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The supersymmetric solutions and extensions of ungauged matter-coupled $N = 1, d = 4$ supergravity

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Abstract

We find the most general supersymmetric solutions of ungauged $N = 1, d = 4$ supergravity coupled to an arbitrary number of vector and chiral supermultiplets, which turn out to be essentially *pp*-waves and strings. We also introduce magnetic 1-forms and their supersymmetry transformations and 2-forms associated to the isometries of the scalar manifold and their supersymmetry transformations. Only the latter can couple to BPS objects (strings), in agreement with our results.

1 Introduction

Supersymmetric classical solutions of supergravity theories (low-energy superstring theories) are a key tool in the current research on many topics ranging from AdS/CFT correspondence to stringy black-hole physics. Not all locally supersymmetric solutions are necessarily interesting or useful in the end, but, clearly, it is an important goal to find them all for every possible supergravity theory.

This goal has been pursued and reached in several lower-dimensional theories and families of theories. The pioneering work [1] was done in 1983 by Tod in pure, ungauged, $N = 2, d = 4$ supergravity. It has been subsequently extended to the gauged case in Ref. [2], to include the coupling to general (ungauged) vector multiplets and hypermultiplets in Refs. [3] and [4], respectively and some partial results on the theory with gauged vector multiplets have been recently obtained [5]. Research on pure $N = 4, d = 4$ supergravity was started in Ref. [6] and completed in Ref. [7].

In $d = 5$, the minimal $N = 1$ (sometimes referred as $N = 2$) theory was worked out in Ref. [8] and the results were extended to the gauged case in Ref. [9]. The coupling to an arbitrary number of vector multiplets and their Abelian gaugings was considered in Refs. [10, 11][1]. The inclusion of (ungauged) hypermultiplets was considered in [14][2] and the extension to the most general gaugings with vector multiplets and hypermultiplets was worked out in [18].

The minimal $d = 6$ SUGRA was dealt with in Refs. [19, 20], some gaugings were considered in Ref. [21] and the coupling to hypermultiplets has been fully solved in Ref. [22].

All these works are essentially based on the method pioneered by Tod and generalized by Gauntlett *et al.* in Ref. [8], which we will use here. An alternative method is that of spinorial geometry, developed in Ref. [23]. Further works on this subject in 4 or higher dimensions are Refs. [24].

It is somewhat surprising that the simpler $N = 1, d = 4$ theories have not yet been studied. The purpose of this paper is to start filling this gap. We will find all the supersymmetric configurations and solutions of ungauged $N = 1, d = 4$ supergravity and we will relate them to supersymmetric solutions of $N = 2, d = 4$ supergravity theories that we can truncate to $N = 1, d = 4$ theories following [25, 26]. As we are going to see, there are no timelike supersymmetric solutions such as charged, extreme, black holes in these theories and in the null class we find essentially *pp*-waves, cosmic strings and combinations of both. This is, precisely, the kind of supersymmetric solutions of $N = 2, d = 4$ supergravity that would survive the truncation to $N = 1$.

We are also going to study the extension of the set of standard bosonic fields of $N = 1, d = 4$ supergravity along the lines of Ref. [27]. We are going to show that we can add consistently (we can define supersymmetry transformations for them such that the local supersymmetry algebra closes) the magnetic vectors and also 2-forms which are associated to the isometries of the scalar manifold. The electric and magnetic vectors of the theory

[1]Previous work on these theories can be found in Refs. [12, 13].
[2]Previous partial results on that problem were presented in Refs. [15, 16, 17].

transform into the gauginos and not into the gravitino. This makes it impossible to write a κ-symmetric action for 0-branes, in agreement with the absence of supersymmetric black-hole solutions in the theory. The 2-forms do transform into the gravitino and one can, in principle, construct κ-symmetric actions for 1-branes, which agrees with the existence of supersymmetric string solutions.

This paper is organized as follows: in Section 2 we introduce ungauged $N = 1, d = 4$ supergravity coupled to vector and chiral supermultiplets. We obtain this theory by truncation of ungauged $N = 2, d = 4$ supergravity coupled to vector supermultiplets and hypermultiplets in Appendix A. This helps us to fix the conventions and to relate the solutions to $N = 2, d = 4$ solutions. In Section 3 we set up the problem we aim to solve. In Section 4.1 we find all the bosonic field configurations that admit Killing spinors (as we check in Section 4.2) and in Section 4.3 we identify amongst them those that satisfy the classical equations of motion, which solves our problem. In Section 5 we find the bosonic field extensions of the theory. Finally, in Section 6 we discuss our results and give our conclusions.

After completion of this work we became aware that a similar results have been obtained by U. Gran, J. Gutowski and G. Papadopoulos and are about to be published [28].

2 Matter-coupled, ungauged, $N = 1, d = 4$ supergravity

In this section we describe briefly the theory [30], which is obtained by truncation of $N = 2, d = 4$ theories in the appendix. Our conventions are derived from those we use in the study of $N = 2, d = 4$ theories [3, 4, 5]. It contains a supergravity multiplet with one graviton $e^a{}_\mu$ and one chiral gravitino $\psi_{\bullet\mu}$, n_C chiral multiplets with as many chiral dilatini $\chi_\bullet{}^i$ and complex scalars Z^i, $i = 1, \cdots n_C$ that parametrize a Kähler-Hodge manifold with metric $\mathcal{G}_{ij^*}$, and n_V vector multiplets with as many vector fields A^Λ and chiral gaugini $\lambda_\bullet{}^\Lambda$ $\Lambda = 1, \cdots, n_V$.

The action for the bosonic fields is

$$S = \int d^4x \sqrt{|g|}\left[R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i \partial^\mu Z^{*\,j^*} - \Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} - \Re\mathrm{e} f_{\Lambda\Sigma} F^{\Lambda\,\mu\nu\star} F^\Sigma{}_{\mu\nu}\right] , \quad (2.1)$$

where $f_{\Lambda\Sigma}(Z)$ is a $n_V \times n_V$ matrix with entries which are holomorphic functions of the complex scalars and with definite positive imaginary part.

The supersymmetry transformation rules for the bosonic fields are

$$\delta_\epsilon e^a{}_\mu = -\tfrac{i}{4}\bar{\psi}_{\bullet\,\mu}\gamma^a\epsilon^\bullet + \text{c.c.}\,, \tag{2.2}$$

$$\delta_\epsilon A^\Lambda{}_\mu = \tfrac{i}{8}\bar{\lambda}_\bullet{}^\Lambda\gamma_\mu\epsilon^\bullet + \text{c.c.}\,, \tag{2.3}$$

$$\delta_\epsilon Z^i = \tfrac{1}{4}\bar{\chi}_\bullet{}^i\epsilon_\bullet\,, \tag{2.4}$$

and those of the fermions, for vanishing fermions, are

$$\delta_\epsilon\psi_{\bullet\,\mu} = \mathfrak{D}_\mu\epsilon_\bullet = \left(\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu\right)\epsilon_\bullet\,, \tag{2.5}$$

$$\delta_\epsilon\lambda_\bullet{}^\Lambda = \tfrac{1}{2}\not{F}^{\mathbf{\Lambda}+}\epsilon_\bullet\,, \tag{2.6}$$

$$\delta_\epsilon\chi_\bullet{}^i = i\,\not{\partial}Z^i\epsilon^\bullet\,, \tag{2.7}$$

where $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form connection

$$\mathcal{Q} \equiv \tfrac{1}{2i}(dz^i\partial_i\mathcal{K} - dz^{*\,i^*}\partial_{i^*}\mathcal{K})\,, \tag{2.8}$$

where $\mathcal{K}$ is the Kähler potential from which the Kähler metric can be derived in the standard fashion, namely

$$\mathcal{G}_{ij^*} = \partial_i\partial_{j^*}\mathcal{K}\,. \tag{2.9}$$

For convenience, we denote the bosonic equations of motion by

$$\mathcal{E}_a{}^\mu \equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_\mu}\,, \quad \mathcal{E}^i \equiv -\frac{\mathcal{G}^{ij^*}}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^{*j^*}}\,, \quad \mathcal{E}_\Lambda{}^\mu \equiv \frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta A^\Lambda{}_\mu}\,. \tag{2.10}$$

and the Bianchi identities for the vector field strengths by

$$\mathcal{B}^{\Lambda\,\mu} \equiv \nabla_\nu \star F^{\Lambda\,\nu\mu}\,, \quad \star\mathcal{B}^\Lambda \equiv -dF^\Lambda\,. \tag{2.11}$$

Then, using the action Eq. (2.1), we find

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\partial_\mu Z^i \partial_\nu Z^{*\,j^*} - \tfrac{1}{2} g_{\mu\nu} \partial_\rho Z^i \partial^\rho Z^{*\,j^*}] \\
&\quad -4\Im\mathrm{m} f_{\Lambda\Sigma} F^{\Lambda\,+}{}_{\mu}{}^{\rho} F^{\Sigma\,-}{}_{\nu\rho}\,, \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_i &= \mathcal{G}_{ij^*} \mathfrak{D}_\mu \partial^\mu Z^{*\,i^*} + \partial_i [F_\Lambda{}^{\mu\nu} \star F^\Lambda{}_{\mu\nu}] \qquad (2.13) \\
&= \mathcal{G}_{ij^*} \mathfrak{D}_\mu \partial^\mu Z^{*\,i^*} - \tfrac{i}{2} \partial_i f_{\Lambda\Sigma} F^{\Lambda\,+}{}_{\mu\nu} F^{\Sigma\,+\,\mu\nu}\,, \qquad (2.14)
\end{aligned}$$

$$\mathcal{E}_\Lambda{}^\mu = \nabla_\nu \star F_\Lambda{}^{\nu\mu}\,, \qquad (2.15)$$

where we have defined the dual vector field strength F_Λ by

$$F_{\Lambda\,\mu\nu} \equiv -\frac{1}{2\sqrt{|g|}} \frac{\delta S}{\delta^\star F^\Lambda{}_{\mu\nu}} = \Re\mathrm{e} f_{\Lambda\Sigma} F^\Sigma{}_{\mu\nu} - \Im\mathrm{m} f_{\Lambda\Sigma}{}^\star F^\Sigma{}_{\mu\nu} = 2\Re\mathrm{e}\,(f_{\Lambda\Sigma} F^{\Sigma\,+})\,. \qquad (2.16)$$

The Maxwell equations can be read as Bianchi identities for these dual field strengths ensuring the local existence of n_V dual vector potentials A_Λ such that

$$F_\Lambda = dA_\Lambda\,. \qquad (2.17)$$

It is convenient to combine the standard, electric, field strengths and potentials and their duals Eq. (2.16) into a single $2n_V$-dimensional symplectic vector

$$\mathcal{F} \equiv \begin{pmatrix} F^\Lambda \\ F_\Lambda \end{pmatrix} = d\mathcal{A} \equiv d \begin{pmatrix} A^\Lambda \\ A_\Lambda \end{pmatrix}\,. \qquad (2.18)$$

The global symmetries of these theories will be the isometries of the scalar manifold that can be embedded in $Sp(2n_V, \mathbb{R})$ [31].

3 Supersymmetric configurations: general setup

Our first goal is to find all the bosonic field configurations $\{g_{\mu\nu}, F^\Lambda{}_{\mu\nu}, Z^i\}$ for which the Killing spinor equations (KSEs):

$$\delta_\epsilon \psi_{\bullet\,\mu} = \mathfrak{D}_\mu \epsilon_\bullet = 0\,, \qquad (3.1)$$

$$\delta_\epsilon \lambda_\bullet{}^\Lambda = \tfrac{1}{2} \not{F}^{\Lambda+} \epsilon_\bullet = 0\,, \qquad (3.2)$$

$$\delta_\epsilon \chi_\bullet{}^i = i \not{\partial} Z^i \epsilon^\bullet = 0\,, \qquad (3.3)$$

admit at least one solution. It must be stressed that the configurations considered need not be classical solutions of the equations of motion. Furthermore, we will not assume that the Bianchi identities are satisfied by the field strengths of a configuration.

Our second goal will be to identify among all the supersymmetric field configurations those that satisfy all the equations of motion (including the Bianchi identities).

Let us initiate the analysis of the KSEs by studying their integrability conditions.

3.1 Killing Spinor Identities (KSIs)

Using the supersymmetry transformation rules of the bosonic fields Eqs. (2.2–2.4) and using the results of Refs. [32, 33] we can derive following relations (*Killing spinor identities*, KSIs) between the (off-shell) equations of motion of the bosonic fields Eqs. (2.12–2.15) that are satisfied by any field configuration $\{e^a{}_\mu, A^\Lambda{}_\mu, Z^i\}$ admitting Killing spinors:

$$\mathcal{E}^\mu{}_a \gamma^a \epsilon^\bullet = 0\,, \tag{3.4}$$

$$\mathcal{E}_\Lambda{}^\mu \gamma_\mu \epsilon^\bullet = 0\,, \tag{3.5}$$

$$\mathcal{E}_i \epsilon_\bullet = 0\,. \tag{3.6}$$

In this way of finding the KSIs the Bianchi identities are assumed to be satisfied. It is convenient to have KSIs in which they appear explicitly. These can be found through the integrability conditions of the KSEs. The only KSI in which we expect the Bianchi identities to appear is the second one above, which involves the Maxwell equations. The Bianchi identities should combine with the Maxwell equations in a electric-magnetic duality-invariant way. Then, the second KSI above should be replaced by

$$(\mathcal{E}_\Lambda{}^\mu - f_{\Lambda\Sigma}\mathcal{B}^{\Sigma\,\mu})\gamma_\mu \epsilon^\bullet = 0\,. \tag{3.7}$$

This can be explicitly checked via the following integrability condition of the gaugini:

$$\begin{aligned} \not{\mathfrak{D}}\delta_\epsilon \lambda_\bullet{}^\Lambda &= (\Im\mathrm{m} f)^{-1|\Lambda\Sigma}(\not{\mathcal{E}}_\Sigma - f^*_{\Sigma\Omega}\,\not{\mathcal{B}}^\Omega)\epsilon_\bullet \\ &\quad + i(\Im\mathrm{m} f)^{-1|\Lambda\Sigma}\,\not{\partial} f_{\Sigma\Omega}\delta_\epsilon\lambda_\bullet{}^\Omega - \tfrac{1}{4}\,\not{F}^{\Lambda\,-}\delta_\epsilon\chi^{\bullet\,i^*} + \tfrac{1}{2}\gamma^\mu\,\not{F}^{\Lambda\,+}\delta_\epsilon\psi_{\bullet\,\mu}\,. \end{aligned} \tag{3.8}$$

From these identities one can derive identities that involve tensors constructed as bilinears of the Killing spinors. In $N=1$ supergravity there is only one chiral spinor $\epsilon_\bullet$. With it, we can only construct a real null vector $l_\mu = i\sqrt{2}\bar{\epsilon}^\bullet\gamma_\mu\epsilon_\bullet$, one self-dual 2-form $\Phi_{\mu\nu} = \bar{\epsilon}_\bullet\gamma_{\mu\nu}\epsilon_\bullet$ and no scalars. In the $N>1$ cases one can construct a vector which is non-spacelike and, thus, one considers separately the case in which the vector is timelike and the case in which it is null. In $N=1, d=4$ there is no timelike case. It is convenient to introduce an auxiliary chiral spinor $\eta_\bullet$ with normalization

$$\bar{\epsilon}_{\bullet}\eta_{\bullet} = \tfrac{1}{2}\,, \tag{3.9}$$

and with the same chirality but opposite Kähler weight as $\epsilon_{\bullet}$. With both spinors we construct the null tetrad

$$\begin{array}{rclcrcl} l_{\mu} & = & i\sqrt{2}\bar{\epsilon}^{\bullet}\gamma_{\mu}\epsilon_{\bullet}\,, & \qquad & n_{\mu} & = & i\sqrt{2}\bar{\eta}^{\bullet}\gamma_{\mu}\eta_{\bullet}\,, \\ m_{\mu} & = & i\sqrt{2}\bar{\epsilon}^{\bullet}\gamma_{\mu}\eta_{\bullet}\,, & \qquad & m^{*}_{\mu} & = & i\sqrt{2}\bar{\epsilon}_{\bullet}\gamma_{\mu}\eta^{\bullet}\,. \end{array} \tag{3.10}$$

l and n have 0 $U(1)$ charges but m has -2 times the charges of ϵ and m^{*} has $+2$ times the charges of ϵ.

$$\mathcal{E}_{\mu\nu}l^{\nu} = \mathcal{E}_{\mu\nu}m^{\nu} \;=\; 0\,, \tag{3.11}$$

$$(\mathcal{E}_{\Lambda\,\mu} - f_{\Lambda\Sigma}\mathcal{B}^{\Sigma}{}_{\mu})l^{\mu} = (\mathcal{E}_{\Lambda\,\mu} - f_{\Lambda\Sigma}\mathcal{B}^{\Sigma}{}_{\mu})m^{\mu} \;=\; 0\,, \tag{3.12}$$

$$\mathcal{E}_{i} \;=\; 0\,. \tag{3.13}$$

This means that the only independent equations of motion that we have to impose on supersymmetric configurations are

$$\mathcal{E}_{\mu\nu}n^{\mu}n^{\nu} \;=\; 0\,, \tag{3.14}$$

$$(\mathcal{E}_{\Lambda\,\mu} - f_{\Lambda\Sigma}\mathcal{B}^{\Sigma}{}_{\mu})n^{\mu} \;=\; 0\,, \tag{3.15}$$

$$(\mathcal{E}_{\Lambda\,\mu} - f_{\Lambda\Sigma}\mathcal{B}^{\Sigma}{}_{\mu})m^{*\mu} \;=\; 0\,. \tag{3.16}$$

4 Supersymmetric configurations and solutions

4.1 Supersymmetric configurations

Our first goal is to derive from the KSEs consistency conditions expressed in terms of the null tetrad vectors.

Acting on the KSE Eq. (3.2) with $\bar{\epsilon}^{\bullet}\gamma_{\mu}$ and $\bar{\eta}^{\bullet}\gamma_{\mu}$ we get, respectively

$$F^{\Lambda\,+}{}_{\mu\nu}l^{\nu} \;=\; 0\,, \tag{4.1}$$

$$F^{\Lambda\,+}{}_{\mu\nu}m^{*\nu} \;=\; 0\,, \tag{4.2}$$

which imply that

$$F^{\Lambda\,+} = \tfrac{1}{2}\phi^{\Lambda}\hat{l} \wedge \hat{m}^* \,, \tag{4.3}$$

for some functions ϕ^{Λ} to be determined. This form of $F^{\Lambda\,+}$ solves the KSE Eq. (3.2) by virtue of the Fierz identities

$$l_\mu \gamma^{\mu\nu}\epsilon_\bullet = l^\nu \epsilon_\bullet \,, \qquad m^*_\mu \gamma^{\mu\nu}\epsilon_\bullet = m^{*\nu}\epsilon_\bullet \,. \tag{4.4}$$

Acting now on the KSE Eq. (3.3) with $\bar{\epsilon}_\bullet$ and $\bar{\eta}_\bullet$ we get, respectively

$$l^\mu \partial_\mu Z^i = 0 \,, \tag{4.5}$$

$$m^\mu \partial_\mu Z^i = 0 \,, \tag{4.6}$$

which imply

$$dZ^i = A^i \hat{l} + B^i \hat{m} \,, \tag{4.7}$$

for some functions A^i and B^i to be determined. This form of dZ^i solves the KSE Eq. (3.3) by virtue of the Fierz identities

$$\not{l}\epsilon^* = \not{m}\epsilon^* = 0 \,. \tag{4.8}$$

Now, , from the normalization condition of the auxiliary spinor $\eta_\bullet$ we find the condition

$$\mathfrak{D}_\mu \eta_\bullet + a_\mu \epsilon_\bullet = 0 \,, \tag{4.9}$$

for some a_μ with $U(1)$ charges -2 times those of ϵ, *i.e.*

$$\mathfrak{D}_\mu a_\nu = (\nabla_\mu - i\mathcal{Q}_\mu) a_\nu \,, \tag{4.10}$$

to be determined by the requirement that the integrability conditions of this differential equation have to be compatible with those of the differential equation for ϵ.

Taking the covariant derivative of the null tetrad vectors and using the KSE Eq. (3.1), we find

$$\mathfrak{D}_\mu l_\nu = \nabla_\mu l_\nu = 0 \,, \tag{4.11}$$

$$\mathfrak{D}_\mu n_\nu = \nabla_\mu n_\nu = -a^*_\mu m_\nu - a_\mu m^*_\nu \,, \tag{4.12}$$

$$\mathfrak{D}_\mu m_\nu = (\nabla_\mu - i\mathcal{Q}_\mu) m_\nu = -a_\mu l_\nu \,. \tag{4.13}$$

The first of these equations is solved by identifying the most general metric compatible with it: a Brinkmann *pp*-wave metric [34, 35]. One introduces the coordinates u and v such that

$$\hat{l} = l_\mu dx^\mu \equiv du\,, \tag{4.14}$$

$$l^\mu \partial_\mu \equiv \frac{\partial}{\partial v}\,, \tag{4.15}$$

and defines a complex coordinate z by

$$\hat{m} = e^U dz\,, \tag{4.16}$$

where U may depend on z, z^* and u. The most general form that $\hat{n}$ can take in this case is

$$\hat{n} = dv + Hdu + \hat{\omega}\,, \qquad \hat{\omega} = \omega_{\underline{z}} dz + \omega_{\underline{z}^*} dz^*\,, \tag{4.17}$$

where all the functions in the metric are independent of v and where either H or the 1-form $\hat{\omega}$ could, in principle, be removed by a coordinate transformation but we have to check that the tetrad integrability equations (4.11)-(4.13) are satisfied by our choices of e^U, H and $\hat{\omega}$

The above choice of coordinates leads to the metric

$$ds^2 = 2du(dv + Hdu + \hat{\omega}) - 2e^{2U} dz dz^*\,. \tag{4.18}$$

It also implies that the complex scalars Z^i are functions of z and u but not of z^* and v. The same is true for A^i and B^i.

Let us consider the tetrad integrability equations (4.11)-(4.13): the first equation is solved because the metric does not depend on v. The third equation, with the choice of coordinate z, Eq. (4.16), implies

$$\hat{a} = n^\mu(\partial_\mu U - i\mathcal{Q}_\mu)\hat{m} + D\hat{l}\,, \tag{4.19}$$

$$m^\mu \partial_\mu (U - i\mathcal{Q}_\mu) = 0\,, \tag{4.20}$$

where D is a function to be determined.

The second equation can be written using the definition of the Kähler connection and the dependence $Z^i(z,u)$ in the form

$$\partial_{\underline{z}^*}(U + \mathcal{K}/2) = 0 \;\Rightarrow U = -\mathcal{K}/2 + h(u)\,, \tag{4.21}$$

where $h(u)$ can be eliminated by a coordinate redefinition that does not change the general form of the Brinkmann metric.

The second tetrad integrability equation (4.12) implies

$$D = e^{-U}(\partial_{\underline{z}^*} H - \dot{\omega}_{\underline{z}^*})\,, \tag{4.22}$$

$$(d\omega)_{\underline{z}\underline{z}^*} = 2ie^{2U} n^\mu \mathcal{Q}_\mu\,, \tag{4.23}$$

whence $\hat{a}$ is given by

$$\hat{a} = [\dot{U} - \tfrac{1}{2}e^{-2U}(d\omega)_{\underline{z}\underline{z}^*}]\hat{m} + e^{-U}(\partial_{\underline{z}^*} H - \dot{\omega}_{\underline{z}^*})\hat{l}\,. \tag{4.24}$$

4.2 Killing spinor equations

We are now going to see that field configurations given by a metric of the form (Eqs. (4.18) where $\hat{\omega}$ satisfies (Eq. (4.23)) and U satisfies Eq. (4.21), field strengths given by Eqs. (4.3) and scalars of the form (4.7) are always supersymmetric, even though we derived these equations as necessary conditions for supersymmetry.

With the above form of the scalars and vector field strengths the KSE $\delta_\epsilon \chi_\bullet{}^i = 0$ takes the form

$$i[A^i \not{l} + B^i \not{m}]\epsilon^\bullet = 0\,. \tag{4.25}$$

This equation is solved by imposing two conditions on the spinors:

$$\not{l}\epsilon^\bullet = 0\,, \qquad \not{m}\epsilon^\bullet = 0\,. \tag{4.26}$$

As shown in Ref. ([3]) these two constraints are not just compatible but equivalent and only half of the supersymmetries are broken by them.

Let us now consider the KSE $\delta_\epsilon \psi_{\bullet\, a} = 0$. It takes the form

$$\{\partial_a - \tfrac{1}{4}\omega_{abc}\gamma^{bc} + \tfrac{i}{2}\mathcal{Q}_a\}\epsilon_\bullet = 0\,. \tag{4.27}$$

The v component is automatically satisfied for v-independent Killing spinors. The z and z^* components take, after use of the constraints Eq. (4.26) and their consequence $\gamma^{zz^*}\epsilon_\bullet = \epsilon_\bullet$ the form

$$\{\partial_{\underline{z}} + \tfrac{1}{2}\partial_{\underline{z}}(U + \mathcal{K}/2)\}\epsilon_\bullet = 0\,, \tag{4.28}$$

$$\{\partial_{\underline{z}^*} + \tfrac{1}{2}\partial_{\underline{z}^*}(U + \mathcal{K}/2)\}\epsilon_\bullet = 0\,. \tag{4.29}$$

They are solved for z- and z^*-independent spinors once Eq. (4.21) is taken into account. The u component simply implies that the Killing spinors are also u-independent.

Thus, all the configurations identified are supersymmetric with Killing spinors which are constant spinors satisfying Eqs. (4.26). Thus, they generically preserve 1/2 of the supersymmetries (no less).

4.3 Solutions

The Bianchi identities take, in differential-form language, the form

$$\hat{\mathcal{B}}^{\Sigma} = -dF^{\Lambda} = \tfrac{1}{2}d(\phi^{\Sigma}\hat{m} + \text{c.c}) \wedge \hat{l}\,, \tag{4.30}$$

and are solved by

$$A^{\Lambda} = \varphi^{\Lambda}(z,u)du + \text{c.c.}\,, \qquad e^{\mathcal{K}/2}\partial_{\underline{z}}\varphi^{\Lambda}(z,u) = \phi^{*\Lambda}\,. \tag{4.31}$$

The Maxwell equations take the form

$$\hat{\mathcal{E}}_{\Lambda} = d(f_{\Lambda\Sigma}F^{\Lambda\,+} + \text{c.c.}) = -\tfrac{1}{2}d(f_{\Lambda\Sigma}\phi^{\Sigma}\hat{m}^{*} + \text{c.c}) \wedge \hat{l}\,, \tag{4.32}$$

which is solved by holomorphic functions $\varphi_{\Lambda}(z,u)$ such that

$$\partial_{\underline{z}}\varphi_{\Lambda}(z,u) = f^{*}_{\Lambda\Sigma}\phi^{*\Sigma}e^{-\mathcal{K}/2}\,. \tag{4.33}$$

Using the solution of the Bianchi identities, we get

$$\partial_{\underline{z}}\varphi_{\Lambda}(z,u) = f^{*}_{\Lambda\Sigma}\partial_{\underline{z}}\varphi^{\Sigma}(z,u)\,. \tag{4.34}$$

Taking now into account that $f_{\Lambda\Sigma}$ is a holomorphic function of the Z^is which are, themselves, holomorphic functions of z (and standard functions of u), we arrive to the conclusion that the above equation can be solved in two ways: either the Z^is are z-independent or

$$\partial_{\underline{z}}\varphi_{\Lambda}(z,u) = f^{*}_{\Lambda\Sigma}\partial_{\underline{z}}\varphi^{\Sigma}(z,u) = 0\,. \tag{4.35}$$

In general $f_{\Lambda\Sigma}$ will not have null eigenvectors and, therefore, the only generic solutions are z-independent φ^{Σ} and, therefore, trivial vector fields.

Taking into account Eq. (4.35), the only non-automatically satisfied component of the Einstein equations is[3]

$$\partial_{\underline{z}}\partial_{\underline{z}^*}H - e^{-\mathcal{K}/2}\partial^2_{\underline{u}}e^{-\mathcal{K}/2} - e^{-\mathcal{K}}\mathcal{G}_{ij^*}\partial_{\underline{u}}Z^i\partial_{\underline{u}}Z^{*\,j^*} - \tfrac{1}{2}\Im\mathrm{m}f_{\Lambda\Sigma}\partial_{\underline{z}}\varphi^{\Lambda}\partial_{\underline{z}^*}\varphi^{*\Sigma} = 0\,. \tag{4.36}$$

There are two cases to be considered:

- When the Z^is are z-independent. Then

$$H = \Re\mathrm{e}f(z) + [e^{-\mathcal{K}/2}\partial^2_{\underline{u}}e^{-\mathcal{K}/2} + e^{-\mathcal{K}}\mathcal{G}_{ij^*}\partial_{\underline{u}}Z^i\partial_{\underline{u}}Z^{*\,j^*}]|z|^2 + \tfrac{1}{2}\Im\mathrm{m}f_{\Lambda\Sigma}\varphi^{\Lambda}\varphi^{*\Sigma}\,. \tag{4.37}$$

These solutions describe gravitational, electromagnetic and scalar *pp* waves.

[3] For simplicity we choose the gauge $\omega = 0$.

- When the Z^is are not z-independent. The vector fields are trivial, but the above equation is not easy to integrate. In the special case in which the Z^is are u-independent holomorphic functions of z

$$H = \Re\mathrm{e} f(z)\,. \tag{4.38}$$

These solutions describe a superposition of a *pp*-wave and cosmic strings such as those studied in Refs. [36, 37, 38, 27] and found in $N = 4, d = 4$ [6, 33] and $N = 2, d = 4$ [3, 4] theories.

5 Extensions

In this section we are going to explore the possible extensions of the standard formulation of $N = 1, d = 4$ supergravity, using our previous results on the supersymmetric solutions of the theory. These suggest the possible addition of 2-forms associated to the isometries of the Kähler scalar manifold. These should couple to the cosmic string solutions exactly in the form discussed in Ref. [27] for $N = 2, d = 4$ supergravity. Since one can define magnetic potentials from the Maxwell equations, it should also be possible to add dual, magnetic, 1-forms. These, however, may not couple to any standard 0-brane since all 1-forms transform into gaugini (and not the gravitino) under supersymmetry.

5.1 1-forms

Given the supersymmetry transformation rule of the standard (electric) potentials Eq. (2.3) and the definition of the dual field strengths Eq. (2.16), the simplest Ansatz for the transformation of the dual (magnetic) potentials A_Λ would be

$$\delta_\epsilon A_{\Lambda\,\mu} = \tfrac{i}{8} f^*_{\Lambda\Sigma}\bar{\epsilon}_\bullet \gamma_\mu \lambda^{\bullet\,\Sigma} + \text{c.c.}\,. \tag{5.1}$$

$$[\delta_\eta, \delta_\epsilon] A_{\Lambda\,\mu} \quad = \quad -2\Re\mathrm{e}[a f^*_{\Lambda\Sigma} F^{\mathbf{\Sigma}-}{}_{\mu\nu}]\xi^\nu\,, \tag{5.2}$$

where

$$\xi^\nu \equiv \tfrac{i}{4}\bar{\epsilon}_\bullet \gamma^\nu \eta^\bullet + \text{c.c.}\,. \tag{5.3}$$

In absence of the functions $f_{\Lambda\Sigma}$, we have

$$[\delta_\eta, \delta_\epsilon] A^\Lambda{}_\mu = -2\Re\mathrm{e}[F^{\Lambda\,-}{}_{\mu\nu}]\xi^\nu = -F^\Lambda{}_{\mu\nu}\xi^\nu = [\delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda^\Lambda)]A^\Lambda{}_\mu\,, \tag{5.4}$$

where

$$\delta_{\text{g.c.t.}}(\xi) A^\Lambda{}_\mu = \xi^\nu \partial_\nu A^\Lambda{}_\mu + \partial_\mu \xi^\nu A^\Lambda{}_\nu\,, \tag{5.5}$$

and

$$\delta_{\text{gauge}}(\Lambda)A^{\Lambda}{}_{\mu} = \partial_{\mu}\Lambda^{\Lambda}\,, \qquad \Lambda^{\Lambda} \equiv -\xi^{\nu}A^{\Lambda}{}_{\nu}\,. \tag{5.6}$$

In presence of the functions $f_{\Lambda\Sigma}$, we have

$$[\delta_{\eta}, \delta_{\epsilon}]A_{\Lambda\,\mu} = -2\Re\text{e}[f^{*}_{\Lambda\Sigma}F^{\Sigma\,-}{}_{\mu\nu}]\xi^{\nu} = -F_{\Lambda\,\mu\nu}\xi^{\nu} = [\delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda_{\Lambda})]A_{\Lambda\,\mu}\,, \tag{5.7}$$

where the g.c.t.s and gauge transformations have the same form and the parameter of the gauge transformations is now

$$\Lambda_{\Lambda} \equiv \xi^{\nu}A_{\Lambda\,\nu}\,. \tag{5.8}$$

5.2 2-forms

2-forms can be introduced in the theory by dualizing the Noether currents associated to those isometries of the scalar manifold that are symmetries of the whole theory [27]. We are always talking, then, of a subgroup of $Sp(2n_V, \mathbb{R})$ [31]. The action of these symmetries on the fields is

$$\delta Z^{i} = \alpha^{A}k_{A}{}^{i}(Z)\,, \tag{5.9}$$

$$\delta\mathcal{F} = \alpha^{A}T_{A}\mathcal{F}\,, \tag{5.10}$$

$$\tag{5.11}$$

where $\mathcal{F}$ is defined in Eq. (2.16) and where the T_A are matrices of $\mathfrak{sp}(2n_V)$ that generate the Lie algebra of the symmetry group:

$$[k_A, k_B] = -f_{AB}{}^{C}k_C\,, \qquad [T_A, T_B] = +f_{AB}{}^{C}T_C\,. \tag{5.12}$$

The computation of the Noether current proceeds as in Ref. [27] and the result is identical, up to the difference between the period matrix and $f_{\Lambda\Sigma}$:

$$J_{N\,\mu} = \alpha^{A}J_{N\,A\,\mu}\,, \qquad J_{N\,A\,\mu} = 2k^{*}_{A\,i}\partial_{\mu}Z^{i} + \text{c.c.} - 2\langle \star\mathcal{F}^{\mu\nu} \mid T_{A}\mathcal{A}_{\nu}\rangle\,. \tag{5.13}$$

These Noether currents are covariantly conserved, i.e.

$$d\star J_{N\,A} = 0\,, \tag{5.14}$$

which implies the local existence of 2-forma B_A such that

$$dB_{A} \equiv \star J_{N\,A} = 2k^{*}_{A\,i}\star dZ^{i} + \text{c.c.} - 2\langle \mathcal{F} \mid T_{A}\mathcal{A}\rangle\,. \tag{5.15}$$

The second term in the r.h.s. is not invariant under the gauge transformations of the vector potentials, and the same is therefore true for the 2-forms B_A, which transform as

$$\delta_{\text{gauge}}\mathcal{A} \;=\; d\Lambda\,, \tag{5.16}$$

$$\delta_{\text{gauge}}(\Lambda,\Lambda_{1\,A})B_A \;=\; d\Lambda_{1\,A} - 2\langle\,\mathcal{F}\mid T_A\Lambda\,\rangle\,. \tag{5.17}$$

One, then, defines the gauge-invariant 3-form field strengths

$$H_A \equiv dB_A + 2\langle\,\mathcal{F}\mid T_A\mathcal{A}\,\rangle = 2k^*_{A\,i}\star dZ^i + \text{c.c.}\,. \tag{5.18}$$

Inspired by the results of Ref. [27] it is not difficult to guess the form of the supersymmetry transformation rules of these 2-forms:

$$\begin{aligned}\delta_\epsilon B_{A\,\mu\nu} \;&=\; -\tfrac{i}{2}k^*_{A\,i}\bar{\epsilon}_\bullet\gamma_{\mu\nu}\chi_\bullet{}^i + \text{c.c.}\\ &\quad +i\mathcal{P}_A\bar{\epsilon}^\bullet\gamma_{[\mu|}\psi_{\bullet|\nu]} + \text{c.c.}\\ &\quad -4\langle\,\mathcal{A}_{[\mu|}\mid T_A\delta_\epsilon\mathcal{A}_{|\nu]}\,\rangle\,,\end{aligned} \tag{5.19}$$

where $\mathcal{P}_A$ is the momentum map associated to the Killing vector k_A.

We find

$$[\delta_\eta,\delta_\epsilon]B_{A\,\mu\nu} = [\delta_{\text{g.c.t.}}(\xi) + \delta_{\text{gauge}}(\Lambda,\Lambda_{1\,A})]B_{A\,\mu\nu}\,, \tag{5.20}$$

where ξ is defined in Eq. (5.3), Λ in Eqs. (5.6) and (5.8) and $\Lambda_{1\,A}$ is given by

$$\Lambda_{1\,A\,\mu} \equiv -2\mathcal{P}_A\xi_\mu\,. \tag{5.21}$$

A shown in Ref. [27] in $N=2, d=4$ supergravity theories, these 2-forms can be coupled to strings of different species labeled by A whose tensions would be proportional to $\mathcal{P}_A$.

6 Conclusions

We have found all the supersymmetric configurations and solutions of ungauged $N = 1, d = 4$ with arbitrary couplings to vector and chiral supermultiplets. It is clear that, qualitatively, these are those of ungauged $N = 2, d = 4$ supergravity whose fields and Killing spinors survive the $N = 2 \rightarrow N = 1$ truncation explained in the appendix, although the scalar manifolds of the $N = 1$ theory are more general. In particular, all the $N = 2$ supersymmetric configurations in the timelike class (typically black holes) do not survive to this truncation since their supersymmetry projectors

$$\epsilon_I + i\epsilon_{IJ}\gamma_0\epsilon^J = 0\,, \tag{6.1}$$

involve necessarily the two supersymmetry parameters and one of them is eliminated in the truncation. The fields of extreme, supersymmetric $N = 2, d = 4$ black holes may still survive the truncation to $N = 1$, but they will not be BPS in this theory.

The Killing spinors supersymmetric configurations of the null class obey projections of the form

$$\gamma^u \epsilon_I = 0\,, \quad I = 1, 2\,, \tag{6.2}$$

and, thus, they always survive the truncation.

It is likely that the situation in the most general (gauged) $N = 1, d = 4$ theory is the same, and, again qualitatively, the supersymmetric solutions can be obtained by truncation from the $N = 2, d = 4$ theory on which some partial results are already available [5]. Of course, a direct calculation is necessary and, anyway, the most general supersymmetric solutions of gauged $N = 2, d = 4$ supergravity are not known, although progress in this direction is being made [39]. Work in this direction is already in progress [40].

Further extensions (3- and 4-forms) are clearly possible and a more general study of the possibilities in more general (gauged) $N = 1, d = 4$ supergravities has to be performed [41] to compare the results with those of the Kac-Moody approach.

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A Truncating $N = 2$ to $N = 1$ supergravity in $d = 4$

The purpose of this appendix is to show, following Refs. [25, 26], how ungauged $N = 2, d = 4$ supergravity coupled to vector multiplets can be truncated to ungauged $N = 1, d = 4$ supergravity by decoupling the $N = 1$ supermultiplet that contains the second gravitino $\psi_{2\mu}$. We will only deal with the leading terms in fermions. In doing so, we will obtain $N = 1, d = 4$ supergravity in suitable conventions and the relations between the fields of both theories.

A.1 Ungauged matter-coupled $N = 2, d = 4$ supergravity

We start by a very brief description of ungauged $N = 2, d = 4$ supergravity coupled to vector multiplets referring the reader to Refs. [3, 7] for detailed description of the conventions and further references to the literature.

The gravity multiplet of the $N = 2, d = 4$ theory consists of the graviton $e^a{}_\mu$, a pair of gravitinos $\psi_{I\,\mu}\,,$ $(I = 1, 2)$ which we describe as Weyl spinors, and a vector field A_μ.

Each of the n vector supermultiplets of $N=2, d=4$ supergravity that we are going to couple to the pure supergravity theory contains complex scalar Z^i, $(i=1,\cdots,n_V)$, a pair of gauginos $\lambda^{I\,i}$, which we also describe as Weyl spinors and a vector field $A^i{}_\mu$. In the coupled theory, the n_V+1 vectors can be treated on the same footing and they are described collectively by an array $A^\Lambda{}_\mu$ $(\Lambda=1,\cdots,n_V+1)$. The coupling between the complex scalars is described by a non-linear σ-model with Kähler metric $\mathcal{G}_{ij^*}(Z,Z^*)$, and the coupling to the vector fields by a complex scalar-field-valued matrix $\mathcal{N}_{\Lambda\Sigma}(Z,Z^*)$. These two couplings are related by a structure called special Kähler geometry, described in the references.Each hypermultiplet consists of 4 real scalars q (*hyperscalars*) and 2 Weyl spinors ζ called *hyperinos*. The $4m$ hyperscalars are collectively denoted by q^u , $u=1,\cdots,4m$ and the $2n_H$ hyperinos are collectively denoted by ζ_α , $\alpha=1,\cdots,2n_H$. The $4n_H$ hyperscalars parametrize a quaternionic Kähler manifold with metric $\mathsf{H}_{uv}(q)$.

The action for the bosonic fields of the theory is

$$\begin{aligned} S \;=\; & \int d^4x\sqrt{|g|}\,\big[R+2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*\,j^*}+2\mathsf{H}_{uv}\partial_\mu q^u\partial^\mu q^v \\ & +2\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^\Sigma{}_{\mu\nu}-2\Re\mathrm{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu\star}F^\Sigma{}_{\mu\nu}\big]\ , \end{aligned} \tag{A.1}$$

In these conventions $\Im\mathrm{m}\mathcal{N}_{\Lambda\Sigma}$ is negative definite.

For vanishing fermions, the supersymmetry transformation rules of the fermions are

$$\delta_\epsilon\psi_{I\,\mu} \;=\; \mathfrak{D}_\mu\epsilon_I+\epsilon_{IJ}T^+{}_{\mu\nu}\gamma^\nu\epsilon^J\,, \tag{A.2}$$

$$\delta_\epsilon\lambda^{Ii} \;=\; i\,\not\partial Z^i\epsilon^I+\epsilon^{IJ}\,\not G^{i\,+}\epsilon_J\,, \tag{A.3}$$

$$\delta_\epsilon\zeta_\alpha \;=\; -i\mathbb{C}_{\alpha\beta}\,\mathsf{U}^{\beta I}{}_u\,\varepsilon_{IJ}\,\,\not\partial q^u\,\epsilon^J\,, \tag{A.4}$$

where $\mathfrak{D}_\mu$, the Lorentz- and Kähler- and $SU(2)$-covariant derivative acts on the spinors ϵ_I as

$$\mathfrak{D}_\mu\epsilon_I=(\nabla_\mu\;+\;\tfrac{i}{2}\,\mathcal{Q}_\mu)\,\epsilon_I\;+\;\mathsf{A}_{\mu\,I}{}^J\,\epsilon_J\,. \tag{A.5}$$

and $\mathcal{Q}_\mu$ is the pullback of the Kähler 1-form defined in Eq. (2.8)
and $\mathsf{A}_{\mu\,I}{}^J$ is the pullback of the $SU(2)$ connection $\mathsf{A}_I{}^J$.
The 2-forms T and G^i are the combinations

$$T_{\mu\nu} \;\equiv\; \mathcal{T}_\Lambda F^\Lambda{}_{\mu\nu}\,, \tag{A.6}$$

$$G^i{}_{\mu\nu} \;\equiv\; \mathcal{T}^i{}_\Lambda F^\Lambda{}_{\mu\nu}\,, \tag{A.7}$$

where, in turn, $\mathcal{T}_\Lambda$ and $\mathcal{T}^i{}_\Lambda$ are, respectively, the graviphoton and the matter vector fields projectors, defined by

$$\mathcal{T}_\Lambda \equiv 2i\mathcal{L}_\Lambda = 2i\mathcal{L}^\Sigma \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,, \tag{A.8}$$

$$\mathcal{T}^i{}_\Lambda \equiv -f^*{}_\Lambda{}^i = -\mathcal{G}^{ij^*} f^*{}^\Sigma{}_{j^*} \Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda}\,. \tag{A.9}$$

The supersymmetry transformations of the bosons are

$$\delta_\epsilon e^a{}_\mu = -\tfrac{i}{4}(\bar\psi_{I\,\mu}\gamma^a\epsilon^I + \bar\psi^I{}_\mu\gamma^a\epsilon_I)\,, \tag{A.10}$$

$$\begin{aligned}\delta_\epsilon A^\Lambda{}_\mu &= \tfrac{1}{4}(\mathcal{L}^{\Lambda\,*}\epsilon^{IJ}\bar\psi_{I\,\mu}\epsilon_J + \mathcal{L}^\Lambda\epsilon_{IJ}\bar\psi^I{}_\mu\epsilon^J)\\ &\quad +\tfrac{i}{8}(f^\Lambda{}_i\epsilon_{IJ}\bar\lambda^{Ii}\gamma_\mu\epsilon^J + f^{\Lambda *}{}_{i^*}\epsilon^{IJ}\bar\lambda_I{}^{i^*}\gamma_\mu\epsilon_J)\,,\end{aligned} \tag{A.11}$$

$$\delta_\epsilon Z^i = \tfrac{1}{4}\bar\lambda^{Ii}\epsilon_I\,, \tag{A.12}$$

$$\delta_\epsilon q^u = \tfrac{1}{4}\mathsf{U}_{\alpha I}{}^u(\bar\zeta^\alpha\epsilon^I + \mathbb{C}^{\alpha\beta}\epsilon^{IJ}\bar\zeta_\beta\epsilon_J)\,. \tag{A.13}$$

A.2 Truncation to $N=1, d=4$ supergravity

The truncation to $N=1, d=4$ supergravity consists in setting to zero the supermultiplet that contains the second gravitino $\psi_{2\,\mu}$ and the graviphoton. The remaining fields in the supergravity multiplet $\{e^a{}_\mu, \psi_{1\mu}\}$ will become those of the $N=1, d=4$ supergravity multiplet and the n_V $N=2, d=4$ vector multiplets will be split into n_V chiral multiplets, each of them containing one complex scalar and the first component of one $N=2$ gaugino λ^{1i} and n_V vector multiplets, each of them containing one vector and the second component of one $N=2$ gaugino λ^{2i}. However, not all of them can simultaneously. Finally, only half of the hyperscalars, parametrizing a Kähler manifold will survive the truncation.

We relabel the $N=2$ indices $\Lambda \to \mathbf{\Lambda}$ and $i, i^* \to \mathbf{i}, \mathbf{i}^*$ to label the $N=1$ vector multiplets with Λ and the chiral multiplets with i. We set

$$\psi_{2\,\mu} = \delta_\epsilon\psi_{2\,\mu} = \epsilon_2 = 0\,, \tag{A.14}$$

and define

$$\psi_{\bullet\,\mu} \equiv \psi_{1\,\mu}\,, \qquad \epsilon_\bullet \equiv \epsilon_1\,. \tag{A.15}$$

The supersymmetry transformations of the two gravitini become

$$\delta_\epsilon\psi_{\bullet\,\mu} = \left(\nabla_\mu + \tfrac{i}{2}\mathcal{Q}_\mu + \mathsf{A}_{\mu\,1}{}^1\right)\epsilon_\bullet\,, \tag{A.16}$$

$$\delta_\epsilon\psi_{2\,\mu} = \mathsf{A}_{\mu\,1}{}^2\epsilon_\bullet - T^+{}_{\mu\nu}\gamma^\nu\epsilon^\bullet = 0\,. \tag{A.17}$$

This means that the component $\mathsf{A}_{\mu\,1}{}^{1}$ of the $SU(2)$ connection has to be integrated into the Kähler connection and the component $\mathsf{A}_{\mu\,1}{}^{2}$ and the graviphoton field strength has to be set to zero

$$\mathsf{A}_{\mu\,1}{}^{2} \;=\; 0\,, \tag{A.18}$$

$$T^{+}{}_{\mu\nu} \;=\; 0\,. \tag{A.19}$$

The supersymmetry transformation rule of the graviton becomes, simply

$$\delta_\epsilon e^a{}_\mu = -\tfrac{i}{4}\bar{\psi}_{\bullet\,\mu}\gamma^a\epsilon^\bullet + \text{c.c.} \tag{A.20}$$

Let us now consider the $N = 2$ vector multiplets.

The most general solution to the constraint Eq. (A.19) is to see it as an orthogonality condition between the graviphoton projector and the vector fields [25, 26]. The $N = 2$ vector index is split $\mathbf{\Lambda} = (\Lambda, X)$, where $\Lambda = 1, \cdots, n$ and $X = 0, 1, \cdots, \mathbf{n_V} - n_V \equiv n_C$ and

$$\mathcal{T}_\Lambda = 2i\mathcal{L}^{\mathbf{\Sigma}}\Im\mathrm{m}\,\mathcal{N}_{\mathbf{\Sigma}\Lambda} = 0\,, \qquad F^{X+}{}_{\mu\nu} = 0\,. \tag{A.21}$$

The $N = 2$ vector multiplets in the range Λ give only $N = 1$ vector multiplets (the chiral multiplets have to be truncated) and those in the range X give only chiral $N = 1$ multiplets (the $N = 1$ vector multiplets must be truncated). Since the dual vector field strengths

$$F_X{}^{+}{}_{\mu\nu} = \mathcal{N}_{X\mathbf{\Lambda}}F^{\mathbf{\Lambda}+}{}_{\mu\nu} = \mathcal{N}_{X\Lambda}F^{\Lambda+}{}_{\mu\nu} + \mathcal{N}_{XY}F^{Y+}{}_{\mu\nu}\,, \tag{A.22}$$

must also vanish for consistency, the off-diagonal blocks of the period matrix must also vanish

$$\mathcal{N}_{X\Lambda} = 0\,, \tag{A.23}$$

and, therefore

$$\mathcal{T}_\Lambda = 2i\mathcal{L}^{\Sigma}\Im\mathrm{m}\,\mathcal{N}_{\Sigma\Lambda} = 0 \;\; \Rightarrow \mathcal{L}^\Lambda = 0\,. \tag{A.24}$$

Only the components $\mathcal{L}^X$ survive, and, together with the period matrix $\mathcal{N}_{XY}$, define a special Kähler manifold of dimension $\mathbf{n_V} - n_V = n_C$ and with Kähler metric

$$\mathcal{G}_{ij^*} = -2\Im\mathrm{m}\,\mathcal{N}_{XY}f^X{}_i f^Y{}_{j^*}\,, \qquad i,j = 1, \cdots, n_C\,. \tag{A.25}$$

The diagonal block

$$\mathcal{N}_{\Lambda\Sigma} \equiv \tfrac{1}{2}f^*_{\Lambda\Sigma}\,, \tag{A.26}$$

determines the couplings of the scalars of the chiral multiplets to the vectors. It can be shown that $f_{\Lambda\Sigma}$ is a holomorphic function of the Z^is.

The consistency of these conditions leads to several conditions that the special Kähler manifold has to satisfy on order to be reducible to $N = 1$ and can be found in [25, 26].

It is convenient to study the supersymmetry transformations of the two gaugini in the form

$$f^{\mathbf{\Lambda}}{}_{\mathbf{i}}\delta_\epsilon\lambda^{I\mathbf{i}} = if^{\mathbf{\Lambda}}{}_{\mathbf{i}}\ \not{\partial}Z^{\mathbf{i}}\epsilon^I + \tfrac{1}{2}\ \not{F}^{\mathbf{\Lambda}+}\epsilon^{IJ}\epsilon_J\,, \tag{A.27}$$

where we have used the constraint Eq. (A.19). Then, splitting the index $\mathbf{i} = (\alpha, i)$ with $\alpha = 1, \cdots, n$ and $i = 1, \cdots, n_C$, the above equation splits as follows

$$f^{\Lambda}{}_{\mathbf{i}}\delta_\epsilon\lambda^{1\mathbf{i}} \;=\; 0\,, \tag{A.28}$$

$$f^{X}{}_{i}\delta_\epsilon\lambda^{1i} \;=\; if^{X}{}_{i}\ \not{\partial}Z^i\epsilon^{\bullet}\,, \tag{A.29}$$

$$f^{\Lambda}{}_{\alpha}\delta_\epsilon\lambda^{2\alpha} \;=\; \tfrac{1}{2}\ \not{F}^{\mathbf{\Lambda}+}\epsilon_{\bullet}\,, \tag{A.30}$$

$$f^{X}{}_{i}\delta_\epsilon\lambda^{2i} \;=\; 0\,. \tag{A.31}$$

Then, we define the $N = 1$ gaugini and dilatini

$$\lambda_{\bullet}{}^{\Lambda} \;\equiv\; -f^{\Lambda}{}_{\alpha}\lambda^{2\alpha}\,, \tag{A.32}$$

$$\chi_{\bullet}{}^{i} \;\equiv\; \lambda^{1i}\,, \tag{A.33}$$

and set to zero all the other components. Their resulting supersymmetry transformation rules are

$$\delta_\epsilon\lambda_{\bullet}{}^{\Lambda} \;=\; \tfrac{1}{2}\ \not{F}^{\mathbf{\Lambda}+}\epsilon_{\bullet}\,, \tag{A.34}$$

$$\delta_\epsilon\chi_{\bullet}{}^{i} \;=\; i\ \not{\partial}Z^i\epsilon^{\bullet}\,. \tag{A.35}$$

The supersymmetry transformation rules of the vector fields are split in

$$\delta_\epsilon A^{\Lambda}{}_{\mu} \;=\; \tfrac{i}{8}\bar{\lambda}_{\bullet}{}^{\Lambda}\gamma_\mu\epsilon^{\bullet} + \text{c.c.}\,, \tag{A.36}$$

$$\delta_\epsilon A^{X}{}_{\mu} \;=\; 0\,. \tag{A.37}$$

Finally, the supersymmetry transformation rules of the scalars split into

$$\delta_\epsilon Z^i \;=\; \tfrac{1}{4}\bar{\chi}_\bullet{}^i \epsilon_\bullet\,, \tag{A.38}$$

$$\delta_\epsilon Z^\alpha \;=\; 0\,. \tag{A.39}$$

Let us now consider the truncation in the hypermultiplet sector. The $4n_H$ real dimensional quaternionic-Kähler manifold has to be truncated to a n_H complex dimensional Kähler manifold [25, 26]. The truncation can only be done in some quaternionic-Kähler manifold: if we split the $Sp(2n_H)$ index α into $A, \dot{A} = 1, \cdots, n_H$ and the undotted indices correspond to the sector which will survive, the components $\Omega_{\dot{A}\dot{B}\dot{C}\dot{D}}$ must vanish identically. If this condition is satisfied, then one can set

$$\mathsf{U}^{2A} = \mathsf{A}^1 = \mathsf{A}^2 = \Delta^A{}_{\dot{B}} = \zeta_{\dot{A}} = 0\,, \tag{A.40}$$

consistently. The surviving components of the Quadbein are U^{1A} and its complex conjugate $\mathsf{U}^{2\dot{A}}$ which can be expressed in terms of just n_H holomorphic coordinates w^s.

The independent non-vanishing supersymmetry transformation rules of the hyperscalars and the hyperinos are

$$(\mathsf{U}_{1Au})^* \delta_\epsilon q^u \;=\; \tfrac{1}{4}\bar{\zeta}^{\bullet A}\epsilon^\bullet\,, \tag{A.41}$$

$$\mathsf{U}^{1Au}\delta_\epsilon \zeta_{\bullet A} \;=\; i\,\not{\partial} q^u \epsilon^\bullet\,. \tag{A.42}$$

Using the holomorphic coordinates w^s we now define the n_H $N = 1$ dilatini ζ^s

$$\zeta_\bullet{}^s \equiv \mathsf{U}^{1As}\zeta_{\bullet A}\,, \tag{A.43}$$

and the above supersymmetry transformation rules take the standard form

$$\delta_\epsilon w^s \;=\; \tfrac{1}{4}\bar{\zeta}_\bullet{}^s \epsilon_\bullet\,, \tag{A.44}$$

$$\delta_\epsilon \zeta_\bullet{}^s \;=\; i\,\not{\partial} w^s \epsilon^\bullet\,. \tag{A.45}$$

The quaternionic Kähler manifolds that can be truncated to $N = 1$ chiral multiplets are precisely those in which one can construct cosmic string solutions (*hyperstrings*): in Ref. [4] the supersymmetry equations were solved by choosing a metric of the form

$$ds^2 = dt^2 - (dx^3)^2 - 2e^{\Phi(z,z^*)}dzdz^*\,, \tag{A.46}$$

hyperscalars which are real functions of the complex coordinate z and its complex conjugate $q^u(z, z^*)$. In these conditions, the supersymmetry conditions take the form

$$\mathsf{U}^{\alpha 2}{}_u \partial_{\underline{z}} q^u = \mathsf{U}^{\alpha 1}{}_u \partial_{\underline{z}^*} q^u = 0\,, \tag{A.47}$$

$$\varpi_{\underline{z}}{}^{zz^*} = \mathsf{A}^3{}_u\, \partial_{\underline{z}} q^u\,, \tag{A.48}$$

$$\mathsf{A}^1{}_u\, \partial_{\underline{m}} q^u = \mathsf{A}^2{}_u\, \partial_{\underline{m}} q^u = 0\,. \tag{A.49}$$

They are clearly solved by setting $\mathsf{A}^1{}_u = \mathsf{A}^2{}_u = \mathsf{U}^{\alpha 2}{}_u = 0$ and, then, taking the hyperscalars to be holomorphic functions of z $\partial_{\underline{z}^*} q^u = 0$.

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