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Quantum time uncertainty in a gravity's rainbow formalism

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The existence of a minimum time uncertainty is usually argued to be a consequence of the combination of quantum mechanics and general relativity. Most of the studies that point to this result are nonetheless based on perturbative quantization approaches, in which the effect of matter on the geometry is regarded as a correction to a classical background. In this paper, we consider rainbow spacetimes constructed from doubly special relativity by using a modification of the proposals of Magueijo and Smolin. In these models, gravitational effects are incorporated (at least to a certain extent) in the definition of the energy-momentum of particles without adhering to a perturbative treatment of the backreaction. In this context, we derive and compare the expressions of the time uncertainty in quantizations that use as evolution parameter either the background or the rainbow time coordinates. These two possibilities can be regarded as corresponding to perturbative and nonperturbative quantization schemes, respectively. We show that, while a nonvanishing time uncertainty is generically unavoidable in a perturbative framework, an infinite time resolution can in fact be achieved in a nonperturbative quantization for the whole family of doubly special relativity theories with unbounded physical energy.

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I. INTRODUCTION

In quantum mechanics, the passage of time can be tracked by studying the evolution of the probability densities of observables in a given quantum state [1]. Nevertheless, every observable $\hat{A}$ of the system has a characteristic time $\Delta_A t$ that limits the ability to detect its evolution, and that can be estimated as the lapse needed by its expectation value $\langle\hat{A}\rangle$ to change an amount equal to its root-mean-square (rms) deviation ΔA, namely $\Delta_A t \geq \Delta A/|d_t\langle\hat{A}\rangle|$. On the other hand, the quantum evolution of any explicitly time-independent observable is given by the Heisenberg equation $i\hbar d_t\hat{A} = [\hat{A}, \hat{H}]$, where $\hat{H}$ is the Hamiltonian. Taking into account these expressions, together with the uncertainty principle applied to the pair of observables $\hat{A}$ and $\hat{H}$, and allowing the choice of any observable $\hat{A}$ of the system, one easily concludes that any measurement of time made with our quantum state will have an uncertainty Δt (at least equal to the minimum of all characteristic times $\Delta_A t$) that satisfies the inequality $\Delta t\Delta H \geq \hbar/2$ [1]. This is usually called the fourth Heisenberg relation.

Therefore, to improve the time sensitivity, states with a larger and larger energy uncertainty must be allowed. However, in general relativity, an uncertainty in the energy of the system implies an uncertainty in the geometry. The latter introduces in turn an uncertainty in the physical (or proper) time, if this corresponds to a unit (asymptotic) timelike Killing vector of the metric [2,3]. In this way, the time uncertainty gets contributions both from a purely quantum mechanical and from a gravitational origin [3]. As a consequence, an infinite time resolution seems impossible, unless both types of contributions are related in a very specific manner. Moreover, since the energy of the system is generally defined in terms of the (assumed) unit timelike Killing vector, the backreaction leads also to a redefinition of the physical energy, thus giving rise to new energy uncertainties. This nontrivial intertwining between time and energy uncertainties in the presence of gravity complicates the analysis of quantum measurements.

A way to face this problem is by adopting perturbative approaches, in which one starts with a flat background and introduces in it the matter content of the system, deforming hence the spacetime geometry. This deformation subsequently results in a change of the physical matter energy, leading to successive corrections in a feedback mechanism. Several arguments strongly support the idea that this type of perturbative quantization always leads to a minimum time uncertainty (at least in the next-to-leading-order approximation) [3–5]. However, it is not clear at all whether a minimum time structure would emerge if one performed the quantization of the gravitational system by adopting nonperturbative schemes. These kind of schemes, for instance, could allow one to encode in the theory, from the very beginning, the modification of the physical energy-momentum of the matter content owing to the process of backreaction.

In a recent paper [2], the quantum limits for time resolution have been studied from both (perturbative and nonperturbative) points of view in a family of gravitational models that include the Einstein-Rosen (ER) cylindrical waves [6–9]. It has been shown that, in these models, a minimum time uncertainty always exists if the physical energy is bounded from above, as it happens to be the case at least for ER waves [8,10]. Nonetheless, the possibility was open that there could exist gravitational systems with similar properties as those analyzed in that work but with an unbounded physical energy. In these circumstances, it was argued that an infinite time resolution could be reached in a nonperturbative quantum description.

Moreover, for the systems considered in Ref. [2], the behavior of the time uncertainty is radically different depending on whether the quantization employs as evolution parameter either a fixed time coordinate T associated with a classical (Minkowski) background or, alternatively, the physical time t, which (for ER waves) coincides with the proper time in the asymptotic region at spatial infinity. In the following, we will understand by perturbative and nonperturbative quantizations those quantum theories whose evolution is described, respectively, in terms of these two types of time parameters, T and t. The motivation for this terminology is clear, since the time T is linked to a background solution, while t is the physical time whose definition includes the effects of the energy content on the geometry. For the models considered in Ref. [2], the relation between these two times is given by a scaling that depends only on the energy of the solution (the energy of the gravitational waves in the case of the ER spacetimes [8,9]).

It has also been proven recently [11] that, from the perspective of an equivalent formulation of the ER geometries as a massless scalar field coupled to gravity in $2 + 1$ dimensions [7], these cylindrical waves can be viewed as an example of the so-called doubly special relativity (DSR) theories [12]. Such theories incorporate modifications to the expressions of the energy and momentum of relativistic particles owing to (possibly quantum) gravitational effects in such a way that Lorentz symmetry is maintained but its implementation becomes nonlinear, so that it may be compatible with the presence of an invariant scale in energy and/or momentum, ultimately related to the Planck scale [12–15]. Because of these properties and the commented connection with ER waves, DSR theories are natural candidates when trying to extend the discussion presented in Ref. [2] about the emergence of a minimum time uncertainty in the presence of gravity.

In order to carry out this extension, an extra piece of information must be added to the usual formulation of DSR theories in momentum space, namely, the dual realization of these relativity theories in position space. We will introduce a modification of the gravity's rainbow proposal put forward by Magueijo and Smolin [16]. This modification will ensure the invariance of the symplectic structure defined in standard special relativity, which can then be interpreted as corresponding to a Minkowski background before switching on any gravitational interaction. In this way, we will arrive at flat spacetime coordinates that are related to those of the background by means of a linear transformation which depends on the matter energy-momentum. As a result, the metric associated with them can be regarded as energy and momentum dependent. It is in this sense that the so-constructed DSR theories can be considered a kind of gravity's rainbow [16].

We will show that, for this gravity's rainbow formalism, the uncertainty in the physical time (conjugate to the physical energy) is always strictly positive in perturbative quantization schemes that employ as evolution parameter the time coordinate of the auxiliary, flat background. However, an infinite time resolution can actually be reached in a nonperturbative quantization if the DSR theory involves an invariant momentum scale, but not an energy scale. This example should clarify that the emergence of a minimum time uncertainty in gravity is not ineluctable in principle if one adopts a nonperturbative quantization.

The rest of the paper is organized as follows. In Sec. II we briefly review some results about DSR theories, formulated in momentum space. We describe the relation between the physical energy-momentum and the pseudo energy-momentum, on which the Lorentz transformations act linearly. This relation is provided by a nonlinear map U whose properties we discuss. Sec. III deals with the dual realization of the DSR theories in position space. We derive the expressions for the spacetime coordinates that are conjugate to the physical energy-momentum. Assuming an underlying Hamiltonian framework, we then analyze the quantization of this gravity's rainbow formalism. In Sec. IV we obtain the uncertainty in the physical time for a perturbative quantization, proving that it cannot vanish under very mild hypotheses. In Sec. V we demonstrate that, on the contrary, the uncertainty in the physical time can be as small as desired in a nonperturbative quantization, provided that the DSR theory has no invariant energy scale corresponding to a maximum of the physical energy. Finally, Sec. VI contains the conclusions and some further discussion. In the following, all dimensionful quantities will be expressed in Planck units. In particular, we set $\hbar = c = 1$.

II. DSR IN MOMENTUM SPACE

DSR theories are characterized by a nonlinear action of the Lorentz transformations in momentum space that preserves an energy or momentum scale (besides respecting the role of the speed of light as a fundamental scale) [12–15]. A way to understand this nonlinear action is by mapping the physical energy-momentum $P^a = (E, p^i)$ into a standard Lorentz 4-vector $\Pi^a = (\epsilon, \pi^i)$, which transforms in a linear way [17]. The involved nonlinear map is generally denoted by U, and the 4-vector Π^a is called the pseudo energy-momentum. Lowercase Latin indices from the beginning and the middle of the alphabet denote, respectively, Lorentz and (flat) spatial indices. The map U must be invertible; then the transformation of the physical energy-momentum is given by [17,18]

$$L(P) = (U^{-1} \circ \mathcal{L} \circ U)(P), \qquad (1)$$

where $\mathcal{L}$ is the standard linear action of the Lorentz transformation.

In the sector of small energies and momenta compared to the DSR scale, the physical and pseudo variables must coincide and, therefore, the map U must reduce to the identity, a property that will be used in the following. In addition, it is usually assumed that the standard action of rotations is not modified in DSR theories [18,19]. As a consequence, the most general functional form of U (and of its inverse) is [19]

$$\Pi = U(P) \Rightarrow \begin{cases} \epsilon = \tilde{g}(E, p), \\ \pi^i = \tilde{f}(E, p)\frac{p^i}{p}, \end{cases}$$
$$P = U^{-1}(\Pi) \Rightarrow \begin{cases} E = g(\epsilon, \pi), \\ p^i = f(\epsilon, \pi)\frac{\pi^i}{\pi}, \end{cases} \qquad (2)$$

where $p := |\vec{p}|$ and $\pi := |\vec{\pi}|$. So the map U is totally determined by two scalar functions $\tilde{g}$ and $\tilde{f}$ (or g and f).

Since standard Lorentz boosts run over the whole range $[0, \infty)$ for both energy and (the norm of the) momentum, the image of U must equal this range, so that the inverse of $\mathcal{L} \circ U$ can always exist in Eq. (1). Furthermore, in order to have a finite energy scale E^* (and/or momentum p^*) invariant under the Lorentz transformations (1), it is necessary that the map U sends it to infinity in the space of pseudo energy-momentum vectors, since this is the only invariant scale in standard special relativity. Therefore, the map U must be singular at E^* (and/or p^*) and the domain of definition of U (assumed to contain the sector of low energies) is bounded by that scale [18]. We then have three possible types of DSR theories, depending on whether one has only a bounded physical momentum (DSR1 type), a bounded physical energy (DSR3 type), or bounds in both physical quantities (DSR2 type).

More explicitly, if we consider a particle with pseudo mass $\mu \geq 0$ (namely, the Casimir invariant of the pseudo momentum space $\mu^2 = \epsilon^2 - \pi^2$, related to the rest mass m_0 by $\mu = \tilde{g}(m_0, 0)$ [17]), then, in the limit of infinite momentum on the mass shell (denoted by $\pi|_\mu \to \infty$), the existence of an invariant scale, where the map U is singular, implies one (or both) of the following possibilities:

$$\text{(a)} \lim_{\pi|_\mu \to \infty} g = E^* < \infty, \qquad \text{(b)} \lim_{\pi|_\mu \to \infty} f = p^* < \infty. \qquad (3)$$

Possibility (a) is realized for DSR2 and DSR3 types of theories, but not for DSR1. On the other hand, the behavior (b) is found only in the DSR1 and DSR2 classes. In general, the invariant scale is assumed to be of the Planck order, but this supposition, motivated by quantum considerations, can be relaxed.

III. A GRAVITY'S RAINBOW PROPOSAL

The recent interest in deformed dispersion relations, justified by their potential observational consequences in fields like astrophysics [20], explains why DSR theories are usually formulated in momentum space. Within this formulation, the transformation laws in position space are not determined. There exist different proposals for constructing a modified spacetime geometry consistent with DSR [19,21]. One of them, suggested by several hypotheses concerning quantum gravity, consists of introducing a noncommutative geometry, namely, admitting that spacetime coordinates no longer commute [15,19]. An example of this is the κ-deformed Minkowski spacetime. However, noncommuting spacetime coordinates are not a necessary consequence of DSR theories: the realization in position space can be achieved in the framework of commutative geometries [19,21,22].

For instance, a way to specify this realization was recently proposed by Magueijo and Smolin [16]. By demanding that the contraction between the energy-momentum and an infinitesimal spacetime displacement be a linear invariant, they derived modified expressions for the spacetime coordinates that are linear in the original (Minkowski) background coordinates q^a, but depend nontrivially on the energy-momentum. Owing to this dependence, a rainbow of metrics emerged in the formalism, each particle being associated with a different metric according to its energy-momentum.

Here, we will adopt a related kind of proposal, but, instead of the above contraction, we will demand the invariance of the symplectic form $\mathbf{d}q^a \wedge \mathbf{d}\Pi_a$ [where $\Pi_a = (-\epsilon, \pi^i)$ and the wedge denotes the exterior product for differential forms]. The modified position variables x^a obtained in this way are then conjugate to the physical energy-momentum P_a, i.e., the map from (q^a, Π_a) to (x^a, P_a) is just a canonical transformation. The physical energy-momentum can then be assigned the role of generator of spacetime translations in the coordinates x^a. In fact, the same requirement of covariance, ensuring that the space of coordinates can be identified with the cotangent space for the physical energy-momentum, was already put forward by Mignemi [22] (though introduced in a different manner).

An additional reason supporting the suggested change with respect to Ref. [16] is that it leads to the correct expression for the physical time (and spatial coordinates) in the case of ER waves (formulated in 2 + 1 dimensions) [2,9], as we will in part discuss later. Since this and other physical implications of our proposal significantly differ from those of the formalism presented in Ref. [16], one can view our construction as a distinct realization of DSR theories in position space, rather than simply as a modification. Nevertheless, it is worth commenting that the essential feature employed in the rest of our analysis is that the relation between the background and the physical (rainbow) spacetime coordinates is a linear transformation that depends only on the energy-momentum. This property persists even if one adheres exactly to the Magueijo and Smolin proposal, the only difference being the detailed form of the transformation.

It is straightforward to complete the map U in momentum space into a contact canonical transformation providing position variables conjugate to P_a. Employing the form of this map from $P_a = (-E, p_i)$ to

$$\Pi_a = \left(-\tilde{g}(E, p), \tilde{f}(E, p)\frac{p_i}{p}\right), \tag{4}$$

it is easy to see that the desired transformation is generated by the function

$$F(q^a, P_b) = -\tilde{g}(E, p)q^0 + \tilde{f}(E, p)\frac{p_j q^j}{p}. \tag{5}$$

Then, $x^a = \partial F/\partial P_a$. Making use of the implicit function theorem (and the identity $p_j/p = \pi_j/\pi$), we finally get the expressions for the new spacetime coordinates:

$$x^0 = \frac{1}{\det J(\epsilon, \pi)}\left[\frac{\partial f(\epsilon, \pi)}{\partial \pi} q^0 + \frac{\partial f(\epsilon, \pi)}{\partial \epsilon}\frac{\pi_i}{\pi} q^i\right],$$

$$x^i = \frac{1}{\det J(\epsilon, \pi)}\left[\frac{\partial g(\epsilon, \pi)}{\partial \pi}\frac{\pi^i}{\pi} q^0 + \frac{\partial g(\epsilon, \pi)}{\partial \epsilon}\frac{\pi^i \pi_j}{\pi^2} q^j\right] + \frac{\pi}{f(\epsilon, \pi)}\left(q^i - \frac{\pi^i \pi_j}{\pi^2} q^j\right). \tag{6}$$

Here, g and f are the two functions that fix the inverse map U^{-1}, and

$$\det J = \frac{\partial g}{\partial \epsilon}\frac{\partial f}{\partial \pi} - \frac{\partial g}{\partial \pi}\frac{\partial f}{\partial \epsilon}. \tag{7}$$

In the following, we will call physical variables the canonical set formed by x^a and the physical energy-momentum, whereas we will refer to q^a and π_a as background or auxiliary variables. In addition, to simplify in part our index notation, we will designate q^0 by T and x^0 by t (this type of notation reproduces that employed in Ref. [2]). Finally we note that, as it happens for the energy-momentum, the physical and background coordinates coincide in the limit where energies and momenta are small compared to the DSR scale, since in this regime $g(\epsilon, \pi) \approx \epsilon$ and $f(\epsilon, \pi) \approx \pi$.

IV. PHYSICAL TIME UNCERTAINTY: PERTURBATIVE CASE

Let us assume that our system possesses an underlying Hamiltonian formalism such that the values of the physical and pseudo energies are determined, respectively, by a physical Hamiltonian H and a background one H_0. In agreement with our previous discussion, in this Hamiltonian system the physical and pseudo momenta p_i and π_i are conjugate to the position variables x^i and q^i, whose translations they generate. In addition, motivated in part by the fact that DSR theories are supposed to provide effective descriptions of free particles, we also assume that our system is free, so that the energy and momentum are conserved (had one to consider composite systems, the physical energy and momentum would not be additive). In this way, apart from being time independent, the Hamiltonian must indeed commute under Poisson brackets with the momentum, both for the physical and the background variables.

From Eq. (2), we then have that $E \rightarrow H = g(H_0, \pi)$ and $\epsilon \rightarrow H_0 = \tilde{g}(H, p)$. In this section, we will analyze the quantization of the system with evolution generated by the background Hamiltonian H_0. In such a quantization, the evolution parameter is the corresponding time coordinate $q^0 = T$, namely, the background time. We leave for Sec. V the analysis of the alternative quantization with evolution parameter given by $x^0 = t$.

A. Calculation of the time uncertainty

Let us admit that a quantization of the system with evolution generated by the background Hamiltonian H_0 is feasible. In this perturbative quantization, the background time T plays the role of evolution parameter, whereas the physical time is in fact promoted to an operator $\hat{t}$ [2]. Taking into account the expression obtained in Eq. (6) for $x^0 = t$, and replacing energies by Hamiltonians, we can write

$$\hat{t} = \hat{A}(H_0, \pi)T + \hat{C}_T, \tag{8}$$

$$\hat{C}_T = \frac{\hat{B}(H_0, \pi)\hat{Q}_T + \hat{Q}_T\hat{B}(H_0, \pi)}{2}, \tag{9}$$

where

$$A(H_0, \pi) = \frac{1}{\det J(H_0, \pi)}\frac{\partial f(H_0, \pi)}{\partial \pi}, \tag{10}$$

$$B(H_0, \pi) = \frac{1}{\det J(H_0, \pi)}\frac{\partial f(H_0, \pi)}{\partial H_0}, \tag{11}$$

$$Q_T = \frac{\pi_i q^i}{\pi}. \tag{12}$$

In Eq. (9) we have symmetrized the product of $\hat{B}$ and $\hat{Q}_T$ (although our results are insensitive to the actual choice of factor ordering for this product) and the operators $\hat{A}$ and $\hat{B}$ can be defined, using the spectral theorem, in terms of those for the background Hamiltonian and momentum (H_0 and π) which, according to our comments above, are assumed to commute (so that the momentum is conserved quantum mechanically). As for the operator representing Q_T, we will analyze its form in brief. Let us simply remark for the moment that it will generically be time dependent since, under quantization, the auxiliary spatial variables q^i will not commute with the Hamiltonian. This explains the subindex notation employed for the operators $\hat{Q}_T$ and $\hat{C}_T$. Note that, by contrast, our assumptions guarantee that $\hat{A}$ and $\hat{B}$ are time independent.

Given a quantum state, we can measure the probability densities of the operators $\hat{A}$ and $\hat{C}_T$ [23]. Let us call ΔA and ΔC_T their rms deviations. In order to evaluate the operator $\hat{t}$, we still need to determine the value of the parameter T. The passage of this time parameter can be tracked by analyzing the evolution of probability densities of observables in the quantum state. This process leads to a statistical measurement of T, with probability density $\rho(T)$. We denote the associated mean value by $\bar{T}$. Obviously, the corresponding uncertainty in T must satisfy the fourth Heisenberg relation $\Delta T \Delta H_0 \geq 1/2$. With this measurement procedure, the physical time uncertainty would be

$$\begin{aligned}(\Delta t)^2 &= \int dT \rho(T) \langle (\hat{A}T + \hat{C}_T - \langle \hat{A} \rangle \bar{T} - \langle \hat{C}_{\bar{T}} \rangle)^2 \rangle \\ &= \int dT \rho(T) \{ T^2 (\Delta A)^2 + \langle \hat{A} \rangle^2 (T^2 - \bar{T}^2) \\ &\quad + T \langle \hat{A}\hat{C}_T + \hat{C}_T \hat{A} \rangle - 2\bar{T} \langle \hat{A} \rangle \langle \hat{C}_T \rangle + \langle \hat{C}_T^2 \rangle \\ &\quad + \langle \hat{C}_{\bar{T}} \rangle^2 - 2 \langle \hat{C}_T \rangle \langle \hat{C}_{\bar{T}} \rangle \}. \end{aligned} \tag{13}$$

Here, $\langle \hat{O} \rangle$ denotes the expectation value in our quantum state of any operator $\hat{O}$. In addition, in the estimation of the mean value of the physical time, we have substituted the parameter T by its corresponding mean value $\bar{T}$ (in particular, $\hat{C}_{\bar{T}}$ is the operator $\hat{C}_T$ at the instant $\bar{T}$) [24].

This expression becomes relatively simple when the dependence of $\hat{C}_T$ on T is linear. In fact, this is the case with our hypothesis that the system is free. To be more specific let us accept, according to our hypothesis, that the Hamiltonian H_0 is a scalar function of the pseudo momentum π (and some parameters). The assumed canonical symplectic structure for the background variables implies that Q_T [given by Eq. (12)] and π are canonically conjugate, i.e., their Poisson bracket is $\{Q_T, \pi\} = 1$. Since H_0 generates the evolution in T, one then has that, classically, $dQ_T/dT = \{Q_T, H_0\} = dH_0/d\pi$. Obviously dQ_T/dT is constant (because $dH_0/d\pi$ depends only on the pseudo momentum, which is a conserved quantity), and therefore $Q_T = Q_0 + T(dH_0/d\pi)$. We can then promote Q_T to a linearly T-dependent observable by representing Q_0 as a time-independent operator and defining $dH_0/d\pi$ in terms of the pseudo momentum operator by means of the spectral theorem. Taking into account that $\hat{B}(H_0, \pi)$ is constant in time, Eq. (9) shows then that $\hat{C}_T$ is linear in T.

The above analysis allows us to write the operator $\hat{t}$ in the alternative form

$$\hat{t} = \hat{V}(H_0, \pi)T + \hat{W}(H_0, \pi, Q_0), \tag{14}$$

$$\hat{V}(H_0, \pi) = \hat{A}(H_0, \pi) + \hat{B}(H_0, \pi)\widehat{\frac{dH_0}{d\pi}}(\pi), \tag{15}$$

$$\hat{W}(H_0, \pi, Q_0) = \frac{\hat{B}(H_0, \pi)\hat{Q}_0 + \hat{Q}_0 \hat{B}(H_0, \pi)}{2}. \tag{16}$$

In Eq. (15) we have employed that H_0 and π commute as operators. We emphasize that, since $\hat{Q}_0$ is time independent, so are $\hat{V}$ and $\hat{W}$.

Equation (13) for the time uncertainty in the physical time still applies, but now with $\hat{A}$ identified with $\hat{V}$, and $\hat{C}_T$ and $\hat{C}_{\bar{T}}$ substituted by $\hat{W}$. The result can be expressed in the form

$$(\Delta t)^2 = [\Delta(V\bar{T} + W)]^2 + \langle \hat{V} \rangle^2 (\Delta T)^2 + (\Delta T \Delta V)^2. \tag{17}$$

Since we have the sum of three positive terms in this equation, for the physical time uncertainty to vanish it is necessary that all of them be zero.

Let us show that this will not generically happen. From the first term in Eq. (17), one can easily see that the uncertainty in the physical time vanishes for $T \gg 1$ if and only if ΔV becomes equal to zero at large values of T. Since the operator $\hat{V}$ is time independent, its rms deviation vanishes then at any instant of time T. Assume now that the expression of the Hamiltonian in terms of π is invertible in the whole range of auxiliary energies, i.e., $\pi = \pi(H_0)$ [25], and define $\mathcal{V}[H_0] := V[H_0, \pi(H_0)]$. An alternative possibility is that V is independent of π, in which case we straightforwardly identify $\mathcal{V}$ with V. In any of these cases, assume finally that $d\mathcal{V}/dH_0 \neq 0$ for all the allowed values of H_0, so that the correspondence between H_0 and its image under $\mathcal{V}$ is one-to-one (a similar assumption was made in Ref. [2]). Making use of the spectral theorem, the requirement that $\Delta V = \Delta \mathcal{V}$ vanish implies then that $\Delta H_0 = 0$, because our assumption guarantees that the eigenstates of these two operators coincide. In these circumstances, the fourth Heisenberg relation states that ΔT is unbounded.

We will now show that the product of uncertainties $\Delta T \Delta V = \Delta T \Delta \mathcal{V}$ that appears in Eq. (17) cannot vanish when ΔH_0 approaches zero, thus concluding the proof that Δt is strictly positive. Expanding $\mathcal{V}(H_0)$ around the expectation value of H_0, where it is peaked when ΔH_0 is small, we arrive at

$$(\Delta \mathcal{V})^2 = \langle \hat{\mathcal{V}}^2 - \langle \hat{\mathcal{V}} \rangle^2 \rangle \approx \left(\frac{d\mathcal{V}}{dH_0} \bigg|_{\langle \hat{H}_0 \rangle} \Delta H_0 \right)^2. \tag{18}$$

Hence, in the limit of localized energy,

$$\lim_{\Delta H_0 \to 0} \Delta T \Delta \mathcal{V} \geq \lim_{\Delta H_0 \to 0} \frac{\Delta \mathcal{V}}{2\Delta H_0} = \left| \frac{1}{2} \frac{d\mathcal{V}}{dH_0} \bigg|_{\langle \hat{H}_0 \rangle} \right| \neq 0. \tag{19}$$

In conclusion, at least under very mild assumptions, the uncertainty in the physical time cannot be zero for an observer that describes the quantum evolution using as time parameter the background time T.

As a particular example we can analyze the case of ER waves, where the physical and pseudo momenta coincide, and the physical energy is $E = (1 - e^{-4\epsilon})/4$ (for an effective gravitational constant in three dimensions equal to the unity in Planck units) [2]. Introducing Hamiltonians, we thus have $g(H_0, \pi) = (1 - e^{-4H_0})/4$ and $f(H_0, \pi) = \pi$. Since f is energy independent, Eq. (11) leads to $B = 0$, a fact that considerably simplifies the expressions of the physical time and its uncertainty. From Eqs. (10), (15), and (16), we straightforwardly get $A = 1/(dg/dH_0) = e^{4H_0}$, $\hat{V} = e^{4\hat{H}_0}$, and $\hat{W} = 0$. Given that the deduced function A (and hence $\mathcal{V}$) is strictly increasing in H_0, the assumptions introduced above are satisfied, and the conclusion of a nonzero uncertainty in the physical time holds. In this way, one recovers the results obtained in Ref. [2].

B. First order corrections

In this subsection, we will analyze the behavior of the uncertainty in the physical time when one approximates this operator by keeping only up to first order corrections in the energy. We will see that the results lend additional support to the statement that this uncertainty is strictly positive in the perturbative approach to the quantization.

In order to study the desired corrections, we start by expanding the functions $g(H_0, \pi)$ and $f(H_0, \pi)$ around the minimum of the pseudo energy and around vanishing pseudo momentum. We will denote the minimum pseudo energy by μ, motivated by the standard relativity case, where it equals (the square root of) the Casimir invariant, $\mu^2 = \epsilon^2 - \pi^2$. We assume that the functions g and f are smooth and that μ is small compared to the invariant DSR scale(s). In particular, this last fact allows us to employ that, to leading order, $g(H_0, \pi) \approx H_0$ and $f(H_0, \pi) \approx \pi$ in the region of the expansion. To derive the first order corrections to the physical time, in the expansion of g and f it is actually necessary to keep only up to quadratic terms in the variables π and

$$\mathcal{H}_0 := H_0 - \mu. \tag{20}$$

One can then use Eqs. (10) and (11) to obtain the expressions of $A(H_0, \pi)$ and $B(H_0, \pi)$ up to linear terms in those variables:

$$\begin{aligned} A &\approx 1 - \frac{\partial^2 g}{\partial H_0^2}\bigg|_0 \mathcal{H}_0 - \frac{\partial^2 g}{\partial H_0 \partial \pi}\bigg|_0 \pi, \\ B &\approx \frac{\partial^2 f}{\partial H_0^2}\bigg|_0 \mathcal{H}_0 + \frac{\partial^2 f}{\partial H_0 \partial \pi}\bigg|_0 \pi, \end{aligned} \tag{21}$$

where the symbol $|_0$ stands for evaluation at $H_0 = \mu$ and $\pi = 0$.

Next, from Eqs. (15) and (16) one can easily calculate the first order corrections to the leading behavior of $\hat{V}$ and $\hat{W}$. In this step, one needs to introduce the expression of the Hamiltonian in terms of the momentum, $H_0(\pi)$ [see Eq. (15)]. On the one hand, it is natural to assume that the minimum of the pseudo energy is reached for vanishing pseudo momentum, $H_0(0) = \mu$. On the other hand, motivated by the standard relativity case [$\epsilon = \sqrt{\mu^2 + \pi^2} \rightarrow H_0(\pi)$], two cases are worth considering. (1) *"Massive" case*: $\mu \neq 0$, with $(dH_0/d\pi)|_{\pi=0} = 0$.

We get $H_0(\pi) \approx \mu + b\pi^2$, where $2b = (d^2H_0/d\pi^2)|_{\pi=0}$. Assuming that $b > 0$, it follows that $\pi \approx \sqrt{\mathcal{H}_0/b}$. For instance, in standard special relativity one would have $b = 1/(2\mu)$. Corrections linear in $\mathcal{H}_0$ are hence negligible compared to those proportional to π. In addition, $dH_0/d\pi \approx 2b\pi$, which can be neglected compared to the unity. As a consequence, we arrive at the following approximations at next-to-leading order:

$$\hat{V} \approx \hat{A} \approx 1 - \frac{\partial^2 g}{\partial H_0 \partial \pi}\bigg|_0 \frac{\hat{\mathcal{H}}_0^{1/2}}{\sqrt{b}}, \tag{22}$$

$$\hat{W} \approx \frac{\partial^2 f}{\partial H_0 \partial \pi}\bigg|_0 \frac{\hat{\mathcal{H}}_0^{1/2}\hat{Q}_0 + \hat{Q}_0\hat{\mathcal{H}}_0^{1/2}}{2\sqrt{b}}. \tag{23}$$

The physical time uncertainty in this approximation can be obtained from Eq. (17).

Note that the resulting leading term (zeroth order in the energy) is the uncertainty of the background time in standard quantum mechanics. We also point out that the function $\mathcal{V}$, introduced in the previous subsection, is given in the studied approximation just by (the classical counterpart of) Eq. (22). Such a function is clearly monotonic in H_0 (or $\mathcal{H}_0$), provided that the second partial derivative $[\partial^2 g/(\partial H_0 \partial \pi)]|_0$ does not vanish, so that we have really kept the first order energy corrections to $\mathcal{V}$. Then, the assumptions made at the end of Sec. IVA hold, leading to the conclusion that the physical time uncertainty cannot be made zero.

(2) *"Massless" case*: $\mu = 0$, $(dH_0/d\pi)|_{\pi=0} = k \neq 0$. In this case $H_0 \approx k\pi$ and $\mathcal{H}_0 = H_0$. In standard special relativity, for instance, one would have $k = 1$. Corrections linear in H_0 and in π are then of the same order, and $dH_0/d\pi$ is of order unity. Therefore, one obtains in the linear order approximation:

$$\begin{aligned} \hat{V} \approx 1 + \bigg(&-\frac{\partial^2 g}{\partial H_0^2}\bigg|_0 - \frac{1}{k}\frac{\partial^2 g}{\partial H_0 \partial \pi}\bigg|_0 + k\frac{\partial^2 f}{\partial H_0^2}\bigg|_0 \\ &+ \frac{\partial^2 f}{\partial H_0 \partial \pi}\bigg|_0\bigg)\hat{H}_0, \end{aligned} \tag{24}$$

$$\hat{W} \approx \bigg(\frac{\partial^2 f}{\partial H_0^2}\bigg|_0 + \frac{1}{k}\frac{\partial^2 f}{\partial H_0 \partial \pi}\bigg|_0\bigg)\frac{\hat{H}_0\hat{Q}_0 + \hat{Q}_0\hat{H}_0}{2}. \tag{25}$$

At this order, the function $\mathcal{V}$ is approximated by (the classical analog of) Eq. (24). Accepting that the coefficient in front of H_0 in that expression is nonzero, so that we have actually included the next-to-leading-order correction, we arrive again to a monotonic function of H_0. Hence, the line of reasoning discussed in Sec. IVA applies, and we conclude that it is impossible to reach the limit of infinite resolution in the physical time.

V. PHYSICAL TIME UNCERTAINTY: NONPERTURBATIVE CASE

We turn now to the discussion of the physical time uncertainty when one adopts the point of view that the quantum evolution of the system is generated by the physical Hamiltonian H. It is worth commenting that, if the system admits a perturbative quantization where the background Hamiltonian H_0 and momentum π are promoted to self-adjoint operators, a nonperturbative quantization is also possible. To see this, notice that, in the representation employed for the perturbative quantization, the spectral theorem allows one to define as self-adjoint operators the physical Hamiltonian H and momentum p, given by the functions g and f in terms of H_0 and π. The exponentiation of this operator realization of H provides then a unitary evolution operator, that describes the dynamics in a time parameter that can be identified with the physical time t. Clearly, in the so-constructed nonperturbative quantization, the uncertainty of t is only limited by the fourth Heisenberg relation, taking as Hamiltonian the physical one, namely $\Delta t \Delta H \geq 1/2$.

As a consequence, for an observer in the nonperturbative quantum system, the resolution for the physical time is intrinsically bounded if and only if the same happens for the physical energy (i.e., the physical Hamiltonian). The conclusion does not depend on other details of the system. The only relevant point is whether the range of the physical energy is infinite. This range is determined by the image of g, one of the two functions that characterize the DSR theory. But the image of g is bounded from above if and only if the DSR theory possesses an invariant energy scale [remember Eq. (3)]. This is not always the case: it occurs only in the so-called DSR2 and DSR3 types of theories, but not for the DSR1 class. Therefore, a finite time resolution is not a necessary consequence of the quantization of the system, at least in this nonperturbative framework. More specifically, for the whole family of DSR1 theories [12,13,19], where only an invariant scale in momentum exists, the quantum resolution in the physical time can be made (nonpertubatively) as large as desired.

VI. SUMMARY AND DISCUSSION

We have investigated the existence of a minimum time uncertainty in a modified gravity's rainbow formalism, obtained by means of a dual realization of DSR theories in position space. This realization leads to a set of spacetime coordinates that are canonically conjugate to the physical energy-momentum. Such coordinates are constructed from the (Minkowski) background ones by means of a linear transformation that depends on the energy and momentum. Assuming an underlying Hamiltonian formulation, with energy determined by the value of the generator of the evolution, and concentrating our attention on free systems, we have discussed the differences in adopting as dynamical generator either the physical or the background Hamiltonian, the latter corresponding to the pseudo energy.

If the dynamics is dictated by this last Hamiltonian, the evolution parameter of the quantum theory is the background time T, and the physical time t is described by a T-dependent family of operators. We have shown that its uncertainty cannot be made to vanish, at least under very mild assumptions about the features of the background Hamiltonian and the DSR theory. In fact, these assumptions are only sufficient, but not necessary in order to prove that the studied uncertainty is greater than zero. For instance, one can show that the resolution in the physical time is finite as well for all those cases in which the function $\mathcal{V}$ is strictly positive (so that the background and physical arrows of time coincide) and the ratio $\mathcal{V}(H_0)/H_0$ has a nonzero limit when H_0 tends to infinity (so that, in the high energy sector, $\mathcal{V}$ grows at least like H_0 by a constant). Therefore, an infinite resolution in the physical time cannot (generically) be reached within a quantization framework in which the energy-momentum modifications in the definition of time are not incorporated in the choice of evolution parameter.

By contrast, when the quantum dynamics is generated by the physical energy, the role of evolution parameter is directly assigned to the physical time. In this case, its uncertainty is only limited by quantum mechanics via the fourth Heisenberg relation. As a result, an infinite resolution is possible if and only if the physical energy of the system is unbounded from above, which in turn is equivalent to the absence of an invariant energy scale in the DSR theory. There exists a whole family of DSR theories that possess a momentum scale but not an energy scale of this kind, namely, the so-called DSR1 theories, whose prototype is a model suggested by Amelino-Camelia [12,13]. This clearly demonstrates that, in nonperturbative quantum descriptions, the existence of a minimum uncertainty in the physical time is not generally unavoidable when gravitational effects are taken into account.

An issue for further discussion is whether, in those nonperturbative quantum systems where an infinite time resolution is possible, there emerges, nonetheless, a minimum uncertainty in the spatial position, as could be suggested by the presence of a bound for the physical momentum in DSR1 theories, supplied by the invariant scale. We plan to study this question in the future, as a natural continuation of the analysis carried out here.

Our discussion can be regarded as a generalization of that of Ref. [2]. Apart from the hypotheses concerning the existence of a feasible quantization and the recovery of the standard results in the low energy sector, the rest of conditions assumed for the models studied in Ref. [2] amount to accept a relation between physical and background coordinates of the form (6), but with the DSR functions f and g satisfying the following: (i) f is independent of the pseudo energy, and (ii) g is a convex or concave (invertible) smooth function of only the pseudo energy. In these cases, one can check that, with our notation,

$$\mathcal{V}(H_0) = V(H_0) = A(H_0) = \frac{1}{dg/dH_0} \qquad (26)$$

and

$$\frac{d\mathcal{V}}{dH_0} = \frac{dV}{dH_0} = -\frac{d^2g/dH_0^2}{(dg/dH_0)^2} \neq 0. \qquad (27)$$

Therefore, the assumptions introduced at the end of Sec. IVA hold in these models, and thus Δt cannot be made equal to zero in the perturbative quantization.

Finally, in our analysis we have implicitly kept in mind the case of a relativistic particle, motivated by the formulation of DSR theories as alternatives to special relativity (at least in momentum space). Since a field can be viewed as a combination of particles, one might try to extend the arguments presented here to a quantum field theory context. In perturbative quantum field theories, the background space coordinates q^j should be treated as parameters. Therefore, one would expect that the physical time operator [see Eq. (8)] adopted an expression of the form $\hat{t} = \hat{A}T + \hat{D}_j q^j$. Then, the resulting time uncertainty would be

$$(\Delta t)^2 = \left[\Delta\left(\sum_a D_a \overline{q}^a\right)\right]^2 + \sum_a \langle\hat{D}_a\rangle^2(\Delta q^a)^2 + \sum_a (\Delta q^a)^2(\Delta D_a)^2, \qquad (28)$$

where $q^0 = T$, $\overline{q}^a$ is the mean value of q^a, and $\hat{D}_0$ stands for $\hat{A}$.

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[23] Similarly, we can measure the covariance (or the anticommutator) of these two operators.

[24] When $\hat{C}_T$ depends linearly on T, as it is the case for the free systems that we will consider in this work, the mean

value obtained for the physical time by taking quantum expectation values, and averaging the parameter T with the probability density $\rho(T)$, is exactly $\langle\hat{A}\rangle\bar{T} + \langle\hat{C}_{\bar{T}}\rangle$.

[25] We can think, for instance, of a relativistic particle with $\pi = \sqrt{H_0^2 - \mu^2}$.