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Conservation laws for the voter model in complex networks

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Abstract. – We consider the voter model dynamics in random networks with an arbitrary distribution of the degree of the nodes. We find that for the usual node-update dynamics the average magnetization is not conserved, while an average magnetization weighted by the degree of the node is conserved. However, for a link-update dynamics the average magnetization is still conserved. For the particular case of a Barabási-Albert scale-free network, the voter model dynamics leads to a partially ordered metastable state with a finite-size survival time. This characteristic time scales linearly with system size only when the updating rule respects the conservation law of the average magnetization. This scaling identifies a universal or generic property of the voter model dynamics associated with the conservation law of the magnetization.

Introduction. – Conservation laws play an important role in the characterization and classification of different nonequilibrium processes of ordering dynamics. For example, in Kinetic Ising models one distinguishes between Glauber and Kawasaki dynamics. In Glauber dynamics the individual dynamical step is that of flipping a spin, while in Kawasaki dynamics two nearest-neighbor spins are exchanged. Kawasaki dynamics conserves magnetization and Glauber dynamics does not. As a consequence, the Glauber and Kawasaki dynamics give rise to different scaling laws for domain growth in coarsening processes [1], and they define different nonequilibrium universality classes. The conservation law of Kawasaki dynamics is implemented at each time step of a stochastic dynamics. A different type of conservation law is the one that refers to an ensemble average. An example of such conservation laws is the conservation of the average spin (global magnetization) in the voter model [2, 3]. When studying spin dynamical models in regular lattices, the existence of an ensemble conservation law does not imply an elementary step conservation such as imposed in the Kawasaki dynamics. Recent interest in ordering processes focuses on situations in which the spins are located in the nodes of a complex network, *i.e.* a network with a large heterogeneity in the number of nearest neighbors with which each spin interacts [4]. This does not affect the fulfillment of a conservation law of the type of the Kawasaki dynamics, but the implementation of an

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ensemble average conservation law requires a careful thought of the dynamical rules. As an interesting example, we discuss in this paper the differences between *node* and *link-update* dynamics for a voter model in which spins are located in the nodes of a random network, with an arbitrary degree distribution. Only link-update dynamics respects the global magnetization conservation law, while another conservation law exists for node-update dynamics.

The standard voter model [2] is defined by a set of "voters" with two opinions (spins $\sigma_i = \pm 1$) located in the nodes of a hypercubic lattice. The elementary dynamical step under *node-update* dynamics consists in randomly choosing one node (asynchronous update) and assigning to it the opinion of one of its nearest neighbors, also chosen at random. One time step corresponds to updating a number of nodes equal to the system size, so that each node is on average updated once. In $d = 1$, this dynamics is equivalent to the zero-temperature Glauber kinetic Ising model. In general dimensionality, the global magnetization is conserved in the thermodynamic limit of large systems and the dynamics is dominated by interfacial noise. The infinite system coarsens for $d \leq 2$, with a slow logarithmic decay in the critical dimension $d = 2$. At variance with other ordering dynamics, coarsening takes place here without surface tension and it is driven by interfacial noise as discussed in [5]. The role of the conservation law of the magnetization and of the Z_2 symmetry (± 1 states) in the voter dynamics universality class has been studied in detail in the critical dimension $d = 2$ of regular lattices [5]. We are here interested in situations in which there is no long-time coarsening, as it occurs in regular lattices for $d > 2$ in which a finite system reaches one of the homogeneous attractors in which all the N spins have the same value. The time to reach such consensus τ, or survival time, scales as $\tau \propto N$, so that there is no complete ordering in the thermodynamic limit [7]. This same scaling behavior has also been found for the voter model in a small-world network [8,9] so that it can be identified as a generic property of the voter model dynamics. We show in this paper that for a Barabási-Albert (BA) scale-free network [6] the scaling law $\tau \propto N$ is only obtained when the average magnetization is conserved, that is when link-update dynamics is used.

Node vs. link-update in the voter model. – Generically, in a complex network such as the small-world network, there is a heterogeneous degree distribution with nodes having a different number of links. In this case, as explained in more detail below, node-update dynamics does not guarantee the conservation of the average magnetization. The conservation is guaranteed in a *link-update* dynamics in which the elementary dynamical step consists in randomly choosing a pair of nearest-neighbor spins, *i.e.* a link, and randomly assigning to both nearest-neighbor spins the same opinion when they had different opinion, and leaving them unchanged otherwise. The reported simulations of the voter model in a small-world network seem to use a node-update [8], while the analytical results are obtained in an approximation which enforces conservation of the average magnetization [9]. Since in both cases the scaling law $\tau \propto N$ is obtained, the role of the conservation law in this generic property is unclear. Still, the degree heterogeneity in a small-world network is rather small, and therefore its effect on the conservation law for node-update is probably not significant. The question is much more crucial when considering, for example, a scale-free network which has nodes with large degree heterogeneity, as we do next.

Conservation laws. – Results for the ensemble average normalized magnetization,

$$\langle \sigma(t) \rangle = \frac{\langle \sum_{i=1}^{N} \sigma_i(t) \rangle}{N}, \tag{1}$$

of a voter model dynamics in a BA network [6] are shown in fig. 1 for node-update and link-update dynamics. The ensemble average indicated as $\langle \cdot \rangle$ is an average over the realizations

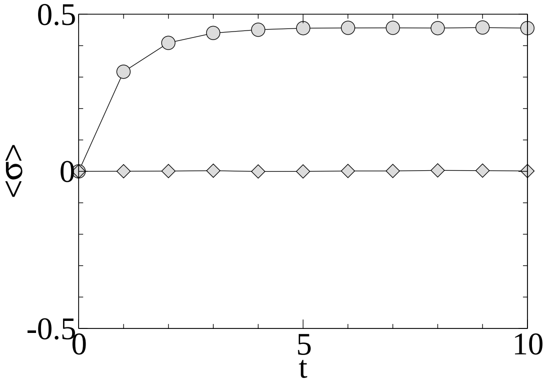

Fig. 1 – Ensemble average magnetization $\langle\sigma\rangle$ in a BA network. The most connected half of the nodes in the network have initial spin $\sigma_i = +1$, while the other half has initial spin $\sigma_i = -1$. System size $N = 1000$, average degree $\overline{k} = 8$, and averaged over 1000 realizations. Circles: node-update; diamonds: link-update.

of the stochastic dynamics and different initial conditions. For both updating rules we have considered the same initial distribution of spins in the initial configuration, with half of the nodes with spin $+1$, and half with spin -1. To provide clearer evidence of the dynamical differences of the two update rules we have chosen an initial configuration in which all the nodes that were given initial spin $+1$ have a higher degree than those that were given initial spin -1. For the node-update dynamics we find that the average spin is not conserved, but rapidly changes towards the positive side, due to the influence of the high-degree nodes in their immediate neighborhood. For the link-update dynamics the average magnetization remains at its vanishing initial value.

For any complex network with a heterogeneous distribution of the degree of the nodes, the differences between the two updating rules are easily understood as follows. The conservation of the average magnetization in a regular lattice relies on the fact that, if nodes i and j have different opinions and are connected, the probability for j to change to the spin of i in a time step of the dynamics is the same as for i to change to the spin of j. Since this is true for all nodes, the ensemble average magnetization is conserved. However, in a network where the nodes have not the same degree, this is no longer true for node-update dynamics. For instance, if i is a highly connected node with degree k_i, and j is a node with a low degree $k_j < k_i$, and they have different spins, then the probability P_{ij} that i changes to the spin of j, $P_{ij} = (Nk_i)^{-1}$, is smaller than the probability P_{ji} that j changes to the spin of i, $P_{ji} = (Nk_j)^{-1}$). This explains the numerical finding in fig. 1 that the average magnetization is not conserved. Choosing the node to be updated preferentially, so that $P_{ji} = P_{ij}$, makes the average spin conserved again. Preferentially choosing the node to be updated in this way is equivalent to randomly choosing a link in the network and updating it in random direction (link-update).

As we have argued, the ensemble average normalized magnetization $\langle\sigma\rangle$ is not conserved in a complex network in the voter model with node-update rule. In order to compensate for the different degrees of the nodes, we consider a degree-weighted normalized magnetization [10,11]

$$\Sigma(t) = \frac{\sum_{i=1}^{N} k_i \sigma_i(t)}{\sum_{i=1}^{N} k_i}\,, \tag{2}$$

where k_i is the degree of the node i. The total number of links is introduced in the denominator to normalize the weighted magnetization between $[-1, 1]$.

If we define $S(t) = \sum_{i=1}^{N} k_i \sigma_i(k, t)$, then for a given configuration and averaging over stochastic realizations, the expected change $\langle \Delta S_{ij} \rangle_c$ in a time step of the dynamics due to node i changing spin to the spin of node j is given by

$$\langle \Delta S_{ij} \rangle_c = \frac{(\sigma_j(t) - \sigma_i(t))}{N k_i} k_i \,, \tag{3}$$

while

$$\langle \Delta S_{ji} \rangle_c = \frac{(\sigma_i(t) - \sigma_j(t))}{N k_j} k_j \,, \tag{4}$$

where $\langle \cdot \rangle_c$ represents an average over stochastic realizations for a given configuration c at time t, and the denominators take into account the probability to select node i (or j). Thus the expected change in S due to changes on the two nodes i and j is $\langle \Delta S_{ij} + \Delta S_{ji} \rangle_c = 0$. As this argument applies for any pair of neighbors, and any configuration, *the ensemble average weighted magnetization* $\langle \Sigma \rangle$ *is conserved for the voter model using node-update in complex networks with arbitrary degree distributions.*

Given the conservation law of $\langle \Sigma \rangle$, we can find the asymptotic value of the average magnetization. To this end, we introduce the normalized magnetization of the nodes with given degree k as

$$\sigma(k, t) = \frac{\sum_{i:k_i=k} \sigma_i(t)}{N_k} \,, \tag{5}$$

where N_k is the number of nodes with degree k and the sum in the numerator is over all nodes with the same degree k.

For a given configuration c, the expected change of spin of a node i with degree k_i due to interaction with its neighbors in a time step of the dynamics is given by

$$\langle \Delta \sigma_i(t) \rangle_c = \sum_{j \in \mathcal{V}_i} \frac{\sigma_j(t) - \sigma_i(t)}{k_i} \,, \tag{6}$$

where $\mathcal{V}_i$ is the neighborhood of node i, that is the nodes connected by a link to node i. From this expression, if we add for all the nodes with the same degree, we obtain

$$\langle \Delta \sigma(k, t) \rangle_c = \sum_{i:k_i=k} \sum_{j \in \mathcal{V}_i} \frac{\sigma_j(t) - \sigma_i(t)}{N_k k_i} \,. \tag{7}$$

We can now split the r.h.s of eq. (7) into two terms. The second term is simply $\sigma(k, t)$. For the first term, we assume a mean-field approximation, *i.e.*, we consider a random network where the sum over neighbors is equivalent to a random sampling over the whole network. Then

$$\sum_{i:k_i=k} \sum_{j \in \mathcal{V}_i} \frac{\sigma_j(t)}{N_k k_i} = \frac{\sum_k P(k) k \sigma(k, t)}{\sum_k P(k) k} = \frac{\sum_{i=1,N} k_i \sigma_i(t)}{\sum_{i=1,N} k_i} = \Sigma(t), \tag{8}$$

where $P(k) = N_k/N$ is the degree distribution of the network, that is, the probability to find a node of degree k.

Thus, $\langle \Delta \sigma(k, t) \rangle_c = \Sigma - \sigma(k, t)$, where $\Sigma(t)$ and $\sigma(k, t)$ are calculated in the given configuration c. Averaging over different configurations, we find the evolution equation of the ensemble average of $\sigma(k, t)$(1):

$$\frac{\mathrm{d} \langle \sigma(k, t) \rangle}{\mathrm{d}t} = \langle \Sigma \rangle - \langle \sigma(k, t) \rangle \,. \tag{9}$$

(1)An independent derivation for random networks has been obtained recently in ref. [12].

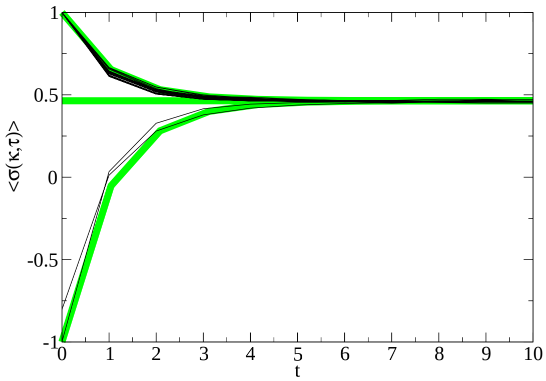

Fig. 2 – Time evolution of $\langle\sigma(k,t)\rangle$ for node-update dynamics in a BA network with $\overline{k}=8$. Initial configuration as in fig. 1. System size $N = 10000$, average degree $\overline{k}=8$ and average taken over 10000 realizations. Thin lines correspond to numerical data; thick lines correspond to the analytical predictions (eq. (10)) for the initial values $\sigma(k,0)=-1$ and $+1$.

Given the conservation law for $\langle\Sigma\rangle$, the solution of eq. (9) is an exponential approach to the asymptotic value

$$\langle\sigma(k,t)\rangle = (\langle\sigma(k,0)\rangle - \langle\Sigma\rangle)e^{-t} + \langle\Sigma\rangle\,. \tag{10}$$

In the long-time limit $\langle\sigma(k,t)\rangle$ approaches a constant value $\langle\Sigma\rangle$ independent of k. Therefore, this constant value coincides with the long-time limit of the ensemble average normalized magnetization $\langle\sigma(t\to\infty)\rangle$. This value is a property of the ensemble of initial configurations.

A numerical check of these general results for the particular case of the BA network is shown in fig. 2, where the fast exponential decay to the final average value is shown. The asymptotic value corresponds to the analytical prediction of $\langle\Sigma\rangle$ calculated in the initial configurations, that is, the ensemble of initial configurations.

Survival times in Barabási-Albert scale-free networks. – We next study the consequences of the two different updating rules and associated conservation laws in the ordering dynamics of the voter model in a BA network. For any of the two updating rules the system falls, after an initial transient, in a metastable partially ordered state until a finite system size fluctuation takes the system to one of the two ordered attractors of the dynamics. This qualitative behavior is similar to the one found in a small-world network [8] and likewise can be characterized in terms of the temporal evolution of the average interface density ρ, defined as the density of links connecting sites with different opinions. In a given realization of the dynamics, ρ initially decreases indicating a partial ordering of the system. After this initial transient, ρ fluctuates randomly around an average value until a fluctuation orders the system leading to an absorbing state with $\rho = 0$. Considering an ensemble of realizations, the ordering of each of them happens randomly with a constant rate. This is reflected in an exponential decay of the average interface density

$$\langle\rho\rangle \propto e^{-\frac{t}{\tau}}, \tag{11}$$

where τ is the survival time of the partially ordered metastable state. This survival time turns out to be a quantity that diverges with growing system size N, so that the system does not order in the thermodynamic limit.

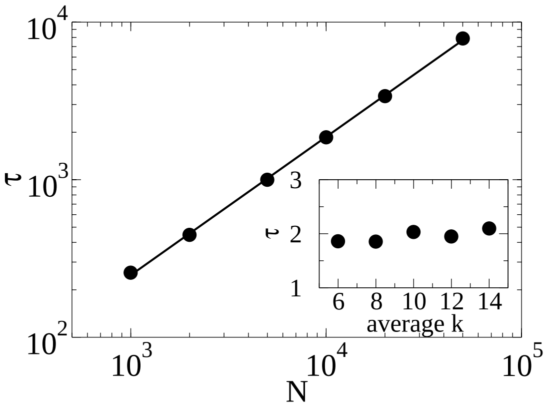

Fig. 3 – Survival time for node-update dynamics in BA networks of different sizes N, and average degree $\overline{k} = 6$. Inset: survival times for different average degree. All data obtained from at least 1000 realizations of networks of size 10000.

We have measured the characteristic time τ for the two updating dynamical rules and for different system sizes N and different mean degree $\overline{k}$. Our results are summarized in figs. 3 and 4. For the node-update rule (fig. 3), in which there is no conservation law of the average magnetization, we have found that τ scales with system size N as

$$\tau \propto N^{\gamma}, \tag{12}$$

where $\gamma = 0.88 \pm 0.01$. For a fixed system size, the value of the average degree $\overline{k}$ does not seem to have any definite influence on τ (see the inset of fig. 3). The value of the exponent γ is consistently and significantly different from $\gamma = 1$ which is the exponent analytically found for regular hypercubic lattices and for an annealed small-world network [9]. In these last two cases the dynamical rules respect the conservation of the average magnetization. When we implement the link-update rule which also conserves the average magnetization, we find a result for γ which is consistent with $\gamma = 1$ (fig. 4).

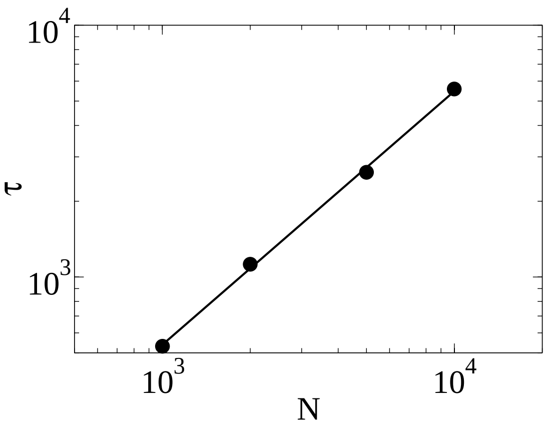

Fig. 4 – Survival time for link-update dynamics in BA networks of different sizes N, and average degree $\overline{k} = 8$. All data obtained from 1000 realizations.

Conclusions. – In summary, we have shown that the voter model dynamics does not lead to an ordered state in a scale-free network in the thermodynamic limit. This is consistent with the results for a small-world network, and in general for networks of dimensionality $d > 2$. Finite-size effects order the system in a time which depends on the updating dynamical rule. Only for the updating rule fulfilling a conservation of the global magnetization does this time scale linearly with the system size. This is also consistent with the result for regular hypercubic lattices of $d > 2$ and for the voter model in annealed small-world networks [9]. Such scaling can then be taken as a proper characterization of universal properties of the dynamics of the voter model.

We note that there are several instances in which the conservation law of the global magnetization is naturally broken independently of the updating rule, as, for example, the consideration of a zealot [13], or the dynamics in a directed network. On the other hand, there are spreading phenomena with "spins" having $N > 2$ states which show voter-like generic properties in $d = 2$ regular lattices and that fulfill the global magnetization conservation law [14]. Further study of these other cases and the implications of other conservation laws, as the one reported here for node-update dynamics, would be useful to identify generic and nongeneric properties of the voter model dynamics.

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REFERENCES

[1] Gunton J. D., San Miguel M. and Sahni P. S., *Phase Transitions and Critical Phenomena*, edited by Domb C. and Lebowitz J., Vol. **8** (Academic Press, London) 1983, pp. 269-466.
[2] Liggett T. M., *Interacting Particle Systems* (Springer, New York) 1985.
[3] Marro J. and Dickman R., *Nonequilirium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge) 1999.
[4] Albert R. and Barabási A.-L., *Rev. Mod. Phys.*, **74** (2002) 47.
[5] Dornic I., Chate H., Chave J. and Hinrichsen H., *Phys. Rev. Lett.*, **87** (2001) 045701.
[6] Barabási A.-L. and Albert R., *Science*, **286** (1999) 509.
[7] Frachenbourg L. and Krapivsky P. L., *Phys. Rev. E*, **53** (1996) 3009.
[8] Castellano C., Vilone D. and Vespignani A., *Europhys. Lett.*, **63** (2003) 153.
[9] Vilone D. and Castellano C., *Phys. Rev. E*, **69** (2004) 016109.
[10] Bianconi C., *Phys. Lett. A*, **313** (2002) 166.
[11] Aleksiejuk A., Holyst J. A. and Stauffer D., *Physica A*, **310** (2002) 260.
[12] Wu F. and Huberman B. A., cond-mat 0407252 (2004).
[13] Mobilia M., *Phys. Rev. Lett.*, **91** (2003) 028701.
[14] Ravasz M., Szabo G. and Szolnoki A., *Phys. Rev. E*, **70** (2004) 012901.