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Inner boundary conditions for black hole initial data derived from isolated horizons

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We present a set of boundary conditions for solving the elliptic equations in the initial data problem for space-times containing a black hole, together with a number of constraints to be satisfied by the otherwise freely specifiable standard parameters of the conformal thin sandwich formulation. These conditions altogether are sufficient for the construction of a horizon that is *instantaneously* in equilibrium in the sense of the isolated horizons formalism. We then investigate the application of these conditions to the initial data problem of binary black holes and discuss the relation of our analysis with other proposals that exist in the literature.

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I. INTRODUCTION

The problem of determining appropriate initial data for binary black holes is of crucial importance in order to construct successful numerical simulations for these astrophysical systems [1]. Starting with Einstein field equations, a specific strategy for this problem consists in solving the relevant elliptic equations on an initial Cauchy surface where a sphere S has been excised [2] for each black hole (inner boundary). The purpose of the present work is to present a set of *inner* boundary conditions inspired by purely geometrical considerations, inasmuch as they are derived from the formalism of isolated horizons (IH) [3–8], and which guarantee that the excised sphere is in fact a section of a quasiequilibrium horizon.

A pioneering work on this inner boundary problem was presented by Cook in Ref. [9], in the context of formulating a definite full prescription for the construction of initial data for binary black holes in quasicircular orbits. The assumptions made in that analysis permit one to determine a proper set of conditions for a quasiequilibrium black hole. However, at least intuitively, one would expect that the isolated horizons formalism, which is mainly a systematic characterization of the notion and properties of quasiequilibrium horizons, could supply a more powerful and consistent framework for discussing the conditions in this black hole regime. Actually, the spirit in Ref. [9] closely resembles that encoded in the isolated horizons scheme, but does not fully capture it. Therefore, in this specific sense, the quasiequilibrium horizon analysis may be refined. With this motivation, we will truly adopt here the isolated horizons formalism as the guideline of a geometrical analysis whose ultimate goal is the *ab initio* numerical construction of an isolated horizon. This strategy provides us with a rigorous mathematical and conceptual framework that systematizes the physical assumptions.

For the sake of clarity, we have considered it important to provide a relatively self-contained presentation, even at the cost of lengthening the article. The rest of the work is organized as follows. As in Refs. [9,10], we use a conformal thin sandwich (CTS) approach to set the initial data problem; thus Sec. II briefly reviews the basics of this approach. Section III introduces the main ideas of the isolated horizons framework and underlines the importance of its hierarchical structure by first introducing nonexpanding horizons (NEH) and then weakly isolated horizons (WIH). Boundary conditions on the horizon are derived in Sec. IV. Section V discusses the relationship of this approach with that of Ref. [9]. Finally, Sec. VI presents the conclusions.

II. CONFORMAL THIN SANDWICH APPROACH TO INITIAL DATA

In this section we formulate the problem that will be analyzed in this work and introduce our notation. We will use Greek letters $(\mu, \nu, \ldots)$ for Lorentzian indices, intermediate Latin letters $(i, j, \ldots)$ for spatial indices on a Cauchy slice, and Latin letters from the beginning of the alphabet $(a, b, \ldots)$ for coordinates on a two-dimensional sphere S^2.

Adopting a standard $3+1$ decomposition for general relativity (see, e.g., Ref. [11]), the space-time $\mathcal{M}$ with Lorentzian metric $g_{\mu\nu}$ is foliated by spacelike hypersurfaces Σ_t parametrized by a scalar function t. The evolution vector t^μ, normalized as $t^\mu \nabla_\mu t = 1$, is decomposed in its normal and tangential parts by introducing the lapse function α and the shift vector β^μ,

$$t^\mu = \alpha n^\mu + \beta^\mu, \tag{1}$$

where $n_\mu = -\alpha \nabla_\mu t$ is the unit timelike vector normal to Σ_t and $n^\mu \beta_\mu = 0$.

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Denoting by γ_{ij} the induced metric on Σ_t, the Lorentzian line element reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \quad (2)$$

The embedding of the hypersurfaces Σ_t in the four-geometry is encoded in the extrinsic curvature:

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu} = -\gamma^\rho{}_\mu \nabla_\rho n_\nu, \quad (3)$$

which can also be expressed as

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - D_i\beta_j - D_j\beta_i), \quad (4)$$

where D_i is the connection associated with γ_{ij}.

Under the 3 + 1 decomposition, Einstein equations split into two sets: evolution and constraint equations. *In vacuo*, the case that we are interested in, the evolution equations are

$$\partial_t K_{ij} - \mathcal{L}_\beta K_{ij} = \alpha(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}) - D_iD_j\alpha, \quad (5)$$

where K is the trace of K_{ij} ($K = \gamma^{ij}K_{ij}$). On the other hand, the constraint equations (respectively, Hamiltonian and momentum constraints) are expressed as

$$R + K^2 - K_{ij}K^{ij} = 0, \quad (6)$$

$$D_j(K^{ij} - \gamma^{ij}K) = 0. \quad (7)$$

In brief, the initial data problem consists in providing pairs (γ_{ij}, K^{ij}) that satisfy the constraints (6) and (7) on an initial Cauchy surface Σ_t.

The discussion of the notion of quasiequilibrium demands a certain control on the time evolution of the relevant fields. The conformal thin sandwich introduced in Refs. [12,13] is particularly well suited, since it provides an approach to the initial data problem that consistently incorporates (a part of) the time derivative of the metric, together with the lapse and the shift.

The CTS approach starts by conformally decomposing the metric and the extrinsic curvature, the latter expressed in terms of its trace K and a traceless part A^{ij},

$$\gamma_{ij} = \Psi^4\tilde{\gamma}_{ij}, \qquad K^{ij} = \Psi^{-4}A^{ij} + \frac{1}{3}K\gamma^{ij}. \quad (8)$$

In this expression the conformal factor Ψ is given by

$$\Psi \equiv \left(\frac{\gamma}{f}\right)^{1/12}, \quad (9)$$

where γ is the determinant of γ_{ij} and f is the determinant of f_{ij}, an auxiliary time-independent metric, $\partial_t f_{ij} = 0$, which captures the asymptotics of γ_{ij} [14].

Substituting the above decomposition of γ_{ij} in relation (4) and taking the trace, we find[1]

$$\partial_t\Psi = \beta^i\tilde{D}_i\Psi + \frac{\Psi}{6}(\tilde{D}_i\beta^i - \alpha K). \quad (10)$$

This expression will play an important role when setting the appropriate boundary conditions in Sec. IV.

With Eq. (8), the Hamiltonian constraint (6) can be written as an elliptic equation for the conformal factor:

$$\tilde{D}_i\tilde{D}^i\Psi = \frac{\Psi}{8}\tilde{R} - \Psi^5\left(\frac{1}{8}\tilde{A}_{ij}A^{ij} - \frac{K^2}{12}\right), \quad (11)$$

whereas the momentum equation is expressed as an elliptic equation for the shift

$$\tilde{D}_j\tilde{D}^j\beta^i + \frac{1}{3}\tilde{D}^i\tilde{D}_j\beta^j + \tilde{R}^i{}_j\beta^j - (\tilde{L}\beta)^{ij}\tilde{D}_j\ln(\alpha\Psi^{-6}) = \frac{4}{3}\alpha\tilde{D}^iK - \tilde{D}_j\partial_t\tilde{\gamma}^{ij} + \partial_t\tilde{\gamma}^{ij}\tilde{D}_j\ln(\alpha\Psi^{-6}), \quad (12)$$

where $\tilde{A}_{ij} \equiv \tilde{\gamma}_{ik}\tilde{\gamma}_{jl}A^{kl}$, $\tilde{D}_i$ is the connection associated with $\tilde{\gamma}_{ij}$, and $(\tilde{L}\beta)^{ij} \equiv \tilde{D}^i\beta^j + \tilde{D}^j\beta^i - \frac{2}{3}\tilde{D}_k\beta^k\tilde{\gamma}^{ij}$.

Only the conformal part $\tilde{\gamma}_{ij}$ of the metric γ_{ij} encodes dynamical degrees of freedom. This suggests solving the trace of the evolution equations (5) together with the constraints, defining in this way an *enlarged* problem on the initial surface [13]. This additional equation turns out to be elliptic in its dependence on the lapse α,

$$\tilde{D}_i\tilde{D}^i\alpha + 2\tilde{D}_i\ln\Psi\tilde{D}^i\alpha = \Psi^4\left\{\alpha\left(\tilde{A}_{ij}A^{ij} + \frac{K^2}{3}\right) + \beta^i\tilde{D}_iK - \partial_tK\right\}. \quad (13)$$

An extra justification to add this equation is because it straightforwardly permits one to impose the condition $\partial_tK = 0$, a good ansatz for quasiequilibrium. In this extended problem the constrained parameters are given by (Ψ, β^i, α) and the free data on the initial Cauchy surface are $(\tilde{\gamma}_{ij}, \partial_t\tilde{\gamma}^{ij}, K, \partial_tK)$, subject to the constraints $\det(\tilde{\gamma}_{ij}) = f$ and $\tilde{\gamma}_{ij}\partial_t\tilde{\gamma}^{ij} = 0$ [in the strict initial data problem $\frac{\alpha}{\Psi^6}$ is a free parameter on the initial slice, but here it is constrained owing to Eq. (13)]. Hence, the inner boundary problem presented in the Introduction reduces to the search for appropriate boundary conditions for Ψ, β^i, and α imposed on the horizon.

III. ISOLATED HORIZONS FORMALISM

In this section we will motivate the introduction of the notion of isolated horizon and summarize the concepts and definitions that will be employed in Sec. IV. We will try to

[1] In the following expressions, objects with an over-tilde are associated with the conformal metric $\tilde{\gamma}_{ij}$. They are consistent with the conformal rescalings in Ref. [14], rather than with those originally introduced in Refs. [12,13].

provide a presentation as accessible as possible for a broad community. For a detailed and more rigorous discussion of the IH formalism see, e.g., Ref. [7] or Ref. [15].

The physical scenario that the IH construction attempts to describe is that of a dynamical space-time containing a black hole in equilibrium, in the sense that neither matter nor radiation cross its horizon. This scenario applies as an approximation for each of the two black holes in a binary before their merger, provided that they are sufficiently separated, therefore justifying the relevance of the IH formalism for the initial data problem of binary black holes.

A very important feature of the IH formalism is its (quasi)local character. In our context, the need of a (quasi)local description is motivated first by the way in which numerical simulations are designed from a $3+1$ approach, in which we do not have *a priori* control on global space-time properties, and secondly by the desire of characterizing physical parameters of the black hole as well as the concept of equilibrium in a (quasi)local manner. The notion of apparent horizon, with a local characterization as an outermost marginal trapped surface[2] in a three-slice, seems an adequate starting point. However, to include the concept of equilibrium we must somehow consider the evolution of this two-dimensional surface. In the (quasi)equilibrium regime, the notion of the world tube of an apparent horizon does in fact make sense (there are no jumps). Actually, an IH implements the idea that an apparent horizon associated with a black hole *in equilibrium* evolves smoothly into apparent horizons of the same area, in such a way that the generated world tube is a null hypersurface. This null character encodes the key quasiequilibrium ingredient, and is essentially linked to the idea of keeping constant the area of the apparent horizon.

Inspired by these considerations, the definition of IH tries to seize the fundamental ingredients of the null world tube of a nonexpanding apparent horizon. In doing this, the world tube is endowed with some additional geometrical structures that are intrinsic to the null hypersurface [8]. The specific amount and nature of these extra structures depend on the physical problem that one wants to address. This introduces a hierarchy of structures in the formalism which turns out to be very useful for keeping track of the hypotheses that are assumed to hold, as will become evident in Sec. IV.

Before describing these structures, let us emphasize the change of strategy with respect to Sec. II: while there the relevant geometry was that of the initial-data spacelike three-surface, the relevance corresponds now to a null three-geometry. The combined use of these two complementary perspectives, each of them suggesting their own natural geometrical objects, will prove to be specially fruitful.

A. Nonexpanding Horizons

1. Definition

A first level in the hierarchy of structures entering the IH formalism is the notion of nonexpanding horizon, which incorporates the idea of quasiequilibrium sketched above. We say that a hypersurface Δ in a vacuum space-time $(\mathcal{M}, g_{\mu\nu})$ is a NEH if [8]

(i) It is a null hypersurface with $S^2 \times (I \subset \mathbb{R})$ topology. That is, there exists a null vector field l^μ on Δ, defined up to rescaling, such that $g_{\mu\nu} l^\mu v^\nu = 0$ for all vectors v^μ tangent to Δ. The degenerate metric induced on Δ by $g_{\mu\nu}$ will be denoted by $q_{\mu\nu}$.

(ii) The expansion $\theta_{(l)} \equiv q^{\mu\nu}\nabla_\mu l_\nu$ of any null normal l^μ vanishes on Δ.[3]

(iii) Einstein equations are satisfied on Δ.

Matter can be included without problems in the scheme, but we will focus here on the vacuum case.

2. Main consequences for our problem

Let us first note that the cross sections $S \simeq S^2$ of the NEH are not necessarily strict apparent horizons since they are not imposed to be outermost surfaces and no condition is enforced on the expansion $\theta_{(k)}$ of the ingoing null vector k^μ (see footnote [2]). Abusing the language, we will, however, refer throughout to the cross sections as apparent horizons, a practice ultimately justified in our problem by a sensible choice of freely specifiable data on the initial surface.

(a) Constant area.—Owing to the null character of Δ, any null generator l^μ defines a *natural* evolution on the hypersurface, in such a way that the area of the apparent horizons ($a = \int_S d^2V = \int_S \sqrt{q} d^2q$, where we use the natural metric q_{ab} on S induced by $q_{\mu\nu}$) does not change, since $\mathcal{L}_l(\ln\sqrt{q}) = \theta_{(l)} = 0$. Therefore, there is a well-defined notion of radius of the horizon, $R_\Delta \equiv \sqrt{a}/(4\pi)$.

(b) Surface gravity.—Since l^μ is null and normal to Δ, it can be shown to be pregeodesic and twist free. Hence,[4]

$$\nabla_l l^\mu \triangleq \kappa_{(l)} l^\mu, \tag{14}$$

where $\kappa_{(l)}$ is a function on Δ that will be referred to as *surface gravity* (see the Appendix).

(c) Second fundamental form $\Theta_{\mu\nu}$ on Δ and evolution Killing vector on Δ.—We introduce the second fundamental form of Δ [16]

$$\Theta_{\mu\nu} \equiv \frac{1}{2} P^\alpha{}_\mu P^\beta{}_\nu \mathcal{L}_l q_{\alpha\beta} = \frac{1}{2} q^\alpha{}_\mu q^\beta{}_\nu \mathcal{L}_l q_{\alpha\beta}, \tag{15}$$

where $P^\alpha{}_\beta = \delta^\alpha_\beta + k^\alpha l_\beta$ and $q^\alpha{}_\beta = \delta^\alpha_\beta + k^\alpha l_\beta + l^\alpha k_\beta$,

[2] That is, a surface S in Σ_t on which the expansions $\theta_{(l)}$ and $\theta_{(k)}$ of the outgoing and ingoing null vectors, l^μ and k^μ, respectively, satisfy $\theta_{(l)} = 0$ and $\theta_{(k)} < 0$.

[3] Here $q^{\mu\nu}$ is any tensor satisfying $q^{\rho\sigma} q_{\rho\mu} q_{\sigma\nu} = q_{\mu\nu}$.

[4] The symbol $\triangleq$ denotes equality on the horizon Δ.

with $k_\mu l^\mu = -1$. This is an essentially two-dimensional object *living* on apparent horizons, such that $\theta_{(l)} = \Theta^\mu{}_\mu$, while

$$\Theta_{ab} \equiv \frac{1}{2}\theta_{(l)} q_{ab} + \sigma_{(l)ab} \qquad (16)$$

defines the shear $\sigma_{(l)ab}$ associated with l^μ. Since Einstein equations hold on Δ, so does the Raychaudhuri equation. *In vacuo*, and since the twist of l^μ cancels, it has the form

$$\mathcal{L}_l \theta_{(l)} = \kappa_{(l)}\theta_{(l)} - \frac{1}{2}\theta_{(l)}^2 - \sigma_{(l)ab}\sigma_{(l)}{}^{ab}. \qquad (17)$$

The vanishing of $\theta_{(l)}$ throughout Δ (so that, in particular, $\mathcal{L}_l\theta_{(l)} \triangleq 0$) implies then the vanishing of the shear $\sigma_{(l)ab}$. As a consequence, $\Theta_{\mu\nu} \triangleq 0$. From Eq. (15) we then see that, on Δ, $q_{\mu\nu}$ is Lie-dragged by the null vector l^μ. Therefore, although in general there is no Killing vector of the full space-time, the induced metric on Δ admits an intrinsic Killing symmetry. This fact extracts from the stronger notion of Killing horizon [17] the relevant part for our problem.

(d) Connection ω_μ.—The vanishing of $\Theta_{\mu\nu}$ and the fact that l^μ is normal to Δ suffice to define a one-form ω_μ intrinsic to Δ, such that

$$v^\nu \nabla_\nu l^\mu \triangleq v^\nu \omega_\nu l^\mu \qquad (18)$$

for any vector v^μ tangent to Δ. This one-form provides a strategy for computing $\kappa_{(l)}$ in Eq. (14):

$$\kappa_{(l)} \triangleq l^\mu \omega_\mu. \qquad (19)$$

In addition, we will see that it plays a central role in introducing the next level of the IH hierarchy of structures.

(e) Transformations under rescaling of l^μ.—For later applications, let us also summarize the transformation of the main geometrical objects under a rescaling of l^μ by a function λ on Δ. Under a change $l^\mu \to \lambda l^\mu$, we find

$$\begin{aligned} q_{\mu\nu} &\to q_{\mu\nu}, & \omega_\mu &\to \omega_\mu + P^\nu{}_\mu \nabla_\nu \ln\lambda, \\ \Theta_{\mu\nu} &\to \lambda\Theta_{\mu\nu}, & \kappa_{(l)} &\to \lambda\kappa_{(l)} + l^\mu\nabla_\mu\lambda. \end{aligned} \qquad (20)$$

It is obvious from these expressions that the characterization of NEH does not depend on the rescaling of l^μ.

3. 3 + 1 *perspective of nonexpanding horizons*

As discussed above, we want to cope with intrinsic evolution properties of apparent horizons. However, in contrast with the previous discussion on NEH, in our initial data problem we only dwell on a given spatial slice (at most, on two infinitesimally closed slices in the CTS), not on the whole world tube. Therefore, we must find a procedure to characterize an apparent horizon as a section of an IH by only using information on the initial spatial slice. From the NEH definition, a NEH of infinitesimal width is implemented if, together with the condition $\theta_{(l)} = 0$, we are able to enforce $\mathcal{L}_l\theta_{(l)} = 0$ on the initial sphere S. The Raychaudhuri equation (17) leads then to the characterization given in Ref. [18], that can be expressed as *The infinitesimal world tube of an apparent horizon S is a NEH if and only if the shear $\sigma_{(l)ab}$ of the outgoing null vector vanishes on S.*

Of course, if we want to extend the NEH character to a finite world tube, we need to find a way to impose these conditions on a finite evolution interval, something that is not possible in the initial data problem. At least, this *instantaneous* notion of equilibrium must be completed with a proper choice of dynamical content in the free data on the initial Cauchy surface. Summarizing, we see from Eq. (16) that the condition that we must impose on the sphere S in Σ_t in order to have a section of a NEH is

$$\Theta_{ab}|_S = 0, \qquad (21)$$

where the symbol $|_S$ stands for evaluation on S.

B. Weakly isolated horizons

A NEH describes a minimal notion of quasiequilibrium, but it is not rich enough for allowing the assignment of well-defined physical parameters to the black hole. In order to do so, we must endow the horizon with extra structure. Noting that the key property of the NEH is that l^μ is a Killing vector of the metric induced on the horizon, a way to introduce new structure consists in enforcing that other objects are Lie dragged by l^μ.

A simple choice in this sense, that permits a Hamiltonian analysis leading to (quasi)local physical quantities associated with the black hole, is to demand that $\mathcal{L}_l\omega_\mu \triangleq 0$. However, the transformation rule of ω_μ in Eq. (20) precludes this condition to hold for every null normal l^μ. Nonetheless, a consistent way to impose it is by introducing the notion of weakly isolated horizon[5]:

A weakly isolated horizon is a NEH endowed with an equivalence class $[l^\mu]$ of null normals (l'^μ and l^μ belong to the same class if and only if $l'^\mu = cl^\mu$ with c a positive constant) such that

$$\mathcal{L}_l\omega_\mu \triangleq 0. \qquad (22)$$

This condition turns out to be equivalent to (see the Appendix)

$$d(\kappa_{(l)}) \triangleq 0, \qquad (23)$$

so that the zeroth law of black hole thermodynamics, $\kappa_{(l)} = \text{const}$, characterizes the WIH notion.

[5]The one-form ω_μ encodes some of the components of a connection $\hat{\nabla}$ on Δ compatible with the degenerate metric $q_{\mu\nu}$ [16]. This connection $\hat{\nabla}$ is in fact unique as a consequence of the NEH definition. The stronger condition $[\mathcal{L}_l, \hat{\nabla}] = 0$ defines a Strongly Isolated Horizon, a much more rigid structure.

It is worth commenting that, given a NEH, it is always possible to select a class of null normals $[l^\mu]$ such that Δ becomes a WIH. Actually there exists an infinite freedom in the construction of the WIH structure [8]. Namely, if the surface gravity $\kappa_{(l)}$ is a (nonvanishing) constant for a certain class of null normals $[l^\mu]$, the same happens for any of the classes obtained by the nonconstant rescaling

$$\hat{l}^\mu \triangleq [1 + B(\theta, \phi)e^{-\kappa_{(l)}v}]l^\mu, \tag{24}$$

where $B(\theta, \phi)$ is an arbitrary function on S and v is a coordinate on Δ compatible with l^μ, i.e., $\mathcal{L}_l v \triangleq 1$. In fact, the above rescaling does not modify the constant value of the surface gravity [this follows from the transformation rule for $\kappa_{(l)}$ in Eq. (20)].

Since it is always possible to find WIH structures on a given NEH, the WIH concept does not correspond to a real restriction on the physics of the system. However, it does impose a restriction on the space-time slicing by the hypersurfaces Σ_t introduced in Sec. II if we tie l^μ to t, i.e., if we impose that there is a member l^μ of the WIH class $[l^\mu]$ such that $\mathcal{L}_l t \triangleq 1$. We call such a slicing a *WIH-compatible slicing*.

The derivation of the mass and angular momentum expressions for a WIH using Hamiltonian techniques is beyond the scope of this work (see Refs. [6,7]). Here, we will simply extract those points which are relevant for our analysis. The general idea is to characterize physical parameters as conserved quantities of certain transformations that are associated with symmetries of the WIH. A vector field V^μ tangent to Δ is said to be a symmetry of the particular WIH under consideration if it preserves its equivalence class of null normals, the metric $q_{\mu\nu}$, and the one-form ω_μ, namely,

$$\mathcal{L}_V l^\mu \triangleq \mathrm{const} \cdot l^\mu, \qquad \mathcal{L}_V q_{\mu\nu} \triangleq 0, \qquad \mathcal{L}_V \omega_\mu \triangleq 0. \tag{25}$$

In Sec. IV we will be interested in nonextremal black holes, for which $\kappa_{(l)} \neq 0$. In that case, the general form of a WIH symmetry is [7]

$$V^\mu = c_V l^\mu + b_V S^\mu, \tag{26}$$

where c_V and b_V are constant on Δ and S^μ is an isometry of the apparent horizon S.

The definition of the conserved quantities goes first through the construction of an appropriate phase space for the problem and then through the analysis of canonical transformations on this phase space [7]. An important point is that the relevant transformations are generated by diffeomorphisms in space-time whose restriction to the horizon Δ are symmetries of the WIH in the sense of (25).

1. Angular momentum

In order to define a conserved quantity that we can associate with a (quasi)local angular momentum, we assume that there exists an azimuthal symmetry on the horizon Δ (actually, this hypothesis can be relaxed; see in this sense Ref. [15]). Therefore, we assume the existence of a vector φ^μ tangent to $S \subset \Delta$, which is a SO(2) isometry of the induced metric q_{ab} with 2π affine length.

The conserved quantity associated with an extension of φ^μ to the space-time is given by [7]

$$J_\Delta = -\frac{1}{8\pi G}\int_S \varphi^\mu \omega_\mu d^2V = \frac{1}{8\pi G}\int_S s^i \varphi^j K_{ij} d^2V, \tag{27}$$

where for convenience we have expressed it in terms of objects in the 3 + 1 decomposition. In particular, s^i is the outward (pointing towards spatial infinity) unit vector field in Σ_t normal to the apparent horizon S.

2. Mass and boundary condition for t^μ

The definition of the mass is related to the choice of an evolution vector t^μ with appropriate boundary conditions, namely, that $t^\mu \to (\partial_t)^\mu$ at spatial infinity and $t^\mu \to l^\mu - \Omega_t \varphi^\mu$ with $l^\mu \in [l^\mu]$ and Ω_t constant on the horizon (note that $t^\mu|_\Delta$ is a WIH symmetry). The determination of the mass expression proceeds in two steps.

First, the vector t^μ has to satisfy certain conditions to induce a canonical transformation on the phase space. This turns out to be equivalent to the first law of black hole thermodynamics [6,7], whose practical consequence for us is that the mass M, the surface gravity $\kappa_{(l)}$, and the angular velocity Ω_t depend only on the radius R_Δ and the angular momentum J_Δ of the black hole,

$$M = M(R_\Delta, J_\Delta), \qquad \kappa_{(l)} = \kappa_{(l)}(R_\Delta, J_\Delta), \qquad \Omega_t = \Omega_t(R_\Delta, J_\Delta), \tag{28}$$

but without determining the specific functional form. It is worth emphasizing that this dependence on R_Δ and J_Δ (though arbitrary in principle) must be the same for all solutions to the Einstein equations containing a WIH.

In a second step, this dependence is fixed to coincide with that found in the stationary Kerr family of black holes. This is not an arbitrary choice, but a normalization consistent with the stationary solutions. Technically, this is accomplished by requiring that, at the horizon Δ, $t^\mu + \Omega_t \varphi^\mu$ reproduces just the null normal (in the considered class $[l^\mu]$) whose surface gravity equals that of the Kerr case, something that is always possible in the nonextremal situation via a constant rescaling. This singles out a vector t_o^μ, satisfying

$$t_o^\mu + \Omega_{\mathrm{Kerr}}(R_\Delta, J_\Delta)\varphi^\mu \triangleq c l^\mu \equiv l_o^\mu, \tag{29}$$

with

$$c \equiv \frac{\kappa_{\mathrm{Kerr}}(R_\Delta, J_\Delta)}{\kappa_{(l)}} \tag{30}$$

as the evolution vector used for the derivation of the mass formula. The final expressions obtained in this way for the physical parameters of the horizon are

$$M_\Delta \equiv M_{\rm Kerr}(R_\Delta, J_\Delta) = \frac{\sqrt{R_\Delta^4 + 4G^2J_\Delta^2}}{2GR_\Delta},$$
$$\kappa_\Delta \equiv \kappa_{\rm Kerr}(R_\Delta, J_\Delta) = \frac{R_\Delta^4 - 4G^2J_\Delta^2}{2R_\Delta^3\sqrt{R_\Delta^4 + 4GJ_\Delta^2}}, \qquad (31)$$
$$\Omega_\Delta \equiv \Omega_{\rm Kerr}(R_\Delta, J_\Delta) = \frac{2GJ_\Delta}{R_\Delta\sqrt{R_\Delta^4 + 4GJ_\Delta^2}}.$$

IV. DERIVATION OF THE BOUNDARY CONDITIONS

We are now in an adequate situation to derive boundary conditions for the elliptic equations in Sec. II. In doing so, we adopt a coordinate system (t, x^i) stationary with respect to the horizon, in the sense that the null tube Δ can be identified as the hypersurface $r(x^i) = \text{const}$ for a certain function r which is independent of t. It can be shown that this happens if and only if t^μ is chosen tangent to Δ, i.e., $l_\mu t^\mu \triangleq 0$.

Since we want to have a notion of angular momentum for the black hole, following the discussion in Sec. III B 1 we make the hypothesis that *our physical regime permits the imposition of an axial isometry* φ^a *on* $S \simeq S^2 \subset \Delta$. Even though this is a strong physical hypothesis (especially when having in mind binary black holes), we must emphasize that the bulk space-time will still be generally dynamical in an arbitrarily close neighborhood of the horizon and that φ^μ does not need to extend to an isometry there.

To construct the equilibrium black hole on S, we follow the steps dictated by the hierarchy of the IH formalism.

A. Adapting the evolution vector to the horizon

Aiming at imposing the NEH structure, but already motivated by the boundary condition for the evolution vector selected by the determination of t_o^μ, we adapt t^μ to the horizon by relaxing to a NEH the particular WIH structure implicit in Eq. (29). That is, we only impose

$$t^\mu + \Omega_\Delta\varphi^\mu \overset{\Delta}{\sim} l^\mu, \qquad (32)$$

where the proportionality needs not be given by a constant on Δ. Using the proportionality

$$l^\mu \sim (n^\mu + s^\mu) \qquad (33)$$

(where s^μ is again the outward unit spatial vector normal to S) and the decomposition (1) of t^μ in terms of the lapse and shift, we conclude

$$\beta^i \triangleq \alpha s^i - \Omega_\Delta\varphi^i, \qquad (34)$$

from which boundary conditions for the shift on S immediately follow.

Actually, the choice of stationary coordinates with respect to the horizon automatically leads to the expression $\beta^i \triangleq \alpha s^i - W^i$, where W^i is an arbitrary vector tangent to S. So, here we enforce W^i to be *precisely* $\Omega_\Delta\varphi^i$, a choice that will in fact simplify the imposition of the NEH structure.[6]

B. Nonexpanding horizon condition

We now properly impose the NEH condition. As mentioned in Sec. III A 3, in this initial data problem we demand S to be a slice of a NEH of infinitesimal width. For this, we impose condition (21). Owing to the rescaling property (20) of $\Theta_{\mu\nu}$ under an arbitrary (not necessarily constant) rescaling of l^μ, and taking advantage of the t^μ adaptation to the horizon implemented by the shift boundary conditions, we can write

$$q^\alpha{}_\mu q^\beta{}_\nu \mathcal{L}_{[t+\Omega_\Delta\varphi]} q_{\alpha\beta} \triangleq 0. \qquad (35)$$

In our stationary coordinates with respect to the horizon, this simply reads

$$0 \triangleq 2\Theta_{ab} \triangleq \partial_t q_{ab} + \Omega_\Delta \mathcal{L}_\varphi q_{ab}. \qquad (36)$$

But, under our hypothesis about the existence of an axial isometry on the horizon, the second term must vanish on its own: $\mathcal{L}_\varphi q_{ab} \triangleq 0$. Using $q_{ab} \triangleq \gamma_{ab}$ (for angular covariant components) we find

$$\partial_t \gamma_{ab} \triangleq 0, \qquad \mathcal{L}_\varphi \gamma_{ab} \triangleq 0. \qquad (37)$$

In particular, the restrictions must hold on S. These are the NEH boundary conditions. Note that their simple form depends critically on the specific choice made for W^i in the previous subsection.

Using now the conformal decomposition of the metric, these conditions translate into

$$(4\tilde{\gamma}_{ab}\partial_t\Psi + \Psi\partial_t\tilde{\gamma}_{ab})|_S = 0, \qquad (38)$$

$$(4\tilde{\gamma}_{ab}\mathcal{L}_\varphi\Psi + \Psi\mathcal{L}_\varphi\tilde{\gamma}_{ab})|_S = 0. \qquad (39)$$

The crucial feature, and the ultimate reason for using the CTS, is that these conditions can be satisfied by an appropriate choice of the free data $\tilde{\gamma}_{ab}$ and $\partial_t\tilde{\gamma}_{ab}$. Condition (39), expressing the axial symmetry of the horizon, must be enforced by a self-consistent selection of the free data $\tilde{\gamma}_{ab}$ on S (Ψ is a functional of $\tilde{\gamma}_{ij}$). Regarding (38), if we first take its trace with respect to the *conformal* counterpart of the metric q_{ab} induced on S, $\tilde{q}_{ab} \equiv \Psi^{-4}q_{ab}$ (satisfying $\tilde{q}^{ac}\tilde{\gamma}_{cb} = \delta^a_b$) and then use Eq. (10) for $\partial_t\Psi$, we find (calling $tr_S\dot{\tilde{\gamma}} \equiv \tilde{q}^{ab}\partial_t\tilde{q}_{ab} \triangleq \tilde{q}^{ab}\partial_t\tilde{\gamma}_{ab}$)

[6] Although the derivation of J_Δ actually involves a WIH structure, its expression can be shown to be already well defined for a NEH.

$$\left[\beta^i \tilde{D}_i \Psi + \frac{\Psi}{6}\left(\tilde{D}_i \beta^i - \alpha K + \frac{3}{4} tr_S \dot{\tilde{\gamma}}\right)\right]\bigg|_S = 0. \quad (40)$$

In addition, from Eq. (38) it follows that the $\tilde{q}$-traceless part of $\partial_t \tilde{\gamma}_{ab}$ must vanish. Therefore, on the boundary S, this part of the free data has the form

$$\left(\partial_t \tilde{\gamma}_{ab} - \frac{1}{2} tr_S \dot{\tilde{\gamma}} \tilde{\gamma}_{ab}\right)\bigg|_S = 0. \quad (41)$$

Condition (40) is an inner boundary condition for Ψ. Since we have imposed on S the Dirichlet boundary conditions (34) on β^i, we have no direct control on the sign of $\tilde{D}_i \beta^i$ there. In order to guarantee the positivity of Ψ via the application of a maximum principle, the factor multiplying Ψ in Eq. (40) should be non-negative. The analytical study of this issue goes beyond the present geometrical derivation (see Ref. [19] for a discussion on this point in a related context). We simply comment that the choice of free data for $tr_S \dot{\tilde{\gamma}}$ [and, more indirectly, that of the radial components of the free data $\tilde{\gamma}^{ij}$, which determine $s^i = \gamma^{ri}/\sqrt{\gamma^{rr}}$ in condition (34)] could play a key role in ensuring that Eq. (40) is a well-posed condition.

Except for the implicit use of a well-defined concept of angular momentum based on the WIH formalism, the notion of quasiequilibrium provided by the NEH structure has proved to be sufficient to set boundary conditions for the initial data problem, since it prescribes boundary values for Ψ (Hamiltonian constraint) and β^i (momentum constraint). If this is the problem that we want to solve, we can stop here. However, if we want to solve also the trace of the evolution equations, we need to find appropriate boundary conditions for the lapse. We will show how the existence of a WIH structure can be exploited with that aim.

C. Weakly Isolated Horizon condition

As we have commented, given a NEH one can always find a class of null normals so that it becomes weakly isolated. In fact, the determination of this class is not unique, but there exists an infinite freedom of choice. In this subsection, we will first discuss the restrictions on the lapse function that follow from the introduction of a WIH-compatible slicing and then employ the freedom in the choice of WIH to suggest possible boundary conditions for the lapse that are specially suitable for numerical integration.

Let us start by choosing l_o^μ as the representative of the class of null normals for the WIH, $[l^\mu]$. The inner boundary condition (29) employed in the determination of the mass formula then singles out an evolution vector t_o^μ on the horizon. We proceed as in Sec. IVA, but imposing t^μ to coincide exactly with t_o^μ, therefore demanding that the (Σ_t) foliation constitutes a WIH-compatible slicing. According to the characterization (23) of the WIH notion, the surface gravity $\kappa_{(l)}$ is constant. We further assume $\kappa_{(l)} \neq 0$, thus restricting the analysis to the nonextremal case. We write l_o^μ in terms of 3 + 1 objects,

$$l_o^\mu = \alpha \tilde{l}^\mu \equiv \alpha(n^\mu + s^\mu), \quad (42)$$

where we have explicitly defined the vector $\tilde{l}^\mu$. Again, the introduction of the lapse and shift decomposition for t_o^μ in Eq. (29) leads to the boundary conditions (34) for the shift. In order to analyze the conditions on α, we calculate the expression for $\kappa_{(l_o)}$. We proceed in several steps.

(1) Contracting Eq. (18), particularized to the one-form $\tilde{\omega}_\mu$ associated with $\tilde{l}^\mu$, with the ingoing null covector $\tilde{k}_\mu = (n_\mu - s_\mu)/2$ and expanding the resulting expression, we find for any vector v^μ tangent to Δ,

$$v^\mu \tilde{\omega}_\mu \triangleq v^\mu s^\nu \nabla_\mu n_\nu. \quad (43)$$

Employing the definition of the extrinsic curvature (3) and the identity $n^\rho \nabla_\rho n_\nu = \gamma^\rho{}_\nu \nabla_\rho \ln\alpha$, we get

$$v^\mu \tilde{\omega}_\mu \triangleq -v^\mu s^\nu (K_{\mu\nu} + n_\mu \gamma^\rho_\nu \nabla_\rho \ln\alpha). \quad (44)$$

(2) Taking $\tilde{l}^\mu$ as the tangent vector v^μ and remembering expression (19), we obtain

$$\kappa_{(\tilde{l})} \triangleq \tilde{l}^\mu \tilde{\omega}_\mu \triangleq s^\mu \nabla_\mu \ln\alpha - s^\mu s^\nu K_{\mu\nu}. \quad (45)$$

(3) Recalling then the transformation (20) of the surface gravity under a rescaling of the null normal,

$$\kappa_{(l_o)} \triangleq \alpha(s^\mu \nabla_\mu \ln\alpha - s^\mu s^\nu K_{\mu\nu}) + l_o^\mu \nabla_\mu \ln\alpha. \quad (46)$$

(4) Finally, imposing that $\kappa_{(l_o)}$ equals $\kappa_{\rm Kerr}(R_\Delta, J_\Delta)$, we find

$$\kappa_{\rm Kerr}(R_\Delta, J_\Delta) \triangleq s^i D_i \alpha - s^i s^j K_{ij} \alpha + \mathcal{L}_{l_o} \ln\alpha. \quad (47)$$

This restriction, arising from the WIH-compatible slicing condition, can be regarded as an evolution equation for the lapse on the horizon. Properly speaking, it is not a boundary condition for the initial data, because it contains the derivative of the lapse in the direction of l_o^μ.

Actually, by exploiting the freedom of choice in the WIH structure, one can freely set the value of the lapse on the initial section of the horizon S. This is a consequence of the fact that, from Eq. (24) and the relation $l_o^\mu = \alpha \tilde{l}^\mu$, a change of WIH structure results in a rescaling of the lapse:

$$\hat{\alpha} \triangleq [1 + B(\theta, \varphi) e^{-\kappa_{\rm Kerr}(R_\Delta, J_\Delta) t}]\alpha. \quad (48)$$

Since S can be identified, e.g., with the section $t = 0$ of Δ, the initial value of the lapse on the horizon gets multiplied by an arbitrary positive function on the sphere. Therefore, it can be chosen at convenience, at least as far as the demand for a WIH-compatible slicing is concerned. In fact, if we were solving an evolution problem by following a constrained scheme (see, e.g., Ref. [14]), this initial choice for α together with Eq. (47) might be used to set inner boundary conditions for the lapse at each time step.

Once one realizes the freedom in the choice of the initial value for α on S that follows from the dynamical character of Eq. (47) on Δ, one may ask whether it is possible to benefit from this arbitrariness and put forward a particular proposal for the choice that could be considered specially advantageous. In this sense, one would like to ensure that, under evolution on Δ, the lapse will neither increase exponentially nor decrease to (zero or) negative values. Apparently, the best way to favor this, at least locally, is to pick up, among the infinite WIH structures, that in which the Lie derivative of the lapse with respect to the null normal vanishes initially: $(\mathcal{L}_{l_o}\alpha)|_S = 0$. Using Eq. (48), one can prove under very mild assumptions that such a choice of WIH structure exists. Adopting it, Eq. (47) becomes a true boundary condition for the lapse on S:

$$[s^i D_i \alpha - s^i s^j K_{ij}\alpha]|_S = \kappa_{\rm Kerr}(R_\Delta, J_\Delta). \tag{49}$$

Note in fact that, to deduce this condition, one only needs to demand [in the passage from Eq. (46) to Eq. (47)] that $\kappa_{(l_o)}$ coincides with the constant $\kappa_{\rm Kerr}(R_\Delta, J_\Delta)$ on S, and not on the whole of Δ, because one finally restricts his attention just to the initial section of the horizon. As a consequence, and in contrast with the situation found for the NEH conditions, the above prescription for the lapse on the boundary is only a *necessary* condition for specifying a WIH of infinitesimal width. The extra condition that $\kappa_{(l_o)}$ be constant in the rest of Δ, namely $(\mathcal{L}_{l_o}\kappa_{(l_o)})|_S = 0$, would involve the evolution equations and the second time derivative of the lapse, and therefore cannot be imposed in terms of the initial data.

Finally, we comment that an alternative way of dealing with the WIH condition $\kappa_{(l)} = \text{const}$ would consist in choosing *a priori* the values of α and $\mathcal{L}_{l_o}\alpha$ on S and then interpreting Eq. (47) as a constraint on the free data on the inner boundary.

D. Binary quasicircular orbits

In the previous subsections we have characterized the quasiequilibrium state of each horizon exclusively in local terms. However, the study of a binary black hole in quasicircular orbits requires, in addition, a global notion of quasiequilibrium. In the general case, such a global quasistationary situation is described by the existence of a global quasi-Killing vector L^μ. In the binary black hole case, this is a helical vector (see Refs. [10,20]) that Lie drags the horizons, i.e., $L^\mu|_\Delta$ is tangent to each horizon Δ. Imposing asymptotic flatness, we have at spatial infinity $L^\mu \to t^\mu_\infty + \Omega_{\rm orb}\phi^\mu_\infty$, where t^μ_∞ and ϕ^μ_∞ are vectors associated with an asymptotic inertial observer and $\Omega_{\rm orb}$ is the orbital angular velocity. We can adapt the coordinate system, introducing an evolution parameter t' such that $L^\mu = (\partial_{t'})^\mu$. In such a case, with the 3 + 1 decomposition $L^\mu = \alpha' n^\mu + \beta'^\mu$, outer *corotating* boundary conditions follow

$$\lim_{r\to\infty}\beta'^i \approx \Omega_{\rm orb}\phi^i_\infty, \qquad \lim_{r\to\infty}\alpha' \approx 1, \qquad \lim_{r\to\infty}\Psi \approx 1. \tag{50}$$

In these coordinates, one chooses the *time derivative* part of the CTS free data to vanish

$$\partial_{t'}\tilde{\gamma}_{ij} = 0, \qquad \partial_{t'}K = 0. \tag{51}$$

In the general case, $L^\mu|_\Delta$ and t^μ_o defined in Eq. (29) do not coincide. Since they are both tangent to the horizon,

$$L^\mu \triangleq \rho t^\mu_o + \chi^\mu, \tag{52}$$

where ρ is a scaling factor and χ^μ is tangent to Δ with $\chi^\mu n_\mu = 0$. As a consequence, if we adapt coordinates to L^μ, hence using t', the expressions given in the previous subsections must be corrected. We will discuss two possibilities.

(a) Corotating coordinate system (fully adapted to L^μ).—From Eqs. (29) and (52) we can write

$$L^\mu + \rho\Omega_\Delta\varphi^\mu - \chi^\mu \triangleq \rho l^\mu_o. \tag{53}$$

A natural ansatz for χ^μ is given by $\Omega_{\rm orb}\phi^\mu$, where ϕ^μ is the azimuthal vector tangent to each horizon and associated with the normal direction to the orbital plane. Since Ω_Δ provides a well-defined notion of rotation angular velocity, we can define the corotating physical regime in an intrinsic way as the case with $\rho = 1$ and $\Omega_{\rm orb} = \Omega_\Delta$, from which $L^\mu \triangleq l^\mu_o$ follows. More generally, proceeding as in Sec. IV A we find

$$\beta'^i \triangleq \alpha' s^i - \rho\Omega_\Delta\varphi^i + \chi^i. \tag{54}$$

Imposing the axial symmetry on each horizon Δ, we deduce again condition (39). Defining

$$\eta^\mu \equiv \rho\Omega_\Delta\varphi^\mu - \chi^\mu, \tag{55}$$

the requirements $\Theta_{ab} \triangleq 0$ and $\partial_{t'}\tilde{\gamma}_{ij} = 0$ [see Eq. (51)] leads then to the conditions

$$\left[\beta'^i\tilde{D}_i\Psi + \frac{\Psi}{6}(\tilde{D}_i\beta'^i - \alpha' K) + \frac{1}{8\Psi^3}\tilde{q}^{cd}\mathcal{L}_\eta\gamma_{cd}\right]\Big|_S = 0, \tag{56}$$

$$\left[\frac{1}{2}(\tilde{q}^{cd}\mathcal{L}_\eta\gamma_{cd})\tilde{\gamma}_{ab} - \mathcal{L}_\eta\gamma_{ab}\right]\Big|_S = 0, \tag{57}$$

which replace Eqs. (40) and (41), respectively. Finally, the condition for the lapse is still derived as in Sec. IV C. From Eq. (52) it follows that $\alpha' \triangleq \rho\alpha$, and hence it can be shown

$$\left[\frac{s^i D_i\alpha' - s^i s^j K_{ij}\alpha'}{\rho}\right]\Big|_S = \kappa_{\rm Kerr}(R_\Delta, J_\Delta). \tag{58}$$

Convenient ansätze for ρ and χ^μ must be introduced in practice to cope with these conditions.

(b) Warped coordinate system.—An alternative choice consists in adopting an evolution vector t^μ such that its boundary value on each horizon coincides with t_o^μ, but adapts itself to L^μ at a typical distance δ from them (in this sense, the distance between the black holes provides a natural length scale in the binary problem). Hence, this vector t^μ interpolates between t_o^μ and L^μ, *warping* the coordinate system to better accommodate the physical situation in each of the considered spatial regions (note that the vector L^μ follows the translational motion, whereas t_o^μ is adapted to the intrinsic rotation on the horizon). Of course such a coordinate system can remain regular only for a finite amount of time (typically one orbital period).

In practical terms, this coordinate system is defined by the outer boundary conditions (50), without primes in β^i and α, and the inner boundary conditions (34) [on S], (40), (41), and (49) on the constrained and free data. Moreover, the function $tr_S \dot{\tilde{\gamma}}$ does not have to vanish on S, therefore helping to ensure the positivity of Ψ, even though it must be negligible at a distance of order δ. Likewise, $\partial_t K$ and the *radial* components $\partial_t \tilde{\gamma}_{rj}$ become roughly zero at a distance δ of each horizon. Thus, in this coordinate system, the conditions on the horizons are easier to impose, there is no need to worry about the factor ρ, and one gains control over the positivity of Ψ.

Once the time derivative part of the free data has been fixed, either in the corotating or in the warped coordinate system, one would have to consider the rest of the free data. The choice of the conformal metric must be consistent with restriction (39) [and (57) in corotating coordinates] and subject to the constraint $\det(\tilde{\gamma}_{ij}) = f$. The adequate determination of the physical content of $\tilde{\gamma}_{ij}$ goes beyond the limited scope of this paper and must be addressed by means of a proper analysis of the stationary regime for the evolution equations (5).

As for gauge fixing, the Dirac gauge in Ref. [14] appears to be a quite natural choice for the spatial one in the CTS setting, whereas the boundary condition (40) might be viewed to suggest a maximal slicing ($K = 0$) for the temporal gauge in order to improve the control on the positivity of Ψ. However, this latter gauge is not compatible with coordinates of Painlevé-Gullstrand or Kerr-Schild type, which are actually appropriate for the shift boundary condition (34). We do not here subscribe to a particular fixation of the gauge, allowing an optimal adaptation to each case considered.

V. COMMENTS ON PREVIOUS APPROACHES

A. Cook's 2002 proposal

Inner boundary conditions for the elliptic equations (11)–(13) in the quasicircular regime of a binary black hole system were presented in Ref. [9]. The scheme proposed in that work starts by imposing on each excised sphere S the presence of a Killing horizon, together with an *apparent horizon* condition, $\theta_{(\tilde{l})} \triangleq 0$, where $\tilde{l}^\mu$ is defined by Eq. (42). Denoting by $\zeta^\mu = \alpha n^\mu + \beta^\nu s_\nu s^\mu$ the component of t^μ orthogonal to S, the following quasiequilibrium conditions were imposed:

(1) The inner boundary S *remains* an apparent horizon: $\mathcal{L}_\zeta \theta_{(\tilde{l})} \triangleq 0$.

(2) The expansion $\theta_{(\tilde{k})}$ associated with the ingoing null vector $\tilde{k}^\mu = (n^\mu - s^\mu)/2$ does not *change* in time: $\mathcal{L}_\zeta \theta_{(\tilde{k})} \triangleq 0$.

These conditions are enforced under the *approximation*, motivated by the stationary case, that the shear $\sigma_{(\tilde{l})}$ associated with the outgoing null vector vanishes.

Before we actually compare the resulting boundary conditions (or rather some ellaboration of them; see the last part of this section) with those of Sec. IV, we make some general remarks on the involved quasiequilibrium conditions.

Under the vanishing shear approximation, the condition $\mathcal{L}_\zeta \theta_{(\tilde{l})} \triangleq 0$ leads to $\beta^\nu s_\nu \triangleq \alpha$, thus making ζ^μ a null vector parallel to $\tilde{l}^\mu$. In particular, this implies that the underlying coordinate system is stationary with respect to the horizon. Therefore, this condition is either redundant with the vanishing shear approximation (via the Raychaudhuri equation) or must be considered as a gauge choice, and not as an actual quasiequilibrium condition.

More generally, in Ref. [9] the conceptual status of the vanishing shear hypothesis is not clearly stated and an explicit prescription for imposing it in terms of the initial data, such as Eqs. (41) or (57), is missing. The IH analysis shows that the vanishing of the shear is the key quasiequilibrium condition: it guarantees that the world tube of apparent horizons is a null hypersurface. More explicitly, if $\sigma_{(\tilde{l})} \triangleq 0$ is not taken as a quasiequilibrium *characterization* but only as an approximation that might occasionally fail, the vector ζ^μ is no longer necessarily null. As a consequence, $\mathcal{L}_\zeta \theta_{(\tilde{l})} \triangleq 0$ would not really be a quasiequilibrium condition (for instance, $\mathcal{L}_\zeta \theta_{(\tilde{l})}$ vanishes also for dynamical horizons [21], where ζ^μ is spacelike).

Hence, as already pointed out in Refs. [15,18], the approach of Ref. [9] is very close in spirit to that encoded in the IH formalism; in fact, if the approximation of vanishing shear is eventually satisfied, a NEH is actually constructed. However, its quasiequilibrium conditions can be refined[7] (see Ref. [22] and Sec. VA). By contrast, a virtue of our approach, fully based on the IH scheme, is a clear identification and understanding of the physical and

[7] We have obviated the discussion of Ref. [9] about the way to enforce the horizon to remain in the same coordinate location, a discussion that can also be simplified.

mathematical hypotheses that characterize the horizon quasiequilibrium.

The need to clarify, from a conceptual point of view, the quasiequilibrium hypotheses in Ref. [9] can be illustrated as follows. In Ref. [23], the boundary conditions derived in Ref. [9], identified as IH conditions, are disregarded as technically too complicated. They are then substituted by a heuristic set of conditions, involving in particular $0 = \partial_t \ln\sqrt{\gamma} = D_i\beta^i - \alpha K$. This condition, which is equivalent to $\partial_t \Psi = 0$, turns out to be the NEH condition on the conformal factor in the corotating physical regime [make $\eta^\mu = 0$ in Eq. (56)], under the quasiequilibrium bulk condition $\partial_t \tilde{\gamma}_{ij} = 0$ assumed in Ref. [23]. Besides, the Kerr-Schild data (motivating the boundary values for the shift in that reference) are consistent with Eq. (54). At the end of the day, we find that the heuristic choice turns out to be one which is truly in the spirit of the IH scheme. Let us nonetheless mention that these boundary conditions are not imposed on the horizon itself, but in its interior.

B. Addendum

After the first submission of this work, a paper by Cook and Pfeiffer appeared [24] which provides a refinement of the discussion and proposals made by Cook in Ref. [9]. We now comment on the relation between our approach and the quasiequilibrium and boundary conditions proposed in Ref. [24] (and in [9]), in order to facilitate the comparison of our results with those of that reference.

(1) Quasiequilibrium conditions.—Quasiequilibrium is characterized in Ref. [24] by the geometrical conditions $\theta_{(\tilde{l})} \triangleq \sigma_{(\tilde{l})} \triangleq 0$, which are exactly those required to construct a NEH horizon, as discussed in Sec. IV B. Since no other condition is imposed (the requirement $\mathcal{L}_\zeta \theta_{(\tilde{k})} \triangleq 0$ of Ref. [9] is dropped), the analysis remains at the level of a NEH in the IH hierarchy, whereas our approach explores the WIH structure.

(2) Condition on Ψ.—In Refs. [9,24], this boundary condition follows from the requirement of vanishing expansion $\theta_{(\tilde{l})}$ for an apparent horizon. It is therefore essentially equivalent to Eqs. (40) and (56). However, the mathematical expression derived from $\theta_{(\tilde{l})} \triangleq 0$ adopts different forms [see also Eq. (A13) in the Appendix].

(3) Condition on β^i.—In *corotating* coordinates, the boundary condition (79) of Ref. [9] essentially coincides with our Eq. (54). A crucial refinement is introduced in Ref. [24] by actually imposing that the shear vanish: the projection of the shift on S [our vector $-\eta^\mu$ in Eq. (55)] must be a conformal symmetry of $\tilde{q}_{ab}$. This is equivalent to our condition (57) (see also the Appendix).

The main difference between both approaches is our demand of an azimuthal symmetry φ^μ for the metric q_{ab}, namely, Eq. (39). On the one hand, this makes conditions in Ref. [24] more general than ours but, on the other hand, thanks to this symmetry we are able to introduce a definite, intrinsic spinning angular velocity Ω_Δ which, together with $\Omega_{\rm orb}$, permits one to analyze the rotational regime of the system (corotational, irrotational, or general case).

In addition, the availability of Ω_Δ naturally leads us to consider Eq. (54) as a boundary condition on the shift. As a consequence, Eq. (57) becomes a constraint on the free data $\tilde{\gamma}_{ab}$ and $\partial_t \tilde{\gamma}_{ab}$, rather than providing a boundary condition for η^μ as in Ref. [24].

(4) Condition on α.—The analysis of a WIH carried out in Sec. IV C shows that the initial boundary value for the lapse is basically free. This conclusion is also reached in Ref. [24] after a numerical study. It is worth discussing the relation between the proposals that have been made for the choice of this boundary value. Condition $\mathcal{L}_\zeta \theta_{(\tilde{k})} \triangleq 0$ in Ref. [9] can be written as

$$[s^i D_i \alpha - s^i s^j K_{ij} \alpha]|_S = -\frac{\tilde{\mathcal{D}}\alpha}{\theta_{(\tilde{k})}}\bigg|_S, \tag{59}$$

with $\tilde{\mathcal{D}}$ defined in Eq. (85) of Ref. [9]. Our requirement (49) and Eq. (59) are simultaneously satisfied only if $\tilde{\mathcal{D}}\alpha = -\theta_{(\tilde{k})}\kappa_{\rm Kerr}$ on S. This is a nontrivial identity, so that both conditions are generally different.

Insight on their relation is provided by Ref. [8], where the freedom in the construction of a WIH structure $[l^\mu]$ is fixed by imposing that $\mathcal{L}_{(l)}\theta_{(k)} \triangleq 0$ (with $k^\mu l_\mu = -1$), once it is assumed that a certain operator $\mathbf{M}$ which acts on S has a trivial kernel [see Eq. (4.8) of Ref. [8] for the definition of $\mathbf{M}$ and note its close connection with $\tilde{\mathcal{D}}$]. This analysis can be applied to study the possible degeneracy of condition (59) in terms of the invertibility of $\mathbf{M}$ if, in addition, it is satisfied that $\kappa_{(\tilde{l})}$ is constant. If that is the case, employing Eq. (45) one can check that conditions (49) and (59) coincide only if the lapse is constant on the boundary. More details on this issue will appear elsewhere.

Notice that the choice of representative made in a WIH class via the lapse boundary condition (49) determines the initial lapse once γ_{ij} and K_{ij} are given. In this sense, the condition for the lapse is not problematic by itself. However, our full set of boundary conditions for α, Ψ, and β^i, together with the choice of free data ($\tilde{\gamma}_{ij}, \partial_t \tilde{\gamma}^{ij}, K, \partial_t K$), may not be sufficient to single out a unique solution to the initial data problem. In fact, this degeneracy seems to occur when our boundary conditions are implemented in the spherically symmetric, time-independent case if one uses a maximal slicing and a flat conformal metric.[8] Nonetheless, the presence of this degeneracy may depend on the actual choice of initial free data (e.g., isotropic coordinates in the commented example). For each specific choice, it is generally only after a numerical study that one may decide whether a degeneracy exists.

[8]We thank G. B Cook for pointing out this fact that also happens with the boundary conditions of Ref. [9].

VI. CONCLUSIONS

In this work we have explicitly shown how the IH formalism provides a rationale for some aspects of the numerical construction of initial data for a space-time containing a black hole in local quasiequilibrium, with special emphasis in the binary case.

The IH framework sheds light into the justification and implications of already existing quasiequilibrium sets of conditions for the analysis of this problem. The hierarchical structure of the IH formalism permits a control on the hypotheses that arise at each of the considered steps.

Adopting the IH approach fully, we have derived a set of boundary conditions on each black hole horizon for solving the elliptic equations obtained in a CTS scheme, deduced from the constraints and the trace of the evolution part in Einstein equations.

In a first step, the NEH condition characterizing quasiequilibrium ($\Theta_{ab} \triangleq 0 \Leftrightarrow \theta_{(l)} \triangleq \sigma_{(l)ab} \triangleq 0$), together with the choice of spatial coordinates stationary with respect to the horizon, provides boundary conditions for the shift [see Eqs. (34) and (54)] and the conformal factor [see Eqs. (40) and (56)]. These conditions are basically equivalent to those of Ref. [9] (at least in the recently refined form presented in Ref. [24]). In a second step, the requirement for a WIH-compatible slicing ($\kappa_{(l)} \triangleq \text{const} \neq 0$) leads to the evolution Eq. (47) for the lapse on Δ, leaving the choice of its initial value essentially free. Once this point has been acknowledged, we have tentatively suggested a specific boundary condition for the lapse in Eq. (49), obtained by fixing the freedom which is available in the construction of a WIH structure. In addition to these boundary conditions, the NEH requirement entails that the free data on the initial slice fulfill, on the horizon, the constraints (41) [or (57) in corotating coordinates] and (39) (assuming an axially symmetric horizon; see Ref. [25] otherwise). Similarly, Eq. (47) could alternatively be seen as a constraint on the free data if one decided to fix the lapse on Δ.

These inner boundary conditions and constraints are sufficient conditions for constructing a black hole in instantaneous quasiequilibrium. However, in order to obtain black holes in quasiequilibrium during a finite evolution time rather than just instantaneously (as required in the quasicircular binary black hole problem), these conditions must be complemented with appropriate free data that encode the desired dynamical behavior.

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APPENDIX: SOME TECHNICAL DETAILS

In this appendix we explain some calculations and formulas employed in the main text.

(a) Proof of Eq. (14). If Δ is defined as the hypersurface $r = \text{const}$, its normal l^μ takes the form $l_\mu = \lambda \nabla_\mu r$ for certain function λ. Then $\nabla_{[\mu} l_{\nu]} = \lambda^{-1} l_{[\nu} \nabla_{\mu]} \lambda$. Contracting with l^μ, we find

$$l^\mu \nabla_\mu l_\nu = l^\mu \nabla_\mu \ln\lambda\, l_\nu. \tag{A1}$$

Equation (14) follows by identifying $\kappa_{(l)} \equiv l^\mu \nabla_\mu \ln\lambda$.

(b) Proof of Eq. (18). Choosing the normalization of the outgoing and ingoing null vectors l^μ and k^μ on Δ so that $k^\mu l_\mu = -1$, the induced (degenerate) metric on Δ can be written as

$$q_{\mu\nu} = g_{\mu\nu} + k_\mu l_\nu + l_\mu k_\nu = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu. \tag{A2}$$

Expressing $\mathcal{L}_l q_{\mu\nu}$ in terms of the connection ∇_μ and using Eq. (A2), we get for the second fundamental form (15) the formula

$$\Theta_{\alpha\beta} = q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu l_\nu. \tag{A3}$$

Expanding now $q^\mu{}_\alpha q^\nu{}_\beta$ and recalling Eq. (14), we find

$$\nabla_\alpha l_\beta = \Theta_{\alpha\beta} - l_\alpha k^\nu \nabla_\nu l_\beta - (k^\nu \nabla_\alpha l_\nu + l_\alpha k^\mu k^\nu \nabla_\mu l_\nu) l_\beta. \tag{A4}$$

Since, on a NEH, one has $\Theta_{\alpha\beta} \triangleq 0$ (see the text), contraction with a vector v^μ tangent to Δ (so that $v^\mu l_\mu \triangleq 0$) leads to Eq. (18) after defining

$$\omega_\alpha \triangleq -(k^\nu \nabla_\alpha l_\nu + l_\alpha k^\mu k^\nu \nabla_\mu l_\nu) = -P^\mu{}_\alpha k^\nu \nabla_\mu l_\nu. \tag{A5}$$

(c) Proof of Eq. (23). To demonstrate this equation starting from condition (22), the key remark is the proportionality between the exterior derivative of ω_μ and the volume two-form on the sphere $S^2 \simeq S$,

$$\mathrm{d}\,\omega \propto \sqrt{q} d^2 q. \tag{A6}$$

This is a nontrivial result that follows from the definition of NEH, and we refer the reader to Ref. [6] for its proof. As a consequence, $\mathrm{d}\omega$ *lives* on S and its contraction with the null normal of Δ vanishes: $l^\mu \nabla_{[\mu} \omega_{\nu]} \triangleq 0$. Therefore, using the Cartan identity,

$$0 \triangleq \mathcal{L}_l \omega_\nu = l^\nu \nabla_{[\nu} \omega_{\mu]} + \nabla_\mu (l^\nu \omega_\nu) \triangleq \nabla_\mu \kappa_{(l)}. \tag{A7}$$

(d) General expression of Θ_{ab}. In order to enforce the condition $\Theta_{ab} \triangleq 0$, we have made use in the text of a coordinate system that is stationary with respect to the horizon, so that $l^\mu = t^\mu + W^\mu$ with W^μ tangent to S (W^μ is the black hole surface velocity introduced by

Damour [16,26]). Then, we have considered several specific choices for t^μ on Δ, namely, the vector t_o^μ singled out in Eq. (29) for the *warped* coordinate system, or the quasi-Killing vector L^μ for corotating coordinates. For the sake of completeness, we now provide the general expression of the second fundamental form on Δ for arbitrary vectors t^μ and W^μ.

From the definition of $\Theta_{\mu\nu}$ and Eq. (A2) we find

$$\Theta_{ab} = \frac{1}{2}(\mathcal{L}_t q_{ab} + {}^2D_a W_b + {}^2D_b W_a) = \frac{1}{2}(\mathcal{L}_t q_{ab} + \mathcal{L}_W q_{ab}). \quad \text{(A8)}$$

Here, ${}^2D_a$ denotes the connection compatible with the metric q_{ab} induced on the sphere S and $W_a \equiv q_{ab}W^b$. Note that Eqs. (35) and (36) follow straightforwardly from this when one imposes W^μ to be the q_{ab} isometry $\Omega_\Delta \varphi^\mu$.

The conformal decomposition $q_{ab} = \Psi^4 \tilde{q}_{ab}$ leads to

$$\Theta_{ab} = \frac{\Psi^4}{2}\bigg[\theta_{(l)}\tilde{q}_{ab} + \mathcal{L}_t\tilde{q}_{ab} - \frac{1}{2}(\mathcal{L}_t \ln\tilde{q})\tilde{q}_{ab} + {}^2\tilde{D}_a\tilde{W}_b + {}^2\tilde{D}_b\tilde{W}_a - {}^2\tilde{D}_c W^c\tilde{q}_{ab}\bigg], \quad \text{(A9)}$$

and

$$\theta_{(l)} = \frac{1}{2}\mathcal{L}_t \ln\tilde{q} + 4\mathcal{L}_t \ln\Psi +^2 \tilde{D}_a W^a + 4W^{a2}\tilde{D}_a \ln\Psi, \quad \text{(A10)}$$

where ${}^2\tilde{D}_a$ is the connection compatible with $\tilde{q}_{ab}$ and $\tilde{W}_a \equiv \tilde{q}_{ab}W^b$. In particular, with the notation η^μ for W^μ and choosing t^μ to be the quasi-Killing vector L^μ (so that $\partial_t\tilde{q}_{ab} = 0$), the vanishing of the expansion in Eq. (A10) leads to condition (56) [after substituting Eq. (10) for $\partial_t\Psi$]. On the other hand, from Eq. (A9), the traceless part of $\Theta_{ab} \triangleq 0$ is equivalent to condition (57). It is then clear from Eq. (A9) that in corotating coordinates the vector η^μ must be a conformal symmetry generator of $\tilde{q}_{ab}$.

Finally, a more general expression for $\Theta_{\alpha\beta}$ can be obtained if we do not assume a coordinate system stationary with respect to Δ. Substituting $l^\mu = \alpha(n^\mu + s^\mu)$ in Eq. (A3) and expanding the derivative we find

$$\Theta_{\alpha\beta} = \alpha(D_\mu s_\nu - K_{\mu\nu})q^\mu{}_\alpha q^\nu{}_\beta. \quad \text{(A11)}$$

Taking the trace, the standard expression for $\theta_{(l)}$ follows,

$$\theta_{(l)} = \alpha(D_i s^i + K_{ij}s^i s^j - K). \quad \text{(A12)}$$

Hence, after a conformal decomposition,

$$\theta_{(l)} = (4\tilde{s}^i\tilde{D}_i \ln\Psi + \tilde{D}_i\tilde{s}^i + \Psi^{-2}K_{ij}\tilde{s}^i\tilde{s}^j - \Psi^2 K)\alpha\Psi^{-2}, \quad \text{(A13)}$$

where $\tilde{s}^i = \Psi^2 s^i$.

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