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Hyperspherical harmonic formalism for tetraquarks

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We present a generalization of the hyperspherical harmonic formalism to study systems made of quarks and antiquarks of the same flavor. This generalization is based on the symmetrization of the N -body wave function with respect to the symmetric group using the Barnea and Novoselsky algorithm. Our analysis shows that four-quark systems with non-exotic 2^{++} quantum numbers may be bound independently of the quark mass. 0^{+-} and 1^{+-} states become attractive only for larger quarks masses.

The understanding of few-body systems relies in our capability to design methods for finding an exact or approximate solution of the N -body problem. In two-, three-, and four-body problems it is possible to obtain mathematically correct and computationally tractable equations such as the Schrödinger, Faddeev and Yakubovsky equations describing exactly, for any assumed interaction between the particles, the motion of few-body systems¹. However, the exact solution of these equations requires sophisticated techniques whose difficulty increases with the number of particles.

The solution of any few-particle system may be found in a simple and unified approach by means of an expansion of the trial wave function in terms of hyperspherical harmonic (HH) functions. The idea is to generalize the simplicity of the spherical harmonic expansion for the angular functions of a single particle motion to a system of particles by introducing a global length ρ , called the hyperradius, and a set of angles, Ω . For the HH expansion method to be practical, the evaluation of the potential energy matrix elements must be feasible. The main difficulty of this method is to construct HH functions of proper symmetry for a system of identical particles. This may be overcome by means of the HH formalism based on the symmetrization of the N -body wave function with respect to the symmetric group using the Barnea and Novoselsky algorithm². Therefore, this method becomes ideally suited to look for the possible existence of bound multi-quark systems.

The recent series of discoveries of new meson and baryon resonances whose properties do not fit into the predictions of the naive quark model, has reopened the interest on the possible role played by non- $q\bar{q}$ configurations in the hadron

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Table 1. Charmonium spectrum in MeV. Experimental data are taken from PDG¹⁰.

$(nL) J^{PC}$	State	CQM	Exp.
(1S) 0 ⁻⁺	$\eta_c(1S)$	2990	2979.6±1.2
(1S) 1 ⁻⁻	$J/\psi(1S)$	3097	3096.916±0.011
(1P) 0 ⁺⁺	$\chi_{c0}(1P)$	3443	3415.19±0.34
(1P) 1 ⁺⁺	$\chi_{c1}(1P)$	3496	3510.59±0.10
(1P) 2 ⁺⁺	$\chi_{c2}(1P)$	3525	3556.26±0.11
(1P) 1 ^{+−}	$h_c(1P)$	3507	3526.21±0.25
(2S) 0 ⁻⁺	$\eta_c(2S)$	3627	3654±10
(2S) 1 ⁻⁻	$\psi(2S)$	3685	3686.093±0.034
(1D) 1 ⁻⁻	$\psi(3770)$	3776	3770±2.4
(1D) 2 ⁻⁻	$\psi(3836)$	3790	3836±13
(3S) 1 ⁻⁻	$\psi(4040)$	4050	4040±10
(2D) 1 ⁻⁻	$\psi(4160)$	4104	4159±20

spectra. Among them, the existence of four- (two quarks and two antiquarks) and five-quark (four quarks and one antiquark), have been suggested in the low-energy hadron spectroscopy. Four-quark bound states were already suggested theoretically thirty years ago, both in the light-quark sector by Jaffe³ and in the heavy-quark sector by Iwasaki⁴. In the heavy-light sector there seems to be a consensus that four-quark systems containing two-light and two-heavy quarks, $qq\bar{Q}\bar{Q}$, should be stable against dissociation into two mesons, $q\bar{Q}$, if the ratio of the mass of the heavy to the light quark is large enough⁵. Unfortunately, such an agreement do not exist for multi-quarks made of four heavy quarks. These states have not received as much attention in the last years as their light counterparts and the scarce theoretical predictions for the existence of $cc\bar{c}$ systems differ depending basically on the method used to solve the four-body problem and the interaction employed^{4,5,6,7}. It is our aim to make a general study of four-quark systems of identical flavor in an exact way. For this purpose we have generalized the HH method⁸, widely used in traditional nuclear physics for the study of few-body nuclei, to study four-quark systems. This presents two main difficulties, first the simultaneous treatment of particles and antiparticles, and second the additional color degree of freedom. A comprehensive discussion about the generalization of the HH method and the constituent quark model used can be found in Ref.⁹.

The results for the $cc\bar{c}$ states have been obtained in the framework of the hyperspherical harmonic formalism up to the maximum value of K within our computational capabilities (K_{max}). Since the only relevant two-meson decay thresholds for these systems are those made of two $c\bar{c}$ mesons, we summarize in Table 1 the results obtained for the charmonium spectrum compared with the experimental data quoted by the Particle Data Group (PDG)¹⁰. To analyze the stability of these systems against dissociation through strong decay, parity (P), C -parity (C), and total angular momentum (J) must be conserved. In Table 2 we indicate the lowest two-meson threshold for each set of quantum numbers. Four-quark states will be stable under strong interaction if their total energy lies below all possible, and al-

Table 2. $cc\bar{c}\bar{c}$ masses obtained for the maximum value of K computed, $E(K_{max})$, and using the extrapolation, $E(K = \infty)$, compared with the corresponding threshold for each set of quantum numbers, M_1M_2 and $T(M_1, M_2)$. The subindex stands for relative angular momentum in the final state. The value of Δ for each state is also given. All energies are in MeV.

J^{PC}	$E(K_{max})$	$E(K = \infty)$	M_1M_2	$T(M_1, M_2)$	Δ
0^{++}	6115	6038	$\eta_c(1S) \eta_c(1S) _S$	5980	+58
0^{+-}	6606	6515	$\eta_c(1S) h_c(1P) _P$	6497	+18
1^{++}	6609	6530	$\eta_c(1S) \chi_{c0}(1P) _P$	6433	+97
1^{+-}	6176	6101	$J/\psi(1S) \eta_c(1S) _S$	6087	+14
2^{++}	6216	6172	$J/\psi(1S) J/\psi(1S) _S$	6194	-22
2^{+-}	6648	6586	$\eta_c(1S) h_c(1P) _P$	6497	+89
0^{-+}	7051	6993	$J/\psi(1S) J/\psi(1S) _P$	6194	+779
0^{--}	7362	7276	$J/\psi(1S) \eta_c(1S) _S$	6087	+1189
1^{-+}	7363	7275	$J/\psi(1S) J/\psi(1S) _P$	6194	+1081
1^{--}	7052	6998	$J/\psi(1S) \eta_c(1S) _S$	6087	+911
2^{-+}	7055	7002	$J/\psi(1S) J/\psi(1S) _P$	6194	+808
2^{--}	7357	7278	$J/\psi(1S) \eta_c(1S) _S$	6087	+1191

lowed, two-meson thresholds. It is useful to define $\Delta = M(q_1q_2\bar{q}_3\bar{q}_4) - T(M_1, M_2)$ in such a way that if $\Delta > 0$ the four-quark system will fall apart into two mesons, while $\Delta < 0$ will indicate that such strong decay is forbidden and therefore the decay, if allowed, must be weak or electromagnetic, being its width much narrower.

We show in Table 2 all possible J^{PC} quantum numbers with $L = 0$. Let us first of all note that the convergence of the expansion in terms of hyperspherical harmonics is slow, and the effective potential techniques⁸ are unable to improve it. In order to obtain a more adequate value for the energy we have extrapolated it according to the expression $E(K) = E(K = \infty) + a/K^b$, where $E(K = \infty)$, a and b are fitted parameters. The values obtained for $E(K = \infty)$ are stable within ± 10 MeV for each quantum number. A first glance to these results indicates that only three sets of quantum numbers have some probability of being observed. These are the $J^{PC} = 0^{+-}$, 1^{+-} , and $J^{PC} = 2^{++}$, which are very close to the corresponding threshold. Being all possible strong decays forbidden these states should be narrow with typical widths of the order of a few MeV. It is interesting to observe that the quantum numbers 0^{+-} correspond to an exotic state, those whose quantum numbers cannot be obtained from a $q\bar{q}$ configuration. Therefore, if experimentally observed, it would be easily distinguished as a clear signal of a pure non- $q\bar{q}$ state.

To analyze whether the existence of bound states with exotic quantum numbers could be a characteristic feature of the heavy quark sector or it is also present in the light sector we have calculated the value of Δ for different quark masses. We have obtained that only one of the non-exotic states, the 2^{++} , becomes more bound when the quark mass is decreased, $\Delta \approx -80$ MeV for the light quark mass. The 1^{+-} and 0^{++} states, that were slightly above threshold in the charm sector, increase their attraction when the quark mass is increased and only for masses close to the bottom quark mass may be bound. With respect to the exotic quantum numbers, the negative parity 0^{--} and 1^{-+} are not bound for any value of the

quark mass. Only the 0^{+-} four-quark state becomes more deeply bound when the constituent quark mass increases, and therefore only one possible narrow state with exotic quantum numbers may appear in the heavy-quark sector. This state presents an open P -wave threshold only for quark masses below 3 GeV. Let us note that since the heavy quarks are isoscalar states, the flavor wave function of the four heavy-quark states will be completely symmetric with total isospin equal to zero. Therefore, one should compare the results obtained in the light-quark case with a completely symmetric flavor wave function, i.e., the isotensor states.

There are experimental evidences for three states with exotic quantum numbers in the light-quark sector. Two of them are isovectors with quantum numbers $J^{PC} = 1^{-+}$ named $\pi_1(1400)$ and $\pi_1(1600)$, and one isotensor $J^{PC} = 2^{++}$, the $X(1600)$. Taking the experimental mass for the threshold $T(M_1, M_2)$ in the light-quark case together with the values obtained for Δ , one can estimate the energy of these states, being $M(2^{++}) \approx 1500$ MeV and $M(1^{-+}) \approx 2900$ MeV. The large mass obtained for the 1^{-+} four-quark state makes doubtful the identification of the $\pi_1(1400)$ or the $\pi_1(1600)$ with a pure multiquark state, although a complete calculation is needed before drawing any definitive conclusion¹¹. Concerning the $X(1600)$, being its experimental mass 1600 ± 100 MeV, a possible tetraquark configuration seems likely.

The program we have started for an exact study of multiquark systems by means of the HH formalism will be accomplished by implementing the possibility of treating quarks of different masses. When this is done we will have at our disposal a powerful method, imported from the nuclear physics, to study in an exact way systems made of any number of quarks and antiquarks coupled to a color singlet.

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