the different scale used in both plots.

The asymptotic analysis for $\rho = 0$ shows that the symmetry axis is exceptional in the sense that one does not recover the free-field result in the asymptotic limit. This was already noticed in previous studies of the microcausality of the system; in this context the interpretation of this phenomenon is the enhanced probability mentioned above. Another feature that stands out in the figures is the singularity at $\rho_1 = \rho_2$. This is expected on general grounds as a generic behavior in quantum field theory. It can be removed by introducing suitable regulators (see [4]). Here we can, in practice, identify and isolate the distributional behavior of the relevant objects at these points so we will not introduce regulators explicitly.

Other interesting two-point functions that we may study are mixed ones involving both the gravitational and scalar fields. We then consider matrix elements of the form $\langle f|\hat{\phi}_s(R_2; t_2, t_0)\hat{\phi}_g(R_1; t_1, t_0)|\Omega\rangle$ for some state $|f\rangle$ different from the vacuum.[10] The simplest nonzero matrix elements are obtained by choosing

$$|f\rangle = \int_0^\infty \mathrm{d}k_s \int_0^\infty \mathrm{d}k_g f(k_s, k_g) A_s^\dagger(k_s) A_g^\dagger(k_g)|\Omega\rangle,$$

satisfying the normalization condition

$$\int_0^\infty \int_0^\infty |f(k_s, k_g)|^2 \mathrm{d}k_g \mathrm{d}k_s = 1.$$

[10]Diagonal matrix elements of this type are always zero—for both scalar and gravitational modes—because they involve products of an even number of creation operators with an odd number of annihilation operators (or vice versa) that act on the vacuum to give zero. Also, expectation values between the vacuum and one-particle states of either type are easily seen to vanish.

They can be computed easily to give

$$\begin{aligned}
&\langle f|\hat{\phi}_s(R_2;t_2,t_0)\hat{\phi}_g(R_1;t_1,t_0)|\Omega\rangle \\
&= 4G\int_0^\infty \mathrm{d}k_s \int_0^\infty \mathrm{d}k_g J_0(R_2k_s)J_0(R_1k_g)\bar{f}(k_s,k_g) \\
&\quad \times \exp(i[(t_2-t_0)E(k_s)e^{-4Gk_g}+(t_1-t_0)E(k_g)]) \\
&= \frac{1}{4G}\int_0^\infty \mathrm{d}q_s \int_0^\infty \mathrm{d}q_g J_0(\rho_2 q_s)J_0(\rho_1 q_g)\bar{f}\left(\frac{q_s}{4G},\frac{q_g}{4G}\right) \\
&\quad \times \exp(i[\tau_{20}(1-e^{-q_s})e^{-q_g}+\tau_{10}(1-e^{-q_g})]),
\end{aligned} \tag{16}$$

where we have introduced adimensional variables and parameters as above: $k_s=\frac{q_s}{4G}$, $k_g=\frac{q_g}{4G}$, $\rho_1=\frac{R_1}{4G}$, $\rho_2=\frac{R_2}{4G}$, $\tau_{10}=\frac{t_1-t_0}{4G}$, and $\tau_{20}=\frac{t_2-t_0}{4G}$. It is also interesting to write the matrix element of the commutator,

$$\begin{aligned}
&\langle f|[\hat{\phi}_s(R_2;t_2,t_0),\hat{\phi}_g(R_1;t_1,t_0)]|\Omega\rangle \\
&= 4G\int_0^\infty \mathrm{d}k_s \int_0^\infty \mathrm{d}k_g J_0(R_2k_s)J_0(R_1k_g)\bar{f}(k_s,k_g) \\
&\quad \times [e^{i[(t_2-t_0)E(k_s)e^{-4Gk_g}+(t_1-t_0)E(k_g)]} \\
&\quad - e^{i[(t_2-t_0)E(k_s)+(t_1-t_0)E(k_g)e^{-4Gk_s}]}] \\
&= \frac{1}{4G}\int_0^\infty \mathrm{d}q_s \int_0^\infty \mathrm{d}q_g J_0(\rho_2 q_s)J_0(\rho_1 q_g)\bar{f}\left(\frac{q_s}{4G},\frac{q_g}{4G}\right) \\
&\quad \times [e^{i[\tau_{20}(1-e^{-q_s})e^{-q_g}+\tau_{10}(1-e^{-q_g})]} \\
&\quad - e^{i[\tau_{20}(1-e^{-q_s})+\tau_{10}(1-e^{-q_g})e^{-q_s}]}].
\end{aligned} \tag{17}$$

Some conclusions can be reached by considering some simple choices for the function f, although a more detailed analysis would require us to get suitable asymptotic expansions in terms of a general f. To this end let us consider the normalized function

$$f(k_s,k_g)=\frac{1}{\hat{k}}\chi_{[k_{0s}-\hat{k}/2,k_{0s}+\hat{k}/2]}(k_s)\chi_{[k_{0g}-\hat{k}/2,k_{0g}+\hat{k}/2]}(k_g)$$

where χ_V is the characteristic function of the set V, and $\hat{k}$ is a constant with dimensions of inverse length. For values of $\hat{k}$ small enough that

$$J_0(R_2k_s)J_0(R_1k_g)\exp(i[(t_2-t_0)E(k_s)e^{-4Gk_g} + (t_1-t_0)E(k_g)])$$

is essentially constant in the effective integration region, the value of the matrix elements (16) and (17) are, respectively,

$$4G\hat{k}J_0(R_2k_{0s})J_0(R_1k_{0g})e^{i[(t_2-t_0)E(k_{0s})e^{-4Gk_{0g}}+(t_1-t_0)E(k_{0g})]},$$
$$4G\hat{k}J_0(R_2k_{0s})J_0(R_1k_{0g})\{e^{i[(t_2-t_0)E(k_{0s})e^{-4Gk_{0g}}+(t_1-t_0)E(k_{0g})]} - e^{i[(t_2-t_0)E(k_{0s})+(t_1-t_0)E(k_{0g})e^{-4Gk_{0s}}]}\},$$

and their squared amplitudes are

$$16G^2\hat{k}^2J_0^2(R_2k_{0s})J_0^2(R_1k_{0g}),$$
$$8G^2\hat{k}^2J_0^2(R_2k_{0s})J_0^2(R_1k_{0g}) \times\{1-\cos[4G(t_2-t_1)E(k_{0s})E(k_{0g})]\}.$$

The first important point to notice here is the fact that these quantities, and, in particular, the commutator, are generically different from zero. This is an additional indication of the fact that we are dealing with an interacting (i.e. non-free) theory. It is also easy to see that the commutator is zero when $t_1=t_2$ because

$$\begin{aligned}
&\exp(i(t_2-t_0)[E(k_s)e^{-4Gk_g}+E(k_g)]) \\
&= \exp(i(t_2-t_0)E(k_s+k_g)) \\
&= \exp(i(t_2-t_0)[E(k_g)e^{-4Gk_s}+E(k_s)]).
\end{aligned}$$

Finally, it is remarkable that these functions do not display the type of causal behavior that we have found for the commutator of fields of the same type (by looking, for example, at the vacuum-to-vacuum expectation value). Their magnitude is of order $G^2\hat{k}^2$ so, at least in this approximation, they are small in the scale set by the natural length scales of the model.

VI. NEWTON-WIGNER STATES: PROPAGATORS

The main drawback of the two-point functions discussed in detail in the previous section is the fact that their interpretation as probability amplitudes is only approximate. The reason behind this is the fact that the state vectors

$$\hat{\phi}_{s,g}(R;t,t_0)|\Omega\rangle$$

do not constitute an orthonormal set. This is at the root of the well-known problem of localization in relativistic quantum field theory that has been solved in a more or less satisfactory way by the so-called Newton-Wigner states [7]. These constitute an orthonormal basis of position eigenstates for a certain choice of an inertial reference system in a Minkowskian spacetime. The most important interpretive difficulty with them is the fact that they cease to be localized under Lorentz boosts [8] but the assumptions upon which their construction is based are so natural that it is difficult to believe that a better solution to this problem might exist.

The purpose of this section is to build localized states analogous to the Newton-Wigner ones for our model, that we will label here as $|R\rangle$. We now have a reduced spacetime symmetry group so the problem of the behavior of localized states under spacetime symmetry transformations is partially alleviated. The physical interpretation of propagation amplitudes built with this type of states is clear in the sense that they are now proper probability amplitudes. Our point of view here is that the discussion of the two-point function $\langle\Omega|\hat{\phi}_s(R_2,t_2)\hat{\phi}_s(R_1,t_1)|\Omega\rangle$ together with the matrix elements $\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle$ can give us relevant and robust information about the motion of field quanta in position space and provide meaningful information about the transition between quantum and classical geometry in quantum gravity. It should be pointed out here that the availability of a position space orthonormal basis allows us to define position state normalized wave functions

$$|\Psi\rangle = \int_0^\infty \mathrm{d}R\Psi(R)|R\rangle,$$
$$\langle R|\Psi\rangle = \Psi(R),$$
$$\int_0^\infty |\Psi(R)|^2\mathrm{d}R = 1$$

and study their time evolution (in the Schrödinger picture) given by $|\Psi(t)\rangle = \hat{U}(t, t_0)|\Psi(t_0)\rangle$.

Here we will follow a simple procedure inspired in [7]. Let us write

$$|R\rangle = \int_0^\infty \mathrm{d}kf(k)J_0(kR)|k\rangle$$

where $|k\rangle := |0\rangle^g \otimes |k\rangle^s$ are scalar matter field quanta and we have made use of the fact that $J_0(kR)$ is a solution of the radial part of the Schrödinger equation for states with zero angular momentum in two dimensions:

$$\left[\partial_R^2 + \frac{1}{R}\partial_R + k^2\right]J_0(kR) = 0.$$

Once we make this choice, the function $f(k)$ is fixed by the orthogonality condition $\langle R_2|R_1\rangle = \delta(R_2, R_1)$ which implies $|f(k)|^2 = kR$, and hence $f(k) = \sqrt{kR}e^{i\nu(k)}$. Without loss of generality we will take $\nu(k) = 0$. We finally get then

$$|R\rangle = \int_0^\infty \mathrm{d}k(kR)^{1/2}J_0(kR)|k\rangle.$$

In the following we will study matrix elements of the form

$$\begin{aligned}\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle &= \sqrt{R_1R_2}\int_0^\infty kJ_0(kR_1)J_0(kR_2)\\ &\quad\times \exp[-i(t_2-t_1)E(k)]\mathrm{d}k\\ &= \frac{\sqrt{\rho_1\rho_2}}{4G}\int_0^\infty qJ_0(\rho_1 q)J_0(\rho_2 q)\\ &\quad\times\exp[-i\tau(1-e^{-q})]\mathrm{d}q\\ &= \frac{e^{-i\tau}}{4G}\delta(\rho_1,\rho_2) + \frac{\sqrt{\rho_1\rho_2}}{4G}\\ &\quad\times e^{-i\tau}\int_0^\infty qJ_0(\rho_1 q)J_0(\rho_2 q)\\ &\quad\times[\exp(i\tau e^{-q})-1]\mathrm{d}q. \end{aligned}\tag{18}$$

The last integral in the previous expression converges very quickly as the integrand has an exponential decay. Notice also the singularity at $\rho_1 = \rho_2$ that was also present for the two-point functions. As happened before we cannot give a closed form expression for the integral (18) although, again, it can be computed numerically by essentially the same methods used in the previous section. We can also obtain asymptotic approximations of the types discussed above that help us understand precisely the behavior in several important physical regimes and in the "classical limit."

An important question is the meaning of this probability amplitude. In the case of the two-point function, we highlighted the interpretation of (10) as the radial part of a wave function with zero angular momentum for a free two-dimensional particle. Now[11] $|\langle k|R\rangle|^2 = kRJ_0^2(kR)$ and the appearance of the R factor suggests that the correct interpretation for the amplitudes given by the Newton-Wigner states is that they describe the probability to find field quanta inside thin cylindrical shells at a distance R. This means that we will have to introduce appropriate factors of R_1 and R_2 to compare the two-point functions of the previous section with $\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle$. Specifically we will study

$$\frac{(4G)^2}{\sqrt{R_1R_2}}\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle = \int_0^\infty qJ_0(\rho_1 q)J_0(\rho_2 q)\times\exp[-i\tau(1-e^{-q})]\mathrm{d}q. \tag{19}$$

In the following we give the different asymptotic expansions for this propagator in the same regimes described in the previous section.

A. Asymptotic expansions in ρ_1 or ρ_2

For large values of ρ_1 and ρ_2 the asymptotic behaviors of (19) are, respectively,

$$\frac{\tau}{\rho_1^3}\left\{i + \frac{9}{\rho_1^2}\left[-\frac{i}{6} + \frac{i\rho_2^2}{4} + \frac{\tau}{2} + \frac{i\tau^2}{6}\right]\right\} + O(\rho_1^{-7}),$$
$$\frac{\tau}{\rho_2^3}\left\{i + \frac{9}{\rho_2^2}\left[-\frac{i}{6} + \frac{i\rho_1^2}{4} + \frac{\tau}{2} + \frac{i\tau^2}{6}\right]\right\} + O(\rho_2^{-7})$$

obtained, again, by a straightforward application of Mellin-transform techniques. When this is compared to the asymptotic behavior obtained for the two-point function, we see that they qualitatively agree for the imaginary part but they are quite different for the absolute value or the real part. This is not unexpected as the interpretation of the two-point function as a probability amplitude is approximate.

B. Asymptotic expansions in τ

As before, it is convenient to study separately the case in which both ρ_1 and ρ_2 are different from zero and the one in which either ρ_1 or ρ_2 is zero. By using the same methods

[11]Notice that $\langle k|R\rangle$ is *not* a solution to the Shrödinger equation.

as before we find

$$\frac{1}{2\pi\sqrt{\rho_1\rho_2}}\biggl\{\exp\biggl(\frac{\pi}{2}(\rho_1+\rho_2)-i\biggl[\frac{\pi}{2}+\tau-(\rho_1+\rho_2)\log\tau\biggr]\biggr)\Gamma[-i(\rho_1+\rho_2)]+\exp\biggl(-\frac{\pi}{2}(\rho_1+\rho_2)+i\biggl[\frac{\pi}{2}-\tau-(\rho_1+\rho_2)\log\tau\biggr]\biggr)\Gamma[i(\rho_1+\rho_2)]+\exp\biggl(\frac{\pi}{2}(\rho_1-\rho_2)-i[\tau+(\rho_2-\rho_1)\log\tau]\biggr)\Gamma[i(\rho_2-\rho_1)]+\exp\biggl(\frac{\pi}{2}(\rho_2-\rho_1)-i[\tau+(\rho_1-\rho_2)\log\tau]\biggr)\Gamma[i(\rho_1-\rho_2)]\biggr\}+O(1/\log\tau).$$

When either ρ_1 or ρ_2 is equal to zero we, respectively, obtain

$$\sqrt{\frac{\log\tau}{2\pi\rho_2}}\biggl\{\exp\biggl[\frac{\pi}{2}\rho_2-i\biggl(\frac{\pi}{4}+\tau-\rho_2\log\tau\biggr)\biggr]\Gamma(-i\rho_2)+\exp\biggl[-\frac{\pi}{2}\rho_2+i\biggl(\frac{\pi}{4}-\tau-\rho_2\log\tau\biggr)\biggr]\Gamma(i\rho_2)\biggr\}+O(1/\log^{1/2}\tau)$$

and

$$\sqrt{\frac{\log\tau}{2\pi\rho_1}}\biggl\{\exp\biggl[\frac{\pi}{2}\rho_1-i\biggl(\frac{\pi}{4}+\tau-\rho_1\log\tau\biggr)\biggr]\Gamma(-i\rho_1)+\exp\biggl[-\frac{\pi}{2}\rho_1+i\biggl(\frac{\pi}{4}-\tau-\rho_1\log\tau\biggr)\biggr]\Gamma(i\rho_1)\biggr\}+O(1/\log^{1/2}\tau).$$

As we can see, for fixed values of ρ_1 and ρ_2, when one of them is zero the asymptotic behavior of (19) for large values of τ consists of an oscillating function times $\sqrt{\log\tau}$; also, it must be noted that the oscillating part is precisely the one that appeared in the τ asymptotics of the two-point function. For ρ_1 and ρ_2 both different from zero, the asymptotic behavior is given by factors that are purely oscillatory in τ. Again, these coincide with those that appeared in the study of the two-point function. We see now that the value in the axis grows (very slowly) and it has a constant amplitude everywhere else. We see again that there is an enhancement of the probability to find the particle in the axis relative to the rest of the values of ρ_1 and ρ_2. We can compare this result with the one corresponding to the free axially symmetric scalar field in 2 + 1 dimensions given (for $\tau > \rho_1 + \rho_2$) by

$$\frac{2\tau}{\pi[(\rho_1+\rho_2)^2-\tau^2]\sqrt{\tau^2-(\rho_1-\rho_2)^2}}\mathrm{E}\biggl(\sqrt{\frac{4\rho_1\rho_2}{\tau^2-(\rho_1-\rho_2)^2}}\biggr).$$

As we can see, for large values of τ it falls off to zero as $-1/\tau^2$, much faster than the asymptotic expansions that we have already found. Again, we interpret this result as an enhanced probability to find field quanta in the vicinity of the symmetry axis. We see that, although the analytic expressions that we have obtained are different from the ones corresponding to the two-point function, the qualitative conclusions regarding the behavior in the vicinity of the axis are the same. This strongly suggests that we are seeing a genuine quantum gravitational effect. The most significant difference between both types of results is the fact that the probability at the axis decays very slowly if one considers the two-point function whereas it slowly grows if one uses the Newton-Wigner propagator. We do not perceive a contradiction here because of the approximate interpretation of the result for the two-point function and the necessity to look at probabilities over spacetime regions. In fact, we will look at this again in the next section when we consider the evolution of actual wave functions.

C. Asymptotic expansions for ρ_1, ρ_2, and τ simultaneously large

We discuss now the obtention of an asymptotic approximation that is valid in a "classical limit" corresponding to taking ρ_1, ρ_2, and τ large while keeping their relative values as we did for the two-point function. To this end, let us rewrite the integral in (19) as

$$\int_0^\infty qJ_0(\rho_1 q)J_0(\rho_2 q)\exp[-i\tau(1-e^{-q})]\mathrm{d}q=\int_0^\infty qJ_0(\lambda r_1 q)J_0(\lambda r_2 q)\exp[-i\lambda t(1-e^{-q})]\mathrm{d}q$$

where $\rho_1=\lambda r_1$, $\rho_2=\lambda r_2$, and $\tau=\lambda t$ with r_1, r_2, t fixed, and λ taken as a new parameter that we will consider large (we will use it as an asymptotic parameter $\lambda\to\infty$). The last integral can be written as

$$-\frac{e^{-it\lambda}}{4\pi^2}\int_0^\infty\mathrm{d}q\oint_{\gamma_1}\mathrm{d}z_1\oint_{\gamma_2}\mathrm{d}z_2\frac{q}{z_1z_2}\exp\biggl(\lambda\biggl[\frac{qr_1}{2}\biggl(z_1-\frac{1}{z_1}\biggr)+\frac{qr_2}{2}\biggl(z_2-\frac{1}{z_2}\biggr)+ite^{-q}\biggr]\biggr)$$

by using, again, the representation of the Bessel functions as contour integrals. The leading contribution in the different regions is

$$\text{Region I: } \frac{2i\tau}{\pi[(\rho_1-\rho_2)^2-\tau^2]\sqrt{(\rho_1+\rho_2)^2-\tau^2}}\mathrm{E}\left(\sqrt{\frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2-\tau^2}}\right), \tag{20a}$$

$$\begin{aligned}\text{Region II: } &\frac{\tau}{\pi[(\rho_1-\rho_2)^2-\tau^2][(\rho_1+\rho_2)^2-\tau^2]\sqrt{\rho_1\rho_2}}\left\{[(\rho_1+\rho_2)^2-\tau^2]\mathrm{K}\left(\sqrt{\frac{\tau^2-(\rho_1-\rho_2)^2}{4\rho_1\rho_2}}\right)\right.\\ &\left.-4\rho_1\rho_2\mathrm{E}\left(\sqrt{\frac{\tau^2-(\rho_2-\rho_1)^2}{4\rho_1\rho_2}}\right)+i[(\rho_1-\rho_2)^2-\tau^2]\mathrm{K}\left(\sqrt{\frac{(\rho_1+\rho_2)^2-\tau^2}{4\rho_1\rho_2}}\right)+4i\rho_1\rho_2\mathrm{E}\left(\sqrt{\frac{(\rho_2+\rho_1)^2-\tau^2}{4\rho_1\rho_2}}\right)\right\},\end{aligned} \tag{20b}$$

$$\text{Region III: } \frac{2\tau}{\pi[(\rho_1+\rho_2)^2-\tau^2]\sqrt{\tau^2-(\rho_1-\rho_2)^2}}\mathrm{E}\left(\sqrt{\frac{4\rho_1\rho_2}{\tau^2-(\rho_1-\rho_2)^2}}\right), \tag{20c}$$

and the first asymptotic correction is

$$\text{Region I: } \frac{\tau}{2}\frac{\partial^3}{\partial\tau^3}\left[\frac{2}{\pi\sqrt{(\rho_1+\rho_2)^2-\tau^2}}\mathrm{K}\left(\sqrt{\frac{4\rho_1\rho_2}{(\rho_1+\rho_2)^2-\tau^2}}\right)\right], \tag{21a}$$

$$\text{Region II: } \frac{e^{i[(\pi/4)-\tau+|\rho_2-\rho_1|(1+\log(\tau/|\rho_2-\rho_1|))]}}{\sqrt{2\pi\rho_1\rho_2|\rho_1-\rho_2|}}, \tag{21b}$$

$$\text{Region III: } \frac{1}{\sqrt{2\pi\rho_1\rho_2}}\left\{\frac{e^{i[(\pi/4)-\tau+|\rho_2-\rho_1|(1+\log(\tau/|\rho_2-\rho_1|))]}}{\sqrt{|\rho_1-\rho_2|}}+\frac{e^{-i[(\pi/4)+\tau-(\rho_1+\rho_2)(1+\log(\tau/(\rho_1+\rho_2)))]}}{\sqrt{\rho_1+\rho_2}}\right\}. \tag{21c}$$

The previous expression for region I can be explicitly written in terms of complete elliptic integrals of the first and second kinds with coefficients that are square roots of rational functions of ρ_1, ρ_2, and τ; as they are rather lengthy, we do not give them here. As before, the leading contribution corresponds to the Newton-Wigner propagator for a massless axially symmetric scalar field evolving in a Minkowskian background. It is also possible to give asymptotic expansions in the case when either ρ_1 or ρ_2 is zero. They are

$$\theta(\rho-\tau)\left[\frac{i\tau}{(\rho^2-\tau^2)^{3/2}}+\frac{3\tau^2(3\rho^2+2\tau^2)}{2(\rho^2-\tau^2)^{7/2}}\right]+\theta(\tau-\rho)\times\left[-\frac{\tau}{(\tau^2-\rho^2)^{3/2}}+\frac{1}{\rho}e^{i(\rho-\tau+\rho\log(\tau/\rho))}\sqrt{\log\frac{\tau}{\rho}}\right]$$

where, as above, ρ is the remaining nonzero radial parameter. Notice that this last expansion cannot be obtained by simply putting $\rho_1 = 0$ or $\rho_2 = 0$ in (20) and (21).

The most interesting feature of the propagator is its behavior in the axis and at $\rho_1 = \rho_2$. Let us consider first the behavior at $\rho_1 = \rho_2$. Here we find the singularity expected on general grounds due to the orthonormality property of the Newton-Wigner vectors. This can be identified as the delta function appearing in (18). In addition to this, we see a clear tendency of the probability amplitude to remain concentrated around the region $\rho_1 = \rho_2$. We interpret this as an effect of self-gravity that tends to favor the concentration of matter. The spreading of the amplitude, on the other hand, can be interpreted as quantum mechanical diffusion similar to, but less extreme than, the familiar one for particles in ordinary quantum mechanics. At the axis $\rho_1 = 0$ (or $\rho_2 = 0$) we see that, once the amplitude grows as a consequence of the gravitational collapse of the initial matter distribution, there is a tendency to have a large

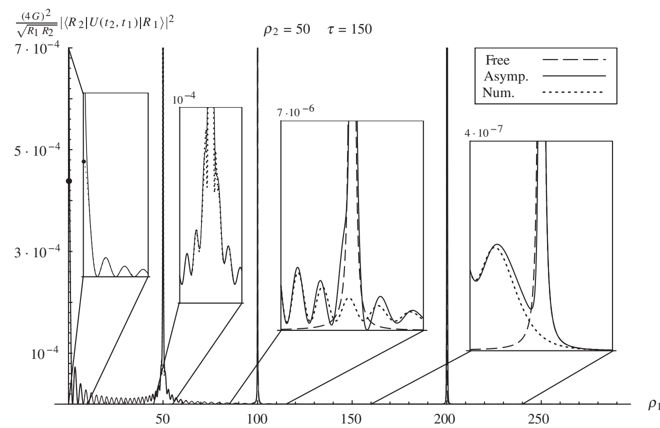

FIG. 3. This figure shows the square modulus of the Newton-Wigner propagator (divided by factors of $R_{1,2}$ introduced in order to compare it with the two-point function discussed in Fig. 1). We can see several interesting features: 1) An enhanced amplitude at the axis similar to the one already seen for the two-point function, 2) A large amplitude at $\rho_1 = \rho_2$, even when the delta function at this position is subtracted (we interpret this as a self-gravity effect in a region of high matter density), 3) The very quick decay of the amplitude beyond the position corresponding to $\rho_1 = 200$ (this marks the position of the light cone). Notice that, even though the amplitudes in the "free case"—corresponding to the propagation of a massless axially symmetric scalar in a Minkowski background—diverge in some regions in the (ρ, τ) plane, they remain finite in our quantum gravity model.

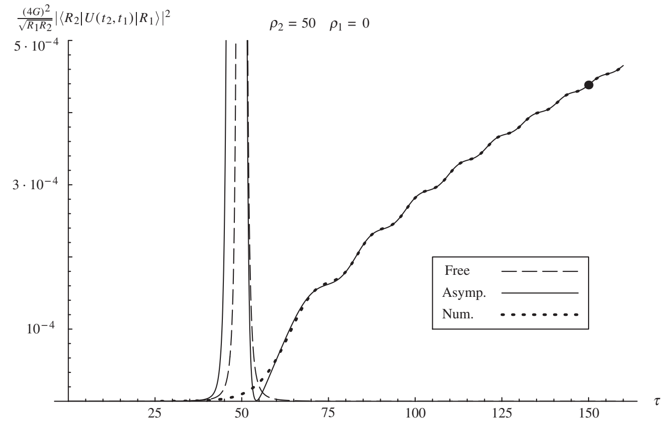

FIG. 4. This figure shows the square modulus of the Newton-Wigner propagator (divided by factors of $R_{1,2}$) at the axis. The dot corresponds to the value shown in Fig. 3.

probability to find field quanta at that position (in fact, the amplitude grows as shown in Fig. 4). To really assess how the probability of finding particles near the symmetry axis evolves, we will consider in the next section the evolution of proper normalizable wave functions and confirm, indeed, that the probability is enhanced but decays very slowly in time. These effects bear some resemblance with what one expects to find in the study of gravitational collapse and black hole evaporation.[12] It is important to compare the results of this section with the ones derived in the study of the two-point function and the free axially symmetric massless scalar field. Regarding the first, it should be pointed out that the qualitative agreement between the two pictures is very good. We see in both cases the enhancement of the probability at the axis, the singular structure in the vicinity of $\rho_1 = \rho_2$, and the motion along null radial geodesics (showing up as a significant probability to find particles at the classical light cone). When the results are compared to the "free" massless case, the gravitational phenomena that we have interpreted as an increased probability due to self-gravity are conspicuously absent but those related to the causal structure and microcausality of the system (i.e. enhanced probabilities on the light cones) are still present.

VII. TIME EVOLUTION OF A (RADIAL) POSITION STATE WAVE FUNCTION

For superpositions of one-particle states, the time evolution of the wave function $\Psi(R, t) = \langle R|\Psi(t)\rangle$ is given by

$$\begin{aligned}\Psi(R, t) &= \langle R|\hat{U}(t, t_0)|\Psi(t_0)\rangle \\ &= \int_0^\infty \int_0^\infty k\sqrt{R\tilde{R}}J_0(kR)J_0(k\tilde{R}) \\ &\quad \times \exp[-i(t - t_0)E(k)]\Psi(\tilde{R}, t_0)\mathrm{d}\tilde{R}\mathrm{d}k.\end{aligned}$$

[12]The main difference is the apparent absence of a horizon or something behaving, at least in an approximate way, as one.

In the following we want to study the evolution of one-particle wave functions of this type. They will be, in fact, the quantum test particles that we will use to describe the quantum geometry of our spacetime model. In principle, we can make any choice of an initial wave function $\Psi(R, t_0)$. However, in order to obtain closed expressions for the wave function (or at least as simple as possible) we will concentrate on a specific choice that satisfies several reasonable requirements: the possibility of having some control on the position of the peak of the probability distribution, the possibility of controlling the width of the wave packet, and the possibility of performing (some) integrations to get a manageable closed form for it. A (normalized) function satisfying these conditions at t_0 is

$$\Psi(R, t_0) = \sqrt{\frac{2R}{r_2^2 - r_1^2}}\chi_{[r_1,r_2]}(R) \qquad (\text{with } r_2 > r_1)$$

which gives

$$\begin{aligned}\Psi(R, t) = \sqrt{\frac{2R}{r_2^2 - r_1^2}} \int_0^\infty J_0(kR)[r_2J_1(kr_2) - r_1J_1(kr_1)] \\ \times \exp[-i(t - t_0)E(k)]\mathrm{d}k.\end{aligned}$$

It is convenient to rewrite this in terms of adimensional objects by introducing the following redefinitions, $\rho = \frac{R}{4G}$, $\sigma_2 = \frac{r_2}{4G}$, $\sigma_1 = \frac{r_1}{4G}$, $\tau = \frac{t-t_0}{4G}$, and the change of integration variable $q = 4Gk$. Writing $\psi(\rho, \tau) := \Psi(4G\rho, 4G\tau + t_0)$ we finally get

$$\begin{aligned}\psi(\rho, \tau) = \sqrt{\frac{2\rho}{4G(\sigma_2^2 - \sigma_1^2)}} \int_0^\infty J_0(q\rho)[\sigma_2J_1(q\sigma_2) \\ - \sigma_1J_1(q\sigma_1)]\exp[-i\tau(1 - e^{-q})]\mathrm{d}q \qquad (22)\end{aligned}$$

which satisfies, for all τ, the normalization condition

$$4G\int_0^\infty |\psi(\rho, \tau)|^2\mathrm{d}\rho = 1.$$

It is straightforward to see that when $\tau = 0$ we recover the initial wave function at $t = t_0$. Our goal now is to extract information about the evolution of this wave function. In particular, we are interested in the behavior near the symmetry axis to see if the enhanced probability suggested by the analysis of the two-point functions and the Newton-Wigner propagator is present. We also want to find out if the evolution of this wave function somehow defines a classical trajectory in the spacetime that can be used to define in an approximate way a physical notion of the geodesic. Finally, we want to compare the result with the one obtained for the axially symmetric massless scalar that we are using to disentangle quantum gravitational effects from more prosaic behaviors. As in the previous sections we will use asymptotic approximations in different regimes to extract the analytic behavior of $\psi(\rho, \tau)$.

A. Asymptotic expansion in ρ

For large values of ρ the asymptotic behavior of (22) is

$$\frac{1}{2}\sqrt{\frac{\sigma_2^2-\sigma_1^2}{4G}}\frac{\tau}{\rho^{3/2}}\Big\{i+\frac{9}{2\rho^2}\Big[-\frac{i}{3}+\tau+\frac{i\tau^2}{3}+\frac{9}{4}(\sigma_1^2+\sigma_2^2)\Big]\Big\} + O(\rho^{-9/2}),$$

obtained, again, by a straightforward application of Mellin-transform techniques. When this is compared to the asymptotic behavior obtained for the evolution of the same wave function for the case of an axially symmetric massless scalar in a Minkowskian background

$$\frac{1}{2}\sqrt{\frac{\sigma_2^2-\sigma_1^2}{4G}}\frac{\tau}{\rho^{3/2}}\Big\{i+\frac{9}{2\rho^2}\Big[\frac{i\tau^2}{3}+\frac{9}{4}(\sigma_1^2+\sigma_2^2)\Big]\Big\} + O(\rho^{-9/2}),$$

we see that the leading behavior far from the axis (large ρ and fixed τ) is the same in both cases.

B. Asymptotic expansions in τ

In principle, we only have to consider the situation in which $\rho \neq 0$ because $\psi(0,\tau)=0$. The asymptotic behavior in τ for the integral

$$\int_0^\infty J_0(\rho q)J_1(\sigma q)\exp[-i\tau(1-e^{-q})]\mathrm{d}q \qquad (23)$$

is obtained by the same methods used in previous sections and is

$$\begin{aligned} S(\rho,\sigma,\tau) = \frac{1}{2\pi\sqrt{\rho\sigma}\log\tau}\Big\{ & -\exp\Big(\frac{\pi}{2}(\rho+\sigma)+i[-\tau+(\rho+\sigma)\log\tau]\Big)\Gamma[-i(\rho+\sigma)] - \exp\Big(-\frac{\pi}{2}(\rho+\sigma) \\ & - i\Big[\frac{\pi}{2}\tau+(\rho+\sigma)\log\tau\Big]\Big)\Gamma[i(\rho+\sigma)] + \exp\Big(\frac{\pi}{2}(\rho-\sigma)+i\Big[-\tau+(\rho-\sigma)\log\tau+\frac{\pi}{2}\Big]\Big)\Gamma[i(\sigma-\rho)] \\ & + \exp\Big(\frac{\pi}{2}(\sigma-\rho)-i\Big[\tau+(\rho-\sigma)\log\tau+\frac{\pi}{2}\Big]\Big)\Gamma[i(\rho-\sigma)]\Big\} + O(1/\log^2\tau), \end{aligned}$$

so that we have that the asymptotic behavior for $\psi(\rho,\tau)$ is given by

$$\sqrt{\frac{2\rho}{4G(\sigma_2^2-\sigma_1^2)}}[\sigma_2 S(\rho,\sigma_2,\tau)-\sigma_1 S(\rho,\sigma_1,\tau)].$$

This displays the slow decay in time that is characteristic of the system. As before one can compare it with the one corresponding to the free, massless, axially symmetric scalar field that is given for large values of τ by

$$\begin{aligned} \frac{2}{\pi}\sqrt{\frac{2\rho}{4G(\sigma_2^2-\sigma_1^2)}}\Bigg[& \frac{(\rho+\sigma_2-\tau)\Pi\big(\frac{2\rho}{\rho-\sigma_2+\tau}\big|\sqrt{\frac{4\rho\sigma_2}{\tau^2-(\rho-\sigma_2)^2}}\big)-\sigma_2\mathrm{K}\big(\sqrt{\frac{4\rho\sigma_2}{\tau^2-(\sigma_2-\rho)^2}}\big)}{\sqrt{\tau^2-(\sigma_2-\rho)^2}} \\ & - \frac{(\rho+\sigma_1-\tau)\Pi\big(\frac{2\rho}{\rho-\sigma_1+\tau}\big|\sqrt{\frac{4\rho\sigma_1}{\tau^2-(\rho-\sigma_1)^2}}\big)-\sigma_1\mathrm{K}\big(\sqrt{\frac{4\rho\sigma_1}{\tau^2-(\sigma_1-\rho)^2}}\big)}{\sqrt{\tau^2-(\sigma_1-\rho)^2}}\Bigg] \end{aligned}$$

and decays to zero as

$$\frac{\sigma_1^2-\sigma_2^2}{2\tau^2}\sqrt{\frac{2\rho}{4G(\sigma_2^2-\sigma_1^2)}}.$$

This behavior in τ means that, if the evolution of the initial wave function is such that at some instant of time τ the probability of finding the particle in the vicinity of the symmetry axis builds up, it will remain high for a large interval in τ as the asymptotic behavior obtained above shows. In fact, this is what happens in this case as can be seen in Fig. 5. Finally, it is interesting to notice that the falloff in the τ direction is much faster than the one given by the τ asymptotic expansion for our system.

C. Asymptotic expansions for ρ, σ, and τ simultaneously large

We discuss now the obtention of an asymptotic approximation that is valid in a "classical limit" corresponding to taking ρ and τ large while keeping their relative values as we did in previous sections. We will also take σ_1 and σ_2, which define the support of the wave function at the initial time, large in comparison with the length scale $4G$. This will allow us to use the same type of asymptotic expansion that we have used in previous sections. The procedure should be clear by now so we skip the details here; we just use the contour integral representation introduced above for the Bessel functions and write the integrals in (22) in terms of them. The asymptotic expansion for (23) is then obtained as the sum of a boundary term contribution that coincides with

$$\int_0^\infty J_0(\rho q)J_1(\sigma q)\exp(-i\tau q)\mathrm{d}q$$

plus some extra contributions. The boundary term is

$$|\rho-\sigma|>\tau:\ \frac{\mathrm{sgn}(\sigma-\rho)}{\sigma}+\frac{2i[\sigma \mathrm{K}(\sqrt{\frac{4\rho\sigma}{(\rho+\sigma)^2-\tau^2}})+(\rho-\sigma-\tau)\Pi(\frac{2\rho}{\rho+\sigma+\tau}|\sqrt{\frac{4\rho\sigma}{(\rho+\sigma)^2-\tau^2}})]}{\pi\sigma\sqrt{(\rho+\sigma)^2-\tau^2}}, \tag{24a}$$

$$\begin{aligned}|\rho-\sigma|<\tau<\rho+\sigma:\ &\frac{1}{\sigma}+\frac{i(\rho+\tau)[i\mathrm{K}(\sqrt{\frac{\tau^2-(\rho-\sigma)^2}{4\rho\sigma}})-\mathrm{K}(\sqrt{\frac{(\rho+\sigma)^2-\tau^2}{4\rho\sigma}})]}{\pi\sigma\sqrt{\rho\sigma}}\\ &+\frac{i[(\rho+\sigma+\tau)\mathrm{K}(\sqrt{\frac{(\rho+\sigma)^2-\tau^2}{4\rho\sigma}})+(\rho-\sigma-\tau)\Pi(\frac{\rho+\sigma-\tau}{2\sigma}|\sqrt{\frac{(\rho+\sigma)^2-\tau^2}{4\rho\sigma}})]}{\pi\sigma\sqrt{\rho\sigma}}\\ &-\frac{(\rho+\sigma-\tau)\Pi(\frac{\rho-\sigma+\tau}{2\rho}|\sqrt{\frac{\tau^2-(\rho-\sigma)^2}{4\rho\sigma}})-2\rho\mathrm{K}(\sqrt{\frac{\tau^2-(\rho-\sigma)^2}{4\rho\sigma}})}{\pi\sigma\sqrt{\rho\sigma}},\end{aligned} \tag{24b}$$

$$\rho+\sigma<\tau:\ \frac{1}{\sigma}+\frac{2[(\rho+\sigma-\tau)\Pi(\frac{2\rho}{\rho-\sigma+\tau}|\sqrt{\frac{4\rho\sigma}{\tau^2-(\rho-\sigma)^2}})-\sigma\mathrm{K}(\sqrt{\frac{4\rho\sigma}{\tau^2-(\sigma-\rho)^2}})]}{\pi\sigma\sqrt{\tau^2-(\sigma-\rho)^2}} \tag{24c}$$

and the extra contribution is given by

$$\begin{aligned}&\theta(\tau-\sigma-\rho)\frac{\exp(i[-\tau+(\rho+\sigma)(1+\log\frac{\tau}{\rho+\sigma})-\frac{3\pi}{4}])}{\sqrt{2\pi\sigma\rho(\sigma+\rho)}\log\frac{\tau}{\rho+\sigma}}+\theta(\tau-\sigma+\rho)\theta(\sigma-\rho)\\ &\times\frac{\exp(i[-\tau+(\sigma-\rho)(1+\log\frac{\tau}{\sigma-\rho})-\frac{\pi}{4}])}{\sqrt{2\pi\sigma\rho(\sigma-\rho)}\log\frac{\tau}{\sigma-\rho}}+\theta(\tau+\sigma-\rho)\theta(\rho-\sigma)\frac{\exp(i[-\tau+(\rho-\sigma)(1+\log\frac{\tau}{\rho-\sigma})+\frac{3\pi}{4}])}{\sqrt{2\pi\sigma\rho(\rho-\sigma)}\log\frac{\tau}{\rho-\sigma}}.\end{aligned}$$

Introducing these expressions in (22) we finally obtain the desired asymptotic approximation for the wave function in the limit when all the lengths are significantly larger than $4G$.

These asymptotic expansions allow us to explore different possibilities as far as the width and the position of the support of the wave function at $t = t_0$ are concerned. We may consider the case in which the support—in the scale defined by $4G$—is wide or narrow.[13] In the first case the wave function evolves in a way that closely resembles the propagation of an initial wave function of this type for a free axially symmetric field in a Minkowski background. One can easily see that most of the probability amplitude for large values of ρ is concentrated along the lines $\tau = \rho_0 + \rho$ and $\tau = \rho - \rho_0$ in the (ρ, τ) plane. These two lines define trajectories that can be interpreted as null geodesics of an emergent spacetime metric. Notice that they are defined with a resolution of the order of the width of the initial support of the wave function (see Fig. 6). The other case defines a situation when the matter density is in some sense high and then displays a behavior that can be interpreted as due to self-gravity effects (see Fig. 5). Also in this case, especially when the initial support is close to the axis, there is a buildup of the probability amplitude at $\rho = 0$ that decays subsequently in the very slow fashion characteristic of the model. This means that the probability to find the particle in the vicinity of the axis remains high for a long time and, as a consequence, the probability of finding it on the "light cone" is much lower. This is shown in Fig. 5.

As we have already discussed in the case of the two-point function and the Newton-Wigner propagator there is significant enhancement of the probability to find field quanta close to the axis. This can easily be seen by comparing this wave function to the one corresponding to quanta of an axially symmetric, massless scalar field in a $2 + 1$-dimensional Minkowskian background. As the asymptotic behavior of both shows (and the figures clearly display) the probability near the axis remains significantly higher in the gravitational case if the support of the initial wave function is narrow. Another interesting feature that can be seen is the persistence of a footprint of the wave function in the range of ρ where the support of the initial wave function is. This can be understood by realizing that it is possible to write the Newton-Wigner propagator (18) as the sum of a delta function, multiplied by a time-dependent phase, and a rapidly convergent integral.

[13]The Newton-Wigner propagator $\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle$ corresponds to the limit when the support is infinitely narrow.

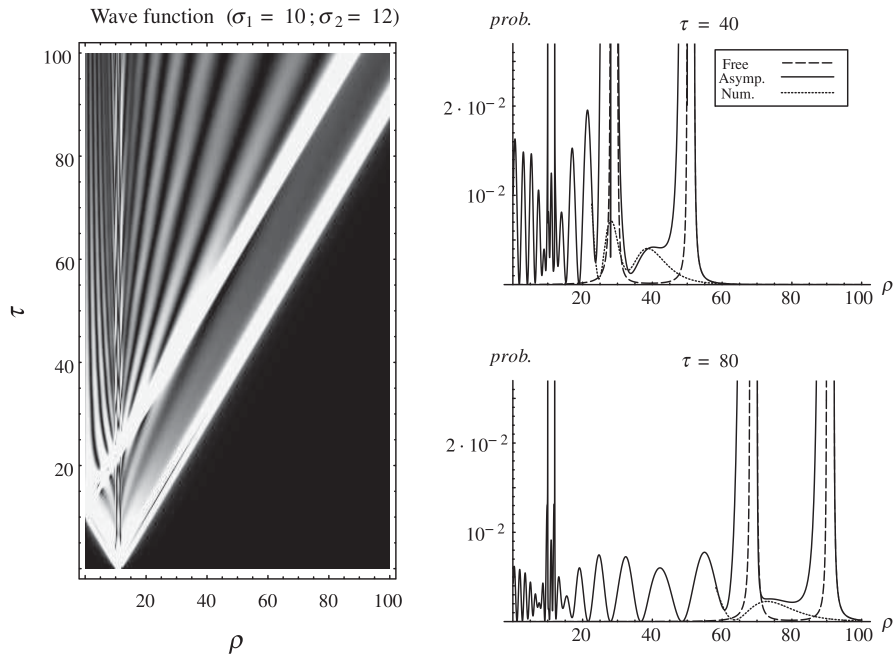

FIG. 5. Probability density for an initial wave function with a narrow support at the initial instant of time. Notice the enhanced probability close to the axis and the position of the light cone structure that appears.

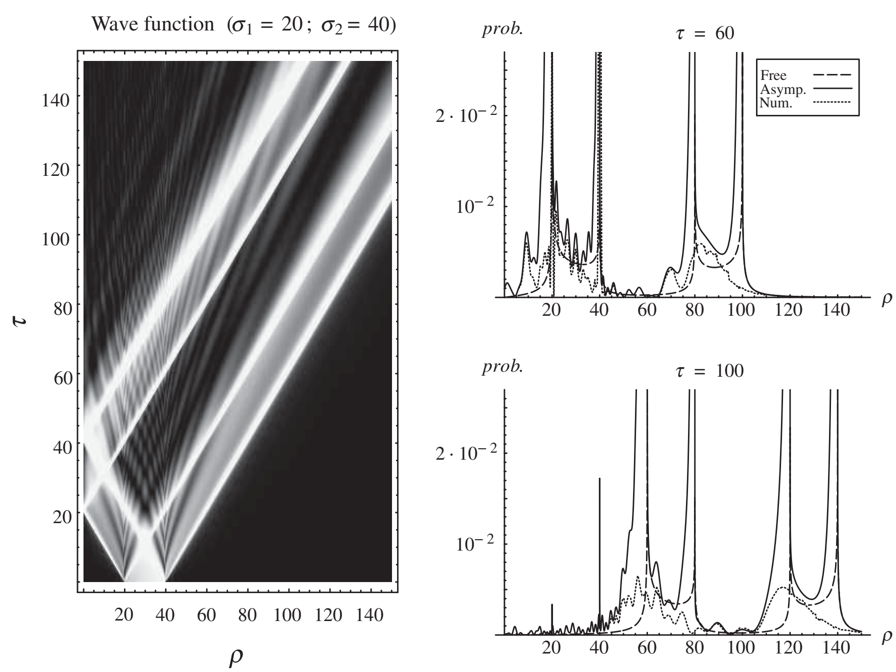

FIG. 6. Probability density for a wave function with a wide support at the initial instant of time. The density plot is made with the asymptotic approximation discussed in the text to avoid long and slow numerical integrations. Saturated (white) areas correspond to divergencies of this asymptotic expansion that are not present in the true wave function. This can be seen in the sections plotted in the right-hand side of the figure. Notice that in this case the two null curves signaling the light cone are well defined—certainly better than in the narrow case shown in Fig. 5—and the probability inside the light cone or at the axis becomes very low as τ grows.

VIII. CONCLUSIONS AND COMMENTS

In the first part of the paper we have discussed in some detail the exact nonperturbative quantization of Einstein-Rosen waves coupled to a massless, axially symmetric scalar field. This provides us with an interesting system where gravity is coupled to a matter field that can be exactly quantized and retains some of the features of full general relativity. In particular, and even though we have performed a symmetry reduction, we still have an infinite number of local degrees of freedom in both the gravitational and matter sectors. We also have (some) diffeomorphism invariance that has been taken care of through gauge fixing in the Hamiltonian formalism.

One of the uses of this toy model is to explore the possibility of getting relevant information on the nature of quantum space time by using the probe provided by the matter field. This is particularly useful in the setting of Einstein-Rosen waves owing to the fact that the object that encodes the gravitational degrees of freedom is a scalar field and, hence, its relation with the four-dimensional metric—especially in the quantum case—is somewhat indirect. Even though we have discussed some specific issues of the combined system (such as two-point functions involving both the scalar and gravitational fields) we have mostly used the matter field in this role of a spacetime probe.

In order to obtain geometric information on quantum spacetime, one must try to work with objects of direct physical interpretation in spacetime terms. One such object is the two-point function; in fact, the field commutator has already been successfully used in our previous work to understand microcausality [2]. Here we have also considered two-point functions with an approximate interpretation (as in ordinary Minkowski space QFT) of propagation amplitudes from one spacetime event to another. As is well known this interpretation is only approximate because the Hilbert space vectors obtained by acting on the vacuum state with the field operator do not form an orthonormal set. In order to overcome this difficulty we have introduced an orthonormal basis of position state vectors, labeled by the radial coordinate, and used them to define radial wave functions. These states are a generalization of the Newton-Wigner states of ordinary QFT. With the aid of this orthonormal basis we have considered first the Newton-Wigner propagator $\langle R_2|\hat{U}(t_2, t_1)|R_1\rangle$ and we have used it later to study the time evolution of a radial position wave function.

The results obtained in all the approaches are different but compatible. They can be summarized as follows:

(i) There are some interesting physical effects happening at the symmetry axis. In particular, all the approaches that we have followed (the approximate ones provided by the two-point functions, the ones given by the Newton-Wigner propagator, and the radial wave functions themselves) suggest a significant enhancement of the probability to find field quanta there. In principle, one could expect such a behavior due to backscattering—as it happens when one considers the classical system. However, we think that the comparison with the massless axially symmetric field propagating in a Minkowskian background suggests that most of it is due to a combination of quantum and gravitational effects.

(ii) The probability amplitudes show an enhanced probability of finding scalar field quanta on some lines of slope 1 in the (ρ, τ) plane that can be interpreted as approximate null geodesics of an emergent metric. Even though we cannot obtain other types of geodesics with our massless fields, it is reassuring to see the emergence of approximate spacetime trajectories with a clear physical interpretation. These approximate null geodesics correspond in this case to the ones of the Minkowski metric in $1 + 1$ (or rather $2 + 1$ with axial symmetry). It is worthwhile to note that, as the one-particle states that we are using are "the closest ones" to the vacuum state (other than the vacuum state itself), we are seeing in an operational way the appearance of a classical flat spacetime as far as (some) of its geometric properties are concerned.

(iii) Another interesting effect is the persistence of the amplitudes in the support of the initial wave function. This is clearly displayed in the behavior of the wave function itself as it evolves in time and shows up in the Newton-Wigner propagator. A somewhat similar effect appears in the two-point function. The divergence at $\rho_1 = \rho_2$ for these objects is, however, a consequence of the fact that we are not regularizing the fields. Even if we regularize, for example, by introducing a cutoff [4], we would observe a large value for this function at $\rho_1 = \rho_2$.

It would be interesting to apply the methods developed in the paper and use matter field quanta to explore quantized geometries for states representing classical configurations corresponding to arbitrary solutions for the Einstein-Rosen waves. In particular, it would be illuminating to compare the results with those obtained by quantizing the matter fields in the curved backgrounds provided by such solutions (where particle creation effects may play a relevant role). This would first require us to find suitable semiclassical states for the system describing a nontrivial gravitational part and a simple matter part and study their quantum evolution. As long as they can be found, our approach should lead to unambiguous answers to questions related to the emergence of classical trajectories for quantum tests particles. We are working on this problem in the present moment. It should be emphasized, however, that a direct comparison between both approaches may be difficult because of the very different Hilbert spaces used in their quantization. In particular, the use of Newton-Wigner states in the present scheme is very useful because n-particle subspaces of our Hilbert space are stable under the quantum evolution of the system. This allows us to rely on a probabilistic interpretation of one-particle wave functions. For a scalar field evolving in a general Einstein-Rosen curved background, this may no longer be possible due to particle creation effects.

In our opinion the model provided by Einstein-Rosen waves, free or coupled to cylindrically symmetric matter, is an excellent test bed to discuss issues in quantum gravity. Of course, there is always the issue as to what extent the results obtained are equivalent (at least in a qualitative way) to real effects in a full theory of quantum gravity or artifacts of the symmetry reduction. In this respect we think that the effects that we have described above admit a sensible interpretation and give interesting hints about the behavior of quantized gravity that we want to explore in the future.

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APPENDIX: NUMERICAL COMPUTATION OF INTEGRALS INVOLVING TWO BESSEL FUNCTIONS

Throughout the paper we have repeatedly encountered a class of improper integrals involving products of Bessel functions. The oscillatory character of the integrand, and the fact that they are defined on the half-real line and are usually conditionally convergent, makes it necessary to find an efficient way to compute them. We provide such a method here. This has been extensively used in the several plots that appear in the paper. The seamless mesh between these numerical results and the analytic approximation provided by the asymptotic expansions is an eloquent proof of the accuracy of both the asymptotic approximations and the numerical results.

Let us consider integrals of the type

$$I(\rho,\sigma,\tau) := \int_0^\infty J_\mu(\rho q)J_\nu(\sigma q)f(q)\exp[-i\tau(1-e^{-q})]\mathrm{d}q$$

with $\mu, \nu \in \mathbb{Z}$, and $f(q)$ a sufficiently regular function such that the integral is, at least, conditionally convergent. We first change variables according to $s = e^{-q}$ to get

$$I(\rho,\sigma,\tau) = e^{-i\tau}\int_0^1 \frac{e^{i\tau s}}{s} J_\mu(-\rho\log s)J_\nu(-\sigma\log s) \times f(-\log s)\mathrm{d}s.$$

We now write it as the sum of two contour integrals, I_1 and I_2, in the complex s plane defined on the paths $C_1 = \{iu{:}u \in [0,\infty)\}$, $C_2 = \{1 + iu{:}u \in [0,\infty)\}$:

$$I_1(\rho,\sigma,\tau) = e^{-i\tau}\int_0^\infty \frac{e^{-\tau u}}{u} J_\mu(-\rho\log iu)J_\nu(-\sigma\log iu) \times f(-\log iu)\mathrm{d}u,$$

$$I_2(\rho,\sigma,\tau) = -i\int_0^\infty \frac{e^{-\tau u}}{1+iu} J_\mu(-\rho\log[1+iu]) \times J_\nu(-\sigma\log[1+iu])f(-\log[1+iu])\mathrm{d}u.$$

For the functions f that appear in the paper, the second integral I_2 is very well behaved, because of the exponential falloff of the integrand and its nonsingular character. It can be computed numerically without difficulty. On the other hand, the integrand in I_1 has a nasty oscillating behavior in the vicinity of $u = 0$, although the integral itself is convergent. A way to turn it into a much tamer object is to use the contour integral representation for the Bessel functions introduced above to write it as the multiple integral

$$I_1(\rho,\sigma,\tau) = e^{-i\tau}\int_0^\infty \mathrm{d}u \oint_{\gamma_1}\frac{\mathrm{d}z_1}{z_1^{\mu+1}}\oint_{\gamma_2}\frac{\mathrm{d}z_2}{z_2^{\nu+1}}\frac{f(-\log(iu))}{u} \times \exp\left(-\left[\frac{\rho}{2}\left(z_1-\frac{1}{z_1}\right)+\frac{\sigma}{2}\left(z_2-\frac{1}{z_2}\right)\right] \times \log(iu) - \tau u\right) \tag{A1}$$

where $\gamma_{1,2}$ are simple closed paths surrounding the origin in $\mathbb{C}$. By choosing these paths appropriately [satisfying, for example, the conditions $\mathrm{Re}(z_1 - 1/z_1) \le 0$ and $\mathrm{Re}(z_2 - 1/z_2) \le 0$] it is possible in many cases to guarantee that

$$\int_0^\infty \frac{f(-\log(iu))}{u}\exp\left(-\left[\frac{\rho}{2}\left(z_1-\frac{1}{z_1}\right)+\frac{\sigma}{2}\left(z_2-\frac{1}{z_2}\right)\right] \times \log(iu) - \tau u\right)\mathrm{d}u$$

is convergent. This allows us to change the integration order in (A1). Furthermore, for specific choices of the function f (most of the cases appearing in the paper.) this last integral in u can be exactly obtained in analytic form thus leaving us with a double integral representation for I_1, defined on a set with the topology of a torus, of a perfectly regular function (except for a measure zero set of values of the parameters ρ, σ, and τ). For example if $\mu = 0$, $\nu = 1$, $f = 1$, we have

$$I_1(\rho, \sigma, \tau) = \frac{e^{-i\tau}}{4\pi^2} \oint_{\gamma_1} \frac{dz_1}{z_1} \oint_{\gamma_2} \frac{dz_2}{z_2^2} \exp\bigg(\bigg[\frac{\rho}{2}\bigg(z_1 - \frac{1}{z_1}\bigg) + \frac{\sigma}{2}\bigg(z_2 - \frac{1}{z_2}\bigg)\bigg]\bigg(i\frac{\pi}{2} - \log\tau\bigg)\bigg)\Gamma\bigg[\frac{\rho}{2}\bigg(z_1 - \frac{1}{z_1}\bigg) + \frac{\sigma}{2}\bigg(z_2 - \frac{1}{z_2}\bigg)\bigg].$$

By using the same methods as above we find the integrand is now well behaved and the integral can be computed numerically in an efficient and quick way by ordinary methods.

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