Synchronization in Complex Networks: a Comment on two recent PRL papers

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I show that the conclusions of [Hwang, Chavez, Amann, & Boccaletti, PRL **94**, 138701 (2005); Chavez, Hwang, Amann, Hentschel, & Boccaletti, PRL **94**, 218701 (2005)] are closely related to those of previous publications.

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Recent interest on dynamical networks regards the synchronizability properties of networks with coupled identical oscillators [1]. According to the standard formalism in the study of synchronization in a class of coupled identical oscillators [1], the eigenratio $R = \lambda_N/\lambda_2$, defined from the eigenvalues of the laplacian matrix is a measure of the synchronizability of the network, and has to be smaller than a certain dynamical ratio corresponding to the oscillators. In [2] it was shown that networks that are heterogeneous in the degree distribution (e.g. scale-free networks (SFNs)) are more difficult to synchronize. In Ref. [3] it was shown that synchronization can be enhanced in SFNs by constructing a weighted network.

More precisely, the idea is weight all the oscillators in the network such that the total strength of input connections is the same for all the oscillators [3]. Thus, in Ref. [3] the strength of connections is weighted such that $w_{i\to j} = 1/k_j^\beta$ (and analogously for $w_{j\to i}$). It was proven [3] that synchronizability is maximum for $\beta = 1$, and for this case it follows immediately that the total strength of input connections $\sum_j w_{j\to i}$ is the same for any node i of the network, while for output connections $\sum_i w_{j\to i}$ the distribution of strengths is identical to the degree distribution (cf. comment reference [30] in [4]). On the other hand, because of the asymmetry $w_{i\to j} = 1/k_j^\beta < w_{j\to i} = 1/k_i^\beta$ when $k_j > k_i$ and $\beta > 0$, then it is clear that the strength of the output connections is positively correlated with the degree of the node, implying that when synchronizability is improved the dominant coupling direction is from high-degree nodes to low-degree ones.

In particular, in the 1024-node random SFN [5] considered in [3] it is shown, Fig. 2, that a *ten-fold*, i.e. 1000% improvement in synchronizability is attained, compared to the unweighted case considered in Ref. [2]. Moreover, as the number of oscillators N grows, this synchronization enhancement improves even further (check the ratio for weighted networks, Eq. [4] of Ref. [3], with the unweighted case, Eq. 2 in Ref. [2]). This improvement in synchronizability is shown to be significant when the total strength of input connections is equal for all the oscillators in the network.

In more recent work, Ref. [6], the same idea has been suggested: a weighting scheme that lowers the eigenratio R by lowering the connection strength of the most highly connected nodes, while satisfying also the condition uncovered in Ref. [3] that the total strength of input connections is the same for all the oscillators in the network. Relaxing constraints on the individual connections, they were able to improve the synchronizability uncovered by Motter *et al.* by a factor of upto 1.2 [8], i.e., 20 % (cf. Fig.2 in Ref. [6]). On the other hand, the dependence on the system size N of the obtained improvement is difficult to ascertain.

In closely connected work, Ref. [7], a more refined version of the weighted and directed coupling of [6] has been introduced. The main conclusion of Ref. [7] is that *propensity for synchronization is enhanced in networks of asymmetrically coupled units* and that in growing SFNs *such enhancement is particularly evident when the dominant coupling direction is from older to younger nodes.* But for the kind of growing SFNs considered in Ref. [7] older nodes have a larger degree (more connections)[9]. So, this means that the concepts of *age* and *number of connections* (or degree) are actually equivalent, and, thus, the improvements in the *propensity for synchronization* and *synchronizability* in Refs. [7] and [3, 4] are, thus, intimately related.

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[8] Notice, Eq. 2 in Ref. [6], that the normalization implied by the denominator makes this scheme very close indeed to the case $\beta = 1$ of Ref. [3]
[9] This was shown analytically in the very first paper about scale-free networks [5]: $k_i(t) = m(t/t_i)^{1/2}$