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A NEW GEOMETRIC SETTING FOR CLASSICAL FIELD THEORIES

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Abstract. A new geometrical setting for classical field theories is introduced. This description is strongly inspired in the one due to Skinner and Rusk for singular lagrangians systems. For a singular field theory a constraint algorithm is developed that gives a final constraint submanifold where a well-defined dynamics exists. The main advantage of this algorithm is that the second order condition is automatically included.

1. Introduction. The search of a convenient setting for classical field theories has been an strong motivation for geometers and physicists in the last forty years. In the end of the sixties it was developed the so-called multisymplectic formalism, which is a natural extension of the symplectic framework for mechanics.

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The multisymplectic approach was developed by the Polish school led by W. Tulczyjew (see [3] for more details), and independently by P.L García and A. Pérez-Rendón [11, 12], and Goldschmidt and Sternberg [13]. This approach leads to a geometric definition of multisymplectic form in [16, 17], and more recently in [4, 5] where a careful study of these structures is developed (see also [26, 27] for previous results, and [2, 24, 28, 29] for recent developments).

There are two different ways to present the evolution equations in a geometric form. One uses the notion of Ehresmann connections [22, 23] which is widely employed along the present paper. The other one uses the notion of multivector field (see [7, 8, 9, 10]). Of course, both are equivalent, and permit to develop a convenient constraint algorithm when we are dealing with singular lagrangians.

Alternative geometric approaches based on the so-called n-symplectic geometry (see [19] for a recent survey), and polysymplectic geometry (see [30, 31]) are also available.

The aim of the present paper is to give a new geometric setting, based in that developed by Skinner and Rusk [32, 33]. In order to treat with singular lagrangian systems, Skinner and Rusk have constructed a hamiltonian system on the Withney sum $T^*Q \oplus TQ$ over the configuration manifold Q. The advantage of their approach lies on the fact that the second order condition of the dynamics is automatically satisfied. This does not happen in the Gotay and Nester formulation, where the second order condition problem has to be considered after the implementation of the constraint algorithm (see [14, 15, 20]).

Here, we start with a lagrangian function defined on Z, where $\pi_{XZ} : Z \longrightarrow X$ is the 1-jet prolongation of a fibration $\pi_{XY} : Y \longrightarrow X$. We consider the fibration $\pi_{XW_0} : W_0 \longrightarrow X$, where $W_0 = \Lambda^n_2 Y \times_Y Z$ is the fibered product. On W_0 we construct a multisymplectic form by pulling back the canonical multisymplectic form on $\Lambda^n_2 Y$, and define a convenient hamiltonian. The solutions of the field equations are viewed as integral sections of Ehresmann connections in the fibration $\pi_{XZ} : Z \longrightarrow X$. The resultant algorithm is compared with the ones developed in the lagrangian and hamiltonian sides. The scheme is applied to an example, the bosonic string. The case of time-dependent mechanics is recovered as a particular case. The paper also contains three appendices exhibiting some notions and properties of Ehresmann connections.

2. Lagrangian formalism. A classical field theory consists of a fibration $\pi_{XY} : Y \longrightarrow X$ (that is, π_{XY} is a surjective submersion) over an orientable n-dimensional manifold X and an n-form Λ (the lagrangian form) defined on the 1-jet prolongation $\pi_{XZ} : J^1\pi_{XY} \longrightarrow X$ along the projection π_{XY}. We will use the notation $Z = J^1\pi_{XY}$. In addition, if η is a fixed volume form on X we have $\Lambda = L\eta$, where L is a function on Z. An additional fiber bundle $\pi_{YZ} : Z \longrightarrow Y$ is also obtained. Here X represents the space-time manifold, and the fields are viewed as sections of π_{XY}. (See [3, 16, 17, 18, 30, 31]).

Definition 2.1. A lagrangian $L : Z \longrightarrow \mathbb{R}$ is said to be regular if the hessian matrix

$$\left(\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right)$$

is regular. Otherwise, L is said to be singular.

Along this paper we will choose fibered coordinates (x^μ, y^i, z^i_μ) on Z such that $\eta = d^n x = dx^1 \wedge \ldots \wedge dx^n$. Here μ runs from 1 to n, and i runs from 1 to m, so that Y has dimension $n+m$. A useful notation will be $d^{n-1}x^\mu = i_{\frac{\partial}{\partial x^\mu}} \eta$.

The volume form η permits to construct a tensor field of type $(1,n)$ on Z:

$$S_\eta = (dy^i - z^i_\mu dx^\mu) \wedge d^{n-1}x^\nu \otimes \frac{\partial}{\partial z^i_\nu}.$$

Next, the *Poincaré-Cartan n-form and $(n+1)$-form* are defined as follows:

$$\Theta_L = \Lambda + S^*_\eta(dL) \ , \quad \Omega_L = -d\Theta_L,$$

where S^*_η is the adjoint operator of S_η. In coordinates, we have

$$\begin{aligned} \Theta_L &= (L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) d^n x + \frac{\partial L}{\partial z^i_\mu} dy^i \wedge d^{n-1}x^\mu \\ \Omega_L &= -d(L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) \wedge d^n x - d(\frac{\partial L}{\partial z^i_\mu}) \wedge dy^i \wedge d^{n-1}x^\mu. \end{aligned}$$

An *extremal* of L is a section ϕ of π_{XY} such that, for any vector ξ_Z on Z,

$$(j^1\phi)^*(i_{\xi_Z}\Omega_L) = 0 \tag{1}$$

where $j^1\phi$ is the first jet prolongation of ϕ.

As is well-known, ϕ is an extremal of L if and only if it satisfies the Euler-Lagrange equations:

$$(j^1\phi)^* \left(\frac{\partial L}{\partial y^i} - \frac{d}{dx^\mu} \left(\frac{\partial L}{\partial z^i_\mu} \right) \right) = 0, \qquad 1 \leq i \leq n. \tag{2}$$

We can consider a more general kind of solutions, those sections ψ of the fiber bundle $\pi_{XZ} : Z \longrightarrow X$ such that

$$\psi^*(i_{\xi_Z}\Omega_L) = 0 \ , \tag{3}$$

for any vector ξ_Z on Z. Equation (3) is referred as the de Donder equations.

Looking at (3) we have an alternative characterization. Let Γ be an Ehresmann connection in $\pi_{XZ} : Z \longrightarrow X$, with horizontal projector $\mathbf{h}$. Consider the equation

$$i_{\mathbf{h}}\Omega_L = (n-1)\Omega_L. \tag{4}$$

The horizontal sections (if they exist) of Γ are just the solutions of the de Donder problem.

Indeed, if

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\nu\mu} \frac{\partial}{\partial z^i_\nu}$$

then a direct computation shows that equation (4) holds if and only if

$$(\Gamma^j_\nu - z^j_\nu) \left(\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right) = 0 \tag{5}$$

$$\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\mu \partial z^i_\mu} - \Gamma^j_\mu \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \Gamma^j_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\nu \partial z^i_\mu} + (\Gamma^j_\nu - z^j_\nu) \frac{\partial^2 L}{\partial y^i \partial z^j_\nu} = 0 \tag{6}$$

(see [22]).

If the lagrangian L is regular, then Eq. (5) implies that $\Gamma^i_\mu = z^i_\mu$ and therefore (6) becomes

$$\frac{\partial L}{\partial y^i} - \frac{\partial^2 L}{\partial x^\mu \partial z^i_\mu} - z^j_\mu \frac{\partial^2 L}{\partial y^j \partial z^i_\mu} - \Gamma^j_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\nu \partial z^i_\mu} = 0. \tag{7}$$

Now, if $\tau(x^\mu) = (x^\mu, \tau^i(x), \tau^i_\mu(x))$ is an integral section of Γ we would have

$$z^i_\mu = \frac{\partial \tau^i}{\partial x^\mu} \qquad \Gamma^i_{\mu\nu} = \frac{\partial \tau^i_\mu}{\partial x^\nu}$$

which proves that Eq. (7) is nothing but the Euler-Lagrange equations for L.

If the lagrangian L is regular, then every solution ψ of the de Donder equations (3) is automatically a 1-jet prolongation, say $\psi = j^1\phi$ and the section ϕ of π_{XY} is a solution of equations (1).

In terms of Ehresmann connections, if L is regular, then any solution Γ of equations (4) is semi-holonomic (see Appendix B).

3. Hamiltonian formulation. Let $\Lambda^n_r Y$, $1 \leq r \leq m$, be the subbundle of the bundle $\Lambda^n Y$ of n-forms on Y consisting of those n-forms which vanish when r of their arguments are vertical. We have a chain of vector bundles over Y:

$$0 \subset \Lambda^n_1 Y \subset \Lambda^n_2 Y \subset \cdots \subset \Lambda^n Y$$

The elements of $\Lambda^n_1 Y$ (resp. $\Lambda^n_2 Y$) are locally expressed as $p(x,y)d^n x$ (resp. $pd^n x + p^\mu_i dy^i \wedge d^{n-1}x^\mu$). Thus, we introduce local coordinates (x^μ, y^i, p) on the manifold $\Lambda^n_1 Y$, and (x^μ, y^i, p, p^μ_i) on $\Lambda^n_2 Y$.

The manifold $\Lambda^n Y$ carries a canonical n-form, Θ_0, which is defined as follows:

$$\Theta_0(\omega)(\xi_1, \xi_2, \ldots, \xi_n) = \omega(\nu(\omega))(\nu_*(\xi_1), \nu_*(\xi_2), \ldots, \nu_*(\xi_n))$$

where $\omega \in \Lambda^n Y$, $\xi_i \in T_\omega(\Lambda^n Y)$, and $\nu : \Lambda^n Y \longrightarrow Y$ is the canonical projection.

This form Θ_0 induces an n-form on Θ_r on $\Lambda^n_r Y$, for each r, $1 \leq r \leq m$.

The closed $(n+1)$-forms $\Omega_r = -d\Theta_r$ (and of course, $\Omega_0 = -d\Theta_0$) are examples of the so-called multisymplectic forms according the following definition.

Definition 3.1. A multisymplectic form on a manifold M is a closed k-form Ω on M such that the linear mapping $v \in T_x M \longrightarrow i_v\Omega \in \Lambda^{k-1}T^*_x M$ is injective for all $x \in M$. The manifold M equipped with a multisymplectic form Ω will be called a multisymplectic manifold, usually denoted by the pair (M, Ω). Two multisymplectic manifolds (M, Ω) and $(\bar{M}, \bar{\Omega})$ will be said multisymplectomorphic if there exists a diffeomorphism $\phi : M \longrightarrow \bar{M}$ preserving the multisymplectic forms, say $\phi^*\bar{\Omega} = \Omega$; ϕ will be called a multisymplectomorphism.

Remark 3.2. It will be useful to write the local expressions of the canonical multisymplectic forms on $\Lambda^n_2 Y$:

$$\Theta_2 = pd^n x + p^\mu_i dy^i \wedge d^{n-1}x^\mu, \quad \Omega_2 = -dp \wedge d^n x - dp^\mu_i \wedge dy^i \wedge d^{n-1}x^\mu.$$

A direct computation shows the following.

Proposition 3.3. Assume that $n \geq 2$. Then, a lagrangian L is regular if and only if the pair (Z, Ω_L) is a multisymplectic manifold.

Since $\Lambda_1^n Y$ is a vector subbundle of $\Lambda_2^n Y$ we can construct the quotient vector bundle $\Lambda_2^n Y/\Lambda_1^n Y$ which will we denoted by Z^*. The projection $\Lambda_2^n Y \longrightarrow Z^*$ will we denoted by λ. We also have a fibration $\pi_{XZ^*} : Z^* \longrightarrow X$.

In this context, a hamiltonian h is a section of λ. Using this hamiltonian we define an n-form Θ_h on Z^* by pulling back the canonical n-form Θ_2, i.e. $\Theta_h = h^*\Theta_2$. We put $\Omega_h = -d\Theta_h$ so that $\Omega_h = h^*\Omega_2$.

A section σ of $\pi_{XZ^*} : Z^* \longrightarrow X$ is said to satisfy the *Hamilton equations* for a given hamiltonian h if

$$\sigma^*(i_{\xi_{Z^*}}\Omega_h) = 0\,, \tag{8}$$

for all vector fields ξ_{Z^*} on Z^*.

In local coordinates (x^μ, y^i, p_i^μ) for Z^*, the section h may be represented by a local function H:

$$p = -H(x^\mu, y^i, p_i^\mu)$$

then

$$\Theta_h = -H d^n x + p_i^\mu dy^i \wedge d^{n-1}x^\mu, \quad \Omega_h = dH \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1}x^\mu, \tag{9}$$

and the Hamilton equations for a section σ become:

$$\frac{\partial y^i}{\partial x^\mu} = \frac{\partial H}{\partial p_i^\mu}\,, \qquad \frac{\partial p_i^\mu}{\partial x^\mu} = -\frac{\partial H}{\partial y^i}\,. \tag{10}$$

As in the precedent section, we can consider a connection $\widetilde{\Gamma}$ in $\pi_{XZ^*} : Z^* \longrightarrow X$, with horizontal projector $\widetilde{\mathbf{h}}$. An intrinsic version of equations (10) is then the following:

$$i_{\widetilde{\mathbf{h}}}\Omega_h = (n-1)\Omega_h. \tag{11}$$

Indeed, if $\widetilde{\Gamma}$ is flat, then its integral sections are solutions of the Hamilton equations.

Remark 3.4. If $n \geq 2$ then, from (9), it follows that Ω_h is a multisymplectic form on Z^*.

4. The Legendre transformation. Let L be a lagrangian function. We define a fiber preserving map

$$leg_L : Z \longrightarrow \Lambda_2^n Y$$

as follows:

$$leg_L(j_x^1\phi)(X_1, \ldots, X_n) = (\Theta_L)_{j_x^1\phi}(\tilde{X}_1, \ldots, \tilde{X}_n)$$

for all $j_x^1\phi \in Z$ and $X_i \in T_{\phi(x)}Y$, where $\tilde{X}_i \in T_{j_x^1\phi}Z$ are such that $(\pi_{YZ})_*(\tilde{X}_i) = X_i$.

In local coordinates, we have

$$leg_L(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, p = L - z_\mu^i\frac{\partial L}{\partial z_\mu^i}, p_i^\mu = \frac{\partial L}{\partial z_\mu^i}).$$

The Legendre transformation $Leg_L : Z \longrightarrow Z^*$ is defined as the composition $Leg_L = \lambda \circ leg_L$, and it is locally expressed as

$$Leg_L(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, \frac{\partial L}{\partial z_\mu^i}). \tag{12}$$

¿From the definitions, we deduce that $(leg_L)^*\Theta_2 = \Theta_L$ and $(Leg_L)^*\Omega_2 = \Omega_L$.

Proposition 4.1. The lagrangian L is regular if and only if the Legendre transformation $Leg_L : Z \longrightarrow Z^*$ is a local diffeomorphism.

The Legendre transformation permits to connect the lagrangian and hamiltonian descriptions as follows.

Assume the lagrangian L be hyper-regular, that is, $Leg_L : Z \longrightarrow Z^*$ is a global diffeomorphism. We define a hamiltonian section $h : Z^* \longrightarrow \Lambda_2^n Y$ by setting

$$h = leg_L \circ (Leg_L)^{-1}.$$

Then, from (12) it follows that

$$Leg_L^* \Theta_h = \Theta_L, \quad Leg_L^* \Omega_h = \Omega_L.$$

Therefore, the solutions of equations (3) and (8) are Leg_L-related. In terms of connections, the solutions of equations (4) and (11) are also Leg_L-related.

If the lagrangian is regular, the equivalence is only at local level. More precisely, if $n \geq 2$, we have that Leg_L is a local multisymplectomorphism between the multisymplectic manifolds (Z, Ω_L) and (Z^*, Ω_h).

For singular lagrangians, a constraint algorithm was developed in [22] (see Section 6).

5. A new geometric setting. Consider the fibered product $W_0 = \Lambda_2^n Y \times_Y Z$ with canonical projections $\mathrm{pr}_1 : W_0 \longrightarrow \Lambda_2^n Y$ and $\mathrm{pr}_2 : W_0 \longrightarrow Z$. We consider fibered coordinates $(x^\mu, y^i, p, p_i^\mu, z_\mu^i)$ on W_0.

Define the n-form $\Theta = \mathrm{pr}_1^* \Theta_2$ and the $(n+1)$-form $\Omega = -d\Theta = \mathrm{pr}_1^* \Omega_2$.

We also define a function $\Phi : W_0 \longrightarrow \mathbb{R}$ as follows. Take an element $(\omega_{\phi(x)}, j_x^1 \phi) \in W_0$, then $\Phi((\omega_{\phi(x)}, j_x^1 \phi)) = a(x)$, where

$$\phi^*(\omega_{\phi(x)}) = a(x) \eta(x).$$

Locally, we have

$$\Phi(x^\mu, y^i, p, p_i^\mu, z_\mu^i) = p + p_i^\mu z_\mu^i.$$

Define also the function $H_0 : W_0 \longrightarrow \mathbb{R}$ by setting

$$H_0 = \Phi - \mathrm{pr}_2^* L.$$

The function H_0 locally reads as

$$H_0(x^\mu, y^i, p, p_i^\mu, z_\mu^i) = p + p_i^\mu z_\mu^i - L(x^\mu, y^i, z_\mu^i).$$

Put

$$\Omega_{H_0} = \Omega + dH_0 \wedge \eta.$$

In local coordinates we have

$$\Omega_{H_0} = -dp \wedge d^n x - dp_i^\mu \wedge dy^i \wedge d^{n-1} x^\mu + dH_0 \wedge d^n x.$$

Let $\bar{\Gamma}$ be an Ehresmann connection in the fibered bundle $\pi_{XW_0} : W_0 \longrightarrow X$, with horizontal projector $\bar{\mathbf{h}}$.

We search for a solution of the equation:

$$i_{\bar{\mathbf{h}}} \Omega_{H_0} = (n-1) \Omega_{H_0} \ . \tag{13}$$

Define

$$\begin{aligned} W_1 &= \{u \in W_0 \;/\; \exists \bar{\mathbf{h}}_u : T_uW_0 \longrightarrow T_uW_0 \quad \text{linear such that } \; \bar{\mathbf{h}}_u^2 = \bar{\mathbf{h}}_u, \\ & \qquad \ker \bar{\mathbf{h}}_u = (V\pi_{XW_0})_u, \; i_{\bar{\mathbf{h}}_u} \Omega_{H_0}(u) = (n-1)\Omega_{H_0}(u)\}. \end{aligned}$$

Suppose that the local expression of $\bar{\mathbf{h}}$ is

$$\begin{aligned} \bar{\mathbf{h}}(\frac{\partial}{\partial x^\mu}) &= \frac{\partial}{\partial x^\mu} + A^i_\mu \frac{\partial}{\partial y^i} + B_\mu \frac{\partial}{\partial p} + C^\nu_{\mu i} \frac{\partial}{\partial p^\nu_i} + D^i_{\mu\nu} \frac{\partial}{\partial z^i_\nu} \\ \bar{\mathbf{h}}(\frac{\partial}{\partial y^i}) &= 0, \qquad \bar{\mathbf{h}}(\frac{\partial}{\partial p}) = 0 \\ \bar{\mathbf{h}}(\frac{\partial}{\partial p^\mu_i}) &= 0, \qquad \bar{\mathbf{h}}(\frac{\partial}{\partial z^i_\mu}) = 0 \end{aligned}$$

We then obtain

$$\begin{aligned} i_{\bar{\mathbf{h}}}\Omega_{H_0} &= i_{\bar{\mathbf{h}}}\left(-dp \wedge d^n x - dp^\mu_i \wedge dy^i \wedge d^{n-1}x^\mu + dH_0 \wedge d^n x\right) \\ &= (n-1)\Omega_{H_0} + \left(C^\mu_{\mu i} - \frac{\partial L}{\partial y^i}\right) dy^i \wedge d^n x \\ & \quad + \left(z^i_\mu - A^i_\mu\right) dp^\mu_i \wedge d^n x + \left(p^\mu_i - \frac{\partial L}{\partial z^i_\mu}\right) dz^i_\mu \wedge d^n x \end{aligned}$$

Therefore, the submanifold W_1 of W_0 is determined by the vanishing of the constraints:

$$p^\mu_i - \frac{\partial L}{\partial z^i_\mu} = 0,$$

and the components of the connection $\bar{\mathbf{h}}$ would verify the following relations:

$$A^i_\mu = z^i_\mu \tag{14}$$

$$C^\mu_{\mu i} = \frac{\partial L}{\partial y^i} \tag{15}$$

¿From the definition of W_1 we know that for each point $u \in W_1$ there exists a "horizontal projector" $\bar{\mathbf{h}}_u : T_uW_0 \longrightarrow T_uW_0$ satisfying equation (13). However, we can not ensure that such $\bar{\mathbf{h}}_u$, for each $u \in W_1$ will take values in T_uW_1.

But notice that the condition $\bar{\mathbf{h}}_u(T_uW_0) \subset T_uW_1$, $\forall u \in W_1$ is equivalent to have

$$\bar{\mathbf{h}}(\frac{\partial}{\partial x^\mu}) \left(p^\kappa_j - \frac{\partial L}{\partial z^j_\kappa}\right) = 0$$

or, equivalently,

$$C^\kappa_{\mu j} = \frac{\partial^2 L}{\partial z^j_\kappa \partial x^\mu} + z^i_\mu \frac{\partial^2 L}{\partial z^j_\kappa \partial y^i} + D^i_{\mu\nu} \frac{\partial^2 L}{\partial z^j_\kappa \partial z^i_\nu}. \tag{16}$$

We remark that if the lagrangian L is regular, then equations (16) have solutions D's for a particular choice of C's satisfying equations (15). Of course, we can take arbitrary values for the B's. A global solution is obtained using partitions of the unity.

In such a case, we obtain by restriction a connection $\bar{\Gamma}$ in the fibre bundle $\pi_{XW_1} : W_1 \longrightarrow X$, which is a solution of equation (13) when it is restricted to W_1 (in fact, we have a family of such solutions). Assume that $\bar{\Gamma}$ is flat, and $\bar{\psi}$ is a horizontal section of $\bar{\Gamma}$. First of all, notice that $\bar{\psi}$ takes values in W_1 which implies that $\psi = \mathrm{pr}_2 \circ \bar{\psi}$ is a jet prolongation. Let us explain better this assertion. If $\bar{\psi}(x^\mu) = (x^\mu, y^i(x), p(x), p_i^\mu(x), z_\mu^i(x))$ then we have

$$z_\mu^i(x) = \frac{\partial y^i}{\partial x^\mu}.$$

Since

$$D_{\mu\nu}^i = \frac{\partial z_\nu^i}{\partial x^\mu}$$

we deduce that along ψ we have

$$\frac{\partial L}{\partial y^j} - \frac{\partial^2 L}{\partial z_\mu^j \partial x^\mu} - \frac{\partial y^i}{\partial x^\mu}\frac{\partial^2 L}{\partial z_\mu^j \partial y^i} - \frac{\partial z_\nu^i}{\partial x^\mu}\frac{\partial^2 L}{\partial z_\mu^j \partial z_\nu^i} = 0$$

that is,

$$\frac{\partial L}{\partial y^j} - \frac{d}{dx^\mu}\left(\frac{\partial L}{\partial z_\mu^j}\right) = 0$$

which are the Euler-Lagrange equations for L.

Up to now, we have no assigned any meaning to the coordinate p. Consider the submanifold $\bar{W}_1$ of W_1 defined by the equation $H_0 = 0$. In other words, $\bar{W}_1$ is locally characterized by the equation

$$p = -(p_i^\mu z_\mu^i - L),$$

which defines a local energy.

We can ask when a solution exists on $\bar{W}_1$. Indeed, it is possible to construct a family of connections in the fibre bundle $\pi_{X\bar{W}_1} : \bar{W}_1 \longrightarrow X$ which solve equation (13) as follows.

We have to choose coefficients B_μ, $C_{\mu i}^\nu$, and $D_{\mu\nu}^i$ verifying (15) and (16), and in addition,

$$\bar{h}(\frac{\partial}{\partial x^\mu})(H_0) = 0. \tag{17}$$

A direct computation shows that (17) is equivalent to the following local conditions

$$B_\mu + C_{\mu i}^\nu z_\nu^i = \frac{\partial L}{\partial x^\mu} + z_\mu^i \frac{\partial L}{\partial y^i}. \tag{18}$$

Now, if we choose appropriate values for $C_{\mu i}^\nu$ satisfying (15) and (16), then we can take the values for B_μ given by equation (18). A global solution is finally obtained using partitions of the unity.

Denote by $\Omega_{\bar{W}_1}$ the restriction of Ω_{H_0} to $\bar{W}_1$.

Proposition 5.1. *If $n \geq 2$ and the Lagrangian L is regular then $\Omega_{\bar{W}_1}$ is a multisymplectic form.*

Proof.

The result follows from a direct computation taking into account that on W_1 we have

$$p_i^\mu = \frac{\partial L}{\partial z_\mu^i}$$

and that the hessian matrix

$$\left(\frac{\partial^2 L}{\partial z^i_\mu \partial z^j_\nu} \right)$$

is regular. ∎

Next, we shall relate the above construction with the precedent ones on the lagrangian and the hamiltonian sides.

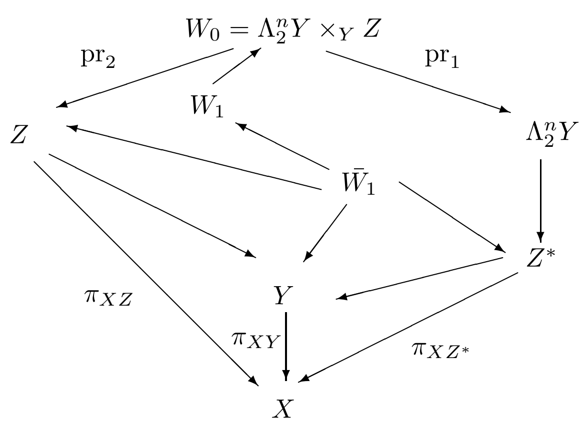
First of all, the following results are quite obvious:

- The submanifold $\bar{W}_1$ is diffeomorphic to Z.
- If $n \geq 2$ and L is (hyper)regular, then the multisymplectic manifolds $(\bar{W}_1, \Omega_{\bar{W}_1})$, (Z, Ω_L) and (Z^*, Ω_h) are (globally) locally multisymplectomorphic. Indeed, the corresponding multisymplectomorphisms are the following ones:

$$\begin{aligned} &(\mathrm{pr}_2)_{|\bar{W}_1} : \bar{W}_1 \longrightarrow Z \\ &Leg_L : Z \longrightarrow Z^* \\ &Leg_L \circ (\mathrm{pr}_2)_{|\bar{W}_1} : \bar{W}_1 \longrightarrow Z^* \end{aligned}$$

  (Note that $\lambda \circ (\mathrm{pr}_1)_{|\bar{W}_1} = Leg_L \circ (\mathrm{pr}_2)_{|\bar{W}_1}$).
- As a consequence, one can choose connections $\mathbf{h}$, $\tilde{\mathbf{h}}$ and $\bar{\mathbf{h}}$ in the fibrations $\pi_{XZ} : Z \longrightarrow X$, $\pi_{XZ^*} : Z^* \longrightarrow X$, and $\pi_{X\bar{W}_1} : \bar{W}_1 \longrightarrow X$, respectively, such that they are solutions of equations (4), (11) and (13), respectively, and, in addition, they are related by the above multisymplectomorphisms.

The following diagram summarizes the above discussion:



6. Singular lagrangians. For a singular lagrangian L, we usually have to go further in the constraint algorithm. Therefore, we will consider a subset $\bar{W}_2$ defined in order to satisfy the tangency condition:

$$\begin{aligned} \bar{W}_2 \;=\; & \{u \in \bar{W}_1 \;/\; \exists \bar{\mathbf{h}}_u : T_u W_0 \longrightarrow T_u \bar{W}_1 \quad \text{linear such that } \; \bar{\mathbf{h}}^2_u = \bar{\mathbf{h}}_u, \\ & \ker \bar{\mathbf{h}}_u = (V\pi_{XW_0})_u, \; i_{\bar{\mathbf{h}}_u} \Omega_{H_0}(u) = (n-1)\Omega_{H_0}(u)\}. \end{aligned}$$

Assume that $\bar{W}_2$ is a submanifold of $\bar{W}_1$. If $\bar{\mathbf{h}}_u(T_uW_0)$ is not contained in $T_u\bar{W}_2$, we go to the third step, and so on.

At the end, and if the system has solutions, we will find a final constraint submanifold $\bar{W}_f$, fibered over X (or over some open subset of X) (see Appendix C) and a connection $\bar{\Gamma}_f$ in this fibration such that $\bar{\Gamma}_f$ is a solution of equation (13) restricted to $\bar{W}_f$.

Similar constraint algorithms can be developed using equations (4) and (11). Our purpose in the following is to relate these three algorithms.

Indeed, we can consider the subset

$$\begin{aligned} Z_2 \quad = \quad & \{z \in Z \;/\; \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ \quad \text{linear such that} \quad \mathbf{h}_z^2 = \mathbf{h}_z, \\ & \ker \mathbf{h}_z = (V\pi_{XZ})_z, i_{\mathbf{h}_z}\Omega_L(z) = (n-1)\Omega_L(z)\}. \end{aligned}$$

If Z_2 is a submanifold, then there are solutions but we have to include the tangency condition, and consider a new step:

$$\begin{aligned} Z_3 \quad = \quad & \{z \in Z_2 \;/\; \exists \mathbf{h}_z : T_zZ \longrightarrow T_zZ_2 \quad \text{linear such that} \quad \mathbf{h}_z^2 = \mathbf{h}_z, \\ & \ker \mathbf{h}_z = (V\pi_{XZ})_z,\, i_{\mathbf{h}_z}\Omega_L(z) = (n-1)\Omega_L(z)\}. \end{aligned}$$

If Z_3 is a submanifold of Z_2, but $\mathbf{h}_z(T_zZ)$ is not contained in T_zZ_3, we go to the third step, and so on. Finally, we will obtain (in the favorable cases) a final constraint submanifold Z_f and a connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f (in fact, a family of connections) with horizontal projector $\mathbf{h}$ which is a solution of equation (4).

There is an additional problem, since our connection would be a solution of the de Donder problem, but not a solution of the Euler-Lagrange equations. This problem is solved constructing a submanifold of Z_f where such a solution exists (see [22, 23] and below for more details).

To develop a hamiltonian counterpart, we need some weak regularity of the lagrangian L.

Definition 6. 1. A lagrangian $L : Z \longrightarrow \mathbb{R}$ is said to be almost regular if $leg_L(Z) = \tilde{Z}$ is a submanifold of $\Lambda_2^n Y$, and $leg_L : Z \longrightarrow \tilde{Z}$ is a submersion with connected fibers.

If L is almost regular, one has:

- $\tilde{Z}_1 = Leg_L(Z)$ is a submanifold of Z^*, and in addition, a fibration over X.
- The restriction $\lambda_1 : \tilde{Z} \longrightarrow \tilde{Z}_1$ of λ is a diffeomorphism.
- The mapping $Leg_1 : Z \longrightarrow \tilde{Z}_1$ is a submersion with connected fibers.

Define a mapping $h_1 = (\lambda_1)^{-1} : \tilde{Z}_1 \longrightarrow \tilde{Z}$, and a $(n+1)$-form $\tilde{\Omega}_1$ on $\tilde{Z}_1$ by $\tilde{\Omega}_1 = h_1^*((\Omega_2)_{|_{\tilde{Z}}})$. Obviously, we have $Leg_1^*\tilde{\Omega}_1 = \Omega_L$.

The hamiltonian description is now based in the equation

$$i_{\tilde{\mathbf{h}}}\tilde{\Omega}_1 = (n-1)\tilde{\Omega}_1 \tag{19}$$

where $\tilde{\mathbf{h}}$ is a connection in the fibration $\pi_{X\tilde{Z}_1} : \tilde{Z}_1 \longrightarrow X$.

Proceeding as above, we construct a constraint algorithm as follows.

First, we define

$$\tilde{Z}_2 \quad = \quad \{\tilde{z} \in \tilde{Z}_1 \;/\; \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}\tilde{Z}_1 \longrightarrow T_{\tilde{z}}\tilde{Z}_1 \quad \text{linear such that} \quad \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}},$$

$$\ker \tilde{\mathbf{h}}_{\tilde{z}} = (V\pi_{X\tilde{Z}_1})_{\tilde{z}}, i_{\tilde{\mathbf{h}}_{\tilde{z}}}\tilde{\Omega}_1(\tilde{z}) = (n-1)\tilde{\Omega}_1(\tilde{z})\}.$$

If $\tilde{Z}_2$ is a submanifold, then there are solutions but we have to include the tangency condition, and consider a new step:

$$\begin{aligned}\tilde{Z}_3 \quad = \quad & \{\tilde{z} \in \tilde{Z}_2 \ / \ \exists \tilde{\mathbf{h}}_{\tilde{z}} : T_{\tilde{z}}\tilde{Z}_1 \longrightarrow T_{\tilde{z}}\tilde{Z}_2 \quad \text{linear such that} \quad \tilde{\mathbf{h}}_{\tilde{z}}^2 = \tilde{\mathbf{h}}_{\tilde{z}}, \\ & \ker \tilde{\mathbf{h}}_{\tilde{z}} = (V\pi_{X\tilde{Z}_1})_{\tilde{z}}, i_{\tilde{\mathbf{h}}_{\tilde{z}}}\tilde{\Omega}_1(\tilde{z}) = (n-1)\tilde{\Omega}_1(\tilde{z})\}.\end{aligned}$$

If $\tilde{Z}_3$ is a submanifold of $\tilde{Z}_2$, but $\tilde{\mathbf{h}}_{\tilde{z}}(T_{\tilde{z}}\tilde{Z}_1)$ is not contained in $T_{\tilde{z}}\tilde{Z}_3$, we go to the third step, and so on. Finally, we will obtain (in the favorable cases) a final constraint submanifold $\tilde{Z}_f$ and a connection in the fibration $\pi_{X\tilde{Z}_1} : \tilde{Z}_1 \longrightarrow X$ along the submanifold $\tilde{Z}_f$ (in fact, a family of connections) with horizontal projector $\tilde{\mathbf{h}}$ which is a solution of equation (11).

The important facts are the following:

- The mapping $Leg_1 : Z \longrightarrow \tilde{Z}_1$ preserves the constraint algorithms, that is, we have $Leg_1(Z_r) = \tilde{Z}_r$ for each integer $r \geq 2$.
- In consequence, both algorithms have the same behavior; in particular, if one of them stabilizes, the same happens with the other, and at the same step, so we have $Leg_1(Z_f) = \tilde{Z}_f$.
- In the latter case, the restriction $Leg_f : Z_f \longrightarrow \tilde{Z}_f$ is a surjective submersion (that is, a fibration) and $Leg_f^{-1}(Leg_f(z)) = Leg_1^{-1}(Leg_1(z))$, for all $z \in Z_f$.

Therefore, the lagrangian and hamiltonian sides can be compared through the fibration $Leg_f : Z_f \longrightarrow \tilde{Z}_f$. Indeed, if we have a connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f with horizontal projector $\mathbf{h}$ which is a solution of equation (4) (the de Donder equation) and, in addition, the connection is projectable via Leg_f to a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold $\tilde{Z}_f$, then the horizontal projector of the projected connection is a solution of equation (11) (the Hamilton equations). Conversely, given a connection in the fibration $\pi_{X\tilde{Z}} : \tilde{Z} \longrightarrow X$ along the submanifold $\tilde{Z}_f$, with horizontal projector $\tilde{\mathbf{h}}$ which is a solution of equation (11), then every connection in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f that projects onto $\tilde{\mathbf{h}}$ is a solution of the de Donder equation (4).

Assume that L is almost regular and construct the above algorithms. Take a Leg_f-projectable connection Γ in the fibration $\pi_{XZ} : Z \longrightarrow X$ along the submanifold Z_f with horizontal projector $\mathbf{h}$ which is a solution of equation (4), and denote by $\tilde{\Gamma}$ its projection. As we have shown, the horizontal projector $\tilde{\mathbf{h}}$ is a solution of equation (11).

In general, Γ is not semi-holonomic, that is, $S_\eta(\mathbf{h}, \ldots, \mathbf{h}) \not\equiv 0$ along Z_f. However, we can define a section β of the fibration $Leg_L : Z_f \longrightarrow \tilde{Z}_f$ such that

$$(S_\eta(\mathbf{h}, \ldots, \mathbf{h}))_{|_{\beta(\tilde{Z}_f)}} = 0.$$

The construction of β is based in the following interpretation of the elements of Z.

Take $z \in Z$, that is, z is a 1-jet of a section ϕ of the fibration $\pi_{XY} : Y \longrightarrow X$. Since $\mathbf{H}_{\phi(x)} = T\phi(x)(T_xX)$ is a horizontal subspace of $T_{\phi(x)}Y$, for every $x \in X$ (in fact, in the domain of ϕ) we can identify z with this horizontal subspace, which in local coordinates means that if $z = (x^\mu, y^i, z^i_\mu)$, then $\mathbf{H}_{\phi(x)}$ is spanned by the tangent vectors $\dfrac{\partial}{\partial x^\mu} + z^i_\mu \dfrac{\partial}{\partial y^i}$.

With the above notations and the obvious identifications, we define

$$\beta(\tilde{z}) = T\pi_{YZ}(\mathbf{h}(T_{z_0}Z)), \tag{20}$$

where $z_0 \in Z_f$ is an arbitrary point projecting onto $\tilde{z}$ through the projection $Leg_f : Z_f \longrightarrow \tilde{Z}_f$.

We have:

- $\beta(\tilde{z})$ is independent of the choice of z_0. This is a consequence of the following two facts: $\mathbf{h}$ projects onto $\tilde{\mathbf{h}}$, and the relation $\pi_{XZ^*} \circ Leg_f = \pi_{XZ}$.
- The point $\beta(\tilde{z})$ belongs to Z_f. Indeed, consider the following local vector field
$$U = (\Gamma^i_\mu - z^i_\mu)\frac{\partial}{\partial z^i_\mu},$$
where Γ^i_μ are the Christoffel components of Γ, that is
$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu}\frac{\partial}{\partial z^i_\nu}.$$
Since Γ is Leg_f-projectable, then Γ^i_μ is constant along the fibre over $\tilde{z}$.
¿From (5) and (12), we deduce that U is a vertical vector field with respect to the fibration $Leg_f : Z_f \longrightarrow \tilde{Z}_f$, and in consequence it is tangent to the fibre over $\tilde{z}$. Consider the curve
$$\alpha(t) = ((x^\mu)_0, (y^i)_0, (\Gamma^i_\mu)_0 - \exp(-t)((\Gamma^i_\mu)_0 - (z^i_\mu)_0)),$$
where $((x^\mu)_0, (y^i)_0, (z^i_\mu)_0)$ are the coordinates of z_0, and $(\Gamma^i_\mu)_0$ are the values of Γ^i_μ at the point z_0 (in fact, along all the fibre). $\alpha(t)$ is an integral curve of U passing through z_0 and totally contained in the fibre over $\tilde{z}$. Thus, the limit point $\lim_{t\to+\infty} \alpha(t)$ is in this fibre, and a direct computation shows that $\lim_{t\to+\infty} \alpha(t) = \beta(\tilde{z})$.
- Now, it is obvious that Γ is semiholonomic at the point $\beta(\tilde{z})$.

Since β is a section, we deduce that $\beta(\tilde{Z}_f)$ is a submanifold of Z_f and hence of Z. In addition, $(Leg_f)_{|_{\beta(\tilde{Z}_f)}} : \beta(\tilde{Z}_f) \longrightarrow \tilde{Z}_f$ is a diffeomorphism.

Next, we define a connection Γ_s in the fibration $\pi_{XZ} : Z \longrightarrow X$ along $\beta(\tilde{Z}_f)$ as follows.

Its horizontal projector is given by

$$(\mathbf{h}_s)_z : T_zZ \longrightarrow T_z\beta(\tilde{Z}_f), \quad (\mathbf{h}_s)_z = (T(Leg_f)_{|_{\beta(\tilde{Z}_f)}}(z))^{-1} \circ \tilde{\mathbf{h}}_{\tilde{z}} \circ TLeg_f(z),$$

for all $z \in \beta(\tilde{Z}_f)$, where $z = \beta(\tilde{z})$. A straightforward computation shows that Γ_s is a solution of (4) and, in addition, is transported onto $\tilde{\Gamma}$ via the diffeomorphism $(Leg_L)_{|_{\beta(\tilde{Z}_f)}} : \beta(\tilde{Z}_f) \longrightarrow \tilde{Z}_f$. Thus, since Γ is semiholonomic along $\beta(\tilde{Z}_f)$, we deduce that Γ_s is also semiholonomic along $\beta(\tilde{Z}_f)$.

Next, we will relate the above constructions with the algorithm developed from equation (13).

To do that, we first develop an alternative constraint algorithm based in the following equation

$$i_{\hat{\mathbf{h}}}\Omega_{\bar{W}_1} = (n-1)\Omega_{\bar{W}_1}, \tag{21}$$

where $\Omega_{\bar{W}_1}$ is the restriction of Ω_{H_0} to $\bar{W}_1$, and $\hat{\mathbf{h}}$ is the horizontal projector of a connection $\hat{\Gamma}$ in the fibration $\pi_{X\bar{W}_1} = (\pi_{XW_0})_{|\bar{W}_1} : \bar{W}_1 \longrightarrow X$.

The algorithm proceed now as in the above cases, and it produces a chain of submanifolds (in the favorable cases). Indeed, we define

$$\begin{aligned} \hat{W}_2 \quad = \quad & \{u \in \bar{W}_1 \;/\; \exists \hat{\mathbf{h}}_u : T_u\bar{W}_1 \longrightarrow T_u\bar{W}_1 \quad \text{linear such that} \;\; \hat{\mathbf{h}}_u^2 = \hat{\mathbf{h}}_u, \\ & \ker \hat{\mathbf{h}}_u = (V\pi_{X\bar{W}_1})_u, \; i_{\hat{\mathbf{h}}_u}\Omega_{\bar{W}_1}(u) = (n-1)\Omega_{\bar{W}_1}(u)\}. \end{aligned}$$

If we assume that $\hat{W}_2$ is a submanifold of $\bar{W}_1$, since in general $\hat{\mathbf{h}}_u(T_u\bar{W}_1)$ is not contained in $T_u\hat{W}_2$, we go to the third step, and so on.

At the end, and if the system has solutions, we will find a final constraint submanifold $\hat{W}_f$, fibered over X (or over some open subset of X) (see Appendix C) and a connection $\hat{\Gamma}_f$ in this fibration such that $\hat{\Gamma}_f$ is a solution of equation (21) restricted to $\hat{W}_f$.

It should be noticed that $\bar{W}_r \subset \hat{W}_r$, for all integer $r \geq 2$. Indeed, any pointwise solution of equation (13) is a solution of equation (21). As a consequence, both algorithms have the same behavior.

This last algorithm can be compared with the lagrangian and hamiltonian ones. In fact, since

$$\tilde{\mathrm{pr}}_2^*\Omega_L = \Omega_{\bar{W}_1}, \quad (\tilde{\mathrm{pr}}_1)^*\tilde{\Omega}_1 = \Omega_{\bar{W}_1},$$

where $\tilde{\mathrm{pr}}_1 = \lambda_1 \circ (\mathrm{pr}_1)_{|\bar{W}_1}$ and $\tilde{\mathrm{pr}}_2 = (\mathrm{pr}_2)_{|\bar{W}_1}$, we have

$$\tilde{\mathrm{pr}}_1(\hat{W}_r) = \tilde{Z}_r, \quad \tilde{\mathrm{pr}}_2(\hat{W}_r) = Z_r,$$

for all $r \geq 2$, and *a fortiori* we deduce that all the algorithms have the same behavior and

$$\tilde{\mathrm{pr}}_1(\hat{W}_f) = \tilde{Z}_f, \quad \tilde{\mathrm{pr}}_2(\hat{W}_f) = Z_f.$$

Thus, the corresponding solutions can be related via the convenient projections. More precisely, we can construct a connection Γ (resp. $\tilde{\Gamma}$, $\hat{\Gamma}$) in the fibration $\pi_{XZ} : Z \longrightarrow X$ (resp. $\pi_{X\tilde{Z}_1} : \tilde{Z}_1 \longrightarrow X$, $\pi_{X\bar{W}_1} : \bar{W}_1 \longrightarrow X$) along the submanifold Z_f (resp. $\tilde{Z}_f$, $\hat{W}_f$) such that they are related by the projections Leg_f, $\tilde{\mathrm{pr}}_1$ and $\tilde{\mathrm{pr}}_2$.

In addition, the connection Γ can be chosen such that its restriction to $\bar{W}_f$ is a solution of equation (13). Making all these selections, and performing the construction of the section β we conclude that $\beta(\tilde{Z}_f) \subset \bar{W}_f$.

The following diagram summarizes the above discussion:

$$
\begin{tikzcd}
\bar W_1 \arrow[rr, equal] \arrow[dr, equal, "(\mathrm{pr}_2)_{|\bar W_1}"'] \arrow[drrr, "(\mathrm{pr}_1)_{|\bar W_1}"] & & \bar W_1 \arrow[dr, "(\mathrm{pr}_1)_{|\bar W_1}"] & \\
& Z \arrow[rr, "Leg_L"] & & \tilde Z_1 \\
\hat W_2 \arrow[uu] \arrow[dr, equal] \arrow[drrr] & & \bar W_2 \arrow[ll, hook'] \arrow[uu] \arrow[dr] & \\
& Z_2 \arrow[uu] \arrow[rr] & & \tilde Z_2 \arrow[uu] \\
\vdots \arrow[uu] & \vdots \arrow[u] & \vdots \arrow[uu] & \vdots \arrow[u] \\
\hat W_f \arrow[u] \arrow[dr, equal, "(\mathrm{pr}_2)_{|\hat W_f}"'] \arrow[drrr, "(\mathrm{pr}_1)_{|\hat W_f}"] & & \bar W_f \arrow[ll, hook'] \arrow[u] \arrow[dr, "(\mathrm{pr}_1)_{|\bar W_f}"] & \\
& Z_f \arrow[uu] \arrow[rr, "Leg_f"] & & \tilde Z_f \arrow[uu]
\end{tikzcd}
$$

REMARK 6.2. According to Appendix C, one has that all the connections considered in this section define *bona fide* connections in the corresponding restricted fibrations

$$
\begin{aligned}
\pi_{X_0 Z_f} &: Z_f \longrightarrow X_0, \\
\pi_{X_0 \tilde Z_f} &: \tilde Z_f \longrightarrow X_0, \\
\pi_{X_0 \bar W_f} &: \bar W_f \longrightarrow X_0, \\
\pi_{X_0 \hat W_f} &: \hat W_f \longrightarrow X_0,
\end{aligned}
$$

where X_0 is an open submanifold of X.

7. Example: The bosonic string (See [1, 16]) Let X be a 2-dimensional manifold, and (B, g) a $d+1$-dimensional spacetime manifold endowed with a Lorentz metric g of signature $(-,+,\dots,+)$. A *bosonic string* is a map $\phi : X \longrightarrow B$.

In the following, we will follow the Polyakov approach to classical bosonic string theory. Let $S_2^{1,1}(X)$ be the bundle over X of symmetric 2-covariant tensors with signature $(-,+)$ or $(1,1)$. We take the vector bundle $\pi : Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$. Therefore, in this formulation, a field ψ is a section (ϕ, h) of the vector bundle $Y = X \times B \times S_2^{1,1}(X) \longrightarrow X$, where $\phi : X \longrightarrow Y$ is the bosonic string and h is a Lorentz metric on X.

Lagrangian description

We have that $Z = J^1(X \times B) \times_X J^1(S_2^{1,1}(X))$. Taking coordinates (x^μ), (y^i) and $(x^\mu, h_{\mu\eta})$ on X, B and $S_2^{1,1}(X)$ then the canonical local coordinates on Z are $(x^\mu, y^i, h_{\eta\xi},$

$y^i_\mu, h_{\eta\xi\mu})$. In this system of local coordinates, the Lagrangian density is given by

$$\Lambda = -\frac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}g_{ij}y^i_\eta y^j_\xi d^2x \ .$$

The Cartan 3-form is

$$\begin{aligned}
\Omega_L &= dy^i \wedge d\left(-\sqrt{-\det(h)}h^{\eta\xi}g_{ij}y^j_\xi\right) \wedge d^1x^\eta \\
&\quad -d\left(\frac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}g_{ij}y^i_\eta y^j_\xi\right) \wedge d^2x \\
&= -\frac{1}{2}\left(\frac{\partial\sqrt{-\det(h)}}{\partial h_{\rho\sigma}}h^{\eta\xi}g_{ij}y^i_\eta y^j_\xi - \sqrt{-\det(h)}h^{\eta\rho}h^{\xi\sigma}g_{ij}y^i_\eta y^j_\xi\right) dh_{\rho\sigma} \wedge d^2x \\
&\quad -\frac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\eta y^j_\xi \, dy^k \wedge d^2x - \sqrt{-\det(h)}h^{\eta\xi}g_{ij}y^i_\eta \, dy^j_\xi \wedge d^2x \\
&\quad +\left(\frac{\partial\sqrt{-\det(h)}}{\partial h_{\rho\sigma}}h^{\eta\xi}g_{ij}y^j_\xi - \sqrt{-\det(h)}h^{\eta\rho}h^{\xi\sigma}g_{ij}y^j_\xi\right) dh_{\rho\sigma} \wedge dy^i \wedge d^1x^\eta \\
&\quad +\sqrt{-\det(h)}h^{\eta\xi}\frac{\partial g_{ij}}{\partial y^k}y^j_\xi \, dy^k \wedge dy^i \wedge d^1x^\eta \\
&\quad +\sqrt{-\det(h)}h^{\eta\xi}g_{ij} \, dy^j_\xi \wedge dy^i \wedge d^1x^\eta.
\end{aligned}$$

If we solve the equation $i_{\mathbf{h}}\Omega_L = \Omega_L$, where

$$\mathbf{h} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \Gamma^i_\mu\frac{\partial}{\partial y^i} + \gamma_{\eta\xi\mu}\frac{\partial}{\partial h_{\eta\xi}} + \Gamma^i_{\eta\mu}\frac{\partial}{\partial y^i_\eta} + \gamma_{\eta\xi\rho\mu}\frac{\partial}{\partial h_{\eta\xi\rho}}\right) \ ,$$

we obtain that:

$$\begin{aligned}
\Gamma^i_\mu &= y^i_\mu \\
0 &= \frac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}\frac{\partial g_{ij}}{\partial y^k}y^i_\eta y^j_\xi - \sqrt{-\det(h)}h^{\eta\xi}\frac{\partial g_{kj}}{\partial y^i}y^i_\eta y^j_\xi - \sqrt{-\det(h)}h^{\eta\xi}g_{kj}\Gamma^j_{\xi\eta} \\
&\quad -\left(\frac{\partial\sqrt{-\det(h)}}{\partial h_{\rho\sigma}}h^{\eta\xi}g_{kj}y^j_\xi - \sqrt{-\det(h)}h^{\eta\rho}h^{\xi\sigma}g_{kj}y^j_\xi\right)\gamma_{\rho\sigma\eta} \ ,
\end{aligned}$$

and the constraints are given by the equations

$$\frac{\partial}{\partial h_{\rho\theta}}\left(\sqrt{-\det(h)}h^{\eta\xi}\right)g_{ij}y^i_\eta y^j_\xi = 0 \ .$$

The previous equation corresponds to the three following constraints

$$\begin{aligned}
\left[h^{\eta 0}h^{\xi 0}(h_{01}^2 - h_{00}h_{11}) + \frac{1}{2}h^{\eta\xi}h_{11}\right]g_{ij}y^i_\eta y^j_\xi &= 0 \\
\left[h^{\eta 1}h^{\xi 1}(h_{01}^2 - h_{00}h_{11}) + \frac{1}{2}h^{\eta\xi}h_{00}\right]g_{ij}y^i_\eta y^j_\xi &= 0 \\
\left[h^{\eta 0}h^{\xi 1}(h_{01}^2 - h_{00}h_{11}) - h^{\eta\xi}h_{01}\right]g_{ij}y^i_\eta y^j_\xi &= 0
\end{aligned}$$

which determine Z_2.

Hamiltonian description

The Legendre transformation is given by

$$Leg_L(x^\mu, y^i, h_{\eta\xi}, y^i_\mu, h_{\eta\xi\mu}) = (x^\mu, y^i, h_{\eta\xi}, -\sqrt{-\det(h)}\, h^{\mu\eta} g_{ij} y^j_\eta, 0)$$

Therefore, the Lagrangian L is almost-regular and, moreover, $\tilde{Z}_1 = \mathrm{Im}\, Leg_L \cong \tilde{Z} = leg_L(Z) \cong J^1(X \times B) \times_X S^{1,1}_2(X)$. Take now coordinates $(x^\mu, y^i, h_{\eta\xi}, p^\mu_i)$ on $\tilde{Z}_1$ and consider the mapping $h_1 : \tilde{Z}_1 \to \tilde{Z}$ given by

$$h_1(x^\mu, y^i, h_{\eta\xi}, p^\mu_i) = (x^\mu, y^i, h_{\eta\xi}, p = \frac{1}{2\sqrt{-\det(h)}} h_{\eta\xi} g^{ij} p^i_\eta p^j_\xi, p^\mu_i)$$

Then, we have

$$\tilde{\Omega}_1 = -d\left(\frac{1}{2\sqrt{-\det(h)}} h_{\eta\xi} g^{ij} p^\eta_i p^\xi_j\right) \wedge d^2x + dy^i \wedge dp^\mu_i \wedge d^1x^\mu$$

and the Hamilton equations are given by $i_{\tilde{\mathbf{h}}}\tilde{\Omega}_1 = \tilde{\Omega}_1$

$$\tilde{\mathbf{h}} = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + \tilde{\Gamma}^i_\mu \frac{\partial}{\partial y^i} + \tilde{\gamma}_{\eta\xi\mu}\frac{\partial}{\partial h_{\eta\xi}} + \tilde{\Gamma}^\eta_{i\mu}\frac{\partial}{\partial p^\eta_i}\right)$$

Solving the above equation, we obtain

$$\begin{aligned} \tilde{\Gamma}^i_\mu &= -\frac{1}{\sqrt{-\det(h)}} h_{\eta\mu} g^{ij} p^\eta_j \\ \tilde{\Gamma}^\mu_{i\mu} &= \frac{1}{2\sqrt{-\det(h)}} h_{\eta\xi} \frac{\partial g^{ij}}{\partial y^k} p^i_\eta p^j_\xi \,, \end{aligned}$$

and the secondary constraints

$$\frac{g^{ij}}{\sqrt{-\det(h)}}\left(\frac{1}{2\det(h)}\frac{\partial \det(h)}{\partial h_{\rho\sigma}} h_{\eta\xi} p^\eta_i p^\xi_j - p^\rho_i p^\sigma_j\right) = 0$$

determining $\tilde{Z}_2$.

The new geometrical setting

We have that $W_0 = \Lambda^2_2 Y \times_Y Z$ with fibered coordinates

$$(x^\mu, y^i, h_{\eta\xi}, p, p^\mu_i, q^{\eta\xi\mu}, y^i_\mu, h_{\eta\xi\mu}).$$

Therefore,

$$\begin{aligned} H_0 &= p + p^\mu_i y^i_\mu + q^{\eta\xi\mu} h_{\eta\xi\mu} + \frac{1}{2}\sqrt{-\det(h)} h^{\eta\xi} g_{ij} y^i_\eta y^j_\xi, \\ \Omega_{H_0} &= -dp \wedge d^2x - dp^\mu_i \wedge dy^i \wedge d^1x^\mu - dq^{\eta\xi\mu} \wedge dh_{\eta\xi} \wedge d^1x^\mu + dH_0 \wedge d^2x. \end{aligned}$$

Consider now an Ehresmann connection in the fibered manifold $\pi_{XW_0} : W_0 \longrightarrow X$ with horizontal projector:

$$\begin{aligned} \bar{\mathbf{h}} = {} & dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + A^i_\mu \frac{\partial}{\partial y^i} + A_{\eta\xi\mu}\frac{\partial}{\partial h_{\eta\xi}} + B_\mu \frac{\partial}{\partial p} + C^\eta_{\mu i}\frac{\partial}{\partial p^\eta_i} + C_{\eta\xi\sigma\mu}\frac{\partial}{\partial q^{\eta\xi\sigma}}\right. \\ & \left. + D^i_{\eta\mu}\frac{\partial}{\partial y^i_\eta} + D^{\eta\xi\sigma\mu}\frac{\partial}{\partial h_{\eta\xi\sigma}}\right) \end{aligned}$$

Solving $i_{\bar{\mathbf{h}}}\Omega_{H_0} = \Omega_{H_0}$ we obtain that the submanifold W_1 is determined by the constraints:

$$\begin{array}{rcl} p_i^\mu & = & -\sqrt{-\det(h)}h^{\mu\eta}g_{ij}y_\eta^j \\ q^{\eta\xi\mu} & = & 0 \end{array}$$

Let $\bar{W}_1$ be the submanifold of W_1 defined by the equation $H_0 = 0$, that is

$$p = \frac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}g_{ij}y_\eta^i y_\xi^j.$$

$\bar{W}_1$ is locally defined by coordinates $(x^\mu, y^i, h_{\eta\xi}, y_\mu^i, h_{\eta\xi\mu})$.

In this coordinates, the solutions of equation (21) are exactly the same than the ones obtained in the lagrangian setting, and $\hat{W}_2$, as a submanifold of W_0, is determined by the vanishing of the constraints functions

$$\begin{array}{rcl} p_i^\mu + \sqrt{-\det(h)}h^{\mu\eta}g_{ij}y_\eta^j & = & 0 \\ q^{\eta\xi\mu} & = & 0 \\ p - \dfrac{1}{2}\sqrt{-\det(h)}h^{\eta\xi}g_{ij}y_\eta^i y_\xi^j & = & 0 \\ \dfrac{\partial\sqrt{-\det(h)}}{\partial h_{\rho\sigma}}h^{\eta\xi}g_{ij}y_\eta^i y_\xi^j - \sqrt{-\det(h)}h^{\eta\rho}h^{\xi\sigma}g_{ij}y_\eta^i y_\xi^j & = & 0 \end{array}$$

It is easy to show that $\bar{W}_2 = \hat{W}_2$ and the solutions of equation (13) are the solutions of equation (21) which, in addition, are semi-holonomic.

8. Time-dependent mechanics. The jet bundle description of time-dependent mechanical systems takes $X = \mathbb{R}$ and $\eta = dt$, where t is the usual coordinate on $\mathbb{R}$ (see, for instance, [21]).

If $L : Z \longrightarrow \mathbb{R}$ is a lagrangian function, Ω_L is the Poincaré-Cartan 2-form on Z and η_Z is the 1-form on Z defined by $\eta_Z = (\pi_{\mathbb{R}Z})^*(\eta)$, then the de Donder equation (4) can be written as

$$i_{\xi_Z}\Omega_L = 0, \quad i_{\xi_Z}\eta_Z = 1, \tag{22}$$

where ξ_Z is a vector field on Z. The integral curves of ξ_Z are the solutions of the de Donder problem.

The lagrangian function L is regular if and only if the pair (Ω_L, η_Z) is a *cosymplectic structure* on Z. We recall that a cosymplectic structure on a manifold M of odd dimension $2n+1$ is a pair which consists of a closed 2-form Ω and a closed 1-form η such that $\eta\wedge\Omega^n$ is a volume form.

If L is regular then there exists a unique vector field ξ_Z which satisfies (22). In fact, ξ_Z is the *Reeb vector field* of the cosymplectic structure (Ω_L, η_Z) and it is a second order differential equation, that is, $S_{dt}\xi_Z = 0$. The trajectories of ξ_Z are the solutions of the Euler-Lagrange equations.

On the other hand, in this case, $\Lambda_2^1 Y$ is the cotangent bundle T^*Y of the manifold Y and Ω_0 is the canonical symplectic structure of T^*Y. Moreover, if $h : Z^* \longrightarrow \Lambda_2^1 Y = T^*Y$ is a hamiltonian and $\eta_{Z^*} = (\pi_{\mathbb{R}Z^*})^*(dt)$, then: i) the pair (Ω_h, η_{Z^*}) is a cosymplectic structure on Z^* and ii) the solutions of the Hamilton equations are just the integral curves of the Reeb vector field ξ_h of the cosymplectic structure (Ω_h, η_{Z^*}).

It should be noticed that if the lagrangian L is regular and $\eta_{\bar{W}_1} = (\pi_{\mathbb{R}_{\bar{W}_1}})^*(dt)$, we have that the pair $(\Omega_{\bar{W}_1}, \eta_{\bar{W}_1})$ is again a cosymplectic structure on $\bar{W}_1$ and there exists a unique solution of equation (13) restricted to $\bar{W}_1$, namely, the Reeb vector field of the cosymplectic structure $(\Omega_{\bar{W}_1}, \eta_{\bar{W}_1})$. Furthermore, if L is (regular) hyper-regular then the maps $(\mathrm{pr}_2)_{|\bar{W}_1} : \bar{W}_1 \longrightarrow Z$, $Leg_L : Z \longrightarrow Z^*$ and $Leg_L \circ (\mathrm{pr}_2)_{|\bar{W}_1} : \bar{W}_1 \longrightarrow Z^*$ are (local) cosymplectomorphisms between the cosymplectic manifolds $(\bar{W}_1, \Omega_{\bar{W}_1}, \eta_{\bar{W}_1})$, (Z, Ω_L, η_Z) and $(Z^*, \Omega_h, \eta_{Z^*})$, where $h = leg_L \circ (Leg_L)^{-1}$. Thus, the Reeb vector fields $\xi_{\bar{W}_1}$, ξ_Z and ξ_{Z^*} are related by the above cosymplectomorphisms.

When the lagrangian L is singular, we can develop the two algorithms using equations (13) and (21) and we obtain the corresponding constraint submanifolds

$$\begin{array}{rcl} \bar{W}_i &=& \{u \in \bar{W}_{i-1} \ \ / \ \ \exists \xi \in T_u\bar{W}_{i-1},\ i_\xi \Omega_{H_0}(u) = 0,\ \eta_{\bar{W}_1}(\xi) = 1\}, \\ \hat{W}_i &=& \{u \in \hat{W}_{i-1} \ \ / \ \ \exists \xi \in T_u\hat{W}_{i-1},\ i_\xi \Omega_{\bar{W}_1}(u) = 0,\ \eta_{\bar{W}_1}(\xi) = 1\}, \end{array}$$

for all $i \geq 2$, with $\bar{W}_1 = \hat{W}_1$ (see Section 6).

If L is almost regular, then we have that

$$\begin{array}{lcl} \bar{W}_i \subset \hat{W}_i, && \\ \tilde{\mathrm{pr}}_1(\hat{W}_i) = \tilde{Z}_i &=& \{\tilde{z} \in \tilde{Z}_{i-1} / \exists \tilde{\xi} \in T_{\tilde{z}}\tilde{Z}_{i-1},\ i_{\tilde{\xi}}\tilde{\Omega}_1(\tilde{z}) = 0,\ \eta_{Z^*}(\tilde{z})(\tilde{\xi}) = 1\}, \\ \tilde{\mathrm{pr}}_2(\hat{W}_i) = Z_i &=& \{z \in Z_{i-1} / \exists \xi \in T_z Z_{i-1},\ i_\xi \Omega_L(z) = 0,\ \eta_Z(z)(\xi) = 1\}, \end{array}$$

for all $i \geq 2$. Moreover, one can construct the section β of $Leg_f : Z_f \longrightarrow \tilde{Z}_f$ and the submanifold $\beta(\tilde{Z}_f)$ of Z_f where a solution of the Euler-Lagrange equations exists.

The constraint algorithms using equations (4) and (11) and the construction of the corresponding constraint submanifolds Z_i and $\tilde{Z}_i$ and of the submanifold $\beta(\tilde{Z}_f)$ has been done in [21] (see also [6, 25]). We remark that, in this case, there exists a unique solution of the Euler-Lagrange equations on the submanifold $\beta(\tilde{Z}_f)$ (for more details, see [21]).

Appendices

A. Projectable connections. A connection Γ in the fibration $\pi_{XY} : Y \longrightarrow X$ is given by a horizontal distribution $\mathbf{H}$ which is complementary to the vertical one $V\pi_{XY}$, that is

$$TY = \mathbf{H} \oplus V\pi_{XY}.$$

Associated to the connection there exist a horizontal projector $\mathbf{h} : TY \longrightarrow \mathbf{H}$ defined in the obvious manner.

If (x^μ, y^i) are fibered coordinates, then $\mathbf{H}$ is locally spanned by the local vector fields

$$(\frac{\partial}{\partial x^\mu})^h = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu(x, y)\, \frac{\partial}{\partial y^i}\ ;$$

$(\frac{\partial}{\partial x^\mu})^h$ is called the horizontal lift of $\frac{\partial}{\partial x^\mu}$, and Γ^i_μ are the Christoffel components of the connection.

Along the paper we repeatedly use the following construction.

Assume that $\pi_{XZ} : Z \longrightarrow X$ and $\pi_{XY} : Y \longrightarrow X$ are two fibrations with the same base manifold X, and that $\Phi : Z \longrightarrow Y$ is a surjective submersion (in other words, a fibration as well) preserving the fibrations, say, $\pi_{XY} \circ \Phi = \pi_{XZ}$.

Let Γ be a connection in $\pi_{XZ} : Z \longrightarrow X$ with horizontal projector $\mathbf{h}$.

Definition A.1. Γ is said to be projectable if $T\Phi(z)(\mathbf{H}_z) = T\Phi(z')(\mathbf{H}_{z'})$, for all $z, z' \in Z$ in the same fibre of Φ.

If Γ is projectable, then we define a connection Γ' in the fibration $\pi_{XY} : Y \longrightarrow X$ as follows: The horizontal subspace at $y \in Y$ is given by

$$\bar{\mathbf{H}}_y = T\Phi(z)(\mathbf{H}_z) \ ,$$

for an arbitrary z in the fibre of Φ over y. It is routine to prove that $\bar{\mathbf{H}}$ defines a horizontal distribution in the fibration $\pi_{XY} : Y \longrightarrow X$.

We can choose fibered coordinates (x^μ, y^i, z^a) on Z such that (x^μ, y^i) are fibered coordinates on Y. The Christoffel components of Γ are obtained by computing the horizontal lift

$$(\frac{\partial}{\partial x^\mu})^h = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu(x,y,z)\,\frac{\partial}{\partial y^i} + \Gamma^a_\mu(x,y,z)\,\frac{\partial}{\partial z^a} \ .$$

A simple computation shows that Γ is projectable if and only if the Christoffel components Γ^i_μ are constant along the fibres of Φ, say $\Gamma^i_\mu = \Gamma^i_\mu(x,y)$. In this case, the horizontal lift of $\dfrac{\partial}{\partial x^\mu}$ with respect to Γ' is just

$$(\frac{\partial}{\partial x^\mu})^h = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu(x,y)\,\frac{\partial}{\partial y^i} \ .$$

As an exercise, the reader can easily check that, conversely, given a connection Γ' in the fibration $\pi_{XY} : Y \longrightarrow X$ and a surjective submersion $\Phi : Z \longrightarrow Y$ preserving the fibrations, one can construct a connection Γ in the fibration $\pi_{XZ} : Z \longrightarrow X$ which projects onto Γ'.

B. Semiholonomic connections. Let $\pi_{XY} : Y \longrightarrow X$ be a fibration and $\pi_{XZ} : Z \longrightarrow X$ its 1-jet prolongation, that is, $Z = J^1\pi_{XY}$. Assume that X is orientable with volume form η.

Definition B.2. A connection Γ in the fibration $\pi_{XZ} : Z \longrightarrow X$ is said to be semiholonomic if

$$S_\eta(\mathbf{h}, \ldots, \mathbf{h}) = 0, \tag{23}$$

where $\mathbf{h}$ is the horizontal projector of Γ. If (23) holds at a point $z \in Z$, then Γ is said to be semiholonomic at z.

Assume that

$$\mathbf{h}(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} + \Gamma^i_{\mu\nu} \frac{\partial}{\partial z^i_\nu}$$

in fibered induced coordinates. Then Γ is semiholonomic if and only if we have $\Gamma^i_\mu = z^i_\mu$.

C. Connections on submanifolds. The notion of connection in a fibration admits a useful generalization to submanifolds of the total space.

Let $\pi_{XY} : Y \longrightarrow X$ be a fibration and P a submanifold of Y.

Definition C.1. A connection in $\pi_{XY} : Y \longrightarrow X$ along the submanifold P consists of a family of linear mappings

$$\mathbf{h}_y : T_yY \longrightarrow T_yP$$

for all $y \in P$, satisfying the following properties

$$\mathbf{h}_y^2 = \mathbf{h}_y, \quad \ker \mathbf{h}_y = (V\pi_{XY})_y,$$

for all $y \in P$. The connection is said to be differentiable (flat) if the n-dimensional distribution $\operatorname{Im}\mathbf{h} \subset TP$ is smooth (integrable), where $n = \dim X$.

We have the following.

Proposition C.2. Let $\mathbf{h}$ a connection in $\pi_{XY} : Y \longrightarrow X$along a submanifold P of Y. Then:

(1) $\pi_{XY}(P)$ is an open subset of X.
(2) $(\pi_{XY})_{|P} : P \longrightarrow \pi_{XY}(P)$ is a fibration.
(3) The 1-jet prolongation $J^1(\pi_{XY})_{|P}$ is a submanifold of Z.
(4) There exists an induced true connection Γ_P in the fibration $(\pi_{XY})_{|P} : P \longrightarrow \pi_{XY}(P)$ with the same horizontal subspaces.
(5) Γ_P is flat if and only if $\mathbf{h}$ is flat.

Proof.

(1) and *(2)* First of all, we shall prove that $(\pi_{XY})_{|P} : P \longrightarrow X$ is a submersion.

Let $y \in P$ such that $\pi_{XY}(y) = x \in X$. We define a linear mapping

$$\mathcal{A}(y) : T_xX \longrightarrow T_yP$$

as follows:

$$\mathcal{A}(y)(U) = \mathbf{h}_y(\bar{U}),$$

where $\bar{U} \in T_yY$ and $T\pi_{XY}(\bar{U}) = U$. The mapping $\mathcal{A}(y)$ is well-defined since if $\bar{U}'$ is another tangent vector in T_yY satisfying $T\pi_{XY}(\bar{U}') = U$, then $\bar{U} - \bar{U}' \in (V\pi_{XY})_y$, and therefore $\mathbf{h}_y(\bar{U}') = \mathbf{h}_y(\bar{U})$.

In addition, $\mathcal{A}(y)$ is injective. In fact, if $U \in T_xX$ is such that $\mathcal{A}(y)(U) = 0$, then $\mathbf{h}_y(\bar{U}) = 0$, that implies $\bar{U} \in (V\pi_{XY})_y$, and therefore $U = T\pi_{XY}(\bar{U}) = 0$.

Finally, $\mathcal{A}(y)$ is a section of $T\pi_{XY}(y) : T_yP \longrightarrow T_xX$. Indeed, take $U \in T_xX$; we have $\mathcal{A}(y)(T\pi_{XY}(\mathcal{A}(y)(U))) = \mathbf{h}_y(\mathcal{A}(y)(U)) = \mathbf{h}_y^2(\bar{U}) = \mathbf{h}_y(\bar{U}) = \mathcal{A}(y)(U)$. Thus, we have proved that $T\pi_{XY} \circ \mathcal{A}(y) = \mathrm{Id}_{T_xX}$. This shows that $(\pi_{XY})_{|P} : P \longrightarrow X$ is a submersion.

Therefore, $\pi_{XY}(P)$ is an open submanifold of X, and $(\pi_{XY})_{|P} : P \longrightarrow \pi_{XY}(P)$ is a fibration.

(3) is obvious.

(4) The induced connection Γ_P is defined by restricting the horizontal subspaces of $\mathbf{h}$, that is,

$$\mathbf{h}'_y = (\mathbf{h}_y)_{|T_yP}, \quad \text{for all } \ y \in P.$$

Since $\operatorname{Im}\mathbf{h}' = \operatorname{Im}\mathbf{h}$ then *(5)* follows. ∎

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