NONHOLONOMIC CONSTRAINTS IN k-SYMPLECTIC CLASSICAL FIELD THEORIES

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Abstract. A k-symplectic framework for classical field theories subject to nonholonomic constraints is presented. If the constrained problem is regular one can construct a projection operator such that the solutions of the constrained problem are obtained by projecting the solutions of the free problem. Symmetries for the nonholonomic system are introduced and we show that for every such symmetry, there exist a nonholonomic momentum equation. The proposed formalism permits to introduce in a simple way many tools of nonholonomic mechanics to nonholonomic field theories.

Contents

1. Introduction

During the past decades, much effort has been devoted to the differential geometric treatment of mechanical systems subject to nonholonomic constraints. To a large extent the growing interest in this field has been stimulated by its close connection to problems in control theory (see, for instance, [7, 10]). In the literature, one can distinguish mainly two different approaches in the study of systems

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subjected to a nonholonomic constraints. The first one, commonly called ***nonholonomic mechanics***, is based on the d'Alembert's principle. This principle specifies from the constraints a subbundle of the tangent bundle, representing the admissible infinitesimal virtual displacements. The second one is a constrained variational approach called *vakonomic mechanics* [2]. As is well know, the dynamical equations generated by both approaches are in general not equivalent [11].

In this paper we will study an extension of nonholonomic mechanics to classical field theories with external constraints. Nonholonomically constrained field theories have already been studied in the literature. The mathematical framework for a nonholonomic field theory that has been proposed in [6] involves, among others, a generalization of d'Alembert's principle and of the so-called Chetaev rule that is commonly used in nonholonomic mechanics to characterize the bundle of constraint forms representing the admissible reaction forces. The constrained field equations for classical field theories, are then derived in a finite-dimensional multisymplectic setting. In [42] the authors continue and extend the work described in [6].

The multisymplectic formalism, was developed by Tulczyjews school in Warsaw (see, for instance, [21]), and independently by García and Pérez-Rendón [13, 14] and Goldschmidt and Sternberg [16]. This approach was revised, between others, by Martin [30, 31] and Gotay et al [17] and, more recently, by Cantrijn et al [9] (see also [25] and references therein).

An alternative way to derive certain types of the field equations is to use the k-symplectic formalism. The k-symplectic formalism is the generalization to field theories of the standard symplectic formalism in Mechanics, which is the geometric framework for describing autonomous dynamical systems. In this sense, the k-symplectic formalism is used to give a geometric description of certain kinds of field theories: in a local description, those theories whose Lagrangian does not depend on the base coordinates, denoted by $(t^1, \dots, t^k)$ (in many of the cases defining the space-time coordinates); that is, the k-symplectic formalism is only valid for Lagrangians $L(q^i, v^i_A)$ and Hamiltonians $H(q^i, p^A_i)$ that depend on the field coordinates q^i and on the partial derivatives of the field v^i_A, or the corresponding moment p^A_i. A more general approach has been given in [29] using the k-cosymplectic formalism.

Günther's paper [19] gave a geometric Hamiltonian formalism for field theories. The crucial device is the introduction of a vector-valued generalization of a symplectic form, called a polysymplectic form. One of the advantages of this formalism is that one only needs the tangent and cotangent bundle of a manifold to develop it. In [34] Günther's formalism has been revised and clarified. It has been shown that the polysymplectic structures used by Günther to develop his formalism could be replaced by the k-symplectic structures defined by Awane [3, 4]. So this formalism is also called k-symplectic formalism (see also [26, 27, 28, 29]).

Let us remark here that the polysymplectic formalism developed by Sardanashvily [15, 39, 40], based on a vector-valued form defined on some associated fiber bundle, is a different description of classical field theories of first order than the polysymplectic (or k-symplectic) formalism proposed by Günther (see also [20] for more details). We must also remark that the soldering form on the linear frames bundle is a polysymplectic form, and its study and applications to field theory, constitute the n-symplectic geometry developed by L. K. Norris in [32, 35, 36, 37, 38].

The purpose of this paper is to give a k-symplectic setting for first-order classical field theories subject to nonholonomic constraints. In the k-symplectic setting we will construct, under an appropriate additional condition, a kind of projection operator that maps solutions of the free problem into solutions of the constrained problem. Nonholonomic symmetries are introduced and we show that for every such symmetry, there exist a nonholonomic momentum equation which reduces to a conservation law when the constraints are absent.

We analyze some particular cases, for instance, the case of a constraint submanifold $\mathcal{M}$ obtained as k-copies of a distribution D in Q has special interest. In this particular case, we construct a distribution H on $T^1_kQ = TQ\oplus \stackrel{k}{\dots} \oplus TQ$ (i.e. the Whitney sum of k copies of TQ) along $\mathcal{M}$ such that for each $w_q \in \mathcal{M}$, H_{w_q} is a k-symplectic subspace of the k-symplectic vector space $(T_{w_q}(T^1_kQ), \omega^1_L(w_q), \dots, \omega^k_L(w_q); V(w_q))$ where $(\omega^1_L, \dots, \omega^k_L; V)$ is the k-symplectic structure obtained from L. Thus if we restrict the k-symplectic structure of T^1_kQ to H, the equations of the constrained problem take the usual form for a free problem at each fibre of H. This procedure extends that by Bates and Sniatycki [5] for the linear case.

The scheme of the paper is as follows. In Section 2 we recall some basic elements from the k-symplectic approach to (unconstrained) Lagrangian classical field theories. In Section 3 we discuss the construction of a nonholonomic model for first-order Lagrangian Classical field theories with external constraints and we obtain the corresponding nonholonomic field equations. Next, in Section 4 we construct, under an appropriate additional condition, a projection operator which maps solutions of the free problem into solutions of the constrained problem. In Section 5 we derive the nonholonomic momentum equation. In Section 6 we analyze some particular cases and in Section 7 we briefly analyze the Hamiltonian case. Finally in Section 8 we conclude with some general comments.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood.

2. k-symplectic Lagrangian field theory

2.1. **Geometric elements.**

2.1.1. *The tangent bundle of k^1-velocities of a manifold.* Let $\tau_Q : TQ \to Q$ be the tangent bundle of Q. Let us denote by T^1_kQ the Whitney sum $TQ\oplus \stackrel{k}{\dots} \oplus TQ$ of k copies of TQ, with projection $\tau : T^1_kQ \to Q$, $\tau(v_{1\mathbf{q}}, \dots, v_{k\mathbf{q}}) = \mathbf{q}$, where $v_{A\mathbf{q}} \in T_{\mathbf{q}}Q$, $1 \leq A \leq k$.

T^1_kQ can be identified with the manifold $J^1_0(\mathbb{R}^k, Q)$ of the k^1-*velocities of* Q, that is, 1-jets of maps $\sigma : \mathbb{R}^k \to Q$ with source at $0 \in \mathbb{R}^k$, say

$$\begin{array}{ccc} J^1_0(\mathbb{R}^k, Q) & \equiv & TQ\oplus \stackrel{k}{\dots} \oplus TQ \\ j^1_{0,\mathbf{q}}\sigma & \equiv & (v_{1\mathbf{q}}, \dots, v_{k\mathbf{q}}) \end{array}$$

where $\mathbf{q} = \sigma(0)$, and $v_{A\mathbf{q}} = \sigma_*(0)(\dfrac{\partial}{\partial t^A}\Big|_0)$. T^1_kQ is called *the tangent bundle of k^1-velocities of Q* (see [33]).

If (q^i) are local coordinates on $U \subseteq Q$ then the induced local coordinates (q^i, v^i), $1 \leq i \leq n$, on $TU = \tau_Q^{-1}(U)$ are given by

$$q^i(v_{\mathbf{q}}) = q^i(\mathbf{q}), \qquad v^i(v_{\mathbf{q}}) = v_{\mathbf{q}}(q^i)$$

and the induced local coordinates (q^i, v^i_A), $1 \leq i \leq n$, $1 \leq A \leq k$, on $T^1_kU = \tau^{-1}(U)$ are given by

$$q^i(v_{1\mathbf{q}}, \dots, v_{k\mathbf{q}}) = q^i(\mathbf{q}), \qquad v^i_A(v_{1\mathbf{q}}, \dots, v_{k\mathbf{q}}) = v_{A\mathbf{q}}(q^i)\,.$$

A. Vertical lifts of vector fields from Q to T^1_kQ.

Definition 2.1. *For a vector $X_{\mathbf{q}} \in T_qQ$, and for $A = 1, \ldots, k$, we define its vertical A-lift $(X_{\mathbf{q}})^{V_A}$ as the local vector field on the fiber $\tau^{-1}(\mathbf{q}) \subset T^1_kQ$ given by*

$$(X_{\mathbf{q}})^{V_A}(w_{\mathbf{q}}) = \frac{d}{ds}(v_{1\mathbf{q}}, \ldots, v_{A-1\mathbf{q}}, v_{A\mathbf{q}} + sX_{\mathbf{q}}, v_{A+1\mathbf{q}}, \ldots, v_{k\mathbf{q}})\Big|_{s=0}$$

for all points $w_{\mathbf{q}} = (v_{1\mathbf{q}}, \ldots, v_{k\mathbf{q}}) \in \tau^{-1}(\mathbf{q}) \subset T^1_kQ$.

In local coordinates, if $X_{\mathbf{q}} = a^i \dfrac{\partial}{\partial q^i}\Big|_{\mathbf{q}}$ then

$$(X_{\mathbf{q}})^{V_A}(w_{\mathbf{q}}) = a^i \frac{\partial}{\partial v^i_A}\Big|_{w_{\mathbf{q}}} \quad . \tag{2.1}$$

If X is a vector field on Q then we define its vertical A-lift to T^1_kQ, $1 \leq A \leq k$, as the vector field X^{V_A} given by

$$X^{V_A}(w_{\mathbf{q}}) = (X^i(\mathbf{q})\frac{\partial}{\partial q^i}\Big|_{\mathbf{q}})^{V_A}(w_{\mathbf{q}}) = X^i(\mathbf{q})\frac{\partial}{\partial v^i_A}\Big|_{w_{\mathbf{q}}} = (X^i \circ \tau)(w_{\mathbf{q}})\frac{\partial}{\partial v^i_A}\Big|_{w_{\mathbf{q}}} \quad ,$$

then

$$X^{V_A} = (X^i \circ \tau)\frac{\partial}{\partial v^i_A}$$

where $X = X^i \dfrac{\partial}{\partial q^i}$.

B. Complete lift of vector fields from Q to T^1_kQ.

Let $\Phi : Q \to Q$ be a differentiable map then the induced map $T^1_k\Phi : T^1_kQ \to T^1_kQ$ defined by $T^1_k\Phi(j^1_0\sigma) = j^1_0(\Phi \circ \sigma)$ is called the *canonical prolongation* of Φ and, it is also given by

$$T^1_k\Phi(v_{1\mathbf{q}}, \ldots, v_{k\mathbf{q}}) = (\Phi_*(\mathbf{q})(v_{1\mathbf{q}}), \ldots, \Phi_*(\mathbf{q})(v_{k\mathbf{q}})) \quad ,$$

where $v_{1\mathbf{q}}, \ldots, v_{k\mathbf{q}} \in T_{\mathbf{q}}Q$, $\mathbf{q} \in Q$.

If Z is a vector field on Q, with local 1-parametric group of transformations $h_s : Q \to Q$ then the local 1-parametric group of transformations $T^1_k(h_s) : T^1_kQ \to T^1_kQ$ which is the flow of the vector field Z^C on T^1_kQ, called the complete lift of Z. Its local expression is

$$Z^C = Z^i\frac{\partial}{\partial q^i} + v^j_A \frac{\partial Z^k}{\partial q^j}\frac{\partial}{\partial v^k_A} \, , \tag{2.2}$$

where $Z = Z^i \dfrac{\partial}{\partial q^i}$.

C. Canonical k-tangent structure.

The *canonical k-tangent structure* on T^1_kQ is the set $(S^1, \ldots, S^k)$ of tensor fields of type $(1,1)$ defined by

$$S^A(w_{\mathbf{q}})(Z_{w_{\mathbf{q}}}) = (\tau_*(w_{\mathbf{q}})(Z_{w_{\mathbf{q}}}))^{V_A}(w_{\mathbf{q}}), \quad \text{for all } Z_{w_{\mathbf{q}}} \in T_{w_{\mathbf{q}}}(T^1_kQ), \quad w_{\mathbf{q}} \in T^1_kQ,$$

for each $A = 1, \ldots, k$.

From (2.1) we have in local coordinates

$$S^A = \frac{\partial}{\partial v^i_A} \otimes dq^i \tag{2.3}$$

The tensors S^A can be regarded as the $(0,\ldots,0,\overset{A}{1},0,\ldots,0)$-lift of the identity tensor on Q to T^1_kQ defined in Morimoto [33].

In the case $k=1$, S^1 is the well-known canonical tangent structure (also called vertical endomorphism) of the tangent bundle (see [12, 18, 22]).

D. Canonical vector fields.

Let us denote by Δ the canonical vector field (Liouville vector field) of the vector bundle $\tau : T^1_kQ \to Q$. This vector field Δ is the infinitesimal generator of the following flow

$$\psi : \mathbb{R}\times T^1_kQ \longrightarrow T^1_kQ \quad , \quad \psi(s, v_{1\mathbf{q}},\ldots,v_{k\mathbf{q}}) = (e^s v_{1\mathbf{q}},\ldots,e^s v_{k\mathbf{q}})\,,$$

and in local coordinates it has the form

$$\Delta = \sum_{i=1}^{n}\sum_{B=1}^{k} v^i_B \frac{\partial}{\partial v^i_B}.$$

Δ is a sum of vector fields $\Delta_1+\ldots+\Delta_k$, where each Δ_A is the infinitesimal generator of the following flow $\psi^A : \mathbb{R}\times T^1_kQ \longrightarrow T^1_kQ$:

$$\psi^A(s, v_{1\mathbf{q}},\ldots,v_{k\mathbf{q}}) = (v_{1\mathbf{q}},\ldots,v_{A-1_\mathbf{q}}, e^s\, v_{A\mathbf{q}}, v_{A+1_\mathbf{q}},\ldots,v_{k\mathbf{q}})$$

and in local coordinates each $\triangle_A$ has the form

$$\Delta_A = \sum_{i=1}^{n} v^i_A \frac{\partial}{\partial v^i_A} \qquad 1\leq A\leq k\,. \tag{2.4}$$

2.1.2. *Second-order partial differential equations in T^1_kQ.*

k-vector fields and integral sections.

Let M be an arbitrary smooth manifold.

Definition 2.2. *A section $\mathbf{X} : M \longrightarrow T^1_kM$ of the projection τ will be called a* k-vector field *on M.*

Since T^1_kM is the Whitney sum $TM\oplus \overset{k}{\ldots} \oplus TM$ of k copies of TM, we deduce that to give a k-vector field $\mathbf{X}$ is equivalent to give a family of k vector fields $X_1,\ldots,X_k$ on M defined by projection onto each factor. For this reason we will denote a k-vector field by $(X_1,\ldots,X_k)$.

Definition 2.3. *An* integral section *of the k-vector field $\mathbf{X}=(X_1,\ldots,X_k)$, passing through a point $\mathbf{q}\in M$, is a map $\psi\colon U_0\subset\mathbb{R}^k\to M$, defined on some neighborhood U_0 of $0\in\mathbb{R}^k$, such that*

$$\psi(0)=\mathbf{q},\ \psi_*(\mathbf{t})\left(\frac{\partial}{\partial t^A}\Big|_t\right) = X_A(\psi(t)) \quad , \quad \text{for } \mathbf{t}\in U_0,\ 1\leq A\leq k$$

or, what is equivalent, ψ satisfies that $X\circ\psi=\psi^{(1)}$, where $\psi^{(1)}$ is the first prolongation of ψ to T^1_kM defined by

$$\begin{array}{rccl} \psi^{(1)}: & U_0\subset\mathbb{R}^k & \longrightarrow & T^1_kM \\ & \mathbf{t} & \longrightarrow & \psi^{(1)}(\mathbf{t}) = j^1_0\psi_\mathbf{t} \equiv \left(\psi_*(\mathbf{t})\left(\frac{\partial}{\partial t^1}\Big|_\mathbf{t}\right),\ldots,\psi_*(\mathbf{t})\left(\frac{\partial}{\partial t^k}\Big|_\mathbf{t}\right)\right)\,. \end{array}$$

A k-vector field $\mathbf{X}=(X_1,\ldots,X_k)$ on M is integrable *if there is an integral section passing through every point of M.*

In local coordinates, we have

$$\psi^{(1)}(t^1,\dots,t^k) = \left(\psi^i(t^1,\dots,t^k), \frac{\partial\psi^i}{\partial t^A}(t^1,\dots,t^k)\right), \; 1\leq A\leq k, \; 1\leq i\leq n\,, \quad (2.5)$$

and ψ is an integral section of $(X_1,\dots,X_k)$ if and only if the following equations holds:

$$\frac{\partial\psi^i}{\partial t^A} = (X_A)^i\circ\psi \quad 1\leq A\leq k, \; 1\leq i\leq n\,.$$

In the k-symplectic formalism, the solutions of the field equations are described as integral sections of some k-vector fields. Observe that, in case $k=1$, the definition of integral section coincides with the usual definition of integral curve of a vector field.

Second-order partial differential equations in T^1_kQ.

The aim of this subsection is to characterize the integrable k-vector fields on T^1_kQ whose integral sections are first prolongations $\phi^{(1)}$ of maps $\phi:\mathbb{R}^k\to Q$.

Definition 2.4. *A k-vector field $\xi=(\xi_1,\dots,\xi_k)$ on T^1_kQ, is called a* second order partial differential equation (SOPDE) *if it is a section of the vector bundle $T^1_k\tau : T^1_k(T^1_kQ)\to T^1_kQ$; that is,*

$$T^1_k\tau\circ(\xi_1,\dots,\xi_k) = Id_{T^1_kQ} \quad ,$$

or equivalently

$$\tau_*(w_{\mathbf{q}})(\xi_A(w_{\mathbf{q}})) = v_{A_{\mathbf{q}}} \qquad \textit{for all } A=1,\dots,k,$$

where $w_{\mathbf{q}}=(v_{1_{\mathbf{q}}},\dots,v_{k_{\mathbf{q}}})\in T^1_kQ$.

In the case $k=1$, this is just the definition of a second order differential equation (SODE).

From a direct computation in local coordinates we obtain that the local expression of a SOPDE $\xi=(\xi_1,\dots,\xi_k)$ is

$$\xi_A(q^i,v^i_A) = v^i_A\frac{\partial}{\partial q^i} + (\xi_A)^i_B\frac{\partial}{\partial v^i_B}, \quad 1\leq A\leq k\,, \qquad (2.6)$$

where $(\xi_A)^i_B$ are functions on T^1_kQ.

If $\psi:\mathbb{R}^k\to T^1_kQ$ is an integral section of $\xi=(\xi_1,\dots,\xi_k)$, locally given by $\psi(\mathbf{t})=(\psi^i(\mathbf{t}),\psi^i_B(\mathbf{t}))$, then from Definition 2.3 and (2.6) we deduce

$$\left.\frac{\partial\psi^i}{\partial t^A}\right|_{\mathbf{t}} = \psi^i_A(\mathbf{t})\,, \qquad \left.\frac{\partial\psi^i_B}{\partial t^A}\right|_{\mathbf{t}} = (\xi_A)^i_B(\psi(\mathbf{t}))\,. \qquad (2.7)$$

From (2.5) and (2.7) we obtain the following proposition.

Proposition 2.5. *Let $\xi=(\xi_1,\dots,\xi_k)$ be an integrable SOPDE. If ψ is an integral section of ξ then $\psi=\phi^{(1)}$, where $\phi^{(1)}$ is the first prolongation of the map $\phi=\tau\circ\psi : \mathbb{R}^k\xrightarrow{\psi}T^1_kQ\xrightarrow{\tau}Q$, and ϕ is solution to the system of second order partial differential equations*

$$\frac{\partial^2\phi^i}{\partial t^A\partial t^B}(\mathbf{t}) = (\xi_A)^i_B\left(\phi^i(\mathbf{t}),\frac{\partial\phi^i}{\partial t^C}(\mathbf{t})\right) \quad 1\leq i\leq n\,;1\leq A,B\leq k. \qquad (2.8)$$

Conversely, if $\phi:\mathbb{R}^k\to Q$ is any map satisfying (2.8) then $\phi^{(1)}$ is an integral section of $\xi=(\xi_1,\dots,\xi_k)$. ∎

From (2.8) we deduce that if ξ is an integrable SOPDE then $(\xi_A)^i_B = (\xi_B)^i_A$ for all $A, B = 1, \ldots, k$.

The following characterization of the SOPDE's, using the canonical k-tangent structure of T^1_kQ, can be obtained from (2.3), (2.4) and (2.6).

Proposition 2.6. *A k-vector field $\xi = (\xi_1, \ldots, \xi_k)$ on T^1_kQ is a* SOPDE *if, and only if, $S^A(\xi_A) = \Delta_A$, for all $A = 1\ldots, k$.*

(*Proof*) This is a direct consequence of the local expressions (2.3), (2.4) and (2.6) of S^A, Δ_A and ξ_A, respectively. ∎

2.2. **Lagrangian formalism: k-symplectic Lagrangian systems.** Consider a Lagrangian function $L : T^1_kQ \to \mathbb{R}$. We now define the *action integral*

$$\mathfrak{J}(\phi) = \int_{U_0} (L \circ \phi^{(1)})(\mathbf{t}) d^k\mathbf{t}\,,$$

where $d^k\mathbf{t} = dt^1 \wedge \ldots \wedge dt^k$ is a volume form on $\mathbb{R}^k$, $\phi : U_0 \subset \mathbb{R}^k \to Q$ is a map, with compact support, defined on an open set U_0 and $\phi^{(1)} : U_0 \subset \mathbb{R}^k \to T^1_kQ$ denotes the first prolongation of ϕ. A map ϕ is called an extremal for the above action if

$$\frac{d}{ds}\mathfrak{J}(\tau_s \circ \phi)\Big|_{s=0} = 0$$

for every flow τ_s on Q such that $\tau_s(\mathbf{q}) = \mathbf{q}$ for all $\mathbf{q}$ in the boundary of $\phi(U_0)$. Since such a flow τ_s is generated by a vector field $Z \in \mathfrak{X}(Q)$ vanishing on the boundary of $\phi(U_0)$, then we conclude that ϕ is an extremal if and only if

$$\int_{U_0} \left((\mathcal{L}_{Z^c} L) \circ \phi^{(1)} \right) (\mathbf{t}) d^k\mathbf{t} = 0\,,$$

for all Z satisfying the above conditions, where Z^c is the complete lift of Z to T^1_kQ. Putting $Z = Z^i \dfrac{\partial}{\partial q^i}$, taking into account the expression (2.2) for the complete lift Z^c and integrating by parts we deduce that $\phi(\mathbf{t}) = (\phi^i(\mathbf{t}))$ is an extremal of $\mathfrak{J}$ if and only if

$$\int_{U_0} \left[\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) - \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} \right] Z^i d^k\mathbf{t} = 0\,,$$

for all values of Z^i. Thus, ϕ will be an extremal of $\mathfrak{J}$ if and only if

$$\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})}\,. \tag{2.9}$$

The equations (2.9) are called the *Euler-Lagrange equations* for L.

We will now give a geometric version of the above equations.

We introduce a family of 1-forms θ^A_L on T^1_kQ, $1 \leq A \leq k$, using the k-tangent structure, as follows

$$\theta^A_L = dL \circ S^A \quad 1 \leq A \leq k\,,$$

which are locally given by

$$\theta^A_L = \frac{\partial L}{\partial v^i_A}\, dq^i \quad .$$

If we denote by $\omega^A_L = -d\theta^A_L$, in local coordinates we have

$$\omega^A_L = dq^i \wedge d\left(\frac{\partial L}{\partial v^i_A} \right) = \frac{\partial^2 L}{\partial q^j \partial v^i_A} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} dq^i \wedge dv^j_B\,. \tag{2.10}$$

Definition 2.7. *The Lagrangian $L : T^1_kQ \longrightarrow \mathbb{R}$ is said to be regular if, and only if, the matrix* $\left(\dfrac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right)$ *is not singular.*

Remark 2.8. Let us observe that L regular if and only if $(\omega^1_L, \dots, \omega^k_L)$ is a polysymplectic form and $(\omega^1_L, \dots, \omega^k_L; V = \operatorname{Ker}\tau_*)$, is a k-symplectic structure (see [34]) (See appendix for the introduction of some basic concepts on k-symplectic vector spaces). $\diamond$

Since $(T^1_kQ, \omega^1_L, \dots, \omega^k_L; V)$ is a k-symplectic manifold (see Appendix), we can define the vector bundle morphism,

$$\begin{array}{rcl} \Omega^\sharp_L : T^1_k(T^1_kQ) & \longrightarrow & T^*(T^1_kQ) \\ \mathbf{X} = (X_1, \dots, X_k) & \mapsto & \Omega^\sharp_L(X_1, \dots, X_k) = \operatorname{trace}(\imath_{X_B}\omega^A_L) = \displaystyle\sum_{A=1}^k \imath_{X_A}\omega^A_L \,. \end{array}$$

We now denote by $\mathfrak{X}^k_L(T^1_kQ)$ the set of k-vector fields $\xi_{\mathbf{L}} = (\xi^1_L, \dots, \xi^k_L)$ on T^1_kQ which are solutions to the equation

$$\Omega^\sharp_L(\xi^1_L, \dots, \xi^k_L) = \mathrm{d}E_L \,, \tag{2.11}$$

where $E_L = \Delta(L) - L$. The family $(T^1_kQ, \omega^A_L, E_L)$ is called a k-symplectic Lagrangian system. If each ξ^A_L is locally given by

$$\xi^A_L = (\xi^A_L)^i \frac{\partial}{\partial q^i} + (\xi^A_L)^i_B \frac{\partial}{\partial v^i_B} \,,$$

then $(\xi^1_L, \dots, \xi^k_L)$ is a solution to (2.11) if, and only if, $(\xi^A_L)^i$ and $(\xi^A_L)^i_B$ satisfy the system of equations

$$\begin{array}{rcl} \left(\dfrac{\partial^2 L}{\partial q^i \partial v^j_A} - \dfrac{\partial^2 L}{\partial q^j \partial v^i_A}\right)(\xi^A_L)^j - \dfrac{\partial^2 L}{\partial v^i_A \partial v^j_B}(\xi^A_L)^j_B & = & v^j_A \dfrac{\partial^2 L}{\partial q^i \partial v^j_A} - \dfrac{\partial L}{\partial q^i} \,, \\ \dfrac{\partial^2 L}{\partial v^j_B \partial v^i_A}(\xi^A_L)^i & = & \dfrac{\partial^2 L}{\partial v^j_B \partial v^i_A} v^i_A \quad . \end{array}$$

If the Lagrangian is regular, the above equations are equivalent to the equations

$$\frac{\partial^2 L}{\partial q^j \partial v^i_A} v^j_A + \frac{\partial^2 L}{\partial v^i_A \partial v^j_B}(\xi^A_L)^j_B = \frac{\partial L}{\partial q^i} \,, \quad 1 \le i \le n,\, 1 \le A \le k, \tag{2.12}$$

and

$$(\xi^A_L)^i = v^i_A \,. \tag{2.13}$$

Thus, if L is a regular Lagrangian, we deduce:

- If $\xi_{\mathbf{L}} = (\xi^1_L, \dots, \xi^k_L)$ is solution of (2.11), then it is a SOPDE, (see (2.13)).
- There are solutions of (2.11) in a neighborhood of each point of T^1_kQ and, using a partition of unity, global solutions to (2.11).
- Since $\xi_{\mathbf{L}} = (\xi^1_L, \dots, \xi^k_L)$ is a SOPDE, from Proposition 2.5 we know that if it is integrable then its integral sections are first prolongation $\phi^{(1)} : \mathbb{R}^k \to T^1_kQ$ of maps $\phi : \mathbb{R}^k \to Q$, and from (2.12) we deduce that ϕ is solution of the Euler-Lagrange equations (2.9).

  So, equations (2.11) can be considered as a geometric version of the Euler-Lagrange field equations.
- The equation (2.11) in the case $k = 1$ is $\imath_\xi\omega_L = dE_L$, that is the dynamical equation of the Lagrangian formalism in Mechanics.

Along this paper the family (T^1_kQ,ω^A_L,E_L) will be called a *k-symplectic Lagrangian system.*

3. Nonholonomic Lagrangian Classical field theory

We now bring constraints into the picture. Suppose we have a Lagrangian k-symplectic system on T^1_kQ, with a regular Lagrangian L. Let $\mathcal{M}\hookrightarrow T^1_kQ$ be a submanifold of T^1_kQ of codimension m, representing some external constraints imposed on the system. Although one can consider more general situations, for the sake of clarity we will confine ourselves to the case that $\mathcal{M}$ projects onto the whole of Q, i.e. $\tau(\mathcal{M})=Q$ and, the restriction $\tau|_{\mathcal{M}}\colon\mathcal{M}\to Q$ of τ to $\mathcal{M}$ is a (not necessarily affine) fibre bundle.

Since $\mathcal{M}$ is a submanifold of T^1_kQ, one may always find a covering $\mathcal{U}$ of $\mathcal{M}$ consisting in open subsets U of T^1_kQ, with $\mathcal{M}\cap U\neq\emptyset$, such that on each $U\in\mathcal{U}$ there exist m functionally independent smooth functions Φ_α that locally determine $\mathcal{M}$, i.e.

$$\mathcal{M}\cap U=\{w_{\mathbf{q}}=(v_{1_{\mathbf{q}}},\dots,v_{k_{\mathbf{q}}})\in T^1_kQ|\;\Phi_\alpha(w_{\mathbf{q}})=0\quad\text{for}\quad 1\leq\alpha\leq m\}\,.$$

The assumption that $\tau|_{\mathcal{M}}$ is a fibre bundle implies, in particular, that the matrix $(\partial\Phi_\alpha/\partial v^i_A)(w_{\mathbf{q}})$ has maximal rank m at each point $w_{\mathbf{q}}\in\mathcal{M}\cap U$.

3.1. The bundle of constraint forms.

We now introduce a special subbundle F of rank m of the bundle of $\mathbb{R}^k$-valued 1-forms on T^1_kQ defined along the constraint submanifold $\mathcal{M}$. The elements η of F are $\mathbb{R}^k$-valued 1-forms defined along $\mathcal{M}$ which are semi-basic, i.e. η vanishes on the τ-vertical vector fields.

The bundle F is locally generated by m independent $\mathbb{R}^k$-valued 1-forms η_α that locally read

$$\eta_\alpha=(\eta^1_\alpha,\dots,\eta^k_\alpha)=(\eta^1_{\alpha\,i}dq^i,\dots,\eta^k_{\alpha\,i}dq^i)\,,\tag{3.1}$$

for some smooth functions $\eta^A_{\alpha\,i}$ on $\mathcal{M}\subset T^1_kQ$. The independence of the forms η_α clearly implies that the $m\times kn$-matrix whose elements are the functions $\eta^A_{\alpha\,i}$, has constant maximal rank m (let us observe that m is exactly the codimension of $\mathcal{M}$). F is called the *bundle of constraints forms.*

Remark 3.1. One interesting case is when F is determined by $\mathcal{M}$ through application of a "Chetaev principle". If the constraint submanifold is giving by the vanishing of m functionally independent functions Φ_α on T^1_kQ, F is generated by the following $\mathbb{R}^k$-valued forms:

$$\eta_\alpha=({S^1}^*(d\Phi_\alpha),\dots,{S^k}^*(d\Phi_\alpha))=(\frac{\partial\Phi_\alpha}{\partial v^i_1}\,dq^i,\dots,\frac{\partial\Phi_\alpha}{\partial v^i_k}\,dq^i)\,.$$

◇

3.2. The constraint distribution.

In the sequel, we will show that the constraint bundle F gives rise to a distribution $\mathcal{S}$ along $\mathcal{M}$, called the *constraint distribution.* As above, take F generated by m $\mathbb{R}^k$-valued 1-forms $\eta_\alpha=(\eta^1_\alpha,\dots,\eta^k_\alpha)$ of the form (3.1).

Firstly we introduce the following vector bundle morphisms

$$\begin{array}{rcl}\Omega^\flat_L:T(T^1_kQ) & \longrightarrow & (T^1_k)^*(T^1_kQ)\\ X & \mapsto & \Omega^\flat_L(X)=(\imath_X\omega^1_L,\dots,\imath_X\omega^k_L)\,,\end{array}$$

where for an arbitrary manifold M we denote by $(T^1_k)^*M$ the Whitney sum $T^*M\oplus\overset{k}{\dots}\oplus T^*M$ of k-copies of T^*M.

For each α , ($\alpha = 1,\dots,k$), let $Z_\alpha \in \mathfrak{X}(T^1_kQ)$ be the unique local vector field on T^1_kQ defined by

$$\tau_*(Z_\alpha) = 0 \quad \text{and} \quad \Omega_L^\flat(Z_\alpha) = -\,\eta_\alpha\,. \tag{3.2}$$

If we write

$$Z_\alpha = (Z_\alpha)^j \frac{\partial}{\partial q^j} + (Z_\alpha)^j_B \frac{\partial}{\partial v^j_B}$$

we deduce from (2.10) and (3.2) that

$$(Z_\alpha)^i = 0\,, \quad (Z_\alpha)^j_B \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} = \eta^A_{\alpha\, i}\,,$$

which determines the $(Z_\alpha)^j_B$ uniquely, since L is supposed to be regular.

One obtains that:

$$Z_\alpha = W^{ij}_{AB}\, \eta^A_{\alpha\, i} \frac{\partial}{\partial v^j_B}\;,$$

where (W^{ij}_{AB}) denotes the inverse matrix of the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right)$.

The independence of the vector fields Z_α is consequence of the independence of the 1-forms η_α. Thus the vector fields Z_α span a m-dimensional distribution $\mathcal{S}$, which we will called *the constraint distribution*. This distribution $\mathcal{S}$ will be used in Section 4.

3.3. **The nonholonomic field equations.** Summarizing, we are looking for a nonholonomic field theory built on the following objects:

(i) a regular Lagrangian L;
(ii) a constraint submanifold $\mathcal{M} \hookrightarrow T^1_kQ$ which can be locally represented by equations of the form $\Phi_\alpha(q^i, v^i_A) = 0$ for $\alpha = 1,\dots,m$, where the matrix $(\partial\Phi_\alpha/\partial v^i_A)$ has maximal rank m;
(iii) a bundle F of constraint forms and an induced constraint distribution $\mathcal{S}$, both defined along $\mathcal{M}$, where F is generated by the m independent semibasic $\mathbb{R}^k$-valued 1-forms (3.1).

To complete our model for nonholonomic field theory, we now have to specify the field equations. We will now introduce the definition of a solution of the nonholonomic constrained problem using a generalization of d'Alembert's principle.

3.3.1. *The d'Alembert's principle.* Proceeding as in the case of unconstrained field theories, we introduce the following definition:

Definition 3.2. *A map $\phi : U_0 \subset \mathbb{R}^k \to Q$ defined on an open set $U_0 \subset Q$ with compact support, is a solution of the constrained problem under consideration if $\phi^{(1)}(U_0) \subset \mathcal{M}$ and*

$$\int_{U_0} \left((\mathcal{L}_{Z^c} L) \circ \phi^{(1)} \right)(\mathbf{t}) d^k\mathbf{t} = 0\,,$$

for each vector field Z on Q that vanish on the boundary of $\phi(U_0)$ and such that

$$\imath_{Z^c}\eta = 0 \tag{3.3}$$

for all η of the bundle F of constraint forms.

Putting $Z = Z^i \dfrac{\partial}{\partial q^i}$ and taking into account the expression (2.2) for the complete lift Z^c, it is easily seen that the condition (3.3) translates into

$$\eta^A_{\alpha\, i} Z^i = 0\,, \quad 1 \le \alpha \le m\,, \quad 1 \le A \le k\,,$$

where $\eta^A_{\alpha\, i}$ are the coefficients of the constraint forms introduced in (3.1).

One can verify that $\phi(\mathbf{t}) = (\phi^i(\mathbf{t}))$ is a solution of the constrained problem if and only if

$$\int_{U_0} \left[\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) - \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} \right] Z^i d^k\mathbf{t} = 0\,,$$

for all values of Z^i satisfying (3.3).

Therefore, a solution ϕ would satisfy the following system of partial differential equations:

$$\begin{array}{rcl} \dfrac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} - \displaystyle\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) & = & \lambda^\alpha_A\, \eta^A_{\alpha\, i}(\phi^{(1)}(\mathbf{t})) \quad (i = 1, \ldots, n)\,, \\ \Phi_\alpha(\phi^{(1)}(\mathbf{t})) & = & 0 \quad (\alpha = 1, \ldots, m)\,. \end{array} \tag{3.4}$$

As usual, the (a priori) unknown functions λ^α_A play the role "Lagrange multipliers". The equations (3.4) are called the *nonholonomic Lagrangian field equations* for the constrained problem (compare with [42]).

3.3.2. *Geometric description of the nonholonomic Lagrangian field equations.* Consider the following system of equations:

$$\Omega^\sharp_L(X_1, \ldots, X_k) - dE_L \in \langle \eta^B_\alpha \rangle\,, \quad X_A \in T\mathcal{M},\ 1 \le A \le k \tag{3.5}$$

along $\mathcal{M}$.

One obtains

Proposition 3.3. *Let $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable k-vector field solution of (3.5). We have*

(i) $\mathbf{X} = (X_1, \ldots, X_k)$ *is a* SOPDE.
(ii) *If $\phi^{(1)} = (\phi^i(\mathbf{t}), \partial\phi^i/\partial t^A)$ is an integral section of $\mathbf{X}$ then ϕ is solution of the nonholonomic Lagrangian field equations (3.4).*

(*Proof*) Let $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable k-vector field solution of (3.5). Taking into account that $\eta^B_\alpha = \eta^B_{\alpha\, i} dq^i$, (see (3.1)) one obtains that the equation (3.5) can be written as follow:

$$\sum_{A=1}^k \imath_{X_A} \omega^A_L - \mathrm{d}E_L = \lambda^\alpha_B\, \eta^B_{\alpha\, i}\, dq^i \tag{3.6}$$

where $E_L = \Delta L - L$.

If each X_A is locally given by

$$X_A = (X_A)^i \frac{\partial}{\partial q^i} + (X_A)^i_B \frac{\partial}{\partial v^i_B}\,,$$

then $(X_1, \ldots, X_k)$ is a solution to (3.6) if and only if $(X_A)^i$ and $(X_A)^i_B$ satisfy the system of equations

$$\begin{aligned} & \left(\frac{\partial^2 L}{\partial q^i \partial v^j_A} - \frac{\partial^2 L}{\partial q^j \partial v^i_A} \right) ((X_A)^j - v^j_A) \\ & \qquad - \left(\frac{\partial^2 L}{\partial q^j \partial v^i_A} (X_A)^j + \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} (X_A)^j_B - \frac{\partial L}{\partial q^i} \right) = \lambda^\alpha_B\, \eta^B_{\alpha\, i}\,, \\ & \frac{\partial^2 L}{\partial v^j_B \partial v^i_A} ((X_A)^i - v^i_A) = 0\ . \end{aligned}$$

If the Lagrangian is regular, then $(X_A)^j = v^j_A$, that is $(X_1, \ldots, X_k)$ is a SOPDE, and the above equations are equivalent to the equations

$$
\begin{aligned}
\frac{\partial^2 L}{\partial q^j \partial v^i_A} v^j_A + \frac{\partial^2 L}{\partial v^i_A \partial v^j_B} (X_A)^j_B - \frac{\partial L}{\partial q^i} &= -\lambda^\alpha_B \, \eta^B_{\alpha\, i} \quad (i = 1, \ldots, n) \\
(X_A)^i &= v^i_A \quad (A = 1, \ldots, k) .
\end{aligned}
\tag{3.7}
$$

We will now prove (ii).

Let $\phi^{(1)} = (\phi^i(\mathbf{t}), \partial\phi^i/\partial t^A)$ be an integral section of $(X_1, \ldots, X_k)$ passing through a point $w_{\mathbf{q}} \in \mathcal{M}$, that is,

$$
\phi^{(1)}(0) = w_{\mathbf{q}} \in \mathcal{M} \, , \quad v^j_A \circ \phi^{(1)} = \frac{\partial \phi^j}{\partial t^A} \, , \quad (X_A)^j_B \circ \phi^{(1)} = \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \, . \tag{3.8}
$$

Substituting (3.8) in the first group of equations (3.7) we obtain the equations

$$
\sum_{A=1}^k \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) - \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} = - \, \lambda^\alpha_B \, \eta^B_{\alpha\, i}(\phi^{(1)}(t)) \quad (i = 1, \ldots, n) \, ,
$$

which are the first group of equations in (3.4).

Finally, from (3.8) and since $X_A\big|_{\mathcal{M}} \in T\mathcal{M}$ we have

$$
0 = X_A(\Phi_\alpha) = v^i_A \frac{\partial \Phi_\alpha}{\partial q^i} + (X_A)^j_B \frac{\partial \Phi_\alpha}{\partial v^j_B} = \frac{\partial \phi^i}{\partial t^A} \frac{\partial \Phi_\alpha}{\partial q^i} + \frac{\partial^2 \phi^j}{\partial t^A \partial t^B} \frac{\partial \Phi_\alpha}{\partial v^j_B} = \frac{\partial (\Phi_\alpha \circ \phi^{(1)})}{\partial t^A}\Big|_{\mathbf{t}}
$$

then $\Phi_\alpha \circ \phi^{(1)}$ is a constant function and since $\phi^{(1)}(0) \in \mathcal{M}$ one obtains $\Phi_\alpha(\phi^{(1)}(0)) = 0$ and thus $\Phi_\alpha(\phi^{(1)}(\mathbf{t})) = 0$, that is, the second group of the equations in (3.4). Therefore, we conclude that ϕ is solution to the equation (3.4). ∎

4. The nonholonomic projector

The purpose of the present section is to show that for a nonholonomic first-order field theory in the sense described above, one can construct, under an appropriate additional condition, a projection operator which maps solutions of the equation (2.11) for the unconstrained Lagrangian problem into solutions of the nonholonomic equations (3.5).

Given a *constrained problem* with regular Lagrangian L, constraint manifold $\mathcal{M} \subset T^1_k Q$ and constraint distribution $\mathcal{S}$, we now impose the following *compatibility condition*: for each $w_{\mathbf{q}} \in \mathcal{M}$

$$
T_{w_{\mathbf{q}}}\mathcal{M} \cap \mathcal{S}(w_{\mathbf{q}}) = \{0\} \, . \tag{4.1}
$$

If $\mathcal{M}$ is locally given by m equations $\Phi_\alpha(q^i, v^i_A) = 0$ and, if $\mathcal{S}$ is locally generated by the vector fields Z_α (see Subsection 3.2), a straightforward computation shows that the compatibility condition is satisfied if and only if

$$
\det(Z_\alpha(\Phi_\beta)(w_{\mathbf{q}})) \neq 0 \, ,
$$

at each point $w_{\mathbf{q}} \in \mathcal{M}$. Indeed, take $v \in T_{w_{\mathbf{q}}}\mathcal{M} \cap \mathcal{S}(w_{\mathbf{q}})$. Then $v = v^\alpha Z_\alpha(w_{\mathbf{q}})$, for some coefficients v^α. On the other hand, $0 = v(\Phi_\beta) = v^\alpha Z_\alpha(\Phi_\beta)(w_{\mathbf{q}})$. Hence, if the matrix $(Z_\alpha(\Phi_\beta)(w_{\mathbf{q}}))$ is regular, we may conclude that $v = 0$ and the compatibility condition holds. The converse is similar: let us suppose that the compatibility condition holds. If the matrix $(Z_\alpha(\Phi_\beta)(w_{\mathbf{q}}))$ is not regular, then there exist some vector $v = v^\alpha Z_\alpha(w_{\mathbf{q}}) \neq 0$ such that $v(\Phi_\beta) = 0$ and thus $v \in T_{w_{\mathbf{q}}}\mathcal{M} \cap \mathcal{S}(w_{\mathbf{q}})$; therefore we conclude that if the compatibility condition holds, then the matrix $(Z_\alpha(\Phi_\beta)(w_{\mathbf{q}}))$ is regular.

We now have the following result.

Proposition 4.1. *If the compatibility condition* (4.1) *holds, then at each point* $w_{\mathbf{q}} \in \mathcal{M}$ *we have the decomposition*

$$T_{w_{\mathbf{q}}}(T^1_k Q) = T_{w_{\mathbf{q}}}\mathcal{M} \oplus \mathcal{S}(w_{\mathbf{q}})\,.$$

(*Proof*) The proof immediately follows from (4.1) and a simple counting of dimensions:

$$\begin{aligned} \dim T_{w_{\mathbf{q}}}\mathcal{M} \oplus \mathcal{S}(w_{\mathbf{q}}) &= \dim T_{w_{\mathbf{q}}}\mathcal{M} + \dim \mathcal{S}(w_{\mathbf{q}}) \\ &= (n + nk - m) + m = n + nk = \dim T_{w_{\mathbf{q}}}(T^1_k Q)\,. \end{aligned}$$

■

We now introduce the following notation: $T_{\mathcal{M}}(T^1_k Q)$ denotes the restriction of $T(T^1_k Q)$ to the submanifold of $T^1_k Q$, $\mathcal{M}$.

The direct decomposition of $T_{\mathcal{M}}(T^1_k Q)$ determines two complementary projection operators P and Q:

$$P : T_{\mathcal{M}}(T^1_k Q) \to T\mathcal{M}\,, \quad Q = I - P : T_{\mathcal{M}}(T^1_k Q) \to \mathcal{S}\,,$$

where I is the identity on $T_{\mathcal{M}}(T^1_k Q)$. The projectors P and Q are respectively written as follows:

$$P = I - \,\mathcal{C}^{\alpha\,\beta} Z_\alpha \otimes d\Phi_\beta\,, \quad Q = \mathcal{C}^{\alpha\,\beta} Z_\alpha \otimes d\Phi_\beta\,,$$

where $(\mathcal{C}^{\alpha\,\beta})$ is the inverse of the matrix $(\mathcal{C}_{\alpha\,\beta} := Z_\alpha(\Phi_\beta))$.

The direct sum decomposition of $T_{\mathcal{M}}(T^1_k Q)$ determines the following decomposition of $T^1_k(T^1_k Q)$ along $\mathcal{M}$:

$$T^1_k(T^1_k Q) = T^1_k\mathcal{M} \oplus \mathfrak{S}$$

where for each $w_{\mathbf{q}} \in \mathcal{M}$, $\mathfrak{S}_{w_{\mathbf{q}}}$ is given by

$$\mathfrak{S}_{w_{\mathbf{q}}} = \mathcal{S}(w_{\mathbf{q}}) \oplus \overset{k}{\dots} \oplus \mathcal{S}(w_{\mathbf{q}})\,.$$

If $\mathcal{S}$ is locally generated by the vector fields Z_α, then

$$\{(Z_\alpha, 0, \dots, 0),\, (0, Z_\alpha, 0, \dots, 0),\, \dots, (0, \dots, 0, Z_\alpha),\, (\alpha = 1, \dots, m)\}$$

is a local basis of $\mathfrak{S}$.

The direct sum decomposition of $T^1_k(T^1_k Q)$ along $\mathcal{M}$ determines two complementary projection operators $\mathcal{P}$ and $\mathcal{Q}$:

$$\mathcal{P} : (T^1_k)_{\mathcal{M}}(T^1_k Q) \to T^1_k\mathcal{M}\,, \quad \mathcal{Q} : (T^1_k)_{\mathcal{M}}(T^1_k Q) \to \mathfrak{S}\,,$$

given by $\mathcal{P}(X_{1 w_{\mathbf{q}}}, \dots, X_{k w_{\mathbf{q}}}) = (P(X_{1 w_{\mathbf{q}}}), \dots, P(X_{k w_{\mathbf{q}}}))$ and $\mathcal{Q} = \mathcal{I} - \mathcal{P}$, where $\mathcal{I}$ is the identity on $(T^1_k)_{\mathcal{M}}(T^1_k Q)$. Here $(T^1_k)_{\mathcal{M}}(T^1_k Q)$ denotes the restriction of $T^1_k(T^1_k Q)$ to the constraint submanifold $\mathcal{M}$.

Proposition 4.2. *Let* $\xi_L = (\xi^1_L, \dots, \xi^k_L)$ *be a solution of the free Lagrangian problem, i.e.,* ξ_L *is solution to the equation* (2.11)*, then* $\xi_{L,\mathcal{M}} = \mathcal{P}(\xi_L\big|_{\mathcal{M}})$ *is a solution to the constraint Lagrangian problem.*

(*Proof*) By definition of $\mathcal{P}$, we know that $\mathcal{P}(\xi_L\big|_{\mathcal{M}}) = \Big(P(\xi^1_L\big|_{\mathcal{M}}), \dots, P(\xi^k_L\big|_{\mathcal{M}})\Big)$ with $P(\xi^A_L\big|_{\mathcal{M}}) \in T\mathcal{M}$. Therefore $\Big(P(\xi^1_L\big|_{\mathcal{M}}), \dots, P(\xi^k_L\big|_{\mathcal{M}})\Big)$ is a solution to (3.5) if and only if $\Omega^\sharp_L\left(P(\xi^1_L\big|_{\mathcal{M}}), \dots, P(\xi^k_L\big|_{\mathcal{M}})\right) - \mathrm{d}E_L \in \langle \eta^A_\alpha \rangle$.

We have

$$\begin{aligned}
&\Omega_L^\sharp\left(P(\xi_L^1\Big|_{\mathcal{M}}),\dots,P(\xi_L^k\Big|_{\mathcal{M}})\right)-\mathrm{d}E_L\\
&=\Omega_L^\sharp\left(\xi_L^1\Big|_{\mathcal{M}}-Q(\xi_L^1\Big|_{\mathcal{M}}),\dots,\xi_L^k\Big|_{\mathcal{M}}-Q(\xi_L^k\Big|_{\mathcal{M}})\right)-dE_L\\
&=-\sum_{A=1}^k i_{\lambda_A^\alpha Z_\alpha}\omega_L^A=-\sum_{A=1}^k\lambda_A^\alpha i_{Z_\alpha}\omega_L^A=\sum_{A=1}^k\lambda_A^\alpha\eta_\alpha^A\in\langle\eta_\alpha^A\rangle\,.
\end{aligned}$$

Thus we conclude that $\xi_{L,M}=\left(P(\xi_L^1\Big|_{\mathcal{M}}),\dots,\mathcal{P}(\xi_L^k\Big|_{\mathcal{M}})\right)$ is a solution to (3.5). ■

Remark 4.3. In the particular case $k=1$ we recover the results in [23] ⋄

5. The nonholonomic momentum equation

In this section, we derive the nonholonomic momentum equation, the nonholonomic counterpart to the well-know Noether theorem. More precisely, we prove that for every *nonholonomic Lagrangian symmetry* there exists a certain partial differential equation which is satisfied by the solutions of the constrained problem, reducing to a conservation law when the constraints are absent.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider an action $\Phi\colon G\times Q\to Q$. The Lie group G acts on T_k^1Q by prolongation of Φ, i.e.

$$T_k^1\Phi_g(v_{1_\mathbf{q}},\dots,v_{k_\mathbf{q}})=((T_\mathbf{q}\Phi_g)(v_{1_\mathbf{q}}),\dots,(T_\mathbf{q}\Phi_g)(v_{k_\mathbf{q}}))\,.$$

Definition 5.1.

(i) *We say that the Lagrangian L is* invariant *under the group action if L is invariant under the induced action of G on T_k^1Q.*

(ii) *We say that the Lagrangian L is* infinitesimally invariant *if for any Lie algebra element $\xi\in\mathfrak{g}$ we have $\xi_Q^C(L)=0$, where for a vector field X on Q, X^C denotes the complete lift of X of Q to T_k^1Q and ξ_Q is the fundamental vector field defined by*

$$\xi_Q(\mathbf{q})=\frac{d}{ds}\,\Phi(exp(s\xi),\mathbf{q})\Big|_{s=0}\qquad\mathbf{q}\in Q\,.$$

When $\xi_Q^C(L)=0$, then ξ_Q will be called an infinitesimal Lagrangian symmetry.

Let us now assume that G leaves invariant L, $\mathcal{M}$ and F:

$$L\circ T_k^1\Phi_g=L,\quad T_k^1\Phi_g(\mathcal{M})\subset\mathcal{M}\quad\text{and}\quad(T_k^1\Phi_g)^*(F)\subset F$$

for all $g\in G$.

We consider the vector bundle $\mathfrak{g}^F$ over Q, defined as follows: denote by $\mathfrak{g}^F(\mathbf{q})$ the linear subspace of $\mathfrak{g}$ consisting of all $\xi\in\mathfrak{g}$ such that

$$\xi_Q^C(w_\mathbf{q})\rfloor F=0\quad\text{for all } w_\mathbf{q}\in\mathcal{M}\cap\tau^{-1}(\mathbf{q})\,.$$

We assume that the disjoint union of all $\mathfrak{g}^F(\mathbf{q})$, for all $\mathbf{q}\in Q$ can be given the structure of a vector bundle $\mathfrak{g}^F$ over Q.

To any section $\widetilde{\xi}$ of $\mathfrak{g}^F$, one can associate a vector field $\widetilde{\xi}_Q$ on Q according to the following definition:

$$\widetilde{\xi}_Q(\mathbf{q}):=[\widetilde{\xi}(\mathbf{q})]_Q(\mathbf{q})\,. \tag{5.1}$$

Definition 5.2. *For each A, the A^{th}-component of the nonholonomic momentum map $(J^{nh})^A$ is the map $(J^{nh})^A : \mathcal{M} \to (Sec(\mathfrak{g}^F))^*$ constructed as follows: let $\widetilde{\xi}$ be any section of $\mathfrak{g}^F$, then we define $(J^{nh})^A_{\widetilde{\xi}}$ along $\mathcal{M}$ as*

$$(J^{nh})^A_{\widetilde{\xi}} = \imath_{\widetilde{\xi}^C_Q}\theta^A_L \, , \tag{5.2}$$

where $\widetilde{\xi}_Q$ is the vector field associated to $\widetilde{\xi}$ according to (5.1).

Remark 5.3. In the particular case $k = 1$, corresponding to the Classical Mechanics, the above definition coincides with the definition of nonholonomic momentum map introduced by Marsden *et al* in [8]. $\diamond$

Remark 5.4. The map $(J^{nh})^A_{\xi}$ is the nonholonomic version of the A^{th}-component $\widehat{J}(0, \ldots, \overset{A}{\xi}, \ldots, 0) = \theta^A_L(\xi_{T^1_kQ})$, of the momentum map on the polysymplectic manifolds T^1_kQ defined in [34] when we consider the polysymplectic structure given by $\omega^A_L = -d\theta^A_L \, , 1 \leq A \leq k$. $\diamond$

The relevant role of the nonholonomic momentum map lies in the nonholonomic momentum equation.

Definition 5.5.

(i) *A* nonholonomic Lagrangian symmetry *is a section $\widetilde{\xi}$ of $\mathfrak{g}^F$ such that $\widetilde{\xi}^C_Q(L) = 0$.*

(ii) *A* horizontal nonholonomic symmetry *is a constant section of $\mathfrak{g}^F$.*

Theorem 5.6. *If $\phi \colon U_0 \subset \mathbb{R}^k \to Q$ is a solution of the nonholonomic field equations, then for any nonholonomic Lagrangian symmetry $\widetilde{\xi}$ the associated components of the momentum map $(J^{nh})^A_{\widetilde{\xi}}$ $(A = 1, \ldots, k)$ satisfies the following* nonholonomic momentum equation*:*

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left((J^{nh})^A_{\widetilde{\xi}(\phi(\mathbf{t}))}\right) = \sum_{A=1}^{k} (J^{nh})^A_{\frac{d}{dt^A}\widetilde{\xi}(\phi(\mathbf{t}))} \, .$$

along $\mathcal{M}$.

(*Proof*)

$$0 = \widetilde{\xi}^C_Q(L) = \widetilde{\xi}^i_Q \frac{\partial L}{\partial q^i} + v^j_A \frac{\partial \widetilde{\xi}^i_Q}{\partial q^j}\frac{\partial L}{\partial v^i_A} \, .$$

From (3.4) and taking into account that $\widetilde{\xi}_Q(\phi(\mathbf{t})) \in \mathfrak{g}^F$ one obtains that the above identity is equivalent to

$$0 = \widetilde{\xi}^i_Q \frac{d}{dt^A}\left(\frac{\partial L}{\partial v^i_A}\right) + v^j_A \frac{\partial \widetilde{\xi}^i_Q}{\partial q^j}\frac{\partial L}{\partial v^i_A} = \frac{d}{dt^A}\left(\widetilde{\xi}^i_Q \frac{\partial L}{\partial v^i_A}\right) - \left(\frac{d}{dt^A}\,\widetilde{\xi}\right)^i_Q \frac{\partial L}{\partial v^i_A} \, , \tag{5.3}$$

where the latter equality is consequence of

$$\begin{aligned} \frac{d}{dt^A}(\widetilde{\xi}^i_Q) &= \frac{\partial}{\partial t^A}(\widetilde{\xi}^i_Q(\phi(\mathbf{t}))) = \frac{\partial}{\partial t^A}\left(\frac{d}{ds}\Big|_{s=0} \exp(s\widetilde{\xi}(\phi(\mathbf{t}))) \cdot \phi(\mathbf{t})\right) \\ &= \frac{d}{ds}\Big|_{s=0} \exp(s\frac{\partial}{\partial t^A}\widetilde{\xi}(\phi(\mathbf{t}))) \cdot \phi(\mathbf{t}) + \frac{\partial \widetilde{\xi}^i_Q}{\partial q^j} v^j_A = \frac{\partial \widetilde{\xi}^i_Q}{\partial q^j} v^j_A + \left(\frac{d}{dt^A}\widetilde{\xi}\right)^i_Q \, . \end{aligned}$$

Finally, from (5.2) and (5.3) one obtains

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left((J^{nh})^A_{\widetilde{\xi}(\phi(\mathbf{t}))}\right) = \sum_{A=1}^{k} (J^{nh})^A_{\frac{d}{dt^A}\widetilde{\xi}(\phi(\mathbf{t}))} \, .$$

■

Corollary 5.7. *If $\widetilde{\xi}$ is a horizontal nonholonomic symmetry, then the following conservation laws holds:*

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left((J^{nh})^A_{\widetilde{\xi}(\phi(\mathbf{t}))} \right) = 0\,.$$

Remark 5.8. If we rewrite this section in the particular case $k = 1$ we reobtain the Section 4.2 in [8]. ◇

6. Particular cases

6.1. Holonomic constraints.

A distribution D on Q of codimension m induces an submanifold $\mathcal{M} \hookrightarrow T^1_kQ$ defined as follows: $(v_{1\mathbf{q}}, \ldots, v_{k\mathbf{q}})$ is an element of $\mathcal{M}$ if $v_{A\mathbf{q}} \in D(\mathbf{q})$ for each $A\,(A = 1, \ldots, k)$. In coordinates, if the annihilator D^0 is spanned by the 1-forms $\varphi_\alpha = \varphi_{\alpha i} dq^i\,(\alpha = 1, \ldots, m)$, then $\mathcal{M}$ is the set of solutions to the mk equations $\Phi^A_\alpha = \varphi_{\alpha i} v^i_A = 0$.

If D is integrable, the constraints induced by D are said to be *holonomic*: in this case, $\phi^{(1)}$ takes values in $\mathcal{M}$ if and only if ϕ takes values in a fixed leaf of the foliation induced by D, and we conclude that the constraints can be integrated to constraints on Q.

6.2. Linear constraints induced by distributions on Q.

The constrained problem: Let $D_1, \ldots, D_k$ be k distributions on Q and we consider the constraint submanifold $\mathcal{M} = D_1 \oplus \ldots \oplus D_k$ of T^1_kQ

If we will assume, for each A $(A = 1, \ldots, k)$, that D_A is defined by the vanishing of m_A functionally independent functions φ_{α_A} on Q, then proceeding as above we obtain that the constraint submanifold is given by the vanishing of $m = m_1 + \ldots + m_k$ independent functions $\Phi^A_{\alpha_A}$ where

$$\Phi^A_{\alpha_A}(v_{1_\mathbf{q}}, \ldots, v_{k_\mathbf{q}}) = \tau^*\varphi_{\alpha_A}(v_{A_\mathbf{q}}) = (\varphi_{\alpha_A})_i v^i_A\,.$$

For the bundle F of constraints forms we take the bundle along $\mathcal{M}$, generated by the m $\mathbb{R}^k$-valued 1-forms

$$\eta^A_{\alpha_A} = (S^{1^*}(d\Phi^A_{\alpha_A}), \ldots, S^{k^*}(d\Phi^A_{\alpha_A})) = (0, \ldots, \overset{A}{\tau^*\varphi_{\alpha_A}}, \ldots, 0)\,.$$

Nonholonomic field equations: In this particular case, a straightforward computation shows that the equations (3.4) become:

$$\left\{ \begin{array}{rcl} \displaystyle\sum_{A=1}^{k} \frac{\partial}{\partial t^A}\Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v^i_A}\Big|_{\phi^{(1)}(\mathbf{t})} \right) - \frac{\partial L}{\partial q^i}\Big|_{\phi^{(1)}(\mathbf{t})} & = & \lambda^{\alpha_A}\,(\varphi_{\alpha_A})_i(\phi(t)) \quad (i = 1, \ldots, n)\,, \\ & & \\ \Phi^A_{\alpha_A}(\phi^{(1)}(\mathbf{t})) & = & 0 \quad (\alpha_A = 1, \ldots, m_A\,, A = 1, \ldots, k)\,. \end{array} \right.$$

The constraint submanifold $\mathcal{M} = D \oplus \overset{k}{\ldots} \oplus D$. Let D be a distribution on Q. The particular case $D_1 = \ldots = D_k = D$ has special interest. As above, if we assume that D to be defined by the vanishing of m functionally independent functions φ_α on Q, then the constraint submanifold $\mathcal{M} = D \oplus \overset{k}{\ldots} \oplus D$ is given by the constraint functions

$$\Phi^A_\alpha(v_{1_\mathbf{q}}, \ldots, v_{k_\mathbf{q}}) = (\varphi_\alpha)_i\, v^i_A = 0\,.$$

We will denote by D^v the distribution on T^1_kQ defined by $(D^v)^0 = \langle \tau^*\varphi_\alpha\rangle$ (see [24, 23] for the case $k=1$). Next, we will prove the following two results (see Appendix for technical definitions).

Lemma 6.1. *$D^v_{w_{\mathbf{q}}}$ is k-coisotropic in $(T_{w_{\mathbf{q}}}(T^1_kQ), \omega^1_L(w_{\mathbf{q}}), \dots, \omega^k_L(w_{\mathbf{q}}), V(w_{\mathbf{q}}))$ for all $w_{\mathbf{q}} \in \mathcal{M}$, i.e. $(D^v)^{\perp}_{w_{\mathbf{q}}} \subset D^v_{w_{\mathbf{q}}}$.*

(*Proof*) In fact, since $(D^v)^0$ is locally generated by semi-basic 1-forms, we deduce that

$$(D^v)^{\perp}_{w_{\mathbf{q}}} = S(w_{\mathbf{q}}) \subset V_{w_{\mathbf{q}}}(T^1_kQ) \subset D^v_{w_{\mathbf{q}}}$$

for all $w_{\mathbf{q}} \in \mathcal{M}$, where $(D^v)^{\perp}_{w_{\mathbf{q}}} = \{U_{w_{\mathbf{q}}} \in T^1_kQ : \omega^A_L(U_{w_{\mathbf{q}}}, W_{w_{\mathbf{q}}}) = 0, \quad \forall \quad W_{w_{\mathbf{q}}} \in D^v_{w_{\mathbf{q}}}\}$ denotes the k-symplectic orthogonal of $D^v_{w_{\mathbf{q}}}$ and $V_{w_{\mathbf{q}}}(T^1_kQ)$ the vertical distribution of T^1_kQ at the point $w_{\mathbf{q}}$. ∎

Proposition 6.2. *The following properties are equivalent:*

(i) *The compatibility condition holds, that is, $T\mathcal{M} \cap \mathcal{S} = \{0\}$.*
(ii) *The distribution $H = TM \cap D^v$ along $\mathcal{M}$ is k-symplectic in the k-symplectic vector bundle $(T(T^1_kQ), \omega^1_L, \dots, \omega^k_L, V)$*

(*Proof*) If $T\mathcal{M} \cap S = \{0\}$ then

$$T\mathcal{M} \cap \mathcal{S} = T\mathcal{M} \cap (D^v)^{\perp} = 0$$

and

$$T_{w_{\mathbf{q}}}(T^1_kQ) = T_{w_{\mathbf{q}}}\mathcal{M} \oplus (D^v)^{\perp}_{w_{\mathbf{q}}} \quad \forall w_{\mathbf{q}} \in \mathcal{M}\,.$$

Hence from Lemma 6.1 we obtain

$$(D^v)_{w_{\mathbf{q}}} = (T_{w_{\mathbf{q}}}\mathcal{M} \cap (D^v)_{w_{\mathbf{q}}}) \oplus (D^v)^{\perp}_{w_{\mathbf{q}}} = H_{w_{\mathbf{q}}} \oplus (D^v)^{\perp}_{w_{\mathbf{q}}} = H_{w_{\mathbf{q}}} \oplus \mathcal{S}(w_{\mathbf{q}})\,.$$

Therefore, from a straightforward computation, we obtains that $H_{w_{\mathbf{q}}} \cap H^{\perp}_{w_{\mathbf{q}}} = \{0\}$ or, equivalently, that $H_{w_{\mathbf{q}}}$ is a k-symplectic vector subspace of $(T_{w_{\mathbf{q}}}(T^1_kQ), \omega^1_L(w_{\mathbf{q}}), \dots, \omega^k_L(w_{\mathbf{q}}), V_{w_{\mathbf{q}}})$.

Conversely, assume that for each $w_{\mathbf{q}} \in \mathcal{M}$, $H_{w_{\mathbf{q}}}$ is a k-symplectic subspace in $(T_{w_{\mathbf{q}}}(T^1_kQ), \omega^1_L(w_{\mathbf{q}}), \dots, \omega^k_L(w_{\mathbf{q}}), V_{w_{\mathbf{q}}})$, that is $H_{w_{\mathbf{q}}} \cap H^{\perp}_{w_{\mathbf{q}}} = \{0\}$. Take

$$Z \in T_{w_{\mathbf{q}}}\mathcal{M} \cap \mathcal{S}(w_{\mathbf{q}}) = T_{w_{\mathbf{q}}}\mathcal{M} \cap (D^v)^{\perp}_{w_{\mathbf{q}}} \subset T_{w_{\mathbf{q}}}\mathcal{M} \cap (D^v)_{w_{\mathbf{q}}} = H_{w_{\mathbf{q}}}\,.$$

Since $\omega^A_L(w_{\mathbf{q}})(Z, Y) = 0$ for all A ($A = 1, \dots, k$) and for all $Y \in H_{w_{\mathbf{q}}}$, we conclude that $Z \in H^{\perp}_{w_{\mathbf{q}}}$. Thus $Z \in H_{w_{\mathbf{q}}} \cap H^{\perp}_{w_{\mathbf{q}}} = \{0\}$ and therefore $Z = 0$. ∎

Consider now, the restrictions ω^A_H and d_HE_L to H of ω^A_L and dE_L, respectively. Since $H_{w_{\mathbf{q}}}$ is k-symplectic for each $w_{\mathbf{q}} \in \mathcal{M}$, there exist a solution on H of the equation

$$\sum_{A=1}^{k} \imath_{X_A}\omega^A_H = d_HE_L\,. \tag{6.1}$$

The above equation may be considered as the k-symplectic version of the characterization of nonholonomic mechanics in the case of linear constraints given by [5].

Proposition 6.3. *If $\xi_{L,\mathcal{M}}$ is a solution to the constrained problem, i.e. $\xi_{L,\mathcal{M}} = (\xi^1_{L,\mathcal{M}}, \dots, \xi^k_{L,\mathcal{M}})$ satisfies* (3.5), *if and only if $\xi_{L,\mathcal{M}}$ is solution of the equation* (6.1).

(*Proof*) $\xi_{L,\mathcal{M}}$ is solution of (3.5) then $\xi_{L,\mathcal{M}}$ is a SOPDE and $\xi^A_{L,\mathcal{M}} \in T_{w_{\mathbf{q}}}\mathcal{M}$.

On the other hand, since $\xi_{L,\mathcal{M}}$ is a SOPDE we have

$$\tau^*\varphi_\alpha(\xi^A_{L,\mathcal{M}}) = (\varphi_\alpha)_i v^i_A = 0 \,,$$

along $\mathcal{M}$, and thus $\xi^A_{L,\mathcal{M}} \in (D^v)_{w_{\mathbf{q}}}$. Therefore $\xi^A_{L,\mathcal{M}} \in H$ and $\xi_{L,\mathcal{M}}$ is trivially a solution of equation (6.1). Conversely, if $(X_1,\dots,X_k)$ is solution of the equation (6.1) then for each A $(A = 1,\dots,k)$, $X_A \in H \subset T\mathcal{M}$ and it is evident that $(X_1,\dots,X_k)$ is solution of (3.5). ∎

6.3. Linear constraints.

In the local picture, L is subjected to linear constraints defined by m local functions $\Phi_\alpha : T^1_kQ \to \mathbb{R}$ of the form

$$\Phi_\alpha(v_{1_{\mathbf{q}}},\dots,v_{k_{\mathbf{q}}}) = \Phi_\alpha(q^i, v^i_A) = \sum_{B=1}^k (\mu^B_\alpha)_i(\mathbf{q})\, v^i_B = \sum_{B=1}^k \mu^B_\alpha(v_{B_{\mathbf{q}}}) \,,$$

where μ^B_α be mk 1-forms on Q, $1 \leq \alpha \leq m$, $1 \leq B \leq k$, locally given by $\mu^B_\alpha = (\mu^B_\alpha)_i\, dq^i$.

We consider the constraint submanifold

$$\mathcal{M} = \{w_{\mathbf{q}} = (v_{1_{\mathbf{q}}},\dots,v_{k_{\mathbf{q}}}) \in T^1_kQ : \Phi_\alpha(w_{\mathbf{q}}) = 0 \quad \forall\alpha\}$$

of dimension $nk - m$.

We now denote by D the distribution on Q given by $D^0 = \langle \mu^B_\alpha \rangle$.

Proposition 6.4. *Let L be a regular Lagrangian and $\mathbf{X} = (X_1,\dots,X_k)$ is an integrable k-vector field which is a solution of*

$$\sum_{A=1}^k i_{X_A}\omega^A_L - dE_L \in (D^V)^0 \,, \quad X_A\big|_{\mathcal{M}} \in T\mathcal{M} \tag{6.2}$$

where $(D^V)^0 =< \tau^*\mu^B_\alpha >$. We have

i) $\mathbf{X} = (X_1,\dots,X_k)$ *is a* SOPDE.

ii) If $\phi^{(1)} = (\phi^i(\mathbf{t}), \partial\phi^i/\partial t^A)$ is an integral section of $\mathbf{X}$, then ϕ satisfy the equations (3.4)*.*

(*Proof*) Let us observe that $\eta^A_\alpha = S^{A^*}(d\Phi_\alpha) = \dfrac{\partial\Phi_\alpha}{\partial v^i_A}dq^i = (\mu^A_\alpha)_i dq^i = \tau^*\mu^A_\alpha$ and thus, in this particular case, $(D^V)^0 = \langle \eta^A_\alpha \rangle$.

Therefore, the equations (6.2) are equivalent to equations (3.5) for the case of linear constraints. ∎

6.4. Constraints defined by connections

Suppose that Q is a fibred manifold over a manifold M, say, $\rho : Q \to M$ is a surjective submersion. Assume that a connection Γ in $\rho : Q \to M$ is given such that

$$TQ = H \oplus V\rho \,,$$

where $V\rho = \ker T\rho$. We take fibred coordinates (q^a, q^α), $1 \leq a \leq n - m$, $1 \leq \alpha \leq m$, $n = \dim Q$. The horizontal distribution is locally spanned by the local vector fields

$$H_a = (\frac{\partial}{\partial q^a})^H = \frac{\partial}{\partial q^a} - \Gamma^\alpha_a(q^b, q^\beta)\frac{\partial}{\partial q^\alpha} \,,$$

where Y^H stands for the horizontal lift to Q of a vector field Y on M, and $\Gamma_a^\alpha(q^b, q^\beta)$ are the Christoffel symbols of Γ. Thus, we obtain a local basis of vector fields on Q,

$$\{H_a, V_\alpha = \frac{\partial}{\partial q^\alpha}\}\,.$$

Its dual basis of 1-forms is

$$\{\eta_a = dq^a\,,\, \eta_\alpha = \Gamma_a^\alpha\, dq^a + dq^\alpha\}\,.$$

We deduce that H^0 is locally spanned by the 1-forms $\{\eta_\alpha\}$.

In this situation we have

$$T_k^1 Q = H \oplus \stackrel{k}{\ldots} \oplus H \oplus V\rho \stackrel{k}{\ldots} \oplus V\rho\,,$$

and for each vector v_A, $1 \leq A \leq k$ we can write

$$v_A = v_A^a \frac{\partial}{\partial q^a} + v_A^\alpha \frac{\partial}{\partial q^\alpha} = v_A^a\left(\frac{\partial}{\partial q^a} - \Gamma_a^\alpha \frac{\partial}{\partial q^\alpha}\right) + (v_A^a \Gamma_a^\alpha + v_A^\alpha)\frac{\partial}{\partial q^\alpha} = v_A^H + v_A^V$$

We define

$$\mathcal{M} = H \oplus \stackrel{k}{\ldots} \oplus H \subset T_k^1 Q\,;$$

then $w_{\mathbf{q}} = (v_{1_{\mathbf{q}}}, \ldots, v_{k_{\mathbf{q}}}) \in \mathcal{M}$ if and only if $v_{A_{\mathbf{q}}} \in H$ for all $A = 1, \ldots, k$, which means that

$$v_A^\alpha = -v_A^a \Gamma_a^\alpha \quad \text{for all} \quad A = 1, \ldots, k$$

Thus,

$$\begin{aligned} \mathcal{M} &= \{w_{\mathbf{q}} \in T_k^1 Q \,:\, v_A^\alpha = -v_A^a \Gamma_a^\alpha,\, 1 \leq A \leq k\} \\ &= \{w_{\mathbf{q}} \in T_k^1 Q \,:\, \varphi_\alpha(v_{A_{\mathbf{q}}}) = 0,\, 1 \leq A \leq k\}\,. \end{aligned}$$

With the 1-forms $\varphi_\alpha = \Gamma_a^\alpha\, dq^a + dq^\alpha$ on Q we shall consider the 1-forms $\tau^*\varphi_\alpha$ on $T_k^1 Q$. We now consider the equations

$$\begin{aligned} &\Omega_L^\sharp(X_1, \ldots, X_k) - dE_L \in \langle \tau^*\varphi_\alpha \rangle \\ &X_A\big|_{\mathcal{M}} \in T\mathcal{M}\,, \quad 1 \leq A \leq k\,. \end{aligned} \tag{6.3}$$

As in the above example, in this particular situation, these equations are equivalent to equations (3.5).

We write the first equation in (6.3) as follows

$$\sum_{A=1}^k i_{X_A} \omega_L^A - dE_L = \sum_{\alpha=1}^m \lambda^\alpha\, \tau^*\varphi_\alpha$$

and each X_A as

$$X_A = (X_A)^a \frac{\partial}{\partial q^a} + (X_A)^\alpha \frac{\partial}{\partial q^\alpha} + (X_A)_B^a \frac{\partial}{\partial v_B^a} + X_A)_B^\alpha \frac{\partial}{\partial v_B^\alpha}\,.$$

From (6.3) we deduce the three following identities:

$$v_A^b \frac{\partial^2 L}{\partial q^b \partial v_A^a} + v_A^\beta \frac{\partial^2 L}{\partial q^\beta \partial v_A^a} + (X_A)_B^b \frac{\partial^2 L}{\partial v_B^b \partial v_A^a} + (X_A)_B^\beta \frac{\partial^2 L}{\partial v_B^\beta \partial v_A^a} - \frac{\partial L}{\partial q^a} = -\lambda_\beta \Gamma_a^\beta \tag{6.4}$$

$$v_A^b \frac{\partial^2 L}{\partial q^b \partial v_A^\alpha} + v_A^\beta \frac{\partial^2 L}{\partial q^\beta \partial v_A^\alpha} + (X_A)_B^b \frac{\partial^2 L}{\partial v_B^b \partial v_A^\alpha} + (X_A)_B^\beta \frac{\partial^2 L}{\partial v_B^\beta \partial v_A^\alpha} - \frac{\partial L}{\partial q^\alpha} = -\lambda_\beta \Gamma_\alpha^\beta \tag{6.5}$$

$$(X_A)^a = v_A^a\,, \quad (X_A)^\alpha = v_A^\alpha \tag{6.6}$$

If $\psi : U_0 \subset \mathbb{R}^k \to T^1_k Q$, $\psi(\mathbf{t}) = (\psi^a(\mathbf{t}), \psi^\alpha(\mathbf{t}), \psi^a_A(\mathbf{t}), \psi^\alpha_A(\mathbf{t}))$, is an integral section of $\mathbf{X} = (X_1, \dots, X_k)$, then from (6.4), (6.5) and (6.6) we deduce that ψ is solution to the equations

$$\left\{\begin{array}{l} v^a_A(\psi(\mathbf{t})) = \dfrac{\partial \psi^a}{\partial t^A}\Big|_{\mathbf{t}} \qquad ; \quad v^\alpha_A(\psi(\mathbf{t})) = \dfrac{\partial \psi^\alpha}{\partial t^A}\Big|_{\mathbf{t}} \\ \displaystyle\sum_{A=1}^k \frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial v^a_A}\Big|_{\psi(\mathbf{t})}\right) \quad - \quad \frac{\partial L}{\partial q^a}\Big|_{\psi(\mathbf{t})} = -\lambda_\alpha\, \Gamma^\alpha_a \\ \displaystyle\sum_{A=1}^k \frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial v^\alpha_A}\Big|_{\psi(\mathbf{t})}\right) \quad - \quad \frac{\partial L}{\partial q^\alpha}\Big|_{\psi(\mathbf{t})} = -\lambda_\alpha \end{array}\right.$$

or, in other words,

$$\left\{\begin{array}{l} v^a_A(\psi(\mathbf{t})) = \dfrac{\partial \psi^a}{\partial t^A}, \qquad v^\alpha_A(\psi(\mathbf{t})) = \dfrac{\partial \psi^\alpha}{\partial t^A} \\ \displaystyle\sum_{A=1}^k \frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial v^a_A}\Big|_{\psi(\mathbf{t})} - \Gamma^\alpha_a \frac{\partial L}{\partial v^\alpha_A}\Big|_{\psi(\mathbf{t})}\right) - \left(\frac{\partial L}{\partial q^a}\Big|_{\psi(\mathbf{t})} - \Gamma^\alpha_a \frac{\partial L}{\partial q^\alpha}\Big|_{\psi(\mathbf{t})}\right) = -\frac{d\Gamma^\alpha_A}{dt}\frac{\partial L}{\partial v^\alpha_A}\Big|_{\psi(\mathbf{t})} \end{array}\right.$$

These equations are the nonholonomic Euler-Lagrange equations (3.4) for this particular case.

6.5. The nonholonomic Cosserat rod

The nonholonomic Cosserat rod is an example of a nonholonomic field theory studied in [41] (see also [43]). It describes the motion of a rod which is constrained to roll without sliding on a horizontal surface.

A Cosserat rod can be visualized as specified by a curve $s \to \mathbf{r}(t)$ in $\mathbb{R}^3$, called the *centerline*, to which is attached a frame $\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$ called *director frame*. We consider an inextensible Cosserat rod of lenght l. If we denote the centerline at time t as $s \to \mathbf{r}(t,s)$, inextensibility allows us to assume that the parameter s is the arc length. The description of the Cosserat rod can be see in [41].

The nonholonomic second-order model. The model described in [41] fits into the multisymplectic framework developed on $J^1\pi$, where we have a fiber bundle $\pi : Y \to X$, where usually X plays the role of the space-time and the sections of this fiber bundle are the fields of the theory. In this particular case the base space X is $\mathbb{R} \times [0,l]$ (time and space), with coordinates (t,s) and the total space Y is $X \times \mathbb{R}^2 \times \mathbb{S}^1$, with fibre coordinates (x,y,θ). In this model, the fields are the coordinates of the centerline $(x(t,s), y(t,s))$ and the torsion angle $\theta(t,s)$

Its Lagrangian is given by

$$\mathcal{L} = \frac{\rho}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha}{2}\dot{\theta}^2 - \frac{1}{2}(\beta(\theta')^2 + Kk^2)\,,$$

where $k = (x'')^2 + (y'')^2$, while the constraints are given by

$$\dot{x} + R\dot{\theta}y' = 0 \quad \text{and} \quad \dot{y} - R\dot{\theta}x' = 0\,.$$

Here ρ, α, β, K and R are real parameters and $\dot{x} = \partial x/\partial t,\ x'\partial x/\partial s$ (analogous for y and θ). This model is a mathematical simplification of the real physical problem.

We now modify this model, by a lowering process to obtain a first-order Lagrangian: we introduce new variables $z = x'$ and $v = y'$ and obtain the modified Lagrangian

$$L = \frac{\rho}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha}{2}\dot{\theta}^2 - \frac{1}{2}(\beta(\theta')^2 + K((z')^2 + (v')^2)) + \lambda(z - x') + \mu(v - y')\,,$$

where λ and μ are Lagrange multiplier associated to the constraint $z = x'$ and $v = y'$. This Lagrangian can be thought as a mapping defined on $T^1_2 Q$ where $Q = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^4 \equiv \mathbb{R}^7$, and if we rewrite this with the notation introduced in the section 1 we obtain a k-symplectic model where the Lagrangian $L : T^1_2 Q \to \mathbb{R}$ is given by

$$L = \frac{\rho}{2}((v^1_1)^2 + (v^2_1)^2) + \frac{\alpha}{2}(v^3_1)^2 - \frac{\beta}{2}(v^3_2)^2 - \frac{K}{2}((v^4_2)^2 + (v^5_2)^2) + q^6(q^4 - v^1_2) + q^7(q^5 - v^2_2)$$

subject to constraints

$$v^1_1 + Rv^3_1 v^2_2 = 0 \quad \text{and} \quad v^2_1 - Rv^3_1 v^1_2 = 0\,. \tag{6.7}$$

In this case the bundle of reaction forces F is generated by the following forms:

$$\eta_1 = (dq^1 + Rv^2_2 dq^3, 0) \quad \text{and} \quad \eta_2 = (dq^2 - Rv^1_2 dq^3, 0)\,.$$

The nonholonomic fields equations associated to L are given by

$$\left\{\begin{array}{rcl}
\rho \dfrac{\partial^2 \phi^1}{\partial t^1 \partial t^1}\Big|_{\mathbf{t}} - \dfrac{\partial \phi^6}{\partial t^2}\Big|_{\mathbf{t}} & = & \lambda \\
\rho \dfrac{\partial^2 \phi^2}{\partial t^1 \partial t^1}\Big|_{\mathbf{t}} - \dfrac{\partial \phi^7}{\partial t^2}\Big|_{\mathbf{t}} & = & \mu \\
\alpha \dfrac{\partial^2 \phi^3}{\partial t^1 \partial t^1}\Big|_{\mathbf{t}} - \beta \dfrac{\partial^2 \phi^3}{\partial t^2 \partial t^2}\Big|_{\mathbf{t}} & = & R\left(\lambda \dfrac{\partial \phi^3}{\partial t^1}\Big|_{\mathbf{t}} - \mu \dfrac{\partial \phi^3}{\partial t^2}\Big|_{\mathbf{t}}\right) \\
K \dfrac{\partial^2 \phi^4}{\partial t^2 \partial t^2}\Big|_{\mathbf{t}} + \phi^6(\mathbf{t}) & = & 0 \\
K \dfrac{\partial^2 \phi^5}{\partial t^2 \partial t^2}\Big|_{\mathbf{t}} + \phi^7(\mathbf{t}) & = & 0 \\
\phi^4(\mathbf{t}) - \dfrac{\partial \phi^1}{\partial t^2}\Big|_{\mathbf{t}} & = & 0 \\
\phi^5(\mathbf{t}) - \dfrac{\partial \phi^2}{\partial t^2}\Big|_{\mathbf{t}} & = & 0\,,
\end{array}\right. \tag{6.8}$$

where λ and μ are Lagrange multipliers associated with the nonholonomic constraints, $\mathbf{t} = (t^1, t^2) = (t, s)$ are the coordinates time and space and the field $\phi : U_0 \subset \mathbb{R}^2 \to \mathbb{R}^7$ are the coordinates of the centerline $(\phi^1(\mathbf{t}), \phi^2(\mathbf{t}))$ and the torsion angle $\phi^3(\mathbf{t})$. As one can see in the equation (6.8) the components $\phi^i,\ i \geq 4$ are determined by (ϕ^1, ϕ^2, ϕ^3). These equations are supplemented by the constraint equations (6.7).

Consider the action of $\mathbb{R}^2 \times \mathbb{S}^1$ on Q according to the following definition: for each $(a, b, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ we consider the map $\Phi_{(a,b,\theta)}(q^1, \ldots, q^7) = (q^1 + a, q^2 + b, q^3 + \theta, q^4, \ldots, q^7)$.

It is easy to see that the following vector field annihilates F along $\mathcal{M}$:

$$\widetilde{\xi} = -Rv^2_2 \frac{\partial}{\partial q^1} + Rv^1_2 \frac{\partial}{\partial q^2} + \frac{\partial}{\partial q^3}\,.$$

This generalized vector field corresponds with the section $\widetilde{\xi} = (-Rv^2_2, Rv^1_2, 1)$ of $\tau^* \mathfrak{g}^F$.

As $\widetilde{\xi}(L) = 0$ the Theorem 5.6 can be applied and the *nonholonomic momentum equation* (see [43]) hence becomes

$$R\left(\rho \frac{\partial^2 \phi^1}{\partial t^1 \partial t^1} - \frac{\partial \phi^6}{\partial t^2}\right) \frac{\partial \phi^2}{\partial t^2} - R\left(\rho \frac{\partial^2 \phi^2}{\partial t^1 \partial t^1} - \frac{\partial \phi^7}{\partial t^2}\right) \frac{\partial \phi^1}{\partial t^2} = \alpha \frac{\partial^2 \phi^3}{\partial t^1 \partial t^1} - \beta \frac{\partial^2 \phi^3}{\partial t^2 \partial t^2}\,.$$

This nonholonomic conservation law can also be derived from the nonholonomic equations substituting the two first equation in (6.8) into the three equation. Unfortunately, the knowledge of this nonholonomic conservation law does not help us in solving the field equations.

Remark 6.5. Rewriting the nonholonomic momentum equation in the notation used in the description of the second order model, we obtain the nonholonomic momentum equation for spacial symmetries given in [43] into the multisymplectic setting. $\diamond$

7. Non-holonomic Hamiltonian field theory

We now turn to the Hamiltonian description of the nonholonomic system on the bundle of k^1-covelocities $(T^1_k)^*Q$ of Q.

The ***Legendre map*** $FL\colon T^1_kQ \to (T^1_k)^*Q$ is defined (see [19, 29]) as follows: if $(v_{1_\mathbf{q}},\dots,v_{k_\mathbf{q}}) \in (T^1_k)_\mathbf{q}Q$,

$$[FL(v_{1_\mathbf{q}},\dots,v_{k_\mathbf{q}})]^A(u_\mathbf{q}) = \frac{d}{ds}\Big|_{s=0} L(v_{1_\mathbf{q}},\dots,v_{A_\mathbf{q}}+su_\mathbf{q},\dots,v_{k_\mathbf{q}})\,,$$

for each $A=1,\dots,k$ and $u_\mathbf{q}\in T_\mathbf{q}Q$. Locally FL is given by

$$FL(q^i,v^i_A) = (q^i,\frac{\partial L}{\partial v^i_A})\,.$$

Assuming the regularity of the Lagrangian, we have that the Lagrangian and Hamiltonian formulations are locally equivalent. If we suppose that the Lagrangian L is hyperregular, the Legendre transformation is a global diffeomorphism.

The constraint function on $(T^1_k)^*Q$ becomes $\Psi_\alpha = \Phi_\alpha\circ FL^{-1} : (T^1_k)^*Q \to \mathbb{R}$, that is,

$$\Psi_\alpha(q^i,p^A_i) = \Phi_\alpha(q^i,\frac{\partial H}{\partial p^A_i})\,,$$

where the Hamiltonian function $H:(T^1_k)^*Q\to\mathbb{R}$ is defined by $H = E_L\circ FL^{-1}$. Since locally $FL^{-1}(q^i,p^A_i) = (q^i,\dfrac{\partial H}{\partial p^A_i})$, then

$$H = v^i_A\circ FL^{-1}\,p^A_i - L\circ FL^{-1}\,.$$

Thus, from (3.4), one obtains

$$\begin{aligned}
\frac{\partial H}{\partial p^A_i} &= v^i_A\circ FL^{-1}\\
\frac{\partial H}{\partial q^i} &= -\frac{\partial L}{\partial q^i}\circ FL^{-1} = -\lambda^\alpha_C\frac{\partial\Psi_\alpha}{\partial p^B_k}\mathcal{H}^{ki}_{BC} - \sum_{A=1}^k\frac{\partial}{\partial t^A}\left(\frac{\partial L}{\partial v^i_A}\circ FL^{-1}\right)\,,
\end{aligned}$$

where $\mathcal{H}^{ki}_{BC}$ are the components of the inverse of the matrix $(\mathcal{H}^{BC}_{ik}) = (\partial^2H/\partial p^B_i\partial p^C_k)$. Note that

$$\frac{\partial\Psi_\alpha}{\partial p^B_k}\mathcal{H}^{ki}_{BC} = \frac{\partial\varphi_\alpha}{\partial v^i_C}\circ FL^{-1}\,.$$

Therefore, the non-holonomic Hamiltonian equations on $(T^1_k)^*Q$ can be written as follows

$$\left\{\begin{array}{lcl} \dfrac{\partial H}{\partial p^A_i}\Big|_{\psi(\mathbf{t})} & = & \dfrac{\partial \psi^i}{\partial t^A}\Big|_{\mathbf{t}}, \\ \dfrac{\partial H}{\partial q^i}\Big|_{\psi(\mathbf{t})} & = & -\lambda^\alpha_C \dfrac{\partial \Psi_\alpha}{\partial p^B_k}\Big|_{\psi(\mathbf{t})} \mathcal{H}^{ki}_{BC}(\psi(\mathbf{t})) - \displaystyle\sum_{A=1}^{k} \dfrac{\partial \psi^A_i}{\partial t^A}\Big|_{\mathbf{t}} \\ 0 & = & \Psi_\alpha(\psi^i(\mathbf{t}), \psi^A_i(\mathbf{t})) \end{array}\right.$$

where $\psi : U \subset \mathbb{R}^k \to (T^1_k)^*Q$ is locally given by $\psi(\mathbf{t}) = (\psi^i(\mathbf{t}), \psi^A_i(\mathbf{t}))$.

Next we will give a geometrical description of these equations.

Let $M \subset (T^1_k)^*Q$ be the image of the constraint submanifold $\mathcal{M}$ under the Legendre map and let $\mathcal{F}$ the bundle locally generated by the independent $\mathbb{R}^k$-valued 1-form

$$\widetilde{\eta}_\alpha = FL^*\eta_\alpha, \quad 1 \leq \alpha \leq m\,.$$

Thus, the "Hamilton equations" for the nonholonomic problem can be rewritten in intrinsic form as

$$\sum_{A=1}^{k} i_{X_A}\omega^A - dH \in \langle \widetilde{\eta}^A_\alpha \rangle\,, \quad X_A\big|_M \in TM\,,$$

where $\widetilde{\eta}^A_\alpha = FL^*\eta^A_\alpha$.

8. Conclusions

We have studied various aspects of first-order classical field theories subject to nonholonomic constraints in the k-symplectic framework. The study is very similar to the case of particle mechanics (see [24]) and, also, the results are quite similar to those obtained in the multisymplectic framework (see [42]) but the k-symplectic approach seems simpler in many applications. We have shown that in the k-symplectic approach, the solutions to the equations for the constrained problem can be obtained by a projection of the solution to the equations for the unconstrained Lagrangian problem.

We analyze the particular case of a constraint submanifold $\mathcal{M}$ which we obtain as k-copies of a distribution on the configuration space Q. In this particular case, we construct a distribution H on T^1_kQ along $\mathcal{M}$ which is a k-symplectic subspace in $(T(T^1_kQ), \omega^1_L, \dots, \omega^k_L; V)$ where $(\omega^1_L, \dots, \omega^k_L; V)$ is the k-symplectic structure obtained from L. Finally, the nonholonomic momentum map is defined in a similar way than in classical mechanics. The applicability of the theory is shown in some examples and particular cases.

Appendix: k-symplectic vector spaces

Let U be a vector space of dimension $n(k+1)$, V a subspace of U of codimension n and $\omega^1, \dots, \omega^k$, k 2-forms on U. For each A ($A = 1, \dots, k$), $\ker\,\omega^A$ denotes the subspace associated to ω^A given by

$$\ker\,\omega^A = \{u \in U / \omega^A(u, v) = 0 \quad \forall\, v \in U\}\,.$$

Definition 8.1. *$(\omega^1, \dots, \omega^k; V)$ is a k-symplectic structure on U if*

$$\omega^A|_{V\times V} = 0\,, \quad \bigcap_{A=1}^{k} \ker\,\omega^A = 0\,.$$

We say that $(U, \omega^1, \ldots, \omega^k; V)$ *is a* k*-symplectic vector space.*

Let W be a liner subspace of U. The k-symplectic orthogonal of W is the linear subspace of U defined by

$$W^{\perp} = \{u \in U / \, \omega^A(u, w) = 0 \text{ for all } w \in W, A = 1, \ldots, k\} \,.$$

Proposition 8.2. *The* k*-symplectic orthogonal satisfies*

(i) $A \subset B \Rightarrow B^{\perp} \subset A^{\perp}$.
(ii) $W \subset (W^{\perp})^{\perp}$.

Remark 8.3. Unlike what happens in the symplectic vector spaces, in our context in general $\dim W + \dim W^{\perp} \neq \dim U$. In fact, considering for instance, the real space $\mathbb{R}^3$ equipped with the 2-symplectic structure defined by:

$$\omega^1 = e^1 \wedge e^3 \quad \omega^2 = e^2 \wedge e^3 \quad V = \ker e^3$$

where $\{e^1, e^2, e^3\}$ is the dual basis of the canonical basis $\{e_1, e_2, e_3\}$ of $U = \mathbb{R}^3$. We consider $W = \text{span}\,\{e_3\}$, the 2-symplectic orthogonal of W is $W^{\perp} = \text{span}\{e_3\}$. In this case, $\dim W + \dim W^{\perp} = 2 \neq \dim \mathbb{R}^3$. ◇

We can now introduce the following special types of subspaces of a k-symplectic vector space, generalizing the corresponding notions from symplectic geometry.

Definition 8.4. *Let* $(U, \omega^1, \ldots, \omega^k; V)$ *be a* k*-symplectic vector space and* W *a linear subspace of* U*.*

- W *is called* isotropic *if* $W \subset W^{\perp}$*.*
- W *is* coisotropic *if* $W^{\perp} \subset W$*.*
- W *is* Lagrangian *if* $W = W^{\perp}$*.*
- W *is* k-symplectic *if* $W \cap W^{\perp} = 0$*.*

Proposition 8.5. *For every vector subspace* W *of* U *the following properties are equivalent:*

(i) W *is an isotropic subspace.*
(ii) $\omega^A(u, v) = 0$ $(A = 1, \ldots, k)$ *for all* $u, v \in W$*.*

(*Proof*) Let us suppose that W is isotropic, then $W \subset W^{\perp}$. Therefore if u, $v \in W$ then $\omega^A(u, v) = 0$ since $u \in W^{\perp}$.

Conversely, $u \in W$, then for each $v \in W$, $\omega^A(u, v) = 0$ $(A = 1, \ldots, k)$. Therefore $u \in W^{\perp}$. Thus we can conclude that $W \subset W^{\perp}$. ■

Proposition 8.6. *Let* $(U, \omega^1, \ldots, \omega^k; V)$ *be a* k*-symplectic vector space and* W *a linear subspace of* U*.* W *is a* k*-symplectic subspace if and only if* W *with the restriction of the* k*-symplectic structure of* U *to* W *is a* k*-symplectic vector space.*

(*Proof*) Let us suppose that W is a k-symplectic subspace of $(U, \omega^1, \ldots, \omega^k; V)$. Consider the restriction ω^A_W to W of the 2-forms ω^A, we will now prove that $(W, \omega^1_W, \ldots, \omega^k_W, V \cap W)$ is a k-symplectic vector space.

Given $u, v \in V \cap W$ one obtains

$$\omega^A_W(u, v) = \omega^A|_{V \times V}(u, v) = 0 \; (A = 1, \ldots, k) \,.$$

On the other hand, if $u \in \cap \ker \omega^A_W$ then $\omega^A(u, v) = 0$ $(A = 1, \ldots, k)$ for all $v \in W$, then $u \in W^{\perp}$. Therefore, since $u \in W \cap W^{\perp} = \{0\}$ we deduce $\cap \ker \omega^A_W = \{0\}$.

Conversely, if $u \in W \cap W^{\perp}$ then $u \in W^{\perp}$, that is $\omega^A(u, v) = 0$ $(A = 1, \ldots, k)$ for all $v \in W$. Since $u \in W$ we obtain that $u \in \cap \ker \omega^A_W = \{0\}$ an therefore $u = 0$. ■

Definition 8.7. (Awane [3]) *A* k-symplectic structure *on a manifold* M *of dimension* $N = n + kn$ *is a family* $(\omega^A, V; 1 \leq A \leq k)$*, where each* ω^A *is a closed* 2*-form and* V *is an integrable* nk*-dimensional distribution on* M *such that*

$$(i) \quad \omega^A|_{V\times V} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker \omega^A = \{0\} \ .$$

Then (M, ω^A, V) *is called a* k-symplectic manifold.

Let us observe that if $(M, \omega^1, \ldots, \omega^k, V)$ is a k-symplectic manifold, then for each $x \in M$, we have that $(\omega^1_x, \ldots, \omega^k_x, V_x)$ is a k-symplectic structure on the vector space T_xM.

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