

## Depletion in Bose-Einstein condensates using quantum field theory in curved space

Roberto Balbinot,<sup>1,\*</sup> Serena Fagnocchi,<sup>1,2,†</sup> and Alessandro Fabbri<sup>3,‡</sup>

<sup>1</sup>*Dipartimento di Fisica dell'Università di Bologna and INFN sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy*

<sup>2</sup>*Centro Studi e Ricerche "Enrico Fermi," Compendio Viminale, 00184 Roma, Italy*

<sup>3</sup>*Departamento de Física Teórica and IFIC, Universidad de Valencia-CSIC, C. Dr. Moliner 50, 46100 Burjassot, Valencia, Spain*

(Received 22 November 2006; published 30 April 2007)

Using methods developed in quantum field theory in curved space we estimate the effects of the inhomogeneities and of a nonvanishing velocity on the depletion of a Bose-Einstein condensate within the hydrodynamical approximation.

DOI: [10.1103/PhysRevA.75.043622](https://doi.org/10.1103/PhysRevA.75.043622)

PACS number(s): 03.75.Kk, 05.30.Jp, 04.62.+v, 04.70.Dy

### I. INTRODUCTION

There is growing interest in the so-called “analog models,” i.e., condensed matter systems which, under some circumstances, behave as gravitational ones. The common feature of these systems is that the fluctuations on top of the ground state propagate similar to fields moving on a fictitious curved spacetime, whose metric tensor depends on the characteristics of the underlying medium. This analogy holds both at the classical and quantum level. At the moment this setting offers the most realistic possibility to test peculiar effects of quantum field theory (QFT) in curved space which are still almost unobservable within the gravitational context. Among these, the most popular one is surely Hawking’s black hole radiation [1]. To this aim the most promising candidates, among analog models, are the Bose-Einstein condensates (BEC) [2,3], because for them the actual experimental limits are very close to those necessary to observe such effects [4], and the huge recent technological improvement in handling BEC makes us confident that this limit will be soon overtaken.

In this paper we use this analogy as a new tool in analytical calculations of the quantum depletion in dilute weakly interacting nonhomogeneous BEC. The depletion is the number density of the noncondensed particles. In fact, because of quantum fluctuations, even at zero temperature a nonvanishing fraction of particles are noncondensed. This quantity has been analytically calculated for homogeneous BEC only [5] as a function of the (constant) density of the condensate  $n_0$ . The most reliable attempt in extending the calculation to nonhomogeneous BEC consists in considering the condensate locally as almost homogeneous, then replacing  $n_0$  in the result obtained in Ref. [5] by the actual nonconstant density  $n$ . This is the so-called “local density approximation” (LDA) (see, e.g., Ref. [6]), which accounts for inhomogeneities only in first approximation, since no dependence on spatial derivatives of the density is considered. When the velocity is vanishing this approximation turns out to be quite good; however, as shown by the authors of Ref. [7], the LDA for a harmonic trap slightly overestimate the depletion. Moreover,

to our knowledge, nothing is known so far about effects due to the presence of a nonhomogeneous velocity in the condensate. In this paper we propose a framework, borrowed from QFT in curved space, which allows an explicit analytical evaluation of how inhomogeneities and velocity affect the depletion in a BEC.

### II. QUANTUM DEPLETION IN BEC

In the usual  $s$ -wave scattering approximation, a Bose gas is described by a field operator  $\hat{\Psi}$  satisfying

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\vec{x}) + g \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi}, \quad (1)$$

where the coupling constant  $g$  is related to the  $s$ -wave scattering length  $a$  as  $g = 4\pi\hbar^2 a/m$ , with  $m$  the mass of the constituents of the gas and  $V_{\text{ext}}$  the trapping external potential. For a dilute gas, in the mean field limit, the field operator  $\hat{\Psi}$  can be split into

$$\hat{\Psi} = (\Psi + \hat{\phi}) \frac{\hat{a}}{\sqrt{\hat{N}}}, \quad (2)$$

where  $\Psi$  is the order parameter,  $\hat{\phi}$  the one particle fluctuation,  $\hat{a}$  the destruction operator, and  $\hat{N} = \hat{a}^\dagger \hat{a}$  the particles number operator. This splitting has been introduced following the so-called number conserving approach [8,9]. The splitting of the field  $\hat{\Psi}$  as in Eq. (2) leads to  $\langle \hat{\Psi} \rangle = 0$ : this preserves the  $U(1)$  symmetry of Eq. (1). As a consequence, since  $[\hat{\phi}, \hat{a}/\sqrt{\hat{N}}] = 0$ , this framework keeps the total number of particles constant. However, the use of a number conserving approach is not crucial for our purpose, as will be discussed below.

Neglecting the backreaction of fluctuations, the evolution of the condensate is governed by the Gross-Pitaevskii equation, which reads

$$i\hbar \frac{\partial}{\partial t} \Psi = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + gn \right) \Psi, \quad (3)$$

where  $n = |\Psi|^2$ . The fluctuation part satisfies the Bogoliubov–de Gennes equation

\*Email address: balbinot@bo.infn.it

†Email address: fagnocchi@bo.infn.it

‡Email address: afabbri@ific.uv.es

$$i\hbar \frac{\partial}{\partial t} \hat{\phi} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + 2gn \right) \hat{\phi} + gn \hat{\phi}^\dagger. \quad (4)$$

The total particles density can be written as a sum of condensed and noncondensed parts

$$n_{\text{tot}} = \langle \hat{\Psi}^\dagger \hat{\Psi} \rangle = n + \langle \hat{\phi}^\dagger \hat{\phi} \rangle. \quad (5)$$

The last term describes the so-called depletion.

It is worth noting that the depletion depends on the Bogoliubov–de Gennes equation only. Therefore a framework which conserves the total particles number is not strictly necessary (at least at first order in  $1/\sqrt{N}$ , following the expansion of Ref. [8]). In fact the error induced by not using a number conserving approach is of the same order of that implicit in the Bogoliubov linearization of the theory. Instead of Eq. (2) one could split the field operator as  $\hat{\Psi} = \Psi + \delta\hat{\Psi}$ , with  $\Psi = \langle \hat{\Psi} \rangle$  and  $\langle \delta\hat{\Psi} \rangle = 0$ , breaking the U(1) symmetry. This will lead to the same results [10].

For a homogeneous condensate ( $n = n_0 = \text{const}$ ) at zero temperature this quantum depletion can be exactly calculated [5]:

$$\tilde{n}_0^q = \langle \hat{\phi}^\dagger \hat{\phi} \rangle = \frac{8}{3\pi^{1/2}} (n_0 a)^{3/2}. \quad (6)$$

At nonzero temperature one has an additional contribution to the depletion because of thermal fluctuations. At low temperature ( $\kappa_B T \ll n_0 g$ , with  $\kappa_B$  the Boltzmann constant), summing up both contributions, one can write the depletion for a homogeneous condensate as

$$\tilde{n}_0 = \tilde{n}_0^q + \tilde{n}_0^T = \frac{8}{3\pi^{1/2}} (n_0 a)^{3/2} \left[ 1 + \left( \frac{\pi \kappa_B T}{2n_0 g} \right)^2 \right]. \quad (7)$$

It is worth noting that the derivation of the zero temperature quantum depletion (6) is based on the full Bogoliubov dispersion relation for the modes, i.e.,  $\omega_k = ck\sqrt{1 + \xi^2 k^2}$ , where  $c = (gn_0/m)^{1/2}$  is the velocity of sound and  $\xi = \hbar/(\sqrt{2}mc)$  is the healing length of the condensate. In contrast, the low temperature correction comes from the phononic part of the spectrum only, i.e.,  $\omega_k = ck$ . The integral giving the thermal depletion is dominated by modes with wavelength  $\lambda \gg \xi$ , so the result is insensitive to the high  $k$  behavior of the dispersion relation. Therefore, while the correct description of the ground state requires the knowledge of the full “microscopic” theory [11], deviations from the ground state value can, in some circumstances, be evaluated with sufficient accuracy using the low energy “macroscopic” phonon theory. This feature of the low energy theory is familiar in the Casimir effect. The standard Casimir energy density between two conducting plates can be evaluated with sufficient accuracy using the photon dispersion relation, provided the plate separation is much larger than the interatomic distance of the constituents of the plates [12]. Moving to nonhomogeneous condensates, when the current  $\vec{j} \equiv \frac{1}{2im} (\Psi^* \vec{\nabla} \Psi - \text{c.c.})$  vanishes, the depletion is usually approximated by Eq. (7) with  $n_0$  now replaced by the actual space-time varying density  $n(\vec{x}, t)$  in the local density approximation (LDA) [6]. We indicate this as

$$\tilde{n}_{\text{LDA}} = \tilde{n}_{\text{LDA}}^q + \tilde{n}_{\text{LDA}}^T = \frac{8}{3\pi^{1/2}} (na)^{3/2} \left[ 1 + \left( \frac{\pi \kappa_B T}{2ng} \right)^2 \right], \quad (8)$$

where the first term is the zero temperature quantum depletion and the second the thermal depletion, both in LDA.

### III. BEC AND THE GRAVITATIONAL ANALOGY

The aim of this paper is to show how, in view of the analogy described at the beginning, through the use of QFT in curved space and under some hypothesis, it is possible to obtain an estimate of the depletion which explicitly accounts for the inhomogeneities of the condensate (i.e.,  $\partial_t n \neq 0$ ) and for a nontrivial velocity (i.e., a velocity that cannot be simply eliminated by a Galileo transformation).

For BEC the analogy traces back to the hydrodynamical approximation of the basic equations for the condensate (3) and (4). In the density-phase representation for the condensate field  $\Psi = \sqrt{n} e^{i\theta/\hbar}$ , Eq. (3) is split in a continuity and an Euler equation, namely [6],

$$\frac{\partial}{\partial t} n + \vec{\nabla} \cdot (n\vec{v}) = 0, \quad (9)$$

$$m \frac{\partial}{\partial t} \vec{v} + \vec{\nabla} \cdot \left( \frac{m\vec{v}^2}{2} + V_{\text{ext}} + gn - \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}} \right) = 0, \quad (10)$$

where  $\vec{v} = \vec{\nabla} \theta / m$  is the velocity field. The last term in Eq. (10) is called quantum pressure. The hydrodynamical approximation consists in neglecting this compared to the mean field term  $gn$ . For this approximation to be valid, the typical length scale  $L$  for the variations of the condensate density  $n$  has to be much bigger than the healing length ( $L \gg \xi$ ). Decomposing the fluctuation operator  $\hat{\phi}$  as

$$\hat{\phi} = e^{i\theta/\hbar} \left( \frac{1}{2\sqrt{n}} \hat{n}_1 + i \frac{\sqrt{n}}{\hbar} \hat{\theta}_1 \right), \quad (11)$$

where  $\hat{n}_1$  and  $\hat{\theta}_1$  are real fields, one finds that for wavelengths  $\lambda \gg \xi$ , the Bogoliubov–de Gennes Eq. (4) can be written as

$$\hat{\square} \hat{\theta}_1 = 0, \quad (12)$$

$$\hat{n}_1 = -g^{-1} (\partial_t \hat{\theta}_1 + m^{-1} \vec{\nabla} \theta \cdot \vec{\nabla} \hat{\theta}_1), \quad (13)$$

where  $\hat{\square} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$  is the covariant d’Alembertian calculated from the so-called “acoustic metric”

$$g_{\mu\nu} \equiv \frac{n}{mc} \begin{pmatrix} -(c^2 - v^2) & -\vec{v}^T \\ -\vec{v} & 1 \end{pmatrix}, \quad (14)$$

where  $c$  is the local sound speed related to the density by  $mc^2 = gn$ . Hence the fluctuation  $\hat{\theta}_1$  behaves exactly as a massless minimally coupled scalar field propagating on a fictitious curved spacetime described by the acoustic metric  $g_{\mu\nu}$ , whose line element reads

$$ds^2 = \frac{n}{mc} [-c^2 dt^2 + (d\vec{x} - \vec{v} dt) \cdot (d\vec{x} - \vec{v} dt)]. \quad (15)$$

This is the core of the analogy [13].

For a homogeneous condensate with constant velocity  $\vec{v}_0$ , the acoustic spacetime is flat as can be seen performing the Galileo transformation  $d\vec{x} \rightarrow d\vec{x} - \vec{v}_0 dt$ . This condensate configuration is a sort of ground state of the theory. On the other hand, inhomogeneities or a nontrivial velocity produce curvature. We will focus our analysis on the latter kind of configurations, looking for the corrections on the depletion induced by curvature with respect to the homogeneous (let say flat) result.

#### IV. QFT IN CURVED SPACE AND THE DEPLETION IN BEC

As just mentioned, phonons in a BEC propagate as a massless scalar field on a curved background spacetime: therefore they can be described through the gravitational formalism. In the case of nonhomogeneous BEC, the curvature is not zero and we expect modifications with respect to the standard result induced by this curvature.

In this case modes with  $\lambda \lesssim \xi \ll L$  see the condensate as almost homogeneous. After renormalization, they will give the quantum depletion term in LDA  $\tilde{n}_{\text{LDA}}^q$ . Deviations from this come from modes with wavelength  $\lambda \gtrsim L \gg \xi$ . These are indeed the modes who feel the inhomogeneities of the condensate. They behave as a massless scalar field on the curved spacetime  $g_{\mu\nu}$ . So the low energy phonon theory (even if unable to reproduce the ground state value of the depletion) can describe with sufficient accuracy how inhomogeneities influence the depletion. Now, within this spirit, calling  $\tilde{n}_{\text{ph}}$  the depletion induced by the phonons, we can write  $\tilde{n} = \tilde{n}_{\text{LDA}}^q + \tilde{n}_{\text{ph}}$ , where, using Eq. (11):

$$\tilde{n}_{\text{ph}} = \langle \hat{\phi}^\dagger \hat{\phi} \rangle_{\text{ph}} = \frac{\langle \hat{n}_1^2 \rangle_{\text{ph}}}{4n} + \frac{n \langle \hat{\theta}_1^2 \rangle_{\text{ph}}}{\hbar^2} \approx \frac{n \langle \hat{\theta}_1^2 \rangle_{\text{ph}}}{\hbar^2} \left( 1 + O\left(\frac{\xi^2}{L^2}\right) \right), \quad (16)$$

and the expectation values are calculated using the phonon theory. The last approximation in Eq. (16) comes from taking  $\hat{n}_1$  from Eq. (13) and estimating  $\nabla \hat{\theta}_1 \approx \frac{\hat{\theta}_1}{L}$  and  $\partial_t \hat{\theta}_1 \approx \frac{\hat{\theta}_1}{Lc}$ . In this way the calculation of the depletion has been reduced to the calculation of the expectation value of the squared of a massless scalar field minimally coupled to the acoustic spacetime (14).  $n \langle \hat{\theta}_1^2 \rangle_{\text{ph}} / \hbar^2$  is dominated by wavelengths  $\sim L$ , therefore one can safely extrapolate the linear dispersion relation to high frequencies without affecting the finite renormalized result, since the diverging terms appearing are then subtracted away.

In QFT in curved space a useful method of renormalization is point splitting [14]. Within this procedure,  $\langle \hat{\theta}_1^2(x) \rangle$  (here and in the following we will drop the subscript ‘‘ph’’ from the expectation values) is the coincidence limit of the Euclidean Green function

$$\langle \hat{\theta}_1^2(x) \rangle_{\text{unren}} = \text{Re}[\lim_{x' \rightarrow x} G_E(x, x')]. \quad (17)$$

Renormalization is performed by subtraction of the divergent part, the so called De Witt–Schwinger term  $G_{\text{DS}}$ , obtained by the short-distance expansion of the Green function

$$\langle \hat{\theta}_1^2(x) \rangle_{\text{ren}} = \lim_{x' \rightarrow x} [\langle \hat{\theta}_1^2(x, x') \rangle_{\text{unren}} - G_{\text{DS}}(x, x')]. \quad (18)$$

$G_{\text{DS}}(x, x')$  is purely geometrical, state independent and contains all the short-distance divergences of the theory (for technical details see, e.g., Ref. [14]).

##### A. Static spherically symmetric configuration

We now restrict our discussion to a static spherically symmetric configuration [ $n = n(r)$ ,  $\vec{v} = \vec{0}$ ]. The resulting acoustic metric is invariant under time translation and  $\partial_t$  is the Killing vector associated to this symmetry. The acoustic metric (14) can be put in the standard form

$$ds^2 = -f(R)dt^2 + h(R)dR^2 + R^2d\Omega^2, \quad (19)$$

by the coordinate transformation

$$R = \sqrt{\frac{n}{mc}} r. \quad (20)$$

Note that this coordinate transformation is valid only if  $R \neq \text{const}$ . The  $R = \text{const}$  case will be treated separately. Now

$$f = \frac{nc}{m}, \quad h = \left( 1 + \frac{c'}{2c} r \right)^{-2}, \quad (21)$$

and  $' \equiv \partial_r$ .

For configurations of this kind, it is natural to consider thermal equilibrium states, i.e., whose Euclidean Green function is periodic in  $T^{-1}$  in the imaginary time. For these states, Anderson *et al.* [15] have developed a WKB technique to evaluate  $\langle \hat{\theta}_1^2(x, x') \rangle_{\text{unren}}$  in the case of a generic metric of type (19) leading to

$$\langle \hat{\theta}_1^2 \rangle_{\text{ren}}^{\text{WKB}} = \frac{\kappa_B^2 T^2}{12\hbar f} + \hbar \left[ \frac{\mathcal{R}}{96\pi^2} \ln(\mu^2 f) - \frac{f'^2}{96\pi^2 f^2 h} - \frac{f' h'}{192\pi^2 f h^2} + \frac{f''}{96\pi^2 f h} + \frac{f'}{48\pi^2 R f h} \right], \quad (22)$$

where  $\mathcal{R}$  is the curvature scalar, only here the prime denotes  $\partial_R$  and  $\mu$  is an arbitrary scale introduced by renormalization [14]. The first term describes a gas of massless scalar particles in thermal equilibrium at the temperature  $T$  in a curved spacetime. The factor  $1/f$  accounts for the ‘‘gravitational redshift’’ (Tolman law [16]). The other terms describe how the quantum fluctuations are affected by the curvature of the effective spacetime. Entering our expressions (21) in Eq. (22) one gets the following analytic expression for the thermal and inhomogeneity dependent part of the depletion:

$$\tilde{n}_{\text{ph}} = \frac{\kappa_B^2 T^2}{12\hbar^3} \left( \frac{m^3}{gn} \right)^{1/2} - \frac{(mgn)^{1/2}}{96\pi^2 \hbar} \left[ -\frac{7n'^2}{8n^2} + \frac{5n''}{2n} + \frac{5n'}{rn} \right] \ln \mu^2 g^{1/2} n^{3/2} + \frac{(mgn)^{1/2}}{48\pi^2 \hbar} \left[ -\frac{39n'^2}{16n^2} + \frac{3n''}{4n} + \frac{3n'}{2rn} \right]. \quad (23)$$

Note that the first term in Eq. (23) elegantly reproduces  $\tilde{n}_{\text{LDA}}^T$ .

The other terms give the corrections to the depletion in LDA induced explicitly by the spatial derivatives of the condensate density. They all vanish for a homogeneous BEC. One can obtain a numerical estimate of the relative weight of the inhomogeneities induced terms. For  $N=10^6$  Rb atoms trapped by a harmonic potential ( $L_{\text{osc}}=1 \mu\text{m}$ ,  $a=5 \text{ nm}$ ,  $R_0=L(15aN/L)^{1/5}$  the size of the condensate) Eq. (23) gives at  $T=0$  a correction with respect to the LDA result of  $<0.1\%$  at the center of the trap, reaching  $\sim 1\%$  approaching the boundary ( $r\sim 0.9R_0$ ). For  $N=10^3$  atoms the correction will be almost  $2\%$  at the center of the trap and almost  $20\%$  at  $r\sim 0.7R_0$  [17,18]. In both cases the corrections turn out to be negative, therefore lowering the actual number of noncondensed particles. This is in qualitative agreement with what pointed out in Ref. [7].

### B. Radial velocity

In the presence of a nonvanishing radial velocity (i.e.,  $\vec{v}=v(r)\frac{\vec{r}}{r}$ ) [19] the system is no longer in equilibrium and no

analytical expression for  $\langle\hat{\theta}_1^2\rangle_{\text{ren}}$  is known. One can only evaluate how vacuum fluctuations are modified by the presence of the radial velocity. Being the density and the velocity functions of the radial coordinate  $r$  only, one can transform the acoustic metric (14) in the standard spherically symmetric form of Eq. (19) by the following coordinate transformation (assuming again  $R\neq\text{const}$ )

$$\tau=t+\int\frac{vdr}{c^2-v^2}, \quad R=\sqrt{\frac{n}{mc}}r, \quad (24)$$

leading to a diagonal form of the type of Eq. (19) with

$$f=\frac{n}{mc}(c^2-v^2), \quad h=\frac{c^2}{c^2-v^2}\left(1+\frac{c'}{2c}r\right)^{-2}. \quad (25)$$

The vacuum part of Eq. (22) for this new configuration gives

$$\begin{aligned} \tilde{n}_{\text{ph}}^q = & -\frac{m}{96\pi^2\hbar c^2}\left[\frac{3}{2}cc'^2+5c^2c''+-3v^2c''-2cv'^2-2cvv''+\frac{9}{2c}v^2c'^2\right. \\ & \left.-4vc'v'+\frac{2}{r}(5c^2c'-4c'v'-v^2c')-\frac{2}{r^2}v^2c\right]\ln\mu^2\frac{c(c^2-v^2)}{g} \\ & +\frac{m}{48\pi^2\hbar c^2}\left\{-\frac{1}{2c(c^2-v^2)}(3c^2c'-v^2c'-2c'v')(3c^2c'-2v^2c'-c'v')\right. \\ & \left.-\frac{15}{4}cc'^2-\frac{1}{4c}v^2c'^2-\frac{5}{2}vc'v'+\frac{3}{2}c^2c''-\frac{1}{2}v^2c''-cv'^2-cvv''+\frac{1}{r}(3c^2c'-v^2c'-2c'v')\right\}. \quad (26) \end{aligned}$$

This equation generalizes  $\tilde{n}_{\text{ph}}$  of Eq. (23) at  $T=0$  in the presence of a radial velocity. Note that for homogeneous condensate ( $c=\text{const}$ ) this expression does not vanish for  $|\vec{v}|=\text{const}$ , as a consequence of the fact that a radial velocity can not be eliminated by a Galileo transformation. Equation (26) reduces to Eq. (23) (at  $T=0$ ) only for  $v(r)=0$ .

It is important to stress that Eq. (26) is strictly valid only in regions where the fluid velocity does not reach the sound velocity. For  $|\vec{v}|=c$ , Eq. (26) diverges. This local divergence has nothing to do with the ultraviolet divergences already canceled out in deriving Eq. (26). This divergence is quite familiar in the gravitational context: in fact for  $|\vec{v}|=c$  the  $g_{00}$  term in the acoustic metric (14) vanishes and this is a hint of the fact that a horizon forms (the so-called ‘‘sonic horizon’’). This reflects on the modes of the scalar field that turn out to be singular on the horizon [20]. As a direct consequence, a singular vacuum polarization on the horizon of a black hole appears. However it is well known that taking the black hole formation process into account a Hawking radiation

emerges: its presence is responsible for an additional contribution to  $\langle\hat{\theta}_1^2\rangle_{\text{unren}}$  which has an equal and opposite sign divergence, leading to a  $\langle\hat{\theta}_1^2\rangle_{\text{ren}}$  regular on the horizon. This has been explicitly shown for a Schwarzschild black hole [for which  $f(r)=g^{-1}(r)=1-2M/r$ ] [16]. A similar behavior is also expected in the BEC case. Unfortunately for a general metric as in Eq. (19) an explicit analytic form for the  $\langle\hat{\theta}_1^2\rangle_{\text{unren}}$  taking Hawking radiation into account and valid in the near horizon region is still unknown.

### C. $R=\text{const}$ case

For the sake of completeness let us consider the case when the coordinate transformation (20) is singular, i.e., when  $n/c\propto 1/r^2$ . The acoustic metric (19) shows that the surface  $(r,t)=\text{const}$  has constant area. In this case Eq. (22) is replaced by

$$\langle \hat{\theta}_1^2 \rangle_{\text{ren}}^{\text{WKB}} = \frac{\kappa_B^2 T^2}{12\hbar f} + \hbar \left[ \frac{\mathcal{R}}{96\pi^2} \ln(\mu^2 f) - \frac{f'^2}{96\pi^2 f^2 \hbar} - \frac{f' h'}{192\pi^2 f h^2} + \frac{f''}{96\pi^2 f h} \right] \quad (27)$$

with the prime denoting  $\partial_r$ ,  $h = \frac{nc}{m}(c^2 - v^2)^{-1}$ , and  $f$  unchanged as in Eq. (25).

For the standard equation of state for BEC  $mc^2 = gn$ , one can take  $n = n_0 r_0^4 / r^4$ , with  $n_0$  and  $r_0$  two constants having the dimension of density and length, respectively. The resulting quantum depletion reads

$$n_{\text{ph}}^q = \frac{r_0^2}{48\pi^2 \hbar r^4} (n_0 g m)^{1/2} \left\{ - \left( 10 + \frac{v}{c} - \frac{r^2}{c^2} (vv')' \right) \ln \left( \mu^2 g^{1/2} n_0^{3/2} \frac{r_0^6}{m^{3/2} r^6} \right) - \left( 1 - \frac{v}{c} \right)^{-1} \right. \\ \left. \times \left( 21 - 22 \frac{v^2}{c^2} + 5 \frac{v^4}{c^4} + r \frac{vv'}{c^2} \left( 10 - 8 \frac{v^2}{c^2} \right) + r^2 \frac{v^2 v' 2}{c^4} \right) + 21 - 3 \frac{v^2}{c^2} + 4r \frac{vv'}{c^2} - r^2 \frac{(vv')'}{c^2} \right\}. \quad (28)$$

## V. CONCLUSIONS

Summarizing, we have shown how sophisticated renormalization techniques originally developed to study quantum effects in black hole physics, can be extremely useful in a (seemingly) complete different context, namely BEC. In this paper these techniques have been used to give an analytical estimation of the effects of spatial derivatives and a non-trivial velocity on the depletion of a BEC. Our predictions are beyond the present experimental capability. Nonetheless

this work represents an attempt to go beyond the LDA, providing a general analytical scheme that can be used in cold-atom systems within the hydrodynamical approximation.

## ACKNOWLEDGMENTS

The authors would like to thank I. Carusotto, P. Pieri, A. Recati, and S. Stringari for stimulating discussions and for their useful comments on the manuscript. S.F. also thanks ‘‘E. Fermi’’ Center for supporting her research.

- 
- [1] S. W. Hawking, *Nature (London)* **248**, 30 (1974).  
 [2] F. D’Alfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**, 463 (1999).  
 [3] *Bose-Einstein Condensation in Dilute Gases*, edited by C. J. Pethick and H. Smith (Cambridge University Press, Cambridge, 2002).  
 [4] M. Visser, C. Barcelo, and S. Liberati, *Gen. Relativ. Gravit.* **34**, 1719 (2002).  
 [5] T. D. Lee, K. Huang, and C. N. Yang, *Phys. Rev.* **106**, 1135 (1957).  
 [6] See, e.g., *Bose-Einstein Condensation*, edited by L. P. Pitaevskii and S. Stringari (Clarendon Press, Oxford, 2003).  
 [7] F. D’Alfovo, S. Giorgini, M. Guilleumas, L. Pitaevskii, and S. Stringari, *Phys. Rev. A* **56**, 3840 (1997).  
 [8] Y. Castin and R. Dum, *Phys. Rev. A* **57**, 3008 (1998).  
 [9] C. W. Gardiner, *Phys. Rev. A* **56**, 1414 (1997).  
 [10] We thank I. Carusotto for kindly pointing this out to our attention.  
 [11] The linear phonon dispersion relation, characteristic of a massless relativistic free field would lead (after renormalization) to a vanishing quantum depletion.  
 [12] S. Bachmann and A. Kempf, eprint gr-qc/0504076.  
 [13] L. J. Garay, J. R. Anglin, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **85**, 4643 (2000); C. Barcelo, S. Liberati, and M. Visser, *Living Rev. Relativ.* **8**, 12 (2005).  
 [14] See for ex.: *Quantum fields in curved spaces*, N. D. Birrel and P. C. W. Davies, Cambridge Univ. Press (1982).  
 [15] P. R. Anderson, W. A. Hiscock, and D. A. Samuel, *Phys. Rev. D* **51**, 4337 (1995).  
 [16] See, e.g., *Black Hole Physics: Basic Concepts and New Developments*, edited by V. P. Frolov and I. D. Novikov (Kluwer, Dordrecht, The Netherlands, 1997).  
 [17] For this estimation  $\mu^2 g^{1/2} n_0^{3/2} / m^{3/2} \approx 1$ .  
 [18] Note that the order of magnitude of these terms confirms the prediction of E. Timmermans, P. Tommasini, and K. Huang, *Phys. Rev. A* **55**, 3645 (1997).  
 [19] To realize this kind of configurations, it is necessary to add source (well) terms in the continuity Eq. (9) in order to conserve the mass. However, the presence of these terms do not invalidate our treatment since Eqs. (12) and (13) remain unchanged.  
 [20] See e.g., R. Balbinot, A. Fabbri, S. Fagnocchi, and R. Parentani, *Riv. Nuovo Cimento* **28**, 1 (2005).