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Magnons and BFKL

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Abstract

We extract from the double logarithmic contributions to DGLAP anomalous dimensions for twist-two operators up to three-loops the magnon dispersion relation for planar $\mathcal{N}=4$ supersymmetric Yang-Mills. Perturbatively the magnon dispersion relation agrees with the expansion of the anomalous dimension for spin-one as well as with the non-collinear double logarithmic contributions to the BFKL anomalous dimensions analytically extended to negative spin. The all-loop expression for the magnon dispersion relation is determined by the double logarithmic resummation of the corresponding Bethe-Salpeter equation. A potential map relating the spin chain magnon to BFKL eigenfunctions in the double logarithm approximation is suggested.

1 Introduction

Integrable structures in a four-dimensional quantum field theory were first shown to arise in the Regge limit of scattering amplitudes in the planar limit of QCD [1, 2]. In the leading logarithm approximation the reggeized scattering amplitudes are described by a non-compact Heisenberg magnet with $SL(2)$ symmetry group. Integrability survives as the amount of symmetry is increased, because supersymmetric extensions of QCD share the same non-compact sector of operators with covariant derivatives. In fact integrability extends to larger sectors of the gauge theory [3], up to the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills, which is completely integrable at one-loop [4, 5]. There is much evidence that integrability holds beyond one-loop in $\mathcal{N} = 4$, and a long-range Bethe ansatz has in fact been suggested to govern the spectrum of anomalous dimensions of local gauge invariant composite operators to all order [6]. The proposal for a Bethe ansatz only applies to asymptotically long single trace operators, and does not cover wrapping interactions, present beyond a certain order for finite-size operators. For the non-compact $SL(2)$ sector of the $\mathcal{N} = 4$ theory, containing twist-two operators of the form

$$\mathrm{Tr}(D^{s_1}\Phi D^{s_2}\Phi) \ , \tag{1.1}$$

with $s_1+s_2 = N$ the total spin, the expansion of the asymptotic Bethe ansatz (ABA) equations completely agrees with the perturbative computation of the three-loop anomalous dimension of twist-two operators [7]. However for twist-two operators wrapping effects are already present beyond third loop, and the ABA fails to reproduce the four-loop prediction for the anomalous dimension obtained from the BFKL pomeron [8]. The pomeron singularity corresponds to the analytic continuation of the spin to $N = -1$. The purpose of this letter is to explore the $N = 1$ case, which in the spin chain picture amounts to a single magnon excitation. The note is organized as follows. In Section 2 the anomalous dimension for twist-two operators with spin-one is shown to agree, up to three loops, with the perturbative expansion of the dispersion relation for planar $\mathcal{N} = 4$ supersymmetric Yang-Mills. In Section 3 the contribution of double logarithms to the analytic extension to negative spin of the anomalous dimension is shown to correspond to the anomalous dimension at $N = 1$, and we conjecture an interpretation for the spin chain magnon in the BFKL picture. We conclude in Section 4 with some discussion on our results.

2 The single magnon anomalous dimension

In deep inelastic scattering (DIS) processes anomalous dimensions for twist-two operators control the renormalization group behaviour of parton distribution functions under changes of the photon resolution. Let us denote by $F_a(x,\, Q^2)$ the number of partons of type a, with transversal momentum k^2 smaller or equal to Q^2 and with a fraction x of the longitudinal momentum of the nucleon. The meaning of Q^2 in DIS is the virtuality of the photon, $Q^2 = -q^2$, and $x = Q^2/s$ is the Bjorken variable describing the rapidity gap between the photon and the nucleon. Denoting by $F_a(N,\, Q^2)$ the Mellin transform,

$$F_a(N,\, Q^2) \equiv \int_0^1 dx\, x^{N-1} F_a(x,\, Q^2)\ , \tag{2.1}$$

the DGLAP renormalization group equation is given by [9]

$$\frac{\partial F_a(N,\, Q^2)}{\partial \log Q^2} = \gamma_{a,b}(N) F_b(N,\, Q^2)\ , \tag{2.2}$$

where $\gamma_{a,b}(N)$ is the DGLAP anomalous dimension matrix, which coincides with the anomalous dimension of a twist-two operator. Since we will only be interested in scalar twist-two operators in this note we will write $\gamma_{\phi,\phi}(N) \equiv \gamma_2(N)$ for brevity. Using conventions such that

$$g^2 = \frac{\lambda}{8\pi^2}\ , \tag{2.3}$$

where $\lambda \equiv g^2_{\rm YM} N$ is the 't Hooft coupling constant, the anomalous dimension

$$\gamma_L(N) = \sum_{n=1}^{\infty} \gamma_{L,n}(N) g^{2n} \tag{2.4}$$

is given, up to three-loops, by [7]

$$\gamma_{2,1}(N) \ = \ 4\, S_1\ , \tag{2.5}$$

$$\gamma_{2,2}(N) \ = \ -4\left(S_3 + S_{-3} - 2S_{-2,1} + 2S_1(S_2 + S_{-2})\right)\ , \tag{2.6}$$

$$\begin{aligned}\gamma_{2,3}(N) \ = \ & -8\Big(2S_{-3}S_2 - S_5 - 2S_{-2}S_3 - 3S_{-5} + 24S_{-2,1,1,1} + 6(S_{-4,1} + S_{-3,2} + S_{-2,3}) \\ & -12(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) - (S_2 + 2S_1^2)(3S_{-3} + S_3 - 2S_{-2,1}) \\ & -S_1(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2 + 3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1})\Big)\ ,\end{aligned} \tag{2.7}$$

where the harmonic sums are defined through

$$S_a \equiv S_a(N) = \sum_{j=1}^{N} \frac{(\mathrm{sgn}(a))^j}{j^a} \ , \tag{2.8}$$

$$S_{a_1,\ldots,a_n} \equiv S_{a_1,\ldots,a_n}(N) = \sum_{j=1}^{N} \frac{(\mathrm{sgn}(a_1))^j}{j_1^a} S_{a_2,\ldots,a_n}(j) \ . \tag{2.9}$$

The anomalous dimension for these twist-two operators can also be obtained as the energy for the proposed long-range $SL(2)$ integrable spin chain with N magnon excitations through [6]

$$\gamma_2(N) = \sum_{i}^{N} E(p_i) \ , \tag{2.10}$$

where the dispersion relation is

$$E(p_i) = \sqrt{1 + 8g^2 \sin^2\left(\frac{p_i}{2}\right)} - 1 \ , \tag{2.11}$$

with $\{p_i\}$ the set of magnon momenta solving the Bethe ansatz equations. At one-loop this spin chain reduces to the length-two $SL(2)$ $\mathrm{XXX}_{s=-1/2}$ Heisenberg chain [5], and the energy for N magnons can be exactly obtained by solving the corresponding Baxter equation (see for instance [10]). The anomalous dimension obtained for twist-two scalar operators from the ABA coincides with equations (2.5)-(2.8). It also provides a four-loop term [8]

$$\gamma_{2,4}(N) = 16\left(4S_{-7} + 6S_7 + \ldots - \zeta(3)S_1(S_3 - S_{-3} + 2S_{-2,1})\right) \ , \tag{2.12}$$

where the entire expression is presented in table 1 of reference [8].

In this note we are concerned with the value of $\gamma_2(N)$ at $N = 1$. In QCD, $\gamma_{a,b}(1)$ is an interesting quantity, because it provides the normalisation of the splitting functions. In $\mathcal{N} = 4$ Yang-Mills, however, one does not have such an interpretation in terms of splitting functions. Instead, if one were forced to prescribe a value for $\gamma(1)$, the natural choice, obtained from the spin chain picture, would be $\gamma(1) = 0$. The reason is that the translation invariance imposed by the trace on gauge operators implies vanishing momentum on states of the corresponding spin chain. Therefore a single magnon state could only have zero momentum, and therefore zero energy. However, the true value of $\gamma(1)$, as given by the expansions of the anomalous dimension in terms of harmonic sums, turns out to be rather surprising. Plugging $N = 1$ into equations (2.5)-(2.8) and (2.12) gives

$$\gamma_2(1) = 4g^2 - 8g^4 + 32g^6 - 160g^8 + \mathcal{O}(g^{10}) \ . \tag{2.13}$$

This is precisely what is obtained if one expands the dispersion relation (2.11) for a magnon of momentum $p = \pi$. It would thus seem that $\gamma_2(1)$ does not provide the energy of a physical, zero-momentum magnon, but rather of some sort of "non-physical" $p = \pi$ magnon. Extrapolating to all-loops we may conjecture that

$$\gamma_2(1) = E\left(p = \pi\right) \ , \tag{2.14}$$

where $E(p)$ is given by (2.11). For later use, let us write the expansion coefficients of $\gamma_2(1)$, at weak-coupling as $e(i)$. The conjecture thus simply states that $E(p = \pi) = \sum_i e(i) g^{2i}$.

2.1 Twist-L and analytical continuations

Considering now that $p = \pi$ is the smallest non-zero momentum that a magnon can have on a chain of length $L = 2$ it is tempting to speculate that a general expression for the anomalous dimension of twist-L operators at $N = 1$ could be

$$\gamma_L(1) = E\left(p = \frac{2\pi}{L}\right) \ . \tag{2.15}$$

At a first glance it would however seem that the above conjecture for arbitraty twist-L fails for twist-three. In [8, 11] the twist-three anomalous dimensions up to four-loops are given in terms of harmonic sums. These expressions, as opposed to the twist-two formulae, have two distinctive features. Firstly, the harmonic sums only have positive indeces, and secondly they are evaluated at $N/2$. Naively this last property would invalidate the conjecture. For example, the one-loop expression is

$$\gamma_{3,1}(N) = 4\, S_1\left(\frac{N}{2}\right) \ , \tag{2.16}$$

which gives $\gamma_{3,1}(1) = 8(1 - \log 2)$, in obvious conflict with (2.15). However, as mentioned in [8, 11], the twist-three expressions have been derived for physical, even values of N, and do not therefore need to be valid for unphysical, odd values of N. In fact, there is an important subtlety in the evaluation of $\gamma_L(N)$ at unphysical values of N related to the two different prescriptions that exist in QCD for analytically continuing the harmonic sums entering the expansions of $\gamma_L(N)$ to generic values of the Mellin moment N. As discussed in [12], there is a unique way to analytically continue sums with positive indeces. Sums with a negative index, however, such as $S_{-a,b,\ldots}$, have, due to their definition as an alternating series, a $(-1)^N$ factor. The oscillatory nature of this factor would, after analytical continuation, make the sums explode exponentially along the imaginary N axis, and the inverse Mellin transforms would thus be ill-defined. Instead, if one chooses to analytically continue solely from even (or odd) values of N, the $(-1)^N$ factor can be set to a constant $+1$ (or -1), and well-behaved analytical continuations are obtained. The harmonic sums obtained by continuing from even N are denoted $S^{(+)}$ (together with the corresponding indeces) and the sums obtained from negative values of N are written $S^{(-)}$. It should be stressed that $S^{(+)}$ (respectively $S^{(-)}$) give incorrect values for odd (even) integer N. The two prescriptions then define two analytic expressions for the anomalous dimensions, $\gamma^{(+)}(N)$ and $\gamma^{(-)}(N)$.

In QCD, both the positive and negative expressions are present, in the form of the singlet and non-singlet anomalous dimensions (see for instance [13] for a recent discussion). In $\mathcal{N}=4$ supersymmetric Yang-Mills, however, physical states always correspond to even moments, and the $(+)$ prescription is therefore singled out. For example, in [8] it was the $(+)$ prescription that was used to analytically continue the twist-two anomalous dimension obtained from the ABA to $N=-1$, where its singular behaviour could be compared to the predictions from BFKL on the leading singularity. At a pole, the singular behaviour of the two prescriptions, differ in sign. For example, near $\omega \to 0$,

$$S^{(+)}_{-a}(N+\omega) \sim \frac{(-1)^{N+1}}{\omega^a}\ , \quad N=-1,\,-2,\,\ldots\ , \tag{2.17}$$

while

$$S^{(-)}_{-a}(N+\omega) \sim \frac{(-1)^{N}}{\omega^a}\ , \quad N=-1,\,-2,\,\ldots \tag{2.18}$$

The twist-three expressions show no oscillatory behaviour, and since they are extracted for even N, they may well be giving only the $(+)$ analytic continuation. It could then be argued that the twist-three dimension is written in terms of sums with positive indeces, for which the two prescriptions give the same result. However, the formulae in terms of positive indices could very well be an effective description only valid for even N. In contrast, in order to obtain the correct dispersion relation for twist-two, the $(-)$ prescription has to be used, since we are evaluating the harmonic sums at an odd value of N. We believe that in general, it is the $(-)$ prescription that should be used to test (2.15).

3 DGLAP and BFKL

The Regge limit of high energy QCD corresponds to the scattering of two hadrons with the center of mass energy s much larger that the typical transverse scales, Q^2 and Q'^2.

When Q^2 and Q'^2 are much larger than the QCD scale we can work in perturbation theory. In DIS $Q^2 \gg Q'^2$, with Q^2 the virtuality of the photon and Q'^2 the transversal scale of the target hadron. In this limit, the leading contribution to the evolution in Q^2 of the unintegrated parton distribution function $f(x,\, Q^2)$, which is related to the integrated parton distribution function $F(x,\, Q^2)$ through

$$F(x,\, Q^2) = \int dk^2 f(x,\, k^2)\Theta(Q^2 - k^2) \ , \tag{3.1}$$

is determined by the Bethe-Salpeter integral equation

$$f(x,\, Q^2) = f_0(x,\, Q^2) + 2g^2 \int_x^1 \frac{dz}{z} \int_{Q'^2}^{Q^2} \frac{dk^2}{Q^2} f\left(\frac{x}{z},\, k^2\right) \ , \tag{3.2}$$

shown pictorially in figure 1. The Bethe-Salpeter equation takes this form for a kinematic region where we have, not only $x \ll 1$ and $z \ll 1$ corresponding to the Regge limit, but strict ordering in the longitudinal momenta, $z \gg x$, and also an ordering $Q^2 \gg k^2$ along the transversal momenta. For example, the assumption $z \ll 1$ implies that only the $1/x$ part of the gluon splitting function will be relevant, giving the $1/z$ factor in the integration kernel.

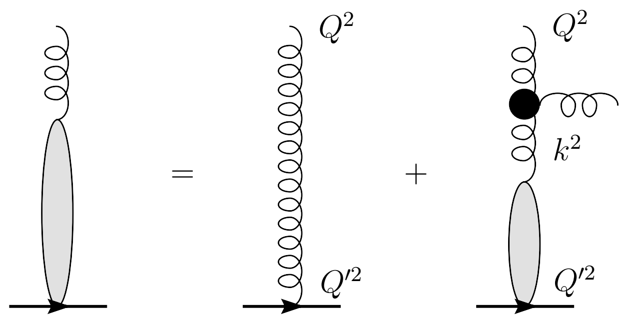

Figure 1: The Bethe-Salpeter equation, performing the resummation of the logarithmic contributions to the evolution of the parton distribution functions in Q^2.

Iterating the integral equation produces a sequence of ladder diagrams where the ordering in the transverse momenta leads to logarithms in the energy, $\log\left(\frac{s}{Q^2}\right) = \log\left(\frac{1}{x}\right)$, while the strict ordering of the transversal momenta produces the logarithmic collinear enhancement factors $\log\left(\frac{Q^2}{Q'^2}\right)$. Thus the previous Bethe-Salpeter integral equation is performing the perturbative resummation of double logarithms of the form

$$\left(g^2 \log\left(\frac{1}{x}\right) \log\left(\frac{Q^2}{Q'^2}\right)\right)^n \ . \tag{3.3}$$

Taking the Mellin transform with respect to both x and Q^2, [1]

$$f(x,\,Q^2) = \int \frac{d\omega}{2\pi i} x^{-\omega} \int \frac{d\gamma}{2\pi i} \frac{1}{Q^2} \left(\frac{Q^2}{Q'^2}\right)^{-\gamma/2} f(\omega,\,\gamma)\ , \tag{3.4}$$

the solution to the Bethe-Salpeter integral equation is given by

$$f(\omega,\,\gamma) = \frac{\omega f_0(\omega,\gamma)}{\omega + 4g^2 \frac{1}{\gamma}}\ , \tag{3.5}$$

which leads to

$$f(x,\,Q^2) \sim \exp\left(\sqrt{8g^2 \log\left(\frac{1}{x}\right) \log\left(\frac{Q^2}{Q'^2}\right)}\right)\ , \tag{3.6}$$

corresponding to the resummation of the double logs (3.3). In this collinear limit the DGLAP kernel in Mellin space is simply given by $-2/\gamma$.

In contrast with the evolution in Q^2 that DGLAP gives, the BFKL equation provides us, in its domain of validity, with the behaviour of unintegrated parton distribution functions under changes of x [14]. The kinematical regime where BFKL is defined corresponds to scattering of two hadronic objects with transversal scales of the same order. In these conditions we cannot impose strict ordering on the transversal momenta in the ladder diagrams, and we have resummations of single logarithms of type $\left(g^2 \log\left(\frac{1}{x}\right)\right)^n$. To a large extent, however, the full leading logarithmic (LLA) BFKL solution is reproduced by requiring that it gives the DIS $-2/\gamma$ pole in the limit $\gamma \to 0$, and by imposing symmetry under the exchange of the scales Q and Q'. From (3.4) we see that, for fixed x, this corresponds to requiring invariance under $-2/\gamma \to (1+\gamma/2)$, which gives the pole

$$\frac{1}{\omega - 2g^2\left(-\frac{2}{\gamma} + \frac{1}{1+\gamma/2}\right)}\ . \tag{3.7}$$

A complete analysis corrects the equation slightly in the region in between the two poles, and implies the LLA BFKL pole

$$\frac{1}{\omega - 2g^2 \chi_{\rm LLA}(\gamma)}\ , \tag{3.8}$$

where

$$\chi_{\rm LLA}(\gamma) = 2\psi(1) - \psi\left(-\frac{\gamma}{2}\right) - \psi\left(1+\frac{\gamma}{2}\right) \tag{3.9}$$

is called the BFKL kernel. Notice that when $\gamma \to 0$ the kernel $\chi_{\rm LLA}(\gamma) \sim -2/\gamma$, in agreement with the DIS result.

[1] If DGLAP conventions were used, the exponent of the $\left(\frac{Q^2}{Q'^2}\right)$-factor would simply be denoted γ. However, as later in this note we will relate γ to the anomalous dimension, which we treat using spin chain conventions, the $-\gamma/2$ factor appears.

3.1 NLLA and scale dependence

In DIS the relevant scale s_0, relating x and s through $x = s_0/s$, is the photon virtuality Q^2. At LLA we do not have dependence on the scale s_0, but this situation changes when we go to next to leading logarithm approximation (NLLA) [15]. In particular if we are working in the BFKL regime the natural scale is the symmetric choice $s_0 = QQ'$. As mentioned above, in the DIS regime we get contributions of the form $\left(g^2 \log\left(\frac{s}{Q^2}\right) \log\left(\frac{Q^2}{Q'^2}\right)\right)^n$. Shifting to the symmetric scale $s_0 = QQ'$ these lead to contributions with more collinear logarithms $\log\left(\frac{Q^2}{Q'^2}\right)$ than powers of g^2, producing non-physical singularities in the $\gamma \to 0$ limit such as g^4/γ^3. Also, from the renormalization group equations it follows immediately that there can not be more powers of $\log Q^2$ than powers of the coupling g^2. These double collinear logarithms, where the term "double" refers to the appearence of two logarithms for each power of the coupling, should therefore be cancelled by higher order corrections to the BFKL kernel. The most straightforward way to substract them is by introducing ω into the arguments of the digamma functions in the LLA BFKL kernel (see for instance [16], and references therein),

$$\chi_{\rm LLA}(\gamma) \to 2\psi(1) - \psi\left(-\frac{\gamma}{2} + \frac{\omega}{2}\right) - \psi\left(1 + \frac{\gamma}{2} + \frac{\omega}{2}\right) \, . \tag{3.10}$$

This shifted kernel coincides with the LLA kernel at lowest order, since ω starts at order g^2, and it resums large parts of the higher order contributions.

The shift in the digamma functions can be easily understood in terms of scale transformations of the Mellin transform. Writing out the scale s_0, the inverse Mellin transform (3.4) is given by

$$f(x, Q^2) = \int \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} \int \frac{d\gamma}{2\pi i} \frac{1}{Q^2} \left(\frac{Q^2}{Q'^2}\right)^{-\gamma/2} f(\omega, \gamma) \, . \tag{3.11}$$

It follows that a change of scale $s_0 \to s_0 \frac{Q'}{Q}$ corresponds to the shift $-\gamma/2 \to -\gamma/2 + \omega/2$. In DIS we have a $-2/\gamma$ pole for small γ, when the scale is Q^2. This imples that the first non-constant digamma of the characteristic function should be $-\psi(-\gamma/2)$ at $s_0 = Q^2$, which implies that it shifts to $-\psi(-\gamma/2 + \omega/2)$ at $s_0 = QQ'$. Requiring symmetry between Q and Q', and therefore a $1/(1 + \gamma/2)$ pole when $s_0 = Q'^2$, provides the argument of the last digamma function.

Now let us recall that the DGLAP anomalous dimensions and their equivalent description in terms of dimensions of twist-two operators arise when studying the parton distribution functions in DIS. When comparing BFKL predictions with the anomalous dimensions obtained from the spin chain picture, we should therefore choose the asymmetric Q^2 scale. With that choice we get

$$\chi(\omega,\,\gamma) = 2\psi(1) - \psi\left(-\frac{\gamma}{2}\right) - \psi\left(1+\frac{\gamma}{2}+\omega\right)\ . \tag{3.12}$$

3.2 The double logarithmic resummation

In what follows we will be interested not only in double logarithms of the type (3.3), but also in purely non-collinear double logarithms, i.e., in contributions to the parton evolution where each power of the coupling g^2 is acompanied by two powers of $\big(\log s\big)$. Contrary to the case of the purely collinear double logarithms discussed in the previous subsection, these non-collinear double logarithms are not compensated for at higher orders in the perturbative expansion.

One way to to resum the entire double logarithmic contribution to the parton evolution, including both $\big(\log s \log Q^2\big)$ and $\big(\log^2 s\big)$ terms, is to modify the Bethe-Salpeter integral equation (3.2) by changing the kinematic region over which one integrates [17]. We still require that $z \gg x$, or equivalently $s \gg s'$, where $s' = \frac{Q^2}{z}$, but we now relax the ordering of the transverse momenta, moving in the direction of BFKL, allowing k^2 to be larger than Q^2, although still much smaller than s or s'. Instead, we require that [2]

$$z \ll \frac{Q^2}{k^2}\ , \tag{3.13}$$

which is automatically satisfied if $k^2 < Q^2$ since $z \ll 1$, but becomes important in the extended kinematic region where $k^2 \gg Q^2$. This additional condition mixes the transverse and longitudinal variables. As a result, collinear $\log Q^2$ logarithms can get substituted for additional logarithms in the energy. The way the double logarithmic contributions are generated from this change of kinematical region is shown in detail in appendix A.

At the level of the Bethe-Salpeter kernel, the change of the integration region leads to a modification of the Mellin space kernel from $-2/\gamma$ to

$$-\frac{2}{\gamma} + \frac{1}{\omega+\gamma/2}\ . \tag{3.14}$$

[2] The equation studied in [17] corresponded to a QED scattering amplitude, with a slighlty different structure than the case at hand, implying that the modification of the integration region performed was different than this one.

From this modified kernel we see that the pole

$$\omega = 2g^2 \left(-\frac{2}{\gamma} + \frac{1}{\omega + \gamma/2} \right) \tag{3.15}$$

in the solution to the Bethe-Salpeter equation in Mellin space can be written [3]

$$\gamma = \omega \sqrt{1 - \frac{8g^2}{\omega^2}} - \omega \ . \tag{3.16}$$

3.3 BFKL anomalous dimensions: analytic continuation

In BFKL anomalous dimensions arise in a different way than for DGLAP. The solutions to the BFKL equation can be related to a four-point Green function of fields defined in impact parameter space. When the impact parameters of two of the fields get close, one can perform an operator product expansion where the anomalous dimensions of the appearing operators are given by the BFKL kernel. One of the labels parameterizing the eigenfunctions of the BFKL equation is the conformal spin n. When $n = 0$, the double logarithm corrected BFKL kernel is given by (3.12). But in general, the BFKL anomalous dimensions depend on n, $\gamma = \gamma(\omega, n)$, and are given as solutions of

$$\omega = 2g^2 \Big(2\psi(1) - \psi\left(-\frac{\gamma}{2}\right) - \psi\left(1 + \frac{\gamma}{2} + \omega + |n|\right)\Big) \ . \tag{3.17}$$

For $\mathcal{N} = 4$ Yang-Mills it was suggested in [17] that by an analytic extension in $|n|$ we can get directly from BFKL the anomalous dimension of formal twist-two operators with negative spin. Defining $j = 1 + |n| + \omega$, we are interested in moving in the $(\omega, |n|)$ plane to points with $|n| = -r - 1$, where r is a positive integer, and with ω going to zero as $-(r + 1 + |n|)$. Next, we should compare this double limit of $\gamma(\omega, |n|)$ with the analytic extension of the DGLAP anomalous dimensions for twist-two operators, $\gamma_2(N)$, analytically continued to $\gamma(-r + \omega)$ for $\omega \to 0$.

When we consider the analytic extension of DGLAP anomalous dimensions beyond one-loop we find terms of type $a_{i,r} g^{2i}/\omega^{2i-1}$, that for $i > 1$ contain one more power of g than powers of ω. These are precisely of the form obtained when expanding the expression (3.16) for the double logarithm pole. The analytic extension of the anomalous dimensions thus contains a piece

$$\gamma(-r + \omega) = \sum_i \frac{a_{i,r}}{\omega^{2i-1}} g^{2i} + \cdots \tag{3.18}$$

[3] This is one of the two poles in γ. However, as noted in the appendix, the integration contour performed when taking the inverse Mellin transform only picks up one of the poles.

which, invoking the relation to DIS, via BFKL, can be traced to the double logarithm contribution.

Using the known perturbative results until four-loops (or three-loops if the ABA result is not trusted), one discovers the following relation between the double logarithm coefficients $a_{i,r}$ and the coefficients of the loop expansion of $\gamma(1) = \sum_i e_i g^{2i}$ at twist-two,

$$a_{i,r} = (-1)^i e_i \ , \tag{3.19}$$

for even values of r. For odd values of r we get $a_2 = a_3 = 0$, which correspond to the typical behaviour of the BFKL pomeron. Assuming the previous relation holds to all-loops we observe that the contribution of the double logarithms to the anomalous dimension analytically extended to negative values of the spin for r even is completely linked to the anomalous dimension $\gamma(1)$.

The double logarithm contribution can also be extracted, as is done in [17], directly from the BFKL kernel. Approximating (3.17) by only keeping the singular parts of the poles at $\gamma = 0$ and $\gamma = -2\omega$ one gets

$$\omega = 2g^2 \left(-\frac{2}{\gamma} + \frac{1}{\omega + \gamma/2} \right) \ , \tag{3.20}$$

which simplifies to (3.16). There is a subtlety in this derivation, however. For fixed coupling, when $\omega \to 0$, the γ does not approach one of the poles, invalidating the pole approximation. [4] This can be seen from equation (3.16) since it implies that γ approaches an imaginary constant when ω tends to zero. The solution is to let $g^2 \ll \omega$. The expression for the double logarithmic pole is thus obtained from BFKL when ω is small, and the coupling is even smaller.

3.4 Magnon dispersion relation and double logarithms

As discussed above the spin chain representation of the anomalous dimensions suggests to interpret $\gamma(1)$ as the energy for a magnon with the minimal non-vanishing momentum in a chain of length-two with periodic boundary conditions. This interpretation of $\gamma(1)$, together with (3.19), leads to

$$\gamma^{(DL)}(-r + \omega) = \omega E \left(p = \pi, \ g \to \frac{ig}{\omega} \right) \ , \tag{3.21}$$

[4] For $|n| = -1$, there is actually a solution of (3.17) where γ approaches 0 as ω does. However, this solution $\gamma(\omega)$ does not seem to be related to the double logarithms.

where by $\gamma^{(DL)}$ we mean the double logarithm contribution to the anomalous dimension and where, as before, we assume r even. Once we have related the double logarithm contribution to the magnon energy, we can use the information about its contribution in DGLAP to determine at all-loops the form of the magnon dispersion relation, $E(ig/\omega)$. The logic flow of the discussion here is first to interpret $\gamma(1)$ as the single magnon energy, secondly to relate $\gamma(1)$ with the double logarithm contribution and finally to get the form of the magnon energy from the DGLAP kernel including the double logarithm pieces. As discussed above, the double logarithm contribution to γ is given by (3.16), and therefore we get

$$E\Big(p=\pi,\, g\to\frac{ig}{\omega}\Big)=\omega\sqrt{1-\frac{8g^2}{\omega^2}}-\omega\ , \tag{3.22}$$

in agreement with the ABA prescription.

In addition, this agreement gives added weight to the currently used form of the $\mathcal{N}=4$ dispersion relation. The algebraic contruction of the ABA [18] introduces a dispersion relation of the form (2.11). However, there is nothing that prevents the algebraically introduced coupling constant from being an arbitrary function of the physical coupling g. [5]

In extracting the dispersion relation from the double logarithmic approximation of BFKL we did not assume that the magnon itself had an interpretation in this formalism. However, we believe that there is a BFKL magnon candidate. The solution to the Bethe-Salpeter equation corresponding to the double logarithmic approximation can be related to a certain t-channel partial wave expansion (see appendix D in [17]). The amplitude for such a partial wave is given by (equation (D2) in [17])

$$f_\omega=\frac{\omega^2}{4g^2}\left(1-\sqrt{1-\frac{8g^2}{\omega^2}}\right)\ . \tag{3.23}$$

We can therefore speculate that the relation between the spin chain magnon and BFKL is as presented in table 1. A single magnon is thus identified with a partial-wave in the double logarithmic approximation. Including subleading terms in the integral equation would then correspond to adding interactions between magnons.

In fact, this relationship is entirely analogous to the approach in [2] linking high energy QCD and the $\mathrm{XXX}_{s=0}$ spin chain. Eigenfunctions of the Bethe-Salpeter kernel, which amount to partial waves in that case, where mapped to magnons of the spin chain, and the spin chain hamiltonian was obtained. The spin 0 construction is, however, limited to

[5] See for instance the related recent proposal in [19].

Spin chain	**BFKL**
Magnon	Partial wave in double logarithmic approximation
E	Partial wave amplitude (re-scaled)
$\sin\left(\frac{p}{2}\right)$	i/ω
g	g

Table 1: BFKL description of the spin chain magnon.

leading order. Here we have possibly the starting point for a map from all-order BFKL to a spin chain. However, obtaining the explicit map may be difficult, because it would entail constructing the complete all-loop dilatation operator, including wrapping effects. Still, a partial map could shed light on both BFKL and the $\mathcal{N}=4$ spin chain.

4 Conclusions

In this note we have put forth a series of conjectures, based on perturbative evidence, on the relation of the dispersion relation for planar $\mathcal{N}=4$ Yang-Mills to the double logarithmic contributions to the anomalous dimension for twist-two operators. Let us briefly recall them: 1. The first conjecture relates the perturbative coefficients e_i in the coupling for the anomalous dimension of twist-two operators, $\gamma_2(N)$, at $N=1$, to the coefficients for the double logarithm contributions to $\gamma_2(N)$ at $N=-r$, for even values of r. We have presented evidence that $e_i=(-1)^i a_{i,r}$. 2. Secondly, we have suggested that the anomalous dimension $\gamma_2(1)$, evaluated using the $(-)$ analytic extension for the harmonic sums, corresponds to the dispersion relation for a single magnon of momentum $p=\pi$, $\gamma_2(1)=E(p=\pi)$. 3. Our last statement is an extension to twist-L operators, $\gamma_L(1)=E(p=\pi/L)$, whenever the $(-)$ analytic extension is defined.

The first conjecture is on firmest footing since it seems that there is some principle restricting the possible harmonic sums which enter the perturbative expansions of the anomalous dimensions, so that their evaluations at $N=1$ and at negative, even integers, are indeed related. Furthermore, only the double logarithm contribution is matched to $\gamma_2(1)$. That is, terms that are subleading in either the coupling, or in $1/\omega$, in the expansion of the anomalous dimensions around $-r$, for r even, do not enter in the anomalous dimension at $N=1$. This is a highly non-trivial statement, since at $N=1$ all harmonic sums contribute to the anomalous dimension, while only the most singular sums contribute to the double logarithm expansion. Notably, nested harmonic sums typically do not affect the double logarithms.

One might then wonder whether wrapping effects could spoil the validity of the first two conjectures. Wrapping is understood as responsible for the mismatch for twist-two operators between the ABA and BFKL at four-loops [8]. From the viewpoint of BFKL, wrapping is never an issue and must automatically be included in the BFKL answer. Since the double logarithmic contribution is, at weak-coupling and at all-loops, determined by equation (3.16), and one could in principle derive also $\gamma_2(1)$ to all orders solely from BFKL [17], there is no reason to believe that something special will happen at fourth loop order that ruins conjecture 1. If one then invokes the intuitive idea of $\gamma_2(1)$ giving the energy of a single magnon as justification for the second conjecture, one is lead to the conclusion that wrapping effects should not modify the single magnon dispersion relation at weak-coupling (strong-coupling is, of course, an entirely different issue). Most likely, before including wrapping effects, the ABA answer should be consistent with the relation $\gamma_2(1) = E(p = \pi)$, implying that wrapping modifications to the ABA should correspond to combinations of harmonic sums respecting transcendentality, and vanishing when evaluated at $N = 1$. This is in fact the case for the ad hoc proposal in [8]. However an important problem that we have not considered in this note is if a potential extension of BFKL to strong 't Hooft coupling (see for instance [20]) would modify the form of the double logarithmic contribution.

The third conjecture, by contrast, is merely a wild idea based on the intuitive notion underlying the second conjecture. It could very well be that it is only valid at even L, or that the relevant magnon momentum is not the minimal $2\pi/L$, but something else.

We have also, by analogy with the emergence of one-loop integrability in high energy QCD, conjectured that the all-loop magnon appears in BFKL in the form of a partial wave in the double logarithmic approximation. It would be very interesting if this correspondence could be extended to a complete BFKL–spin chain map.

As a final comment, let us recall that the magnon dispersion relation for planar $\mathcal{N} = 4$ Yang-Mills is intimately related to the string BMN formula and moreover it can be derived, barring possible differences between the algebraically introduced coupling and the physical coupling, from the centrally extended symmetry algebra [18]. It would be extremely interesting to find glints of these structures in the double logarithmic contributions to the

Bethe-Salpeter equations governing the parton distribution functions. Perhaps one could use this information to extend the BFKL - spin chain map beyond the single magnon.

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A The resummation of double logarithms from the integral equation

We will now show how a modification of the kinematic region used in the Bethe-Salpeter equation

$$f(x,\,Q^2) = f_0(x,\,Q^2) + 2g^2 \int_x^1 \frac{dz}{z} \int_{Q'^2}^{Q^2} \frac{dk^2}{Q^2} f\left(\frac{x}{z},\,k^2\right) \ , \tag{A.1}$$

can produce all types of double logarithmic terms that are consistent with renormalization group constraints, by which we mean that the number of collinear logarithms are not allowed to exceed the order in perturbation theory. Firstly, we will drop the requirement of transversal ordering $Q^2 \gg k^2$. This changes the upper integration limit in the integral over transverse momenta from Q^2 to $s = \frac{Q^2}{x}$. Secondly, we add the condition that

$$z \ll \frac{Q^2}{k^2} \ , \tag{A.2}$$

which is not trivially satisfied when $k^2 > Q^2$. This causes the upper limit of the integral over z to become the smaller of 1 or $\frac{Q^2}{k^2}$. Therefore (A.1) is modified to

$$f(x,\,Q^2) = f_0(x,\,Q^2) + 2g^2 \int_{Q'^2}^{Q^2/x} \frac{dk^2}{Q^2} \int_x^{\min\left(1,\,Q^2/k^2\right)} \frac{dz}{z} f\left(\frac{x}{z},\,k^2\right) \ . \tag{A.3}$$

For the two different cases present in the integration limit $\min\left(1,\,\frac{Q^2}{k^2}\right)$, the integration over transverse momenta comes from different regions ($k^2 > Q^2$ or $k^2 < Q^2$, respectively) and we get

$$f(x,\,Q^2) = f_0(x,\,Q^2) + 2g^2 \int_{Q'^2}^{Q^2} \frac{dk^2}{Q^2} \int_x^1 \frac{dz}{z} f\left(\frac{x}{z},\,k^2\right) + 2g^2 \int_{Q'^2}^{Q^2/x} \frac{dk^2}{Q^2} \int_x^{Q^2/k^2} \frac{dz}{z} f\left(\frac{x}{z},\,k^2\right) . \tag{A.4}$$

By iteration this equation produces the perturbative expansion of the double logarithmic terms. For example, with the simplest possible initial distribution $f_0(x\,Q^2) = \frac{1}{Q^2}$, where the factor $1/Q^2$ has to appear since the integrated parton distribution should be dimensionless, one obtains

$$\begin{aligned} f(x,\,Q^2) = {} & \frac{1}{Q^2} + \frac{2g^2}{Q^2}\left(\log\frac{Q^2}{Q'^2}\log\frac{1}{x} + \frac{1}{2}\log^2\frac{1}{x}\right) + \\ & + \frac{4g^4}{Q^2}\left(\frac{1}{4}\log^2\frac{Q^2}{Q'^2}\log^2\frac{1}{x} + \frac{1}{3}\log\frac{Q^2}{Q'^2}\log^3\frac{1}{x} + \frac{1}{12}\log^4\frac{1}{x}\right) + \mathcal{O}(g^6) . \end{aligned} \tag{A.5}$$

Iteration of the first integral in (A.4), which is the same integral as in (A.1), produces double logarithms of the form $\left(g^2 \log\frac{Q^2}{Q'^2}\log\frac{1}{x}\right)^n$, while iteration of the second integral produces double logarithms in the energy $\left(g^2\log^2\frac{1}{x}\right)^n$. Combining the two terms when iterating leads to mixed cases.

However, usually one introduces (A.4) because the double logarithmic contribution makes the perturbation expansion badly divergent, such as is the case when the energy is so large that $g^2\log^2\frac{s}{Q^2}$ is of order unity or larger. Solving the integral equation provides a resummation to all orders of the double logarithms. This can be done by Mellin transforming the distributions,

$$f(x,\,Q^2) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\omega}{2\pi i} x^{-\omega} \int_{\sigma'-i\infty}^{\sigma'+i\infty} \frac{d\gamma}{2\pi i}\frac{1}{Q^2}\left(\frac{Q^2}{Q'^2}\right)^{\gamma} f(\omega,\,\gamma) , \tag{A.6}$$

where the integration contour for the γ integral runs parallel to the imaginary axis with a positive real part, $\sigma' > 0$, and the ω integration contour is also parallel to the imaginary axis with $\sigma - \sigma' > 0$. As they are much more convenient in performing the following calculations, we are using DGLAP conventions in this appendix for γ as oposed to the spin chain conventions used in the main text. The results obtained can be translated to the spin chain conventions by simply letting

$$\gamma \to -\frac{\gamma}{2} . \tag{A.7}$$

If one introduces (A.6) into (A.4), and performs the integrals over z and k^2, the first integral becomes $f(\omega,\,\gamma)/\omega\gamma$, while the second integral transforms to $f(\omega,\,\gamma)/\omega(\omega-\gamma)$, which gives the double logaritmic pole

$$f(\omega,\,\gamma) \sim \frac{1}{\omega - 2g^2\left[\frac{1}{\gamma} + \frac{1}{\omega-\gamma}\right]}\ . \tag{A.8}$$

The double Mellin transform is full of subtleties, however, and the correct answer is not obtained simply by introducing (A.6) into (A.4). In appendix D of [17] an alternative method is used to pass to Mellin space when solving a similar integral equation. For the simple initial distribution $f_0(x,\,Q^2) = 1/Q^2$ one obtains

$$f(\omega,\,\gamma) = \frac{(\omega-2\gamma)\gamma\omega(2g^2)^{-2}}{\omega - 2g^2\left[\frac{1}{\gamma} + \frac{1}{\omega-\gamma}\right]}\ . \tag{A.9}$$

One can now transform back to the physical variables x and Q^2 by performing the integrals in (A.6). We can re-write (A.9) as

$$\begin{aligned} f(\omega,\,\gamma) &= \frac{(\omega-2\gamma)(\gamma-\omega)\gamma^2(2g^2)^{-2}}{\gamma^2-\omega\gamma+\lambda} \\ &= \frac{(\omega-2\gamma)(\gamma-\omega)\gamma^2(2g^2)^{-2}}{\left(\gamma-\frac{1}{2}\left(\omega+\sqrt{\omega^2-8g^2}\right)\right)\left(\gamma-\frac{1}{2}\left(\omega-\sqrt{\omega^2-8g^2}\right)\right)}\ . \end{aligned} \tag{A.10}$$

Now, performing the integral over γ we will only pick up the pole at

$$\gamma = \frac{1}{2}\left(\omega - \sqrt{\omega^2-8g^2}\right)\ , \tag{A.11}$$

since the two poles lie on either side of the γ contour, and $\left(\frac{Q^2}{Q'^2}\right) > 1$ implies that we must close the contour towards the left.

After having performed the γ integral we are left with

$$\begin{aligned} &Q^2 f(x,\,Q^2) = \\ &= \int\frac{d\omega}{2\pi i}x^{-\omega}\frac{\sqrt{\omega^2-8g^2}\left(\omega+\sqrt{\omega^2-8g^2}\right)\left(\frac{\omega}{2}-\frac{1}{2}\sqrt{\omega^2-8g^2}\right)^2}{8g^4\sqrt{\omega^2-8g^2}}\left(\frac{Q^2}{Q'^2}\right)^{\frac{1}{2}\left(\omega-\sqrt{\omega^2-8g^2}\right)} \\ &= \int\frac{d\omega}{2\pi i}x^{-\omega}\frac{1}{4g^2}\left(1-\sqrt{1-\frac{8g^2}{\omega^2}}\right)\exp\left[\frac{\omega}{2}\left(1-\sqrt{1-\frac{8g^2}{\omega^2}}\right)\log\frac{Q^2}{Q'^2}\right]\ . \end{aligned} \tag{A.12}$$

This integral can be evaluated, for example by performing a saddle point approximation. Instead, let us simply note how the perturbative expansion of this expression consists only of double logarithms. The inverse Mellin transform of $1/\omega^{r+1}$ is $\left(\frac{1}{r!}\log^r\frac{1}{x}\right)$, and each instance of the coupling g^2 is acompanied by either $1/\omega^2$ or by $\left(\frac{1}{\omega}\log\frac{Q^2}{Q'^2}\right)$, explaining the double logarithms. Also, at most one factor of $\log\left(\frac{Q^2}{Q'^2}\right)$ can appear at each order in perturbation theory, which must be the case in order for the double logarithmic approximation to be compatible with the renormalization group equations.

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