ON THE GEOMETRY OF MODULI SPACES OF COHERENT SYSTEMS ON ALGEBRAIC CURVES

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Abstract. Let C be an algebraic curve of genus $g \geq 2$. A coherent system on C consists of a pair (E, V), where E is an algebraic vector bundle over C of rank n and degree d and V is a subspace of dimension k of the space of sections of E. The stability of the coherent system depends on a parameter α. We study the geometry of the moduli space of coherent systems for different values of α when $k \leq n$ and the variation of the moduli spaces when we vary α. As a consequence, for sufficiently large α, we compute the Picard groups and the first and second homotopy groups of the moduli spaces of coherent systems in almost all cases, describe the moduli space for the case $k = n - 1$ explicitly, and give the Poincaré polynomials for the case $k = n - 2$. In an appendix, we describe the geometry of the "flips" which take place at critical values of α in the simplest case, and include a proof of the existence of universal families of coherent systems when $\mathrm{GCD}(n, d, k) = 1$.

Dedicated to the memory of Joseph Le Potier

1. Introduction

Let C be a smooth projective algebraic curve of genus $g \geq 2$. A coherent system on C of type (n, d, k) is a pair (E, V), where E is a vector bundle on C of rank n and degree d and V is a subspace of dimension k of the space of sections $H^0(E)$. Introduced in [16], [29] and [20], there is a notion of stability for coherent systems which permits the construction of moduli spaces. This notion depends on a real parameter, and thus leads to a family of moduli spaces. As described in [4], there is a useful relation between these moduli spaces and the Brill-Noether loci in the moduli spaces of semistable bundles of rank n and degree d.

In [6] we began a systematic study of the coherent systems moduli spaces, partly with a view to applications in higher rank Brill-Noether theory. In this paper, we continue our explorations and obtain substantial new information about the geometry and topology of the moduli spaces in the case $k \leq n$. In particular we obtain precise

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conditions for the moduli space to be non-empty and show that each non-empty moduli space has a distinguished irreducible component which has the expected dimension. For sufficiently large values of the parameter, we show that this is the only component. In addition, we obtain some precise information on the way in which the moduli space changes as the parameter varies, and deduce from this some more refined results on Picard groups and Picard varieties and on homotopy and cohomology groups. When $k = n-1$, we obtain a complete geometrical description of the moduli space for sufficiently large values of the parameter; this allows us to give a precise description of certain Brill-Noether loci. This is the only application to Brill-Noether theory which we include here; our results will certainly make further contributions to this theory, but this involves additional technicalities and will be addressed in future papers.

We refer the reader to [6] (and the further references cited therein) for the basic properties of coherent systems on algebraic curves. For convenience, and in order to set notation, we give a short synopsis before proceeding to a more detailed description of our results.

Definition 1.1. Fix $\alpha \in \mathbb{R}$. Let (E, V) be a coherent system of type (n, d, k). The *α-slope* $\mu_\alpha(E, V)$ is defined by

$$\mu_\alpha(E, V) = \frac{d}{n} + \alpha\frac{k}{n} .$$

We say (E, V) is *α-stable* if

$$\mu_\alpha(E', V') < \mu_\alpha(E, V)$$

for all proper subsystems (E', V') (i.e. for every non-zero subbundle E' of E and every subspace $V' \subseteq V \cap H^0(E')$ with $(E', V') \neq (E, V)$). We define *α-semistability* by replacing the above strict inequality with a weak inequality. A coherent system is called *α-polystable* if it is the direct sum of α-stable coherent systems of the same α-slope. We denote the *moduli space* of α-stable coherent systems of type (n, d, k) by $G(\alpha; n, d, k)$.

Definition 1.2.

- We say that $\alpha > 0$ is a *critical value* (or, in the terminology of [6, Definition 2.4], *actual critical value*) if there exists a proper subsystem (E', V') such that $\frac{k'}{n'} \neq \frac{k}{n}$ but $\mu_\alpha(E', V') = \mu_\alpha(E, V)$. We also regard 0 as a critical value.
- We say that α is *generic* if it is not a critical value. If $\mathrm{GCD}(n, d, k) = 1$ and α is generic, then α-semistability is equivalent to α- stability.
- If we label the critical values of α by α_i, starting with $\alpha_0 = 0$, we get a partition of the α-range into a set of intervals (α_i, α_{i+1}). Within the interval (α_i, α_{i+1}) the property of α-stability is independent of α, that is if $\alpha, \alpha' \in (\alpha_i, \alpha_{i+1})$, $G(\alpha; n, d, k) = G(\alpha'; n, d, k)$. We shall denote this moduli space by $G_i = G_i(n, d, k)$.

The construction of moduli spaces thus yields one moduli space G_i for the interval (α_i, α_{i+1}). If $\mathrm{GCD}(n, d, k) \neq 1$, one can define similarly the moduli spaces $\tilde{G}_i$ of semistable coherent systems. The GIT construction of these moduli spaces has been given in [20] and [16]. A previous construction for G_0 had been given in [29] and also in [2] for large values of d.

In the next two propositions, we suppose that $G(\alpha; n, d, k) \neq \emptyset$ for at least one value of α.

Proposition 1.3. ([6, Proposition 4.2]) *Let $0 < k < n$ and let α_L be the biggest critical value smaller than $\frac{d}{n-k}$. The α-range is then divided into a finite set of intervals determined by critical values*

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L < \frac{d}{n-k} \ .$$

If $\alpha > \frac{d}{n-k}$, the moduli spaces are empty.

We shall also be concerned with the case $k = n$. For this we need the following result from [6].

Proposition 1.4. ([6, Proposition 4.6]) *Let $k \geq n$. Then there is a critical value, denoted by α_L, after which the moduli spaces stabilise, i.e. $G(\alpha; n, d, k) = G_L$ if $\alpha > \alpha_L$. The α-range is thus divided into a finite set of intervals bounded by critical values*

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_L < \infty$$

such that, for any two different values of α in the range (α_L, ∞), the moduli spaces coincide.

The difference between adjacent moduli spaces in the family $G_0, G_1, \ldots, G_L$ is accounted for by the subschemes $G_i^+ \subseteq G_i$, $G_i^- \subseteq G_{i-1}$ where G_i^+ consists of all (E, V) in G_i which are not α-stable if $\alpha < \alpha_i$ and $G_i^- \subseteq G_{i-1}$ contains all (E, V) in G_{i-1} which are not α-stable if $\alpha > \alpha_i$. It follows that $G_i - G_i^+$ and $G_{i-1} - G_i^-$ are isomorphic and that G_i is transformed into G_{i-1} by the removal of G_i^+ and the insertion of G_i^-.

Definition 1.5. We refer to such a procedure, i.e. the transformation of G_i into G_{i-1} by the removal of G_i^+ and the insertion of G_i^-, or the inverse transformation, as a *flip*. We say that a flip is *good* if the flip loci (i.e. the subschemes $G_i^{\pm}$) have strictly positive codimension in every component of the moduli spaces G_i, G_{i-1}; under these conditions the moduli spaces G_i, G_{i-1} are birationally equivalent.

Our main results fall into three categories: non-emptiness, smoothness and irreducibility of the moduli spaces; homotopy and Picard groups and Picard varieties; Poincaré polynomials.

In [4] and [6] we obtained non-emptiness and irreducibility results for the moduli space $G_L(n, d, k)$. In this paper we extend these results to other moduli spaces $G(\alpha; n, d, k)$. Our results are summarized in the following theorems.

Theorem A [Lemmas 2.4 and 3.2, and Theorems 3.3 and 3.4] *Suppose that $0 < k \leq n$ and $n \geq 2$. Then the moduli space $G(\alpha; n, d, k)$ is non-empty if and only if*

$$\alpha > 0 \ , \ (n-k)\alpha < d \ , \ k \leq n + \frac{1}{g}(d-n) \ , \ \textit{and } (n, d, k) \neq (n, n, n) \ .$$

Whenever it is non-empty, $G(\alpha; n, d, k)$ contains a Zariski-open subset, $U(\alpha)$, which is smooth, irreducible of dimension

$$\beta(n, d, k) = n^2(g-1) + 1 - k(k - d + n(g-1)),$$

and such that its closure $\overline{U(\alpha)} \subset G(\alpha; n, d, k)$ is birationally equivalent to G_L.

There is a critical value $\alpha_I \in [0, \frac{d}{n-k})$ such that for all $\alpha > \alpha_I$ the moduli space $G(\alpha; n, d, k) = U(\alpha)$. Thus for $\alpha > \alpha_I$, whenever it is non-empty, $G(\alpha; n, d, k)$ is smooth, irreducible of dimension $\beta(n, d, k)$ and birationally equivalent to $G_L(n, d, k)$. The critical value α_I satisfies the bound

$$\alpha_I \leq \max\left\{\frac{(k-1)(d-n) - n\epsilon}{k(n-k+1)}, 0\right\}$$

where $\epsilon = \min\{k-1, g\}$.

One consequence of the above theorem is that, if $k \leq n$, then there are precisely two possibilities for the moduli spaces $G(\alpha; n, d, k)$: for fixed (n, d, k), either $G(\alpha; n, d, k)$ is non-empty for all allowable α, or it is empty for all α. Moreover, the non-empty moduli spaces always contain a distinguished component of the expected dimension (i.e. $\beta(n, d, k)$) – we will identify this component more precisely in section 3.

For the sake of completeness, we recall from [6] that if $k < n$, then, whenever it is non-empty, the moduli space $G_L(n, d, k)$ is birationally equivalent to a fibration over $M(n-k, d)$, the moduli space of stable bundles of rank $n-k$ and degree d, with fibre the Grassmannian $\mathrm{Gr}(k, d + (n-k)(g-1))$. If $\mathrm{GCD}(n-k, d) = 1$, then the birational equivalence is an isomorphism.

Our next result concerns the case $k = n-1$.

Theorem B [Corollary 5.2 and Theorem 5.3] *Suppose that $n \geq 2$ and $d > 0$. For all α such that*

$$\max\{d-n, 0\} < \alpha < d$$

the moduli space $G(\alpha; n, d, n-1)$ is non-empty if and only if

$$d \geq \max\{1, n-g\} .$$

Moreover, whenever it is non-empty, $G(\alpha; n, d, n-1) = G_L(n, d, n-1)$ and is a fibration over the Jacobian J^d, with fibre the Grassmannian $\mathrm{Gr}(n-1, d+g-1)$. (This fibration will be identified more precisely in section 5.)

As a consequence of Theorem B, we identify the Brill-Noether locus $B(n, d, n-1)$, consisting of stable bundles E of rank n and degree d with $h^0(E) \geq n-1$, with a well known classical variety (Theorem 5.7).

Our main results on the Picard groups and varieties and homotopy groups for $k < n$ are summarized in the following theorem.

Theorem C [Theorem 5.3, Theorem 7.2, Corollary 7.3 and Theorems 7.4, 7.6, 7.12 and 7.15] *Let $0 < k < n$ and $d > 0$. Suppose further that $k < n + \frac{1}{g}(d-n)$ and $\max\left\{\frac{d-n}{n-k}, 0\right\} < \alpha < \frac{d}{n-k}$. Then, except possibly when $g = 2$, $k = n-2$ and d is even,*

(a) $\mathrm{Pic}(G(\alpha;n,d,k)) \cong \mathrm{Pic}(M(n-k,d)) \times \mathbb{Z}$;
(b) $\mathrm{Pic}^0(G(\alpha;n,d,k))$ *is isomorphic to the Jacobian* $J(C)$;
(c) $\pi_1(G(\alpha;n,d,k)) \cong \pi_1(M(n-k,d)) \cong H_1(C,\mathbb{Z})$;
(d) *if also* $k \neq n-1$, *there exists an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(G(\alpha;n,d,k)) \longrightarrow \mathbb{Z} \times \mathbb{Z}_p \longrightarrow 0,$$

where $p = \mathrm{GCD}(n-k,d)$.

If $k = n + \frac{1}{g}(d-n)$, (b) *and* (c) *remain true; in* (a), *the factor* $\mathbb{Z}$ *must be deleted; in* (d), $\pi_2(G(\alpha;n,d,k)) \cong \mathbb{Z} \times \mathbb{Z}_p$.

In the case $k = n-1$, *we have*

(e) *if* $\max\{d-n,0\} < \alpha < d$ *and* $d \geq \max\{1, n-g\}$, *then*

$$\pi_i(G(\alpha;n,d,n-1)) \cong \pi_i(\mathrm{Gr}(n-1,d+g-1)) \text{ for } i \geq 0, i \neq 1.$$

Finally we have some results on Poincaré polynomials in the case $k = n-2$. For any space X, we write $P(X)$ for the Poincaré polynomial of X with coefficients in any fixed field. In order to state our results concisely, we write $G(\alpha)$, G_L for $G(\alpha;n,d,n-2)$, $G_L(n,d,n-2)$. We write also, for any r, e,

$$P_{r,e} = P(G_L(r,e,r-1)).$$

Theorem D [Theorem 8.5, Corollary 8.7, Corollary 8.8]

(a) *For any non-critical value* α' *in the interval* $(\max\{\frac{d-n}{2},0\},\frac{d}{2})$,

$$P(G(\alpha'))(t) - P(G_L)(t) = \sum \frac{t^{2C_{21}(n_1,d_1)} - t^{2C_{12}(n_1,d_1)}}{1-t^2} P_{n_1,d_1}(t) P_{n-n_1,d-d_1}(t),$$

where the summation is over all solutions of (34), (35), (36) *for which* $\alpha > \alpha'$. (For the definitions of the polynomials C_{12} and C_{21} and the equations (34), (35), (36), see section 8.)

(b) *Suppose* $n = 3$ *and* d *is odd. Then, for* $\max\{\frac{d-3}{2},0\} < \alpha' < \frac{d}{2}$,

$$\begin{aligned} P(G(\alpha'))(t) &= P(G_L)(t) = P(M(2,d))(t)\frac{1-t^{2(d+2g-2)}}{1-t^2} \\ &= \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)}. \end{aligned}$$

(c) *Suppose* $n = 4$ *and* d *is odd. Then*
(i) *if* $\max\{\frac{d-2}{2},0\} < \alpha' < \frac{d}{2}$,

$$\begin{aligned} P(G(\alpha'))(t) &= P(G_L)(t) \\ &= \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-3)})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)^2}; \end{aligned}$$

(ii) *if* $\max\{\frac{d-4}{2}, 0\} < \alpha' < \frac{d-2}{2}$,

$$P(G(\alpha'))(t) = \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-3)})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)^2} + \frac{(t^{2g} - t^{6g+2d-10})(1-t^{d-3+2g})(1-t^{d-1+2g})(1+t)^{4g}}{(1-t^2)^2(1-t^4)}.$$

We now summarize the contents of the paper. In section 2, we show that, for $k \leq n$ and α sufficiently large, any α-stable coherent system (E, V) is injective, in other words (E, V) gives rise to an injective morphism $V \otimes \mathcal{O} \hookrightarrow E$. In section 3 we obtain precise conditions for non-emptiness of the moduli spaces of injective coherent systems and deduce our first irreducibility results. In section 4 we assume $k < n$ and prove an important structural result (the Diagram Lemma) for torsion-free coherent systems (that is, injective coherent systems with $E/(V \otimes \mathcal{O})$ torsion-free). In section 5 we give a first application of this lemma to the moduli spaces of coherent systems with $k = n-1$ and to the Brill-Noether locus $B(n, d, n-1)$. Following this, in section 6 we compute codimensions for the flips which occur at critical values $\alpha > \frac{d-n}{n-k}$ when $k < n$. In section 7 we obtain results on Picard groups and varieties and homotopy groups. In section 8 we apply the calculations of section 6 to the case $k = n-2$ and investigate the geometry of the flips in a way similar to the work of Thaddeus; this allows a computation of Poincaré polynomials. The necessary extensions to Thaddeus' results are contained in an appendix. We give also in the appendix a proof of the existence of universal families of coherent systems when $\mathrm{GCD}(n, d, k) = 1$, as we were unable to locate a proof in the literature.

We suppose throughout that C is a smooth irreducible projective algebraic curve of genus $g \geq 2$ defined over the complex numbers. The special cases $g = 0$ and $g = 1$ are being investigated elsewhere [17, 18, 19].

2. Coherent Systems with $k \leq n$

Let (E, V) be a coherent system of some fixed type (n, d, k) on C with $k \geq 1$. For most of this section we assume that $k \leq n$. For convenience we introduce the following definition.

Definition 2.1. A coherent system (E, V) is *injective* if the evaluation morphism $V \otimes \mathcal{O} \to E$ is injective as a morphism of sheaves. Moreover (E, V) is *torsion-free* if it is injective and the quotient sheaf $E/(V \otimes \mathcal{O})$ is torsion-free.

Since we are working over a smooth curve, we have for any torsion-free coherent system (E, V) an exact sequence

$$0 \longrightarrow V \otimes \mathcal{O} \longrightarrow E \longrightarrow F \longrightarrow 0, \tag{1}$$

where F is a vector bundle on C.

Lemma 2.2. *Suppose (E, V) is injective and α-stable. Then $G(\alpha; n, d, k)$ is smooth of dimension*

$$\beta(n, d, k) = n^2(g-1) + 1 - k(k - d + n(g-1)) \tag{2}$$

at (E, V).

Proof. This is [6, Proposition 3.12]. □

We recall from [4, Lemma 3.6] and [6, Proposition 4.4] that, for $k \leq n$, every α-semistable coherent system is injective provided α is large enough. Moreover, if $k < n$, then, by [4, Lemma 3.7] every α-semistable coherent system is torsion-free, again for α large enough.

Definition 2.3. We define $\alpha_I \geq 0$ to be the smallest critical value of α such that every α-semistable coherent system is injective for $\alpha > \alpha_I$. If $k < n$, we define $\alpha_T \geq 0$ to be the smallest critical value of α such that every α-semistable coherent system is torsion-free for $\alpha > \alpha_T$.

Of course, α_I and α_T depend on the type (n, d, k) and we have $\alpha_T \geq \alpha_I$. Note that, if $k = 1$, $\alpha_I = 0$. We have

Lemma 2.4. *Let $k \leq n$. Then*

(i) (a) *if $2 \leq k \leq g+1$, $\alpha_I \leq \max\left\{\frac{(k-1)(d-2n)}{k(n-k+1)}, 0\right\}$;*
(b) *if $k > g+1$, $\alpha_I \leq \max\left\{\frac{(k-1)(d-n)-ng}{k(n-k+1)}, 0\right\}$;*

(ii) *if $0 < k < n$, $\alpha_T = \max\left\{\frac{d-n}{n-k}, 0\right\}$.*

Proof. (i) Suppose the evaluation morphism is not injective. Consider the subsheaf Im generated by V in E. It is a standard fact that $Im \simeq F \oplus \mathcal{O}^s$, where $s = h^0(Im^*)$ and F is a vector bundle. Clearly $h^0(F^*) = 0$ and F is generated by the global sections which lie in the image W' of V in $H^0(F)$. We write $l = \operatorname{rk} F$ and $e = \deg F$.

It is well known that a general subspace $W \subset W'$ of dimension $l+1$ generates F. We have therefore an exact sequence

$$0 \longrightarrow (\det F)^* \longrightarrow W \otimes \mathcal{O} \longrightarrow F \longrightarrow 0\ ,$$

and by dualising we obtain that $h^0(\det F) \geq l+1$. It follows easily from Clifford's Theorem and the Riemann-Roch Theorem that

($*$) if $l \leq g$, then $e \geq 2l$;
($**$) if $l \geq g$, then $e \geq l+g$.

Note that by definition we have $l \leq k-1$.

Now the α-semistability criterion implies that $\mu_\alpha(F, W) \leq \mu_\alpha(E, V)$ and hence

$$\frac{e + \alpha(l+1)}{l} \leq \frac{d + \alpha k}{n},$$

or equivalently

$$\alpha(n(l+1) - kl) \leq ld - ne.$$

Since $n \geq k$, we necessarily have $n(l+1) - kl > 0$ and the above inequality gives

$$\alpha \leq \frac{ld - ne}{n(l+1) - kl}\ .$$

Using the inequalities ($*$), ($**$) and the condition $l \leq k-1$, we obtain (i).

(ii) Suppose $0 < k < n$. To show that $\alpha_T \geq \max\left\{\frac{d-n}{n-k}, 0\right\}$, we need to show that, if $d > n$, there exists an α-semistable coherent system with $\alpha = \frac{d-n}{n-k}$ which is not torsion-free. For this we can take

$$\bigoplus_{i=1}^{k}(\mathcal{O}(p_i), H^0(\mathcal{O}(p_i))) \oplus (F, 0),$$

where $p_1, \ldots, p_k \in C$ and F is a semistable bundle of rank $n-k$ and degree $d-k$.

To prove the opposite inequality, let (E, V) be an α-semistable coherent system which is not torsion-free. Either (E, V) is not injective or the quotient sheaf $E/(V \otimes \mathcal{O})$ has non-zero torsion. In both cases, there exists a non-zero section s in V that vanishes at some point of C. The section s generically generates a line subbundle L with $\deg L > 0$. We thus have a coherent subsystem (L, V_L) with $\deg L \geq 1$, $\operatorname{rk} L = 1$, and $\dim V_L \geq 1$. The α-semistability criterion now yields

$$1 + \alpha \leq \frac{d}{n} + \alpha\frac{k}{n},$$

i. e.

$$\alpha \leq \frac{d-n}{n-k}.$$

□

Corollary 2.5. *If $\alpha > 0$, $2 \leq k \leq n$ and*

$$d \leq \min\left\{2n, n + \frac{ng}{k-1}\right\},$$

then every α-semistable coherent system (E, V) is injective.

Proof. This follows immediately from Lemma 2.4. □

Remark 2.6. (i) It is a simple exercise to show that, if $k < n$ and $d > n$, the bounds on α_I of Lemma 2.4(i) are always strictly smaller than the value of α_T given by Lemma 2.4(ii). In particular, if $\alpha_T > 0$, then $\alpha_I < \alpha_T$.

(ii) Note that, for $k < n$ and $d > 0$, $\alpha_T < \frac{d}{n-k}$, which is the maximum value of α for which α-semistable coherent systems exist (see [6, Lemma 4.1]). Moreover, it was proved in [4, Lemmas 3.6 and 3.7] that

$$\alpha_I \leq \frac{d(k-1)}{k(n-k+1)}, \quad \alpha_T \leq \max\left\{\frac{kd-n}{k(n-k)}, 0\right\}.$$

The statements of Lemma 2.4 are stronger.

(iii) Compare also [24, Chapitre 3, Lemme A.2], in which it is proved that, if $d < \min\{2n, n+g\}$ and E is a semistable bundle, then (E, V) is injective. When E is stable, Corollary 2.5 gives the same result with a weaker restriction on d.

Remark 2.7. In proving that, when $d > n$, $\alpha_T \geq \frac{d-n}{n-k}$, we have made use of a coherent system which is α-semistable for $\alpha = \frac{d-n}{n-k}$ only. In fact it is possible to find a coherent system (E,V) which is not torsion-free and is α-stable for α slightly less than $\frac{d-n}{n-k}$. For this, one can take (E,V) to be given by a non-trivial extension

$$0 \longrightarrow (E_1,V_1) \longrightarrow (E,V) \longrightarrow (E_2,V_2) \longrightarrow 0,$$

where $(E_1,V_1) = (\mathcal{O}(p), H^0(\mathcal{O}(p)))$ and (E_2,V_2) is α-stable for $\alpha = \frac{d-n}{n-k}$. For the existence of (E_2,V_2), see Theorem 3.3 below and, for the existence of a non-trivial extension, see equations (16) and (17).

Remark 2.8. For a general curve, the bound in Lemma 2.4(i)(a) is not best possible when $k < g$; we can improve the bound by using an estimate based on the Brill-Noether number rather than Clifford's Theorem. Details are left to the reader.

The following lemma is true without the restriction $k \leq n$.

Lemma 2.9. *Let (E,V) be an α-stable coherent system for some $\alpha > 0$ with $k > 0$. Then*

- $d > 0$, *except in the case $(n,d,k) = (1,0,1)$.*
- $h^0(E^*) = 0$.

If (E,V) is α-semistable, then

- $d \geq 0$,
- $h^0(E^*) = 0$ *except when $d = \alpha(n-k)$.*

Proof. For the fact that $d > 0$ ($d \geq 0$) for an α-stable (α-semistable) coherent system, see [6, Lemmas 4.1 and 4.3]. Now suppose (E,V) is α-semistable and $h^0(E^*) \neq 0$. Then there exists a non-zero homomorphism $E \to \mathcal{O}$. If the induced map $V \otimes \mathcal{O} \to \mathcal{O}$ is not the zero map, then $(\mathcal{O}, H^0(\mathcal{O}))$ is a direct factor of (E,V). This contradicts α-stability always and α-semistability unless

$$\alpha = \frac{d}{n} + \alpha\frac{k}{n},$$

i. e. $d = \alpha(n-k)$. Otherwise let E' be the kernel of $E \to \mathcal{O}$. Then (E',V) is a coherent subsystem of (E,V) with $\deg E' \geq d$, and the α-semistability criterion gives

$$\frac{d}{n-1} + \alpha\frac{k}{n-1} \leq \frac{d}{n} + \alpha\frac{k}{n}.$$

This contradicts the assumption that $\alpha > 0$. □

Returning now to the case $k \leq n$, we have

Lemma 2.10. *Suppose that $k \leq n$ and $d > 0$. Let (E,V) be any injective coherent system of type (n,d,k), i.e. suppose that (E,V) is represented by an extension*

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow E \longrightarrow F \longrightarrow 0 \tag{3}$$

(where F need not be locally free). Let

$$\vec{e} = (e_1,\ldots,e_k) \in \mathrm{Ext}^1(F,\mathcal{O}^{\oplus k}) = \mathrm{Ext}^1(F,\mathcal{O})^{\oplus k}$$

denote the extension class of (3). *If $h^0(E^*) = 0$, then*

- *$e_1,\ldots,e_k$ are linearly independent as vectors in* $\mathrm{Ext}^1(F,\mathcal{O})$;

- $h^0(F^*) = 0$.

Moreover

$$k \leq n + \frac{1}{g}(d-n). \tag{4}$$

Proof. Suppose that $e_1, \dots, e_k$ are linearly dependent. After acting on (3) by an automorphism of $\mathcal{O}^{\oplus k}$, we can suppose some $e_i = 0$. But then $\mathcal{O}$ is a direct factor of E and $h^0(E^*) \neq 0$. This proves the first statement.

The vanishing of $h^0(F^*)$ follows from the long exact sequence of the $\mathrm{Ext}^i(\cdot, \mathcal{O})$ induced by (3). By Riemann-Roch, this implies that

$$\dim \mathrm{Ext}^1(F, \mathcal{O}) = d + (n-k)(g-1)\ .$$

The bound in (4) follows from this and the linear independence of $e_1, \dots, e_k$. □

3. The moduli space for injective coherent systems

In this section we show that the moduli space of injective α-stable coherent systems is always smooth and irreducible of dimension $\beta(n,d,k)$ whenever it is non-empty. We also determine exactly when this space is non-empty. As consequences, we obtain the same properties for $G(\alpha; n, d, k)$ when $\alpha > \alpha_I$ and results on the birational type of some moduli spaces. These results can be seen as an extension of [6, Theorem 5.4].

We begin with a proposition which restates some key results from [4] and [6] and a useful lemma.

Proposition 3.1. (i) *Suppose $n \geq 2$ and $0 < k \leq n$. Then $G_L(n,d,k) \neq \emptyset$ if and only if*

$$d > 0,\ k \leq n + \frac{1}{g}(d-n) \text{ and } (n,d,k) \neq (n,n,n), \tag{5}$$

and it is then always irreducible and smooth of dimension $\beta(n,d,k)$.

(ii) *If $k = n$, every element of $G_L(n,d,k)$ can be represented by an extension of the form*

$$0 \longrightarrow V \otimes \mathcal{O} \longrightarrow E \longrightarrow T \longrightarrow 0, \tag{6}$$

where T is a torsion sheaf.

(iii) *If $0 < k < n$, every element of $G_L(n,d,k)$ is torsion-free and corresponds to an extension* (1) *with F semistable and $h^0(F^*) = 0$; moreover $G_L(n,d,k)$ is birationally equivalent to a fibration over the moduli space $M(n-k,d)$ with fibre the Grassmannian* $\mathrm{Gr}(k, d + (n-k)(g-1))$*. More precisely, if W denotes the subvariety of $G_L(n,d,k)$ consisting of coherent systems for which the bundle F in* (1) *is strictly semistable, then $G_L(n,d,k) \setminus W$ is isomorphic to a Grassmann fibration over $M(n-k,d)$.*

(iv) *If in addition* $\mathrm{GCD}(n-k,d) = 1$*, then $W = \emptyset$ and $G_L(n,d,k) \to M(n-k,d)$ is the Grassmann fibration associated to some vector bundle over $M(n-k,d)$.*

Proof. For the necessity of the condition $d > 0$, see Lemma 2.9. The rest is essentially a restatement of [6, Proposition 5.2 and Theorem 5.4] (see also [4, section 4]) and [6, Theorem 5.6]. (For the last part of (iv), note that there exists a Poincaré bundle $\mathcal{P}$ over $X \times M(n-k,d)$. The sheaf $R^1 p_{2*}\mathcal{P}^*$ is locally free since $h^1(F^*)$ is constant for $F \in M(n-k,d)$. The vector bundle corresponding to this sheaf has the required properties.) □

Lemma 3.2. *Suppose* $0 < k \leq n$ *and let*

$$U = \{(E,V) \in G_L(n,d,k) \mid E \text{ is stable}\}.$$

Then $U \subset G(\alpha; n,d,k)$ *is a Zariski-open subset for all allowable* α *(that is, for* $0 < \alpha < \frac{d}{n-k}$ *if* $k < n$ *and for* $\alpha > 0$ *if* $k = n$*).*

Proof. If $(E,V) \in U$, then (E,V) is α-stable for small α and for large α. Since the set of α for which (E,V) is α-stable is an open interval [4, Lemma 3.14], it follows that (E,V) is α-stable for all allowable α. This proves that $U \subset G(\alpha; n,d,k)$. The fact that U is Zariski-open follows from the openness of the stability condition. □

We come now to the main result of this section.

Theorem 3.3. *Suppose* $0 < k \leq n$*. For any* $\alpha > 0$*, define* $U(\alpha)$ *by*

$$U(\alpha) = \{(E,V) \in G(\alpha; n,d,k) \mid (E,V) \text{ is injective}\}.$$

Then

(i) *$U(\alpha)$ is a Zariski-open subset of $G(\alpha; n,d,k)$;*
(ii) *if $U(\alpha) \neq \emptyset$, then it is smooth of dimension $\beta(n,d,k)$;*
(iii) *if $U(\alpha) \neq \emptyset$, then it is irreducible; hence $\overline{U(\alpha)}$ is an irreducible component of $G(\alpha; n,d,k)$;*
(iv) *if $n \geq 2$, then $U(\alpha) \neq \emptyset$ if and only if the following conditions hold:*

$$(n-k)\alpha < d,\ k \leq n + \frac{1}{g}(d-n) \text{ and } (n,d,k) \neq (n,n,n); \tag{7}$$

(v) *if $n \geq 2$, the set U of Lemma 3.2 is non-empty if and only if conditions* (5) *hold.*

Proof. (i) is standard, while (ii) follows at once from Lemma 2.2.

For (iii), suppose first that $k < n$ and define

$$V(\alpha) = \{(E,V) \in G(\alpha; n,d,k) \mid (E,V) \text{ is torsion-free}\}.$$

Then $V(\alpha)$ is again open in $G(\alpha; n,d,k)$ and every element of $V(\alpha)$ is given by an extension (1) with $h^0(F^*) = 0$ by Lemma 2.10. It now follows, exactly as in the proof of [5, Theorem 4.3] that $V(\alpha)$ is irreducible if it is non-empty.

Recall now that every irreducible component of $G(\alpha; n,d,k)$ has dimension greater than or equal to $\beta(n,d,k)$ (see [6, Corollary 3.6]). To complete the proof of (iii), it is therefore sufficient to show that

$$\dim(U(\alpha) \setminus V(\alpha)) < \beta(n,d,k). \tag{8}$$

The argument for this is the same as that of the corresponding result for Brill-Noether loci [24, Chapitre 3, Théorème A.1]. For the convenience of the reader, we give an outline here.

The points of $U(\alpha) \setminus V(\alpha)$ are represented by extensions

$$0 \longrightarrow V \otimes \mathcal{O} \longrightarrow E \longrightarrow F \oplus T \longrightarrow 0, \tag{9}$$

where F is a vector bundle and T is a torsion sheaf of length $t \geq 1$. The extension classes are defined by k-tuples

$$e_1, \ldots, e_k \in \operatorname{Ext}^1(F \oplus T, \mathcal{O});$$

by Lemmas 2.9 and 2.10, the α-stability of (E, V) implies that $e_1, \ldots, e_k$ are linearly independent and that $h^0(F^*) = 0$. We now simply have to estimate dimensions; for convenience, we will assume that the support of T consists of t distinct points (the other cases are handled similarly).

Since (E, V) is α-stable, the only automorphisms of (E, V) are scalar multiples of the identity [6, Proposition 2.2]; it follows that the dimension of the component of $U(\alpha) \setminus V(\alpha)$ consisting of the coherent systems of the form (9) for a fixed value of t is

$$\begin{aligned}(n-k)^2(g-1) + 1 + t + \dim \operatorname{Gr}(k, \operatorname{Ext}^1(F \oplus T, \mathcal{O})) - \min \dim \operatorname{Aut}(F \oplus T) + 1 \\ = \beta(n, d, k) + t - \min \dim \operatorname{Aut}(F \oplus T) + 1.\end{aligned}$$

To establish (8), we therefore need to show that

$$\dim \operatorname{Aut}(F \oplus T) \geq t + 2.$$

This is clear since $\dim \operatorname{Aut} T = t$ and $\dim \operatorname{Hom}(F, T) = (n-k)t \geq t \geq 1$.

This completes the proof of (iii) when $k < n$. For the case $k = n$, see [6, Theorem 5.6 and its proof].

It remains to prove (iv) and (v). When $k < n$, the necessity of the conditions (7) and (5) has already been proved in Lemma 2.9, [6, Lemma 4.1] and Lemma 2.10. For $k = n$, (7) and (5) both reduce to $d > n$, which is a necessary condition for non-emptiness by [6, Remark 5.7].

Finally, we shall prove that, if (5) holds, then $U \neq \emptyset$. Since $U \subset U(\alpha)$ for all allowable α by Proposition 3.1 and Lemma 3.2, this will show also that $U(\alpha) \neq \emptyset$ whenever (7) holds.

For $k < n$, we note that, by Proposition 3.1, $G_L(n, d, k) \neq \emptyset$ and every element of it is torsion-free. We claim that there exists a torsion-free coherent system (E, V) for which E is stable. Once this is proved, it follows that (E, V) arises from an extension (1) with $h^0(F^*) = 0$. As already noted earlier in the proof, the set of all such extensions is parametrised by an irreducible variety. It follows from the openness of the stability condition that the general extension (1) defines a coherent system $(E, V) \in G_L(n, d, k)$ with E stable. Hence $U \neq \emptyset$.

For $d \leq n$, the claim is proved in [5]. For $d > n$, the result can be deduced from a combination of [5], [24] and [7], and probably also from [32], but is perhaps most easily obtained by using [25]; in fact it follows directly from [25, Théorème A.5].

When $k = n$, the proof is on the same lines but using extensions of the form (6) in place of those of the form (1). Again it is clear that the extensions are parametrised by an irreducible variety. The fact that there exists an extension (6) with E stable is again a consequence of [25, Théorème A.5]. □

As a consequence of Theorem 3.3, we obtain our first important result on the geometry of $G(\alpha; n, d, k)$.

Theorem 3.4. *Suppose $n \geq 2$, $0 < k \leq n$ and $\alpha > \alpha_I$. If the moduli space $G(\alpha; n, d, k)$ is non-empty, then it is smooth and irreducible of dimension $\beta(n, d, k)$ and is birationally equivalent to $G_L(n, d, k)$. Moreover $G(\alpha; n, d, k)$ is non-empty if and only if the conditions* (7) *hold.*

Proof. It follows from Definition 2.1 that, if $\alpha > \alpha_I$, then $G(\alpha; n, d, k) = U(\alpha)$. Hence, if $G(\alpha; n, d, k) \neq \emptyset$, Theorem 3.3(v) implies that $U \neq \emptyset$. The rest of the theorem now follows easily from Theorem 3.3. $\square$

Remark 3.5. If we denote by $G_I(n, d, k)$ the moduli space of coherent systems of type (n, d, k) which are α-stable for α slightly greater than α_I, the theorem can be restated to say that $G_I(n, d, k)$ is birationally equivalent to $G_L(n, d, k)$.

Corollary 3.6. *Suppose $0 < k < n$. If further $\alpha_I < \alpha < \frac{d}{n-k}$ and $k \leq n + \frac{1}{g}(d - n)$, then $G(\alpha; n, d, k)$ is birationally equivalent to a fibration over the moduli space $M(n - k, d)$ with fibre the Grassmannian $\mathrm{Gr}(k, d + (n - k)(g - 1))$.*

Proof. This follows at once from the theorem and Proposition 3.1. $\square$

Corollary 3.7. *Suppose $2 \leq k \leq n$ and $d \leq \min\{2n, n + \frac{ng}{k-1}\}$. Then $G_0(n, d, k)$ is birationally equivalent to $G_L(n, d, k)$.*

Proof. This follows from Remark 3.5 and Corollary 2.5. $\square$

4. The Diagram Lemma

It follows from Theorem 3.3 that the "flips" at critical points $\alpha > \alpha_I$ all have positive codimension. The purpose of the next few sections is to obtain more information about the flips when $k < n$ and α is large.

In this section we give a structural result that applies in particular to all coherent systems which are α-stable for some α in the range

$$\max\left\{\frac{d-n}{n-k}, 0\right\} = \alpha_T < \alpha < \frac{d}{n-k}. \tag{10}$$

By Lemma 2.4, all such coherent systems are torsion-free. The result of this section may thus be viewed as an extension of [5, p.660, diagram (5)] to α-stable coherent systems in the range (10) but without any restriction on d.

Lemma 4.1. (Diagram Lemma) *Suppose that $k < n$ and that* (10) *holds. Let (E, V) be a torsion-free coherent system with $h^0(E^*) = 0$. Suppose further that there exists an exact sequence of coherent systems*

$$0 \to (E_1, V_1) \to (E, V) \to (E_2, V_2) \to 0 \tag{11}$$

with E_1, E_2 both of positive rank, $h^0(E_1^*) = 0$, (E_2, V_2) α-semistable and $\mu_\alpha(E_2, V_2) \leq \mu_\alpha(E, V)$. Then there exists a diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & V_1 \otimes \mathcal{O} & \rightarrow & E_1 & \rightarrow & F_1 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & V \otimes \mathcal{O} & \rightarrow & E & \rightarrow & F & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & V_2 \otimes \mathcal{O} & \rightarrow & E_2 & \rightarrow & F_2 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{12}$$

where

(a) *the quotients F_1, F, and F_2 are all locally free with positive rank,*
(b) $h^0(F_1^*) = h^0(F^*) = h^0(F_2^*) = 0$,
(c) *the extension classes of E_1, E, E_2 are given respectively by k_1, k, k_2 linearly independent vectors in $H^1(F_1^*)$, $H^1(F^*)$, $H^1(F_2^*)$.*

Proof. Note first that, given $k < n$,

$$\alpha < \frac{d}{n-k} \Leftrightarrow \alpha < \mu_\alpha(E, V) \tag{13}$$

and

$$\frac{d-n}{n-k} < \alpha \Leftrightarrow \mu_\alpha(E, V) - 1 < \alpha. \tag{14}$$

The assumption that $\mu_\alpha(E_2, V_2) \leq \mu_\alpha(E, V)$ implies, by (10) and (14), that

$$\mu_\alpha(E_2, V_2) - 1 \leq \mu_\alpha(E, V) - 1 < \alpha.$$

Since (E_2, V_2) is α-semistable, this implies that (E_2, V_2) cannot possess a subsystem (L, V_L) with $\deg L \geq 1$, $\operatorname{rk} L = 1$ and $\dim V_L \geq 1$. Now the proof of Lemma 2.4(ii) shows that (E_2, V_2) is torsion-free. On the other hand, (E_1, V_1) is a subsystem of the torsion-free coherent system (E, V) and is therefore torsion-free. An easy application of the snake lemma now gives us (12).

If $F_1 = 0$, then $E_1 \simeq \mathcal{O}^{\oplus k_1}$ and $\mu_\alpha(E_1, V_1) = \alpha$. But by hypothesis and (13),

$$\mu_\alpha(E_1, V_1) \geq \mu_\alpha(E, V) > \alpha,$$

giving a contradiction. Hence F_1 has positive rank.

If $F_2 = 0$, then $E_2 \simeq \mathcal{O}^{\oplus k_2}$, which contradicts the assumption that $h^0(E^*) = 0$. This completes the proof of (a).

Finally we have $h^0(E^*) = h^0(E_1^*) = 0$ by hypothesis and $h^0(E_2^*) = 0$ by considering the middle column of (12). Condition (b) follows by considering the rows of (12), and (c) by Lemma 2.10. □

Remark 4.2. We have stated Lemma 4.1 under rather general hypotheses. The situation in which we shall be applying it is that of [6, Lemma 6.5]. Slightly modifying the notation of [6], we suppose

- $\alpha > \alpha_T$ is a critical value,
- α^+ denotes a value of α slightly greater than α, while α^- denotes a value slightly less than α,
- $G^\pm(\alpha)$ is the set of points in $G(\alpha^\pm; n, d, k)$ represented by coherent systems which are $\alpha^\pm$-stable but not $\alpha^\mp$-stable.

If now (E, V) is strictly α-semistable and α^+-stable, we have (by [6, Lemma 6.5]) an extension (11) in which

(d) (E_1, V_1) and (E_2, V_2) are α^+-stable, with $\mu_{\alpha^+}(E_1, V_1) < \mu_{\alpha^+}(E_2, V_2)$,
(e) (E_1, V_1) and (E_2, V_2) are α-semistable, with $\mu_\alpha(E_1, V_1) = \mu_\alpha(E_2, V_2)$.

If (E, V) is strictly α-semistable and α^--stable then we have an extension of the same form, but with α^+ replaced by α^- in the condition (d).

It follows from Lemma 2.9 that, in either case, all the hypotheses of the Diagram Lemma are satisfied, so that we have a diagram (12) and conditions (a), (b), (c) hold as well as (d) and (e).

5. The case $k = n - 1$

As a first application of the Diagram Lemma, we consider the case $k = n-1$, where $n \geq 2$. In this case $\alpha_T = \max\{d - n, 0\}$ and (10) becomes

$$\max\{d - n, 0\} < \alpha < d. \tag{15}$$

Proposition 5.1. *Let (E, V) be a coherent system of type $(n, d, n-1)$. If (E, V) is α-stable for some α in the range* (15)*, then (E, V) is α-stable for all α in the range* (15)*.*

Proof. If the result is false, there is a critical value of α in the range (15) and a coherent system (E, V) which is strictly α-semistable. By [6, Lemma 6.5], there exists an extension (11) with (E_1, V_1), (E_2, V_2) either both α^+-stable or both α^--stable and $\mu_\alpha(E_1, V_1) = \mu_\alpha(E_2, V_2)$. It follows now from Lemma 2.9 that all the hypotheses of the Diagram Lemma are satisfied. Applying this lemma, we obtain $k_1 \leq n_1 - 1$ and $k_2 \leq n_2 - 1$. Thus

$$k = k_1 + k_2 \leq n_1 + n_2 - 2 = n - 2\ ,$$

which contradicts our assumption that $k = n - 1$. □

Corollary 5.2. *Suppose that* (15) *holds. Then*

$$G(\alpha; n, d, n-1) = G_L(n, d, n-1).$$

Proof. This follows immediately from the proposition. □

Theorem 5.3. *Suppose that* (15) *holds. Then $G(\alpha; n, d, n-1)$ is non-empty if and only if $d \geq \max\{1, n - g\}$. When this condition holds, $G(\alpha; n, d, n-1)$ is a fibration over the Jacobian J^d, with fibre the Grassmannian* $\mathrm{Gr}(n-1, d+g-1)$*. In particular*

(i) *if* $d > n - g$*, then* $\operatorname{Pic}(G(\alpha; n, d, n-1)) \cong \operatorname{Pic} J^d \times \mathbb{Z}$*;*
(ii) *if* $n > g$*, then* $\operatorname{Pic}(G(\alpha; n, n-g, n-1)) \cong \operatorname{Pic} J^{n-g}$*;*
(iii) *in all cases,* $\operatorname{Pic}^0(G(\alpha; n, d, n-1))$ *is isomorphic to the Jacobian* $J(C)$*.*

Proof. (i) follows from Proposition 3.1 and Corollary 5.2. For (ii), note that $G_L(n, n-g, n-1) \cong J^{n-g}$. Finally, for (iii), recall that $J^d \cong J(C)$ and $\operatorname{Pic}^0(J(C)) \cong J(C)$ because $J(C)$ is a principally polarized abelian variety. □

Remark 5.4. We have presented these results both in terms of the *Picard group* Pic (the group of all algebraic line bundles) and the *Picard variety* Pic^0 (the group of all topologically trivial line bundles). Our reasons for doing this are firstly that the results for Picard varieties are particularly simple (see also Theorem 7.15) and secondly that, in the coprime case, the Picard variety has the structure of a polarized abelian variety and carries a lot of geometrical information; the isomorphism of Theorem 5.3 is certainly an isomorphism of abelian varieties. If one could prove that it is an isomorphism of *polarized* abelian varieties, it would follow that, under the conditions of the theorem, the variety $G(\alpha; n, d, n-1)$ satisfies a *global Torelli theorem*, that is, it determines C as an algebraic curve.

If we now denote by $G_T(n, d, k)$ the moduli space of α-stable coherent systems for α slightly greater than α_T, we have the following immediate corollary of Theorem 5.3.

Corollary 5.5. *If* $d \geq \max\{1, n-g\}$*, then* $G_T(n, d, n-1)$ *is a fibration over the Jacobian* J^d*, with fibre the Grassmannian* $\operatorname{Gr}(n-1, d+g-1)$*.*

Remark 5.6. In fact, we can identify the fibration of Theorem 5.3 and Corollary 5.5 precisely. Let $d > 0$ and let $\mathcal{P}$ be a Poincaré bundle on $C \times J^d$, that is a line bundle whose restriction to $C \times \{j\}$ is the line bundle L_j on C corresponding to the point $j \in J^d$. Let p_C, p_J be the projections of $C \times J^d$ on its factors. Then the direct image $p_{J*}\mathcal{P}^*$ is zero and hence $R^1 p_{J*}\mathcal{P}^*$ is locally free of rank $d + g - 1$. The corresponding vector bundle classifies the extensions

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow L_j \longrightarrow 0,$$

where j is a (variable) point of J^d. It follows that $G_L(n, d, n-1)$ can be identified with the Grassmann bundle $\operatorname{Gr}(n-1, R^1 p_{J*}\mathcal{P}^*)$ whose fibre over j is the Grassmannian of subspaces of $H^1(L_j^*)$ of dimension $n-1$. By relative Serre duality, we can identify this with $\operatorname{Gr}(d+g-n, p_{J*}(\mathcal{P} \otimes p_C^* K_C))$, where K_C is the canonical bundle. This variety is well-known; it is the variety of linear systems of degree $d + 2g - 2$ and dimension $d + g - n - 1$, classically denoted by $G^{d+g-n-1}_{d+2g-2}$. The vector bundle $p_{J*}(\mathcal{P} \otimes p_C^* K_C)$ is an example of a *Picard bundle*; these are well understood (for example, their Chern classes are known) and there is a substantial literature on them (see, for instance [15, 21, 22, 23, 26]).

We finish this section with an application to Brill-Noether loci.

Theorem 5.7. *Let* $d > 0$ *and* $n - g \leq d < n$*. Then* $B(n, d, n-1)$ *can be identified with* $G^{d+g-n-1}_{d+2g-2}$*.*

Proof. Since $d \leq n$, $G_T(n,d,n-1) = G_0(n,d,n-1)$. By [6, section 2.3], we have a morphism

$$\psi : G_0(n,d,n-1) \longrightarrow \widetilde{B}(n,d,n-1).$$

By Remark 5.6, the theorem will follow if we can prove that ψ is an isomorphism onto $B(n,d,n-1)$.

Note first that, by [5, Theorems B and $\tilde{\text{B}}$], the stated conditions imply that

$$B(n,d,n-1) \neq \emptyset \text{ and } \widetilde{B}(n,d,n) = \emptyset$$

(for the second condition here, we need $d < n$). Now suppose that E is a strictly semistable bundle of rank n and degree d with $h^0(E) = n-1$. Then there exists a proper subbundle E_1 of E such that both E_1 and $E_2 = E/E_1$ are semistable and have the same slope. If we let n_1, n_2 denote the ranks of E_1, E_2, then one of the E_i must have $h^0(E_i) \geq n_i$. This contradicts [5, Theorem B] since $\mu(E_i) = \mu(E)$ and hence $0 < \deg E_i < n_i$. So $B(n,d,n-1) = \widetilde{B}(n,d,n-1)$. The result now follows from [6, Corollary 11.5]. □

6. Dimension counts and flips

In this section we will again suppose that $k < n$ and obtain lower bounds on the codimensions of the flips at all critical values $\alpha > \alpha_T$. We use the notation of Remark 4.2. We are interested in the extensions (11) and note that, as in [6, equation (8)],

$$\dim \operatorname{Ext}^1((E_2,V_2),(E_1,V_1)) = C_{21} + \dim \mathbb{H}^0_{21} + \dim \mathbb{H}^2_{21},$$

where

$$\begin{aligned} \mathbb{H}^0_{21} &:= \operatorname{Hom}((E_2,V_2),(E_1,V_1)) \ , \\ \mathbb{H}^2_{21} &:= \operatorname{Ext}^2((E_2,V_2),(E_1,V_1)) \ , \\ C_{21} &:= k_2\chi(E_1) - \chi(E_2^* \otimes E_1) - k_1k_2 \\ &= n_1n_2(g-1) - d_1n_2 + d_2n_1 + k_2d_1 - k_2n_1(g-1) - k_1k_2 \ . \end{aligned} \tag{16}$$

Remark 6.1. Notice that, if $\mu_{\alpha^+}(E_1,V_1) < \mu_{\alpha^+}(E_2,V_2)$ and $\mu_\alpha(E_1,V_1) = \mu_\alpha(E_2,V_2)$, then it follows that $\mu_{\alpha^-}(E_1,V_1) > \mu_{\alpha^-}(E_2,V_2)$. In this case (E_1,V_1) is an α^--destabilizing subsystem, or, equivalently, (E_2,V_2) is an α^--destabilizing quotient system. Similarly, (E_1,V_1) and (E_2,V_2) are α^+-destabilizing for (E,V) if $\mu_{\alpha^-}(E_1,V_1) < \mu_{\alpha^-}(E_2,V_2)$.

Lemma 6.2. *Suppose that we have a diagram* (12) *and that condition* (d) *in Remark 4.2 holds. Then* $\mathbb{H}^0_{21} = 0 = \mathbb{H}^2_{21}$ *and thus*

$$\dim \operatorname{Ext}^1((E_2,V_2),(E_1,V_1)) = C_{21}. \tag{17}$$

Proof. $\mathbb{H}^0_{21} = 0$ follows at once from condition (d). Moreover $\mathbb{H}^2_{21} = 0$ because (see [6, Proposition 3.2])

$$\mathbb{H}^2_{21} = H^0(E_1^* \otimes N_2 \otimes K)^*,$$

where N_2 is the kernel of the map $V_2 \otimes \mathcal{O} \longrightarrow E_2$. But $N_2 = 0$ since (E_2,V_2) is injective.

□

By Lemma 2.2, $G(\alpha^{\pm}; n_i, d_i, k_i)$ have their expected dimensions. Together with Lemma 6.2 this yields lower bounds (see [6, section 6.3])

$$\operatorname{codim} G^{\pm}(\alpha) \geq \min \{C_{12}\} \ , \tag{18}$$

where

$$C_{12} := n_2 n_1 (g-1) - d_2 n_1 + d_1 n_2 + k_1 d_2 - k_1 n_2 (g-1) - k_2 k_1 \tag{19}$$

and the minimum is taken over all possible extension types which can occur for coherent systems in $G^+(\alpha)$ (respectively $G^-(\alpha)$).

Lemma 6.3. *For $G^+(\alpha)$, the minimum in* (18) *is taken over all (n_1, d_1, k_1) and (n_2, d_2, k_2) satisfying*

(i) $n_1 + n_2 = n,\ d_1 + d_2 = d,\ k_1 + k_2 = k,$
(ii) $0 < n_1 < n,$
(iii) $0 \leq \frac{k_1}{n_1} < \frac{k}{n} < \frac{k_2}{n_2} < 1,$
(iv) $\alpha = \dfrac{nd_1 - n_1 d}{n_1 k - n k_1}.$

Similarly, for $G^-(\alpha)$, the minimum is taken over all (n_1, d_1, k_1) and (n_2, d_2, k_2) satisfying

(i)′ $n_1 + n_2 = n,\ d_1 + d_2 = d,\ k_1 + k_2 = k,$
(ii)′ $0 < n_2 < n,$
(iii)′ $0 \leq \frac{k_2}{n_2} < \frac{k}{n} < \frac{k_1}{n_1} < 1,$
(iv)′ $\alpha = \dfrac{nd_2 - n_2 d}{n_2 k - n k_2}.$

Proof. We prove only the first set of constraints, i. e. (i)–(iv). The proof for the other set is similar.

Constraints (i) and (ii) are obvious and (iv) is a restatement of the condition $\mu_\alpha(E_1, V_1) = \mu_\alpha(E, V)$. Since coherent systems in $G^+(\alpha)$ are α-semistable but α^+-stable, constraint (iii) follows from the Diagram Lemma. □

Proposition 6.4. *Let α be a critical value in the range*

$$\alpha_T < \alpha < \frac{d}{n-k} \ .$$

Then

$$C_{12} \geq (n_1 - k_1)(n_2 - k_2)(g-1) + 1 \geq g \tag{20}$$

for all extension types (as in Lemma 4.1) which can occur for coherent systems in $G^-(\alpha)$, and

$$C_{12} \geq (g-1)(n_1 - k_1)(n_2 - k_2) + d_1 n_2 - d_2 n_1 + 1 \geq g+1 \tag{21}$$

for all extension types (as in Lemma 4.1) which can occur for coherent systems in $G^+(\alpha)$.

Proof. Note first that, since the coherent systems (E_i, V_i) satisfy the hypotheses of Lemma 2.10 with appropriate choice of either α^+ or α^-, we have (restating (4))

$$d_i - k_i \geq (k_i - n_i)(g-1). \tag{22}$$

Case 1: $G^-(\alpha)$. The equation $\mu_\alpha(E_1, V_1) = \mu_\alpha(E_2, V_2)$ can be rewritten as

$$\left(\frac{d_1}{n_1 - k_1} - \alpha\right) = \frac{n_1(n_2 - k_2)}{n_2(n_1 - k_1)}\left(\frac{d_2}{n_2 - k_2} - \alpha\right). \tag{23}$$

Note also that

$$\alpha < \frac{d}{n-k} = \frac{d_1 + d_2}{(n_1 - k_1) + (n_2 - k_2)}.$$

It follows that at least one side of (23) is positive. Hence both sides are positive. Since also $n_1(n_2 - k_2) > n_2(n_1 - k_1) > 0$ by (iii)$'$, it follows from (23) that

$$\frac{d_1}{n_1 - k_1} > \frac{d_2}{n_2 - k_2}\ ,$$

i. e.

$$d_1 n_2 - d_2 n_1 + k_1 d_2 > k_2 d_1\ .$$

(19) and (22) for $i = 1$ now give

$$C_{12} > n_2(n_1 - k_1)(g-1) + k_2(d_1 - k_1) \geq (n_1 - k_1)(n_2 - k_2)(g-1).$$

This gives (20).

Case 2: $G^+(\alpha)$. Using (19) and (22) for $i = 2$, we get

$$\begin{aligned} C_{12} &\geq (g-1)(n_2(n_1 - k_1) + k_1(k_2 - n_2)) + d_1 n_2 - d_2 n_1 \\ &= (g-1)(n_1 n_2 - 2n_2 k_1 + k_1 k_2) + d_1 n_2 - d_2 n_1\ . \end{aligned}$$

But, since $n_1 k_2 > n_2 k_1$ by (iii), we get $(n_1 n_2 - 2n_2 k_1 + k_1 k_2) > (n_1 - k_1)(n_2 - k_2)$, and hence

$$C_{12} > (g-1)(n_1 - k_1)(n_2 - k_2) + d_1 n_2 - d_2 n_1.$$

This gives the first inequality in (21). For the second, note that $\frac{k_2}{n_2} > \frac{k_1}{n_1}$ and $\mu_\alpha(E_1, V_1) = \mu_\alpha(E_2, V_2)$ give $\frac{d_1}{n_1} > \frac{d_2}{n_2}$. □

We know already from Theorem 3.4 that all flips in the range $\alpha > \alpha_I$ are good. Proposition 6.4 applies to a more restricted range for α but gives a much stronger conclusion.

Theorem 6.5. *Suppose that* $0 < k < n$ *and that* (10) *holds. Then,* $G(\alpha; n, d, k)$ *and* $G_L(n, d, k)$ *are smooth and irreducible. Moreover these varieties are isomorphic outside subvarieties of codimension at least* g.

Proof. Smoothness and irreducibility have already been proved (Theorem 3.4). The rest follows from Proposition 6.4. □

7. Picard Groups and homotopy groups for $k < n$

We begin with the following key proposition.

Proposition 7.1. *Let $0 < k < n$ and $d > 0$. Suppose further that $k \leq n + \frac{1}{g}(d-n)$. Then, for $\alpha_T < \alpha < \frac{d}{n-k}$,*

(i) $\mathrm{Pic}(G(\alpha; n, d, k)) \cong \mathrm{Pic}(G_L(n, d, k))$;
(ii) $\pi_i(G(\alpha; n, d, k)) \cong \pi_i(G_L(n, d, k))$ *for* $i \leq 2g-2$.

Proof. This follows from Theorem 6.5. □

In order to apply this, we need to calculate the Picard groups and homotopy groups of G_L. We suppose first that $n-k$ and d are coprime.

Theorem 7.2. *Let $0 < k < n$ and $d > 0$. Suppose that $n-k$ and d are coprime and that $k \leq n + \frac{1}{g}(d-n)$. Then*

(i) *if $k < n + \frac{1}{g}(d-n)$, then* $\mathrm{Pic}(G_L(n, d, k)) \cong \mathrm{Pic}(M(n-k, d)) \times \mathbb{Z}$;
(ii) *if $k = n + \frac{1}{g}(d-n)$, then* $\mathrm{Pic}(G_L(n, d, k)) \cong \mathrm{Pic}(M(n-k, d))$.

Proof. (i) follows from the fact that, by Proposition 3.1(iv), G_L is a fibration over $M(n-k, d)$ with fibre $\mathrm{Gr}(k, d+(n-k)(g-1))$. (Note that the assumption on k implies that this Grassmannian has positive dimension.)

(ii) is clear since, in this case $G_L(n, d, k) \cong M(n-k, d)$. □

Combining Theorem 7.2 with Proposition 7.1(i) we have the following.

Corollary 7.3. *Let $0 < k < n$ and $d > 0$. Suppose that $n-k$ and d are coprime and that $k \leq n + \frac{1}{g}(d-n)$ and $\alpha_T < \alpha < \frac{d}{n-k}$. Then*

(i) *if $k < n + \frac{1}{g}(d-n)$, then* $\mathrm{Pic}(G(\alpha; n, d, k)) \cong \mathrm{Pic}(M(n-k, d)) \times \mathbb{Z}$;
(ii) *if $k = n + \frac{1}{g}(d-n)$, then* $\mathrm{Pic}(G(\alpha; n, d, k)) \cong \mathrm{Pic}(M(n-k, d))$.

To compute the homotopy groups of $G_L = G_L(n, d, k)$, one can use the homotopy sequence for G_L as a $\mathrm{Gr}(k, N)$ fibration over $M(n-k, d)$ with $N = d + (n-k)(g-1)$, given by

$$\ldots \longrightarrow \pi_i(\mathrm{Gr}(k, N)) \longrightarrow \pi_i(G_L) \longrightarrow \pi_i(M(n-k, d)) \longrightarrow \pi_{i-1}(\mathrm{Gr}(k, N)) \longrightarrow \ldots \tag{24}$$

Theorem 7.4. *Let $d \geq \max\{1, n-g\}$ and suppose that $\max\{d-n, 0\} < \alpha < d$. Then*

(i) $\pi_i(G(\alpha; n, d, n-1)) \cong \pi_i(\mathrm{Gr}(n-1, d+g-1))$ *for $i \geq 0$, $i \neq 1$;*
(ii) $\pi_1(G(\alpha; n, d, n-1)) \cong H_1(C, \mathbb{Z})$.

Proof. By Theorem 5.3, G_L is a $\mathrm{Gr}(n-1, d+g-1)$ fibration over J^d. The result follows now from (24) (with $k = n-1$) and the fact that $\pi_i(J^d) = 0$ for $i \neq 1$ and $\pi_1(J_d) = H_1(C; \mathbb{Z})$. □

Corollary 7.5. *Let $d > 0$ and $n - g \leq d \leq n$. Then, for $0 < \alpha < d$,*

(i) $\pi_i(G(\alpha; n, d, n-1)) \cong \pi_i(\mathrm{Gr}(n-1, d+g-1))$ *for $i \geq 0$, $i \neq 1$;*

(ii) $\pi_1(G(\alpha; n, d, n-1)) \cong H_1(C, \mathbb{Z})$.

From the above results and [5] one can derive for $n-g \leq d < n$ the Picard group and homotopy groups of the Brill-Noether locus $B(n, d, n-1)$. However, this also follows from the explicit description of $B(n, d, n-1)$ given by Theorem 5.7. Note also that, if $d < n-g$, then $G(\alpha; n, d, n-1) = \emptyset$ for all α by Lemma 2.4 and Theorem 3.4.

Building upon the gauge-theoretic approach of Atiyah and Bott [1] to the moduli space of stable bundles, Daskalopoulos and Uhlenbeck [8, 9] have computed some of the homotopy groups of $M(r, d)$. This information can be used to compute $\pi_1(G_L)$ and $\pi_2(G_L)$.

Theorem 7.6. *Let $0 < k \leq n-2$ and $d > 0$. Suppose further that $n-k$ and d are coprime and that $\alpha_T < \alpha < \frac{d}{n-k}$. Then*

(i) $\pi_1(G(\alpha; n, d, k)) \cong \pi_1(M(n-k, d)) \cong H_1(C, \mathbb{Z})$;
(ii) *if* $k < n + \frac{1}{g}(d-n)$, $\pi_2(G(\alpha; n, d, k)) \cong \mathbb{Z} \times \mathbb{Z}$;
(iii) *if* $k = n + \frac{1}{g}(d-n)$, $\pi_2(G(\alpha; n, d, k)) \cong \mathbb{Z}$.

Proof. From Proposition 3.1(iv), G_L is a Grassmann fibration over $M(n-k, d)$ and we have the homotopy sequence (24). From this, since $\pi_0(\mathrm{Gr}(k, N)) = \pi_1(\mathrm{Gr}(k, N)) = 0$, we deduce that $\pi_1(G_L) \cong \pi_1(M(n-k, d))$. It follows from [1, Theorem 9.12] that $\pi_1(M(n-k, d))$ is isomorphic to $\pi_1(J^d)$, which in turn is isomorphic to $H_1(C, \mathbb{Z})$. Statement (i) now follows from Proposition 7.1(ii).

To compute the second homotopy group, we first note that the condition $k < n + \frac{1}{g}(d-n)$ (resp. $k = n + \frac{1}{g}(d-n)$) is equivalent to $k < N$ (resp. $k = N$). Moreover, since $\pi_1(\mathrm{Gr}(k, N)) = 0$, (24) gives

$$... \longrightarrow \pi_2(\mathrm{Gr}(k, N)) \longrightarrow \pi_2(G_L) \longrightarrow \pi_2(M(n-k, d)) \longrightarrow 0. \tag{25}$$

For $k < N$, we use again the fact that $\pi_1(\mathrm{Gr}(k, N)) = 0$ to deduce that

$$\pi_2(\mathrm{Gr}(k, N)) \cong H_2(\mathrm{Gr}(k, N), \mathbb{Z}) \cong \mathbb{Z}.$$

We claim that the map $f : \mathbb{Z} \longrightarrow \pi_2(G_L)$, induced by these isomorphisms and (25), is injective. This is true because $\mathrm{Gr}(k, N)$ is a subvariety of G_L and the map $H_2(\mathrm{Gr}(k, N), \mathbb{Z}) \longrightarrow H_2(G_L, \mathbb{Z})$ must be injective (since the restriction of an ample line bundle over G_L to $\mathrm{Gr}(k, N)$ must give an ample line bundle) and factors through f. Now, from [9], we have that $\pi_2(M(n-k, d)) \cong \mathbb{Z}$, and from (25) we deduce that $\pi_2(G_L) \cong \mathbb{Z} \times \mathbb{Z}$. Statement (ii) follows now from Proposition 7.1(ii).

Finally, if $k = N$, $\mathrm{Gr}(k, N)$ is a point; (iii) now follows from (25) and the fact that $\pi_2(M(n-k, d)) \cong \mathbb{Z}$. □

Remark 7.7. More information on higher homotopy groups could be obtained from the knowledge of the higher homotopy groups of the moduli space of stable bundles, which in turn are related, as shown in [9], to the homotopy groups of the unitary gauge group.

Remark 7.8. A direct approach, similar to the one in [8, 9], to compute the homotopy groups of the moduli space of coherent systems should also be possible in general. In fact, this has been carried out by Bradlow and Daskalopoulos [3] for $k = 1$, with some additional restrictions on d and α, but no restrictions on the coprimality of $n-1$ and d. They compute the first and second homotopy groups, which coincide with the results given by Theorem 7.6.

We show now how the results on the Picard group and the first and second homotopy groups are also valid, under certain restrictions, when $n-k$ and d are not coprime. The principal assertion of Proposition 3.1(iii) is that every coherent system $(E, V) \in G_L$ defines an extension (1) with F semistable and $h^0(F^*) = 0$. Hence, the key point is to find good estimates for the number of parameters counting strictly semistable bundles. The results of [9] then allow us to carry out the necessary computations.

To obtain our estimates we can use the Jordan-Hölder filtration of a semistable bundle F, given by

$$0 = F_0 \subset F_1 \subset F_2 \subset ... \subset F_r = F, \tag{26}$$

with $Q_i = F_i/F_{i-1}$ stable and $\mu(Q_i) = \frac{d_i}{m_i} = \mu(F)$ for $1 \leq i \leq r$. The bundle $Q = \oplus_i Q_i$ is the *graduation* of F. (Similar computations to the ones given below can be found in [1, 5, 34].)

Proposition 7.9. *Let $\mathcal{S}$ be a set of isomorphism classes of semistable bundles of rank m and degree d, whose Jordan-Hölder filtration* (26) *has graduation $Q = \oplus_{i=1}^r Q_i$, with $m_i = \operatorname{rk} Q_i$ and $d_i = \deg Q_i$. Then $\mathcal{S}$ depends on at most*

$$\left(\sum_i m_i^2 + \sum_{i<j} m_i m_j\right)(g-1) + 1 \tag{27}$$

parameters.

Proof. This is obtained by adding up the numbers

$$\dim M(m_i, d_i) = m_i^2(g-1) + 1$$

for $1 \leq i \leq r$, and the dimensions of the spaces of equivalence classes of extensions

$$0 \longrightarrow F_{j-1} \longrightarrow F_j \longrightarrow Q_j \longrightarrow 0$$

for $2 \leq j \leq r$. By Riemann-Roch and the condition $\mu(Q_i) = \mu(Q_j)$, these dimensions are given by

$$h^1(Q_j^* \otimes F_{j-1}) - 1 = m_j\left(\sum_{i<j} m_i\right)(g-1) - 1.$$

(We have assumed that Q_i and Q_j are not isomorphic, since if they are isomorphic the number of parameters on which $\mathcal{S}$ depends is actually smaller.) □

Corollary 7.10. *Let $0 < k < n$ and suppose that $G_L(n, d, k) \neq \emptyset$. Let W denote the closed subvariety of $G_L(n, d, k)$ consisting of coherent systems which arise from extensions* (1) *in which F is strictly semistable. Then the codimension of W in $G_L(n, d, k)$ is at least*

$$D := \min\left\{\left(\sum_{i<j} m_i m_j\right)(g-1)\right\}, \tag{28}$$

where the minimum is taken over all sequences of positive integers $r, m_1, \dots, m_r$ *such that* $r \geq 2$ *and* $\sum m_i = n - k$.

Proof. For fixed $F \in \mathcal{S}$ with $h^0(F^*) = 0$, the dimension of the variety of coherent systems of the form (1) is

$$k \cdot h^1(F^*) - k^2 = k(d + (n-k)(g-1) - k).$$

Adding this to (27) and subtracting the total from $\dim G_L(n,d,k) = \beta(n,d,k)$ gives the estimate (28). □

Proposition 7.11. *The minimum in* (28) *is attained when* $r = 2$ *and* $\{m_1, m_2\} = \{1, n-k-1\}$. *Hence*

$$D = (n-k-1)(g-1).$$

Proof. The minimum of the expression $\sum_{i<j} m_i m_j$ is achieved for $r = 2$. So

$$D = \min\{m_1(n-k-m_1)(g-1)\} = (n-k-1)(g-1).$$

□

Theorem 7.12. *Let* $0 < k < n$ *and* $d > 0$. *Suppose that* $k < n + \frac{1}{g}(d-n)$ *and*

$$(n-k-1)(g-1) \geq 2 \tag{29}$$

Then, for $\alpha_T < \alpha < \frac{d}{n-k}$,

(i) $\operatorname{Pic}(G(\alpha; n, d, k)) \cong \operatorname{Pic}(M(n-k, d)) \times \mathbb{Z}$;
(ii) $\pi_1(G(\alpha; n, d, k)) \cong \pi_1(M(n-k, d)) \cong H_1(C, \mathbb{Z})$;
(iii) *there is an exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(G(\alpha; n, d, k)) \longrightarrow \mathbb{Z} \times \mathbb{Z}_p \longrightarrow 0,$$

where $p = \operatorname{GCD}(n-k, d)$.

Proof. By Proposition 3.1(iii), $G_L \setminus W$ is isomorphic to a Grassmann fibration over $M(n-k, d)$. It follows from Proposition 7.11 and (29) that

$$\operatorname{Pic}(G_L) \cong \operatorname{Pic}(G_L \setminus W) \cong \operatorname{Pic}(M(n-k, d)) \times \mathbb{Z}.$$

The statement (i) now follows from Proposition 7.1.

To prove (ii) and (iii) we use the same arguments as in Theorem 7.6 taking into account now that, as proved in [9, Theorem 3.1],

$$\pi_1(M(n-k, d)) \cong H_1(C, \mathbb{Z}), \quad \pi_2(M(n-k, d)) \cong \mathbb{Z} \times \mathbb{Z}_p.$$

This gives $\pi_1(G_L \setminus W) \cong \pi_1(M(n-k, d))$ and an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_2(G_L \setminus W) \longrightarrow \mathbb{Z} \times \mathbb{Z}_p \longrightarrow 0$$

(compare (25)). Now use Proposition 7.11 and (29) again. □

Remark 7.13. When $k = n + \frac{1}{g}(d-n)$, $\operatorname{Gr}(k, N)$ is a point; in this case (ii) remains true, but we must delete the factor $\mathbb{Z}$ in (i) and (iii) becomes $\pi_2(G(\alpha; n, d, k)) \cong \mathbb{Z} \times \mathbb{Z}_p$. It is a plausible conjecture, compatible with Theorem 7.6, [3, Theorem 1.12], Theorem 7.12 and the first part of this remark, that, for $0 < k \leq n-2$, $\pi_2(G(\alpha; n, d, k)) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_q$, where $q = \operatorname{GCD}(n, d, k)$, with one factor $\mathbb{Z}$ being dropped when $k = n + \frac{1}{g}(d-n)$.

Remark 7.14. The inequality (29) fails only for $k = n-1$ (any genus) and for $k = n-2$ when $g = 2$. The cases $k = n-1$ and $g = 2$, $k = n-2$, d odd are covered by Theorems 7.2, 7.4 and 7.6; note that, if $k = n-1$, (i) and (ii) are true, but (iii) must be replaced by $\pi_2(G(\alpha; n, d, k)) \cong \mathbb{Z}$. This leaves only the case

$$g = 2,\ k = n-2,\ d \text{ even},$$

for which the theorem may fail. One may note that [9] specifically excludes this case.

We finish with the result for the Picard variety, which takes a very simple form.

Theorem 7.15. *Let $0 < k < n$ and $d > 0$. Suppose that $k \leq n + \frac{1}{g}(d-n)$ and that $\alpha_T < \alpha < \frac{d}{n-k}$. Then, except possibly in the case $g = 2$, $k = n-2$, d even,*

$$\operatorname{Pic}^0(G(\alpha; n, d, k)) \cong J(C).$$

Proof. From [10, Théorèmes A, C] and the fact that the codimension of the strictly semistable locus in $M(n-k, d)$ is at least 2, it follows that $\operatorname{Pic}^0(M(n-k, d)) \cong J(C)$. For the rest, see the proofs of Theorems 7.2 and 7.12. □

8. The case $k = n-2$: Poincaré polynomials

So far, in applying the Diagram Lemma, we have used only the numerical consequences of the construction. In this section we make a first application of the geometry of the flips at critical values in the range $\alpha > \alpha_T$. We shall be mainly concerned with the case $k = n-2$; for values of $k < n-2$, some information can be obtained from the Diagram Lemma, but additional techniques will be necessary to obtain complete results.

We begin, however, with a basic observation which applies to any $k < n$ with $n-k$ and d coprime. We write $P(X)$ for the Poincaré polynomial of a space X (with coefficients in any fixed field).

Proposition 8.1. *Let $0 < k < n$ and $d > 0$. Suppose that $k \leq n + \frac{1}{g}(d-n)$ and that $n-k$ and d are coprime. Then the Poincaré polynomial of G_L is given by*

$$P(G_L(n, d, k)) = P(M(n-k, d)) \cdot P(\operatorname{Gr}(k, N)),$$

where $N = d + (n-k)(g-1)$.

Proof. By Proposition 3.1(iv), $G_L(n, d, k)$ is the Grassmann fibration over $M(n-k, d)$ associated with some vector bundle. The result is now standard. □

Remark 8.2. (i) Note that, when $\operatorname{GCD}(n-k, d) = 1$, $H^*(M(n-k, d); \mathbb{Z})$ is torsion-free [1, Theorem 9.9]. Hence $P(M(n-k, d))$ and $P(G_L(n, d, k))$ are independent of the characteristic of the coefficient field. A closed (though, in general, complicated) formula for $P(M(n-k, d))$ is known [36]. Here we shall need only the special case

$$P(M(2, d))(t) = \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})}{(1-t^2)(1-t^4)} \quad (d \text{ odd}), \tag{30}$$

which is essentially proved in [27].

(ii) For $0 < k \le N$, the Poincaré polynomial of $\mathrm{Gr}(k, N)$ is given by

$$P(\mathrm{Gr}(k,N))(t) = \frac{(1-t^{2(N-k+1)})(1-t^{2(N-k+2)})\cdots(1-t^{2N})}{(1-t^2)(1-t^4)\cdots(1-t^{2k})}. \tag{31}$$

Combining this with (i), one can obtain an explicit formula for $P(G_L(n,d,k))$.

We suppose now that $k = n-2$. We are concerned with critical values in the range (10), which in this case is

$$\max\{d-n, 0\} < 2\alpha < d. \tag{32}$$

For convenience, we will write $G(\alpha)$, G_L and G_T for $G(\alpha; n, d, n-2)$, $G_L(n,d,n-2)$ and $G_T(n,d,n-2)$.

At any such critical value α, we know, either by Lemma 6.3 or directly from the Diagram Lemma, that $k_1 < n_1$ and $k_2 < n_2$. Hence

$$k_1 = n_1 - 1, \quad k_2 = n_2 - 1. \tag{33}$$

Inserting these into Lemma 6.3 for $G^+(\alpha)$, we get

$$2n_1 < n \tag{34}$$

and

$$\alpha = \frac{nd_1 - n_1 d}{n - 2n_1}. \tag{35}$$

Equation (32) now gives

$$\max\{(d-n)(n-2n_1), 0\} < 2(nd_1 - n_1 d) < d(n-2n_1),$$

i.e.

$$\max\left\{d + 2n_1 - n, \frac{2n_1 d}{n}\right\} < 2d_1 < d. \tag{36}$$

Lemma 8.3. (i) *The flip locus $G^+(\alpha)$ is a disjoint union*

$$G^+(\alpha) = \bigsqcup G^+(n_1, d_1),$$

where (n_1, d_1) ranges over the set of possible solutions of (34), (35), (36). *Here $G^+(n_1, d_1)$ is a smooth subvariety of $G(\alpha^+)$ and is isomorphic to a projective bundle over $G_L(n_1, d_1, n_1 - 1) \times G_L(n_2, d_2, n_2 - 1)$ with fibre dimension $C_{21}(n_1, d_1) - 1$, where*

$$C_{21}(n_1, d_1) = n_1(g-1) + dn_1 - d_1(n_1+1) - (n_1-1)(n-n_1-1). \tag{37}$$

Moreover $G^+(n_1, d_1)$ has codimension $C_{12}(n_1, d_1)$ in $G(\alpha^+)$ given by

$$C_{12}(n_1, d_1) = (n-n_1)(g-1) - d + d_1(n-n_1+1) - (n_1-1)(n-n_1-1). \tag{38}$$

(ii) *Similarly $G^-(\alpha)$ is a disjoint union*

$$G^-(\alpha) = \bigsqcup G^-(n_1, d_1)$$

of smooth subvarieties of $G(\alpha^-)$, where $G^-(n_1, d_1)$ is isomorphic to a projective bundle over $G_L(n_1, d_1, n_1 - 1) \times G_L(n_2, d_2, n_2 - 1)$ with fibre dimension $C_{12}(n_1, d_1) - 1$. Moreover $G^-(n_1, d_1)$ has codimension $C_{21}(n_1, d_1)$ in $G(\alpha^-)$.

Proof. For fixed n, d, the inequalities (34) and (36) give rise to a finite number of choices for n_1, d_1, hence also for α. One can check that the values of α lie in the torsion-free range for coherent systems of types $(n_1, d_1, n_1 - 1)$ and $(n_2, d_2, n_2 - 1)$. It follows from Proposition 5.1 that α is not critical for either of these types of coherent systems; hence the filtration (11) of the Diagram Lemma is the unique Jordan-Hölder filtration of (E, V). It is possible that, for a given α, there may be more than one set of values for n_1, d_1 satisfying (34), (35) and (36). However the uniqueness of the Jordan-Hölder filtration implies that the flip locus $G^+(\alpha)$ is a disjoint union of subvarieties as required. The smoothness of these subvarieties is proved as in [33, section 3] (see Appendix for details). The formulae (37) and (38) are obtained from (16) and (19) using (33) and Lemma 6.3(i)).

(ii) is proved by interchanging the subscripts 12 in the proof of (i) □

This situation is analogous to that of Thaddeus [33]. Using the same method as in [33, section 3] (see Appendix), we obtain a smooth variety G which is simultaneously the blow-up of $G(\alpha^+)$ along $G^+(\alpha)$ and the blow-up of $G(\alpha^-)$ along $G^-(\alpha)$. Moreover the exceptional divisors of the blow-ups coincide. If we write

$$S(n_1, d_1) = G_L(n_1, d_1, n_1 - 1) \times G_L(n_2, d_2, n_2 - 1),$$

then the exceptional divisor Y is the disjoint union

$$Y = \bigsqcup G^+(n_1, d_1) \times_{S(n_1,d_1)} G^-(n_1, d_1)$$

for the values of n_1, d_1 which correspond to the critical value α (Appendix, (48)).

Now write for convenience

$$P_{r,e} = P(G_L(r, e, r - 1)).$$

Proposition 8.4. *Let* $e \geq \max\{1, r - g\}$. *Then*

$$P_{r,e}(t) = \frac{(1+t)^{2g}(1 - t^{2(e+g-r+1)})(1 - t^{2(e+g-r+2)}) \cdots (1 - t^{2(e+g-1)})}{(1 - t^2)(1 - t^4) \cdots (1 - t^{2(r-1)})}. \tag{39}$$

Proof. This follows from Proposition 8.1 and (31). □

Theorem 8.5. *For any non-critical value* α' *in the interval* $(\alpha_T, \frac{d}{2})$,

$$P(G(\alpha'))(t) - P(G_L)(t) = \sum \frac{t^{2C_{21}(n_1,d_1)} - t^{2C_{12}(n_1,d_1)}}{1 - t^2} P_{n_1,d_1}(t) P_{n_2,d_2}(t), \tag{40}$$

where the summation is over all solutions of (34), (35), (36) *for which* $\alpha > \alpha'$. *In particular*

$$P(G_T)(t) - P(G_L)(t) = \sum \frac{t^{2C_{21}(n_1,d_1)} - t^{2C_{12}(n_1,d_1)}}{1 - t^2} P_{n_1,d_1}(t) P_{n_2,d_2}(t),$$

the summation being over all solutions of (34), (35), (36) *for which* $\alpha > \alpha_T$.

Proof. We have, by Lemma 8.3

$$\begin{aligned} P(G^+(n_1, d_1))(t) &= P_{n_1,d_1}(t) P_{n_2,d_2}(t) \left(1 + t^2 + \ldots + t^{2(C_{21}(n_1,d_1)-1)}\right) \\ &= \frac{1 - t^{2C_{21}(n_1,d_1)}}{1 - t^2} P_{n_1,d_1}(t) P_{n_2,d_2}(t), \end{aligned}$$

with a similar formula for $P(G^-(n_1,d_1))(t)$. The formula for the Poincaré polynomial of a blow-up (see [12, p.605]) now gives

$$\begin{aligned} P(G)(t) &= P(G(\alpha^+))(t) + \sum \left(t^2 + \ldots + t^{2(C_{12}(n_1,d_1)-1)}\right) P(G^+(n_1,d_1))(t) \\ &= P(G(\alpha^-))(t) + \sum \left(t^2 + \ldots + t^{2(C_{21}(n_1,d_1)-1)}\right) P(G^-(n_1,d_1))(t), \end{aligned}$$

where the summation is over all solutions (n_1,d_1) of (34), (35), (36) for the given α. With a little manipulation, this gives

$$P(G(\alpha^-))(t) = P(G(\alpha^+))(t) + \sum \frac{t^{2C_{21}(n_1,d_1)} - t^{2C_{12}(n_1,d_1)}}{1-t^2} P_{n_1,d_1}(t) P_{n_2,d_2}(t). \tag{41}$$

The theorem now follows by adding the formulae (41) for all relevant α. □

Remark 8.6. So far, this does not depend on any coprimality assumptions, since the flip loci lie strictly inside the moduli spaces of $\alpha^{\pm}$-stable coherent systems. When d is odd, however, we can say a bit more. In this case, we know by Proposition 3.1(iv) that G_L is a fibration over $M(2,d)$ with fibre $\mathrm{Gr}(n-2, d+2g-2)$, so we can write down an explicit formula for $P(G_L)$. Hence (40) gives an explicit formula for $P(G(\alpha'))$ for $\alpha' \in (\alpha_T, \frac{d}{2})$.

Corollary 8.7. *Suppose $n=3$ and d is odd. Then, for $\max\{\frac{d-3}{2}, 0\} < \alpha' < \frac{d}{2}$,*

$$\begin{aligned} P(G(\alpha'))(t) &= P(G_L)(t) \\ &= P(M(2,d))(t) \frac{1-t^{2(d+2g-2)}}{1-t^2} \\ &= \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)}. \end{aligned}$$

Proof. In this case, there are no solutions to (34) and (36), so $G_T = G_L$ by Theorem 8.5. Now use Proposition 8.1 and (30). □

Corollary 8.8. *Suppose $n=4$ and d is odd. Then*

(i) *if* $\max\{\frac{d-2}{2}, 0\} < \alpha' < \frac{d}{2}$,

$$\begin{aligned} P(G(\alpha'))(t) &= P(G_L)(t) \\ &= \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-3)})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)^2}; \end{aligned}$$

(ii) *if* $\max\{\frac{d-4}{2}, 0\} < \alpha' < \frac{d-2}{2}$,

$$\begin{aligned} P(G(\alpha'))(t) &= P(G_L)(t) + \frac{t^{2g} - t^{6g+2d-10}}{1-t^2} P_{1,\frac{d-1}{2}}(t) P_{3,\frac{d+1}{2}}(t) \\ &= \frac{(1+t)^{2g}((1+t^3)^{2g} - t^{2g}(1+t)^{2g})(1-t^{2(d+2g-3)})(1-t^{2(d+2g-2)})}{(1-t^2)^2(1-t^4)^2} \\ &\quad + \frac{(t^{2g} - t^{6g+2d-10})(1-t^{d-3+2g})(1-t^{d-1+2g})(1+t)^{4g}}{(1-t^2)^2(1-t^4)}. \end{aligned}$$

Proof. For $n = 4$, there is no solution to (34), (35) and (36) for $d = 1$, but for $d \geq 3$ there is a unique solution given by

$$n_1 = 1, \quad d_1 = \frac{d-1}{2}, \quad \alpha = \frac{d-2}{2}.$$

Moreover (37) and (38) now give

$$C_{21}(n_1, d_1) = g, \quad C_{12}(n_1, d_1) = 3g + d - 5.$$

So, if $\alpha' > \frac{d-2}{2}$, $G(\alpha') = G_L$, while, if $\alpha' \in (\frac{d-4}{2}, \frac{d-2}{2})$, then, by Theorem 8.5,

$$P(G(\alpha'))(t) = P(G_L)(t) + \frac{t^{2g} - t^{6g+2d-10}}{1-t^2} P_{1,\frac{d-1}{2}}(t) P_{3,\frac{d+1}{2}}(t).$$

The result now follows from Proposition 8.1, (30), [31] and (39). $\square$

Remark 8.9. The above formulae are independent of the coefficient field for the homology groups. It follows that the integral homology of $G(\alpha')$ is also torsion-free. A further observation is that we can use other cohomology theories and get similar results (for example, algebraic cohomology). With some further work, it may also be possible to compute Chow groups.

Appendix. Geometry of flips

Our object in this appendix is to establish the geometric description of a flip in the best-behaved case. This is analogous to Thaddeus' description [33] for the case $n = 2$, $k = 1$. The general approach is the same as that of Thaddeus and is similar to that of He [14] (see also [30] and, for a related problem, [11]). However, He restricts to rank 2 (and also to coherent sheaves on $\mathbb{P}^2$) at a crucial point, although earlier in the paper he discusses coherent systems in great generality.

As in the main part of the paper, we work with coherent systems over a smooth projective irreducible algebraic curve C, defined over the complex numbers. For a fixed type (n, d, k), let $G(\alpha) = G(\alpha; n, d, k)$ and let α_c be a critical value of α. The flip locus $G^+ := G^+(\alpha_c) \subset G(\alpha_c^+)$ is given by non-trivial extensions of the form

$$0 \longrightarrow (E_1, V_1) \longrightarrow (E, V) \longrightarrow (E_2, V_2) \longrightarrow 0, \tag{42}$$

where (E_j, V_j) is of type (n_j, d_j, k_j) and is α_c-semistable and α_c^+-stable for $j = 1, 2$. Moreover

$$\mu_{\alpha_c}(E_1, V_1) = \mu_{\alpha_c}(E_2, V_2), \quad \frac{k_1}{n_1} < \frac{k_2}{n_2}. \tag{43}$$

In particular, we have

$$\alpha_c = \frac{n_2 d_1 - n_1 d_2}{n_1 k_2 - n_2 k_1} > 0,$$

hence $\frac{d_1}{n_1} > \frac{d_2}{n_2}$. The flip locus $G^- := G^-(\alpha_c) \subset G(\alpha_c^-)$ is given similarly by extensions

$$0 \longrightarrow (E_2, V_2) \longrightarrow (E', V') \longrightarrow (E_1, V_1) \longrightarrow 0, \tag{44}$$

where (E_j, V_j) is now α_c^--stable for $j = 1, 2$, the other conditions being as above. Now $G(\alpha_c^-)$ is obtained from $G(\alpha_c^+)$ by deleting G^+ and inserting G^-; in particular

$$G(\alpha_c^-) \setminus G^- = G(\alpha_c^+) \setminus G^+ .$$

It is possible that there is more than one choice of the values n_1, d_1, k_1 for a given critical value α_c. In this case we write $G^+(n_1, d_1)$, $G^-(n_1, d_1)$ for the subsets of the flip loci corresponding to particular values of n_1, d_1; note that, once n_1, d_1 are fixed, so is k_1.

Assumptions A.1. *We assume, for all choices of n_1, d_1, k_1 corresponding to the critical value α_c,*

(a) $\mathrm{GCD}(n_1, d_1, k_1) = \mathrm{GCD}(n_2, d_2, k_2) = 1$ *;*
(b) *α_c is not a critical value for (n_1, d_1, k_1), (n_2, d_2, k_2) ;*
(c) *$G_1 := G(\alpha_c; n_1, d_1, k_1)$ and $G_2 := G(\alpha_c; n_2, d_2, k_2)$ are smooth of the expected dimensions ;*
(d) $\mathrm{Ext}^2((E_1, V_1), (E_2, V_2)) = \mathrm{Ext}^2((E_2, V_2), (E_1, V_1)) = 0$.

Remark A.2. Note that, by Theorem 3.3 and the proof of Lemma 6.2, (c) and (d) always hold if α_c is in the injective range for both (n_1, d_1, k_1) and (n_2, d_2, k_2).

Lemma A.3. *Given Assumptions A.1, we have*

(i) *$(E_j, V_j) \in G_j$ for $j = 1, 2$;*
(ii) *G_1, G_2 are smooth projective varieties ;*
(iii) $\mathrm{Hom}((E_1, V_1), (E_2, V_2)) = \mathrm{Hom}((E_2, V_2), (E_1, V_1)) = 0$.

Proof. (i) follows at once from (a) and (b). (ii) follows from (a) and (c). Finally, for (iii), it follows from (43) that (E_1, V_1), (E_2, V_2) are non-isomorphic with the same α_c-slope; since both are α_c-stable by (i), this implies (iii). □

Corollary A.4. *The sequences (42), (44) are the unique Jordan-Hölder filtrations of (E, V), (E', V') as α_c-semistable coherent systems.*

Proof. This follows from (i) and the non-triviality of (42), (44). □

Our next object is to show that G^+, G^- are smooth subvarieties of $G(\alpha_c^+)$, $G(\alpha_c^-)$ respectively, and to identify their normal bundles. The proofs are identical in the two cases, so we shall work with the + case only. Since we have not assumed that $G(\alpha_c^+)$ is smooth, we need first a lemma.

Lemma A.5. *$G(\alpha_c^+)$ is smooth of the expected dimension at every point of G^+.*

Proof. We must show that, under Assumptions A.1, $G(\alpha_c^+)$ is smooth of the expected dimension at (E, V) for any non-trivial extension (42). For this, we must show that the Petri map is injective, or equivalently that

$$H^0(E^* \otimes N \otimes K) = 0,$$

where N is the kernel of the evaluation map $V \otimes \mathcal{O} \longrightarrow E$. Let N_l be the kernel of the evaluation map $V_l \otimes \mathcal{O} \longrightarrow E_l$, for $l = 1, 2$. Then $H^0(E_m^* \otimes N_l \otimes K) = 0$ for $l = 1, 2$, $m = 1, 2$, by Assumption A.1 (c) and (d), using [6, Proposition 3.2]. The result now follows by diagram chasing. □

Definition A.6. A *family of coherent systems of type (n, d, k) on C parametrised by S* is a pair $(\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is a vector bundle of rank n over $S \times C$ such that $\mathcal{E}_s = \mathcal{E}|_{\{s\} \times C}$ has degree d for all $s \in S$, and $\mathcal{V}$ is a locally free subsheaf of $p_{S*}\mathcal{E}$ (where

$p_S : S \times C \longrightarrow S$ stands for the projection) of rank k such that the fibres $\mathcal{V}_s$ map injectively to $H^0(\mathcal{E}_s)$ for all $s \in S$.

Remark A.7. This definition is more restrictive than that of [14], but is sufficient for our purposes.

We need two facts about families of coherent systems for which we have been unable to locate proofs. We state these as propositions.

Proposition A.8. *Suppose* $\mathrm{GCD}(n, d, k) = 1$. *Then there exists a universal family of coherent systems over* $G(\alpha; n, d, k) \times C$.

Proof. [The proof is on standard lines and is modelled on those of [35] for $M(n, d)$ (see also [28, Theorem 5.12] or [31, Première Partie, Théorème 18]) when $\mathrm{GCD}(n, d) = 1$ and [29, Theorems 2.8 and 3.3] for $G(\alpha; n, d, k)$ when α is small and $\mathrm{GCD}(n, k) = 1$.]

We recall the method of construction of $G(\alpha; n, d, k)$ [20, 29, 16]. We have a family $(\mathcal{E}, \mathcal{V})$ of α-stable coherent systems of type (n, d, k) parametrised by a variety R^s together with an action of $PGL(N)$ on R^s which lifts to an action of $GL(N)$ on $(\mathcal{E}, \mathcal{V})$. The family $(\mathcal{E}, \mathcal{V})$ satisfies the local universal property for α-stable coherent systems of type (n, d, k) and the moduli space $G(\alpha; n, d, k)$ is the geometric quotient of R^s by $PGL(N)$ (indeed R^s is a principal $PGL(N)$-fibration over $G(\alpha; n, d, k)$). Moreover the action of an element λ of the centre $\mathbb{C}^*$ of $GL(N)$ on $(\mathcal{E}, \mathcal{V})$ is multiplication by λ. Suppose now that we can construct a line bundle $\mathcal{L}$ on R^s such that the action of $PGL(N)$ on R^s lifts to an action of $GL(N)$ on $\mathcal{L}$ with the same property. We then consider the coherent system

$$(\mathcal{E} \otimes p_{R^s}^* \mathcal{L}^*, \mathcal{V} \otimes \mathcal{L}^*)$$

over $R^s \times C$. The action of $PGL(N)$ on $R^s \times C$ now lifts to an action of $PGL(N)$ on this coherent system. It follows from the theory of descent (see [13, Theorem 1] or Kempf's descent lemma [10, Theorem 2.3]) that this coherent system is the pull-back of a coherent system over $G(\alpha; n, d, k) \times C$ which satisfies the required universal property.

It remains to construct $\mathcal{L}$. For this, we consider the bundle $\mathcal{E}$ over $R^s \times C$ and let $\mathcal{E}_t$ denote the bundle over C obtained by restricting $\mathcal{E}$ to $\{t\} \times C$. There exists a line bundle L on C such that $H^1(\mathcal{E}_t \otimes L) = 0$ for all $t \in R^s$. It follows by Riemann-Roch that

$$p := h^0(\mathcal{E}_t \otimes L) = d + n(m + 1 - g)$$

for all $t \in R^s$, where $m = \deg L$. Hence

$$\mathcal{F} := p_{R^s *}(\mathcal{E} \otimes p_C^* L)$$

is locally free of rank p.

Now choose a point $x_0 \in C$; then, by the same argument,

$$\mathcal{F}(x_0) := p_{R^s *}(\mathcal{E} \otimes p_C^* L(x_0))$$

is locally free of rank $p + n$. Now

$$\mathrm{GCD}(p + n, p, k) = \mathrm{GCD}(n, p, k) = \mathrm{GCD}(n, d, k) = 1,$$

so there exist integers a, b, c such that

$$a(p+n)+bp+ck=1.$$

We can now define

$$\mathcal{L}:=(\det\mathcal{F}(x_0))^a\otimes(\det\mathcal{F})^b\otimes(\det\mathcal{V})^c.$$

The element $\lambda\in\mathbb{C}^*$ now acts on $\mathcal{L}$ by

$$\lambda^{a(p+n)+bp+ck}=\lambda$$

as required. □

Proposition A.9. *Let $(\mathcal{E}_1,\mathcal{V}_1)$, $(\mathcal{E}_2,\mathcal{V}_2)$ be two families of coherent systems parametrised by S and let*

$$\mathcal{E}xt^{\,q}_{p_S}((\mathcal{E}_2,\mathcal{V}_2),(\mathcal{E}_1,\mathcal{V}_1))$$

be defined as in [14, 1.2]. *Then there exists a spectral sequence with E_2-term*

$$E_2^{pq}=H^p(\mathcal{E}xt^{\,q}_{p_S}((\mathcal{E}_2,\mathcal{V}_2),(\mathcal{E}_1,\mathcal{V}_1))),$$

which abuts to $\mathrm{Ext}^*((\mathcal{E}_2,\mathcal{V}_2),(\mathcal{E}_1,\mathcal{V}_1))$.

Proof. The construction of [14] depends on embedding the category of families of coherent systems over C parametrised by S into an abelian category $\mathcal{C}$ with enough injectives; in He's notation, the objects of this larger category $\mathcal{C}$ are called *algebraic systems on $S\times X$ relative to S*. Both $\mathcal{E}xt_{p_S}$ and Ext can now be defined using an injective resolution of $(\mathcal{E}_1,\mathcal{V}_1)$ in $\mathcal{C}$. To prove the existence of the spectral sequence, it is sufficient to show that, if $\mathcal{I}$ is an injective in $\mathcal{C}$, then $\mathcal{H}om_{p_S}((\mathcal{E}_2,\mathcal{V}_2),\mathcal{I})$ is an acyclic sheaf; this follows from He's description of the injectives [14, Théorème 1.3]. □

Lemma A.10. *There exists a vector bundle W^+ over $G_1\times G_2$ and a morphism*

$$f_+:\mathbb{P}W^+\longrightarrow G(\alpha_c^+),$$

which maps $\mathbb{P}W^+$ bijectively to $G^+(n_1,d_1)$.

Proof. By Assumption A.1 (a) and Proposition A.8, there exist universal families of coherent systems $(\mathcal{E}_1,\mathcal{V}_1)$ over $G_1\times C$ and $(\mathcal{E}_2,\mathcal{V}_2)$ over $G_2\times C$. By Assumption A.1 (d) and Lemma A.3 (iii),

$$\dim\mathrm{Ext}^1((E_2,V_2),(E_1,V_1))$$

is independent of the choice of $(E_1,V_1)\in G_1$, $(E_2,V_2)\in G_2$. It follows from [14, Corollaire 1.20] that there is a vector bundle W^+ over $G_1\times G_2$ whose fibre over

$$((E_1,V_1),(E_2,V_2))\in G_1\times G_2$$

is $\mathrm{Ext}^1((E_2,V_2),(E_1,V_1))$; in fact, in the notation of [14],

$$W^+=\mathcal{E}xt^{\,1}_{\pi}((p_2\times\mathrm{Id})^*(\mathcal{E}_2,\mathcal{V}_2),(p_1\times\mathrm{Id})^*(\mathcal{E}_1,\mathcal{V}_1)),$$

where $\pi:G_1\times G_2\times C\longrightarrow G_1\times G_2$, $p_1:G_1\times G_2\longrightarrow G_1$ and $p_2:G_1\times G_2\longrightarrow G_2$ are the natural projections. Now $\mathbb{P}W^+$ classifies the non-trivial extensions (42) up to scalar multiples. We can therefore define f_+ set-theoretically as the natural map sending (42) to $(E,V)\in G(\alpha_c^+)$. The fact that f_+ maps $\mathbb{P}W^+$ bijectively to $G^+(n_1,d_1)$ follows from Corollary A.4.

In order to prove that f_+ is a morphism, we need to construct a universal extension (42) over $\mathbb{P}W^+ \times C$. This is done in exactly the same way as for extensions of vector bundles. Let $\sigma : \mathbb{P}W^+ \times C \longrightarrow \mathbb{P}W^+$ and $p : \mathbb{P}W^+ \longrightarrow G_1 \times G_2$ be the natural projections. We write, for $m = 1, 2$,

$$(\mathcal{E}_m, \mathcal{V}_m)^+ = (p \times \mathrm{Id})^*(p_m \times \mathrm{Id})^*(\mathcal{E}_m, \mathcal{V}_m).$$

We construct the universal extension as an extension

$$0 \longrightarrow (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1) \longrightarrow (\mathcal{E}, \mathcal{V})^+ \longrightarrow (\mathcal{E}_2, \mathcal{V}_2)^+ \longrightarrow 0 \tag{45}$$

on $\mathbb{P}W^+ \times X$. Extensions of the form (45) are classified by

$$\mathrm{Ext}^1((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1)).$$

By Proposition A.9 we have a spectral sequence whose E_2-term is given by

$$E_2^{pq} = H^p\big(\mathcal{E}xt^{\,q}_{\sigma}((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1))\big)$$

which abuts to $\mathrm{Ext}^*((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1))$. In view of Assumption A.1 (d) and Lemma A.3 (iii), we have $E_2^{pq} = 0$ except for $q = 1$. Hence

$$\begin{aligned}
&\mathrm{Ext}^1((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1))\\
&\qquad \cong H^0\big(\mathcal{E}xt^{\,1}_{\sigma}((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1))\big).
\end{aligned}$$

On the other hand, by base-change [14, Théorème 1.16],

$$\begin{aligned}
&\mathcal{E}xt^{\,1}_{\sigma}((\mathcal{E}_2, \mathcal{V}_2)^+, (\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1))\\
&\qquad \cong p^*\mathcal{E}xt^{\,1}_{\pi}((p_2 \times \mathrm{Id})^*(\mathcal{E}_2, \mathcal{V}_2), (p_1 \times \mathrm{Id})^*(\mathcal{E}_1, \mathcal{V}_1)) \otimes \mathcal{O}_{\mathbb{P}W^+}(1)\\
&\qquad = p^*W^+ \otimes \mathcal{O}_{\mathbb{P}W^+}(1).
\end{aligned}$$

Now $H^0(p^*W^+ \otimes \mathcal{O}_{\mathbb{P}W^+}(1)) = \mathrm{End}\, W^+$. The universal extension is then the extension (45) corresponding to the identity endomorphism of W^+. It is clear that the restriction of (45) to $\{y\} \times X$ is precisely the extension (42) corresponding to $y \in \mathbb{P}W^+$. So the morphism $\mathbb{P}W^+ \longrightarrow G(\alpha_c^+)$, given by (45) and the universal property of $G(\alpha_c^+)$, coincides with f_+. This completes the proof of the lemma. □

It is an immediate consequence of this lemma and Lemma A.5 that $G^+(n_1, d_1)$ is a projective subvariety of the smooth part of $G(\alpha_c^+)$. Moreover it follows from Corollary A.4 that the $G^+(n_1, d_1)$ for different values of n_1 and d_1 are disjoint. The computation of the normal bundle can therefore be carried out independently for each choice of n_1, d_1. Let W^- be the bundle over $G_1 \times G_2$ constructed in an analogous way to W^+ after interchanging the subscripts $1, 2$.

Proposition A.11. *The morphism f_+ is a smooth embedding with normal bundle $p^*W^- \otimes \mathcal{O}_{\mathbb{P}W^+}(-1)$.*

Proof. The proof is exactly analogous to [33, (3.9)]. In our notation, it proceeds as follows. For simplicity, we begin by looking at the infinitesimal deformations at a point ξ of $\mathbb{P}W^+$ represented by an extension (42). We have a short exact sequence of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & \mathrm{Hom}(E, E) & \longrightarrow & \mathrm{Hom}(E_1, E_2) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \mathrm{Hom}(V, E/V) & \longrightarrow & \mathrm{Hom}(V_1, E_2/V_2) & \longrightarrow & 0.
\end{array}$$

Here V, V_1, V_2 are to be interpreted as sheaves of locally constant sections, the right-hand square is the obvious homomorphism of complexes and $A \to B$ is just the kernel of this homomorphism. The middle complex parametrises infinitesimal deformations of (E, V), while the right-hand one is the complex giving rise to the fibre of W^- at the point $((E_1, V_1), (E_2, V_2))$ of $G_1 \times G_2$ (see the proof of Lemma A.10 and [14, Corollaire 1.6]). Taking hypercohomology, we therefore obtain an exact sequence

$$0 \longrightarrow \mathbb{H}^1(A \to B) \longrightarrow TG(\alpha_c^+)_{f_+(\xi)} \longrightarrow W^-_{p(\xi)} \longrightarrow 0. \tag{46}$$

Now we have another natural short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(E_2, E_1) & \longrightarrow & A & \longrightarrow & \mathrm{Hom}(E_1, E_1) \oplus \mathrm{Hom}(E_2, E_2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(V_2, E_1/V_1) & \longrightarrow & B & \longrightarrow & \mathrm{Hom}(V_1, E_1/V_1) \oplus \mathrm{Hom}(V_2, E_2/V_2) & \longrightarrow & 0. \end{array}$$

In this case the hypercohomology gives

$$0 \longrightarrow (T_{fibre}\mathbb{P}W^+)_\xi \longrightarrow \mathbb{H}^1(A \to B) \longrightarrow T(G_1 \times G_2)_{p(\xi)} \longrightarrow 0.$$

This sequence identifies $\mathbb{H}^1(A \to B)$ with $(T\mathbb{P}W^+)_\xi$. The sequence (46) now becomes

$$0 \longrightarrow (T\mathbb{P}W^+)_\xi \xrightarrow{df_+} TG(\alpha_c^+)_{f_+(\xi)} \longrightarrow W^-_{p(\xi)} \longrightarrow 0. \tag{47}$$

Since we know that $G(\alpha_c^+)$ is smooth at $f_+(\xi)$, this shows that f_+ is smooth at ξ and identifies the normal space with $W^-_{p(\xi)}$.

It remains to globalise this construction. For this we need to replace (E, V), (E_1, V_1), (E_2, V_2) by $(\mathcal{E}, \mathcal{V})^+$, $(\mathcal{E}_1, \mathcal{V}_1)^+ \otimes \sigma^*\mathcal{O}_{\mathbb{P}W^+}(1)$, $(\mathcal{E}_2, \mathcal{V}_2)^+$ in accordance with (45). The sequence (47) now becomes

$$0 \longrightarrow T\mathbb{P}W^+ \longrightarrow f_+^*TG(\alpha_c^+) \longrightarrow p^*W^- \otimes \mathcal{O}_{\mathbb{P}W^+}(-1) \longrightarrow 0.$$

Since we already know from Lemma A.10 that f_+ is injective, this completes the proof of the proposition. $\square$

If we now blow up $G(\alpha_c^+)$ along $G^+(n_1, d_1)$, it follows from Proposition A.11 that the exceptional divisor is isomorphic to $\mathbb{P}W^+ \times_{G_1 \times G_2} \mathbb{P}W^-$. This works for each allowable choice of n_1, d_1. To avoid confusion, we label these choices by $1, \dots, r$ and denote the corresponding W^+, W^- by W_j^+, W_j^- for $1 \le j \le r$. Now performing all the blow-ups simultaneously, we obtain a variety $\widetilde{G(\alpha_c^+)}$ with exceptional divisors $Y_1, \dots, Y_r$, all contained in the smooth part of $\widetilde{G(\alpha_c^+)}$. In exactly the same way, we blow-up $G(\alpha_c^-)$ along the various $G^-(n_1, d_1)$ (labelled as before) to obtain $\widetilde{G(\alpha_c^-)}$ with exceptional divisors $Y_1', \dots, Y_r'$. Note that there exist natural isomorphisms

$$Y_j \cong \mathbb{P}W_j^+ \times_{G_1 \times G_2} \mathbb{P}W_j^- \cong Y_j'. \tag{48}$$

We use these isomorphisms to identify Y_j and Y_j'.

The final step in the construction is to show that $\widetilde{G(\alpha_c^+)}$ is naturally isomorphic to $\widetilde{G(\alpha_c^-)}$. It is easy to construct a natural bijection between these varieties. In fact, if we write $Y = Y_1 \cup \dots \cup Y_r$ and $Y' = Y_1' \cup \dots \cup Y_r'$ then

$$\widetilde{G(\alpha_c^+)} \setminus Y = \widetilde{G(\alpha_c^-)} \setminus Y',$$

each variety consisting precisely of the α_c-stable coherent systems. On the other hand, as observed above, Y_j and Y_j' can be identified. It remains to show that there exist morphisms $\widetilde{G(\alpha_c^+)} \longrightarrow \widetilde{G(\alpha_c^-)}$ and $\widetilde{G(\alpha_c^-)} \longrightarrow \widetilde{G(\alpha_c^+)}$ such that the following diagram commutes for each j :

$$\begin{array}{ccccc} \widetilde{G(\alpha_c^+)} \setminus Y & \subset & \widetilde{G(\alpha_c^+)} & \hookleftarrow & Y_j \\ \| & & \downarrow\uparrow & & \| \\ \widetilde{G(\alpha_c^-)} \setminus Y' & \subset & \widetilde{G(\alpha_c^-)} & \hookleftarrow & Y_j'. \end{array} \tag{49}$$

For this purpose, we prove

Proposition A.12. *There exists a morphism $\widetilde{G(\alpha_c^+)} \longrightarrow G(\alpha_c^-)$ making the following diagram commute:*

$$\begin{array}{ccccc} \widetilde{G(\alpha_c^+)} \setminus Y & \subset & \widetilde{G(\alpha_c^+)} & \hookleftarrow & Y_j \\ \downarrow & & \downarrow & & \downarrow q \\ G(\alpha_c^-) \setminus (\mathbb{P}W_1^- \cup \cdots \cup \mathbb{P}W_r^-) & \subset & G(\alpha_c^-) & \hookleftarrow & \mathbb{P}W_j^-, \end{array} \tag{50}$$

where q is the natural projection.

Proof. Suppose first that $\mathrm{GCD}(n, d, k) = 1$, so that there exists a universal coherent system on $G(\alpha_c^+) \times C$. We write $(\mathcal{E}, \mathcal{V})$ for the pull-back of such a coherent system to $\widetilde{G(\alpha_c^+)} \times C$. We want to compare $(\mathcal{E}, \mathcal{V})|_{Y_j \times C}$ with the pull-backs of the extension (45) and the equivalent extension for $\mathbb{P}W_j^- \times C$ to $Y_j \times C$.

Let $r_j^+ : Y_j \longrightarrow \mathbb{P}W_j^+$ and $r_j^- : Y_j \longrightarrow \mathbb{P}W_j^-$ denote the projections. We write (45) for $\mathbb{P}W_j^+ \times C$ as

$$0 \longrightarrow (\mathcal{E}_1, \mathcal{V}_1)_j^+ \otimes (\sigma_j^+)^* \mathcal{O}_{\mathbb{P}W_j^+}(1) \longrightarrow (\mathcal{E}, \mathcal{V})_j^+ \longrightarrow (\mathcal{E}_2, \mathcal{V}_2)_j^+ \longrightarrow 0, \tag{51}$$

where $\sigma_j^+ : \mathbb{P}W_j^+ \times C \longrightarrow \mathbb{P}W_j^+$, and the corresponding extension on $\mathbb{P}W_j^- \times C$ as

$$0 \longrightarrow (\mathcal{E}_2, \mathcal{V}_2)_j^- \otimes (\sigma_j^-)^* \mathcal{O}_{\mathbb{P}W_j^-}(1) \longrightarrow (\mathcal{E}, \mathcal{V})_j^- \longrightarrow (\mathcal{E}_1, \mathcal{V}_1)_j^- \longrightarrow 0, \tag{52}$$

where $\sigma_j^- : \mathbb{P}W_j^- \times C \longrightarrow \mathbb{P}W_j^-$. Since $(\mathcal{E}_1, \mathcal{V}_1)_j^\pm$ are both pulled back from the same coherent system on $G_1 \times G_2 \times C$, we have

$$(r_j^+ \times \mathrm{Id})^* (\mathcal{E}_1, \mathcal{V}_1)_j^+ = (r_j^- \times \mathrm{Id})^* (\mathcal{E}_1, \mathcal{V}_1)_j^-, \tag{53}$$

with a similar statement for $(\mathcal{E}_2, \mathcal{V}_2)_j^\pm$. For every $y \in Y_j$, the restrictions of $(\mathcal{E}, \mathcal{V})$ and $(r_j^+ \times \mathrm{Id})^* (\mathcal{E}, \mathcal{V})_j^+$ to $\{y\} \times C$ are isomorphic and the coherent systems are all α_c^+-stable; it follows that

$$(\mathcal{E}, \mathcal{V})|_{Y_j \times C} \cong (r_j^+ \times \mathrm{Id})^* (\mathcal{E}, \mathcal{V})_j^+ \otimes L_j$$

for some line bundle L_j pulled back from Y_j. We define

$$\widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j := (r_j^+ \times \mathrm{Id})^* (\mathcal{E}_1, \mathcal{V}_1)_j^+ \otimes L_j,$$

with a similar definition for $\widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j$. We write also

$$\mathcal{O}_{Y_j \times C}(a, b) = (r_j^+ \times \mathrm{Id})^* (\sigma_j^+)^* \mathcal{O}_{\mathbb{P}W_j^+}(a) \otimes (r_j^- \times \mathrm{Id})^* (\sigma_j^-)^* \mathcal{O}_{\mathbb{P}W_j^-}(b).$$

Taking account of (53), we can then tensor the pull-backs of (51) and (52) to $Y_j \times C$ by L_j to get

$$0 \longrightarrow \widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j \otimes \mathcal{O}_{Y_j \times C}(1,0) \longrightarrow (\mathcal{E}, \mathcal{V})|_{Y_j \times C} \longrightarrow \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j \longrightarrow 0 \tag{54}$$

and

$$0 \longrightarrow \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j \otimes \mathcal{O}_{Y_j \times C}(0,1) \longrightarrow (r_j^- \times \mathrm{Id})^*(\mathcal{E}, \mathcal{V})_j^- \otimes L_j \longrightarrow \widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j \longrightarrow 0. \tag{55}$$

Now let

$$\widetilde{(\mathcal{E}_1, \mathcal{V}_1)} = \bigsqcup_{j=1}^{r} \widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j, \qquad \widetilde{(\mathcal{E}_2, \mathcal{V}_2)} = \bigsqcup_{j=1}^{r} \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j,$$

and define $(\hat{\mathcal{E}}, \hat{\mathcal{V}})$ on $\widetilde{G(\alpha_c^+)} \times C$ by the exact sequence

$$0 \longrightarrow (\hat{\mathcal{E}}, \hat{\mathcal{V}}) \longrightarrow (\mathcal{E}, \mathcal{V}) \xrightarrow{\psi} \widetilde{(\mathcal{E}_2, \mathcal{V}_2)} \longrightarrow 0, \tag{56}$$

where ψ is given by the composition

$$(\mathcal{E}, \mathcal{V}) \longrightarrow (\mathcal{E}, \mathcal{V})|_{Y_j \times C} \longrightarrow \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j$$

in the neighbourhood of $Y_j \times C$. Since the Y_j are disjoint Cartier divisors in $\widetilde{G(\alpha_c^+)}$, $\hat{\mathcal{E}}$ is locally free and $(\hat{\mathcal{E}}, \hat{\mathcal{V}})$ is a family of coherent systems parametrised by $\widetilde{G(\alpha_c^+)}$. The restriction of (56) to $Y_j \times C$ gives a 4-term exact sequence which can be split into two short exact sequences. The right-hand one coincides by construction with (54), while the left-hand one takes the form

$$0 \longrightarrow \ker \longrightarrow (\hat{\mathcal{E}}, \hat{\mathcal{V}})|_{Y_j \times C} \longrightarrow \widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j \otimes \mathcal{O}_{Y_j \times C}(1,0) \longrightarrow 0. \tag{57}$$

Here

$$\begin{aligned} \ker &\cong \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j \otimes \mathcal{T}or_1(\mathcal{O}_{Y_j \times C}, \mathcal{O}_{Y_j \times C}) \\ &\cong \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j \otimes N^*, \end{aligned}$$

where N is the pull-back to $Y_j \times C$ of the normal bundle of Y_j in $\widetilde{G(\alpha_c^+)}$. Now, by Proposition A.11 and a standard property of blow-ups, we have

$$N \cong \mathcal{O}_{Y_j \times C}(-1, -1).$$

So (57) becomes

$$0 \longrightarrow \widetilde{(\mathcal{E}_2, \mathcal{V}_2)}_j \otimes \mathcal{O}_{Y_j \times C}(1,1) \longrightarrow (\hat{\mathcal{E}}, \hat{\mathcal{V}})|_{Y_j \times C} \longrightarrow \widetilde{(\mathcal{E}_1, \mathcal{V}_1)}_j \otimes \mathcal{O}_{Y_j \times C}(1,0) \longrightarrow 0. \tag{58}$$

In the neighbourhood of $Y_j \times C$, we can interpret this in terms of a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & (\mathcal{E},\mathcal{V})\otimes\mathcal{O}(-Y_j\times C) & = & (\mathcal{E},\mathcal{V})\otimes\mathcal{O}(-Y_j\times C) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & (\hat{\mathcal{E}},\hat{\mathcal{V}}) & \to & (\mathcal{E},\mathcal{V}) & \to & \widetilde{(\mathcal{E}_2,\mathcal{V}_2)}_j & \to & 0 \\
 & & \downarrow & & \downarrow & & || & & \\
0 & \to & \widetilde{(\mathcal{E}_1,\mathcal{V}_1)}_j\otimes\mathcal{O}_{Y_j\times C}(1,0) & \to & (\mathcal{E},\mathcal{V})|_{Y_j\times C} & \to & \widetilde{(\mathcal{E}_2,\mathcal{V}_2)}_j & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here the middle row is (56), the bottom row is (54) and the quotient map in the left-hand column factorises as

$$(\hat{\mathcal{E}},\hat{\mathcal{V}}) \longrightarrow (\hat{\mathcal{E}},\hat{\mathcal{V}})|_{Y_j\times C} \longrightarrow \widetilde{(\mathcal{E}_1,\mathcal{V}_1)}_j\otimes\mathcal{O}_{Y_j\times C}(1,0),$$

where the two maps here are the restriction to $Y_j \times C$ and the quotient map of (58). We are now in the situation of the main diagram of [14, p.583].

Using the identification (48), a point of Y_j can be represented as (t^+,t^-), where $t^+ \in \mathbb{P}W_j^+$, $t^- \in \mathbb{P}W_j^-$ represent the classes of non-trivial extensions (42) and (44) respectively. Moreover the natural projections to $G(\alpha_c^+)$ and $G(\alpha_c^-)$ are given by

$$(t^+,t^-) \mapsto f_+(t^+) = (E,V), \quad (t^+,t^-) \mapsto f_-(t^-) = (E',V'),$$

where f_+ is defined in Lemma A.10 and f_- in a similar way. But, by Proposition A.11, t^- represents a normal direction to $G_+(n_1,d_1)$ at the point $f_+(t^+)$. He's argument [14, pp.583, 584] now shows that t^- is precisely the class of the restriction of (58) to $\{(t^+,t^-)\} \times C$. [He doesn't quite claim this directly, but note the sentence beginning "Mais cette question..." in the middle of p.584.] So, considered as families of extensions over C parametrised by Y_j, the sequences (58) and (55) coincide. By the universality of families of extensions, it follows that (58) can be obtained from (55) by tensoring by $\mathcal{O}_{Y_j\times C}(1,0)$. Hence $(\hat{\mathcal{E}},\hat{\mathcal{V}})$ is a family of α_c^--stable coherent systems, confirming the existence of the required morphism. The commutativity of (50) is obvious.

If $\mathrm{GCD}(n,d,k) \neq 1$, we no longer have a universal family defined on $G(\alpha_c^+) \times C$. However $G(\alpha_c^+)$ is constructed as a geometric quotient of a variety R^s by an action of $PGL(N)$ such that there exists a locally universal family on $R^s \times C$. By pulling everything back to R^s, we see that the argument above determines a blow-up $\widetilde{R^s}$ and a morphism $\widetilde{R^s} \longrightarrow G(\alpha_c^-)$, which (as a map) factors through $\widetilde{G(\alpha_c^+)}$. Since $\widetilde{R^s} \longrightarrow \widetilde{G(\alpha_c^+)}$ is again a geometric quotient, it follows that the map $\widetilde{G(\alpha_c^+)} \longrightarrow G(\alpha_c^-)$ is a morphism as required. This completes the proof. □

By Proposition A.12, we have a morphism

$$\widetilde{G(\alpha_c^+)} \longrightarrow G(\alpha_c^+) \times G(\alpha_c^-),$$

which is injective and is easily seen to be a smooth embedding. Moreover the image of $\widetilde{G(\alpha_c^+)}$ is precisely the closure of the graph of the identification map

$$G(\alpha_c^+) \setminus (\mathbb{P}W_1^+ \cup \cdots \mathbb{P}W_r^+) = G(\alpha_c^-) \setminus (\mathbb{P}W_1^- \cup \cdots \mathbb{P}W_r^-)$$

in $G(\alpha_c^+) \times G(\alpha_c^-)$. By the same argument with $+$ and $-$ interchanged, $\widetilde{G(\alpha_c^-)}$ can also be identified with the closure of the graph of the same identification. This completes the construction of the diagram (49).

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