

MAXIMAL SURFACE GROUP REPRESENTATIONS IN ISOMETRY GROUPS OF CLASSICAL HERMITIAN SYMMETRIC SPACES

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ABSTRACT. Higgs bundles and non-abelian Hodge theory provide holomorphic methods with which to study the moduli spaces of surface group representations in a reductive Lie group G . In this paper we survey the case in which G is the isometry group of a classical Hermitian symmetric space of non-compact type. Using Morse theory on the moduli spaces of Higgs bundles, we compute the number of connected components of the moduli space of representations with maximal Toledo invariant.

1. INTRODUCTION

Given a closed oriented surface, X , and a connected semisimple Lie group G , the moduli space of representations of $\pi_1(X)$ in G is defined as the set

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X), G)/G$$

of reductive homomorphisms from $\pi_1(X)$ to G modulo conjugation. The reductiveness condition ensures that this orbit space is Hausdorff, and in fact $\mathcal{R}(G)$ is a real analytic variety. The geometry and topology of these moduli spaces, though clearly reflective of properties of both X and G , is still far from fully understood, especially in the case where G is non-compact. In this paper we consider the non-compact groups for which the homogeneous space G/H , where $H \subset G$ is a maximal compact subgroup, is a Hermitian symmetric space. By the Cartan classification of irreducible symmetric spaces, it thus suffices for us to consider the groups $G = \text{SU}(p, q)$, $G = \text{Sp}(2n, \mathbb{R})$, $G = \text{SO}^*(2n)$ and $G = \text{SO}_0(2, n)$. We concentrate mainly on the most primitive topological property of $\mathcal{R}(G)$, namely the number of connected components.

The first division of $\mathcal{R}(G)$ into disjoint closed subspaces comes from the correspondence between representations of $\pi_1(X)$ and flat principal bundles over X . Every representation $\rho : \pi_1(X) \rightarrow G$ carries a topological invariant which is the characteristic class

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$d \in \pi_1(G)$ of the flat G -bundle corresponding to ρ . This class measures the obstruction to lift ρ to a representation of $\pi_1(X)$ in the universal cover of G . The subvarieties $\mathcal{R}_d(G) \subset \mathcal{R}(G)$, consisting of representations with a fixed value of the invariant, form disjoint closed subspaces but not necessarily connected components. The problem is to count and understand the distinct components of each of the $\mathcal{R}_d(G)$.

For compact groups G , it is well-known that for every $d \in \pi_1(G)$, the moduli space $\mathcal{R}_d(G)$ is non-empty and connected. One way to see this is to choose a complex structure on X and to use the theory of holomorphic bundles on the resulting Riemann surface. When $G = \mathrm{SU}(n)$, the topological invariant is trivial, since $\mathrm{SU}(n)$ is simply connected, and by a theorem of Narasimhan and Seshadri [31], $\mathcal{R}(\mathrm{SU}(n))$ can be identified with the moduli space of polystable vector bundles of rank n and trivial determinant, which is connected. A similar result was proved by Ramanathan [33] for every connected compact semisimple Lie group G . He identified $\mathcal{R}_d(G)$ with the moduli space of polystable holomorphic principal $G^{\mathbb{C}}$ -bundles over X with topological class $d \in \pi_1(G)$, where $G^{\mathbb{C}}$ is the complexification of G , and showed that this moduli space is connected.

Holomorphic methods can also be used when G is a complex semisimple Lie group. In place of a holomorphic principal bundle, the holomorphic object corresponding to a representation is now a G -Higgs bundle, i.e. a pair consisting of a holomorphic G -bundle and a holomorphic section of the adjoint bundle twisted with the canonical bundle of X . A combination of theorems by Hitchin [25] and Donaldson [15] for $G = \mathrm{SL}(2, \mathbb{C})$ and Simpson [36] and Corlette [13] for general G identify $\mathcal{R}_d(G)$ with $\mathcal{M}_d(G)$, the moduli space of polystable G -Higgs bundles with fixed topological class. Morse-theoretic methods introduced by Hitchin [25] prove the connectedness of $\mathcal{M}_d(G)$ by relating it to that of the moduli space of polystable holomorphic principal G -bundles¹.

The situation is very different if G is a non-compact real form of a semisimple complex Lie group. The simplest case is $G = \mathrm{SL}(2, \mathbb{R})$. In this case $\pi_1(G) = \mathbb{Z}$ and the topological invariant $d \in \mathbb{Z}$ of a representation is the Euler class of the corresponding flat $\mathrm{SL}(2, \mathbb{R})$ -bundle. The Milnor–Wood inequality says that $\mathcal{R}_d(G)$ is empty unless $|d| \leq g - 1$, where g is the genus of X . In [20], Goldman showed that $\mathcal{R}_{\pm(g-1)}(G)$ has 2^{2g} components consisting of discrete faithful representations, each of which can be identified with the Teichmüller space of X . Later in [21] he showed that $\mathcal{R}_d(G)$ is connected for $|d| < g - 1$. This was also proved by Hitchin [25] using Higgs bundle methods.

The results for $G = \mathrm{SL}(2, \mathbb{R})$ can be generalized in two ways. The first goes back to [26] where Hitchin extended the results from $\mathrm{SL}(2, \mathbb{R})$ to $G = \mathrm{SL}(n, \mathbb{R})$. Using Higgs bundles, he counted the number of connected components and, moreover, for any split real form identified a component homeomorphic to $\mathbb{R}^{\dim G(2g-2)}$ and which naturally contains a copy of Teichmüller space. This component, known as the Teichmüller or Hitchin component, has special geometric significance. In the case of $\mathrm{SL}(3, \mathbb{R})$, Choi and Goldman [11, 12] showed that the representations in the Hitchin component are discrete and faithful and correspond to convex projective structures on the surface. More recently, Labourie [28] has shown for $G = \mathrm{SL}(n, \mathbb{R})$ that these representations are discrete and faithful and are related to certain Anosov geometric structures, and

¹Other methods to prove that $\mathcal{M}_d(G)$ is connected had been used by Goldman [21] for $G = \mathrm{SL}(2, \mathbb{C})$ and $G = \mathrm{PSL}(2, \mathbb{C})$ and by J. Li [29] for an arbitrary semisimple complex Lie group G .

analogous results have been obtained for $G = \mathrm{Sp}(2n, \mathbb{R})$ by Burger, Iozzi, Labourie and Wienhard [8].

The second generalization from $G = \mathrm{SL}(2, \mathbb{R})$ exploits the fact that the homogeneous space G/H , where $H \subset G$ is a maximal compact subgroup, is Hermitian symmetric. Indeed, $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ is the hyperbolic plane. This generalization is the main focus of this paper. In particular, we review how the theory of Higgs bundles is used in the general case of a non-compact real form G such that the symmetric space G/H is Hermitian.

We start by introducing the appropriate notion of a G -Higgs bundle when G is any connected reductive real Lie group. The information required to define such a Higgs bundle includes a choice of maximal compact subgroup $H \subset G$ and a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebras. The correct notion of a Higgs bundle then turns out to be a pair (E, φ) , where E is an $H^{\mathbb{C}}$ -bundle (where $H^{\mathbb{C}}$ is the complexification of H) and the Higgs field φ takes values in the complexification $\mathfrak{m}^{\mathbb{C}}$ of \mathfrak{m} .

Topological classes of E are characterized by elements $d \in \pi_1(H) \cong \pi_1(G)$. When G is semisimple and G/H is an irreducible Hermitian symmetric space, the torsion-free part of $\pi_1(H)$ is isomorphic to \mathbb{Z} (in fact, for all the classical groups that we will study $\pi_1(H) \cong \mathbb{Z}$, except in the cases $G = \mathrm{SO}_0(2, n)$ with $n \geq 3$, in which cases $\pi_1(H) \cong \mathbb{Z} \oplus \mathbb{Z}_2$). This gives an integer invariant known as the Toledo invariant. If $\mathcal{M}_d(G)$ is the moduli space of polystable G -Higgs bundles with fixed topological class of E , the theorem of Corlette [13] and an adaptation of the arguments of Simpson in [36] identify $\mathcal{R}_d(G) \cong \mathcal{M}_d(G)$ as real analytic varieties. In a direct generalization of the result for $G = \mathrm{SL}(2, \mathbb{R})$, these moduli spaces are non-empty only for values of d satisfying an inequality of Milnor–Wood type (proved by Domic and Toledo [14] and also by Turaev [40] for the symplectic group) given by

$$|d| \leq \mathrm{rk}(G/H)(g - 1),$$

where $\mathrm{rk}(G/H)$ is the rank of the symmetric space G/H . For the semisimple classical groups defining irreducible Hermitian symmetric spaces, namely $G = \mathrm{SU}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}^*(2n)$ and $\mathrm{SO}_0(2, n)$, we show that this inequality is a consequence of semistability of the Higgs bundle.

Representations with maximal Toledo invariant, so-called *maximal representations*, are of particular geometric interest, as is already clear in the case of $G = \mathrm{SL}(2, \mathbb{R})$ where, as mentioned above, they are just the uniformizing representations. Maximal representations have been the subject of extensive study by Burger, Iozzi and Wienhard [9, 43, 10] using methods of bounded cohomology. Among other things they show that maximal representations are discrete and faithful, generalizing Goldman’s theorem for $\mathrm{SL}(2, \mathbb{R})$ mentioned above. Another important result proved by them is that any maximal representation is in fact reductive, thus the restriction to reductive representations inherent in the Higgs bundle approach is unnecessary in the case of maximal representations.

Our main concern in this paper are maximal representations. In this case one finds that the geometry of G/H is important. In particular, the Shilov boundary \mathring{S} of the realization of G/H as a bounded symmetric complex domain plays a key role. There are two cases to consider, depending on whether G/H is or is not of tube-type. In the first

case, G/H can be realized as tube domain over a symmetric cone $\Omega = G'/H'$. Moreover, the Shilov boundary \check{S} is symmetric space of compact type, and the homogeneous space G'/H' is the non-compact dual of \check{S} . This happens for the groups $G = \mathrm{SU}(n, n)$, $G = \mathrm{Sp}(2n, \mathbb{R})$, $G = \mathrm{SO}^*(2n)$ with n even and $G = \mathrm{SO}_0(2, n)$. In these cases the moduli space $\mathcal{M}_d(G)$ with d maximal can be identified with another moduli space related to G' . This correspondence which we call Cayley correspondence, allows us to detect new topological invariants for the maximal representations.

In the non-tube cases (viz. $G = \mathrm{SU}(p, q)$ with $p \neq q$ and $G = \mathrm{SO}^*(2n)$ with n odd), let $\tilde{G} \subset G$ define the maximal tube-type space isometrically embedded in G/H . Then any maximal representation reduces to a representation in the normalizer $N_G(\tilde{G})$. This leads to a description of the moduli space $\mathcal{M}_d(G)$ with d maximal as a fibration, whose fibres are isomorphic to the moduli space of \tilde{G} -Higgs bundles with maximal Toledo invariant, and whose base is the moduli space of polystable holomorphic $H''^{\mathbb{C}}$ -bundles, for a certain compact group H'' , which can be defined in terms of the Shilov boundaries of G and \tilde{G} . This generalizes the rigidity results of Toledo [39] for $p = 1$ and Hernández [24] for $p = 2$ (and in the case $p > 2$ for representations satisfying a certain non-degeneracy condition). These two results were generalized by the authors to any reductive representation and arbitrary p in [3, 4] and shortly afterwards it was shown in general by Burger, Iozzi and Wienhard [9] that any maximal representation stabilizes a maximal tube type subdomain of G/H . It was this latter result that made us aware of the importance of the tube type condition for the study of maximal representations.

Finally, we count the number of connected components of the moduli spaces $\mathcal{R}_d(G)$ when d corresponds to maximal values of the Toledo invariant. The results are summarized in Table 2 and contain as special cases results obtained for specific groups by Goldman [21], Hitchin [25], Xia [44, 45], Xia–Markman [30], and in joint work involving Mundet i Riera and the authors [19, 22, 4, 18] (in fact many of these references give a complete count, for all values of the Toledo invariant).

The method we use to make the final count of components is based on the Morse function defined by Hitchin, using the L^2 -norm of the Higgs field. The key steps involve characterizing the subvariety of local minima of this function and counting the number of components of these.

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2. SURFACE GROUP REPRESENTATIONS AND G -HIGGS BUNDLES

2.1. Surface group representations. Let X be a closed oriented surface of genus g and let

$$\pi_1(X) = \{a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^g [a_i, b_i] = 1\}$$

be its fundamental group. Let G be a connected reductive real Lie group. By a *representation* of $\pi_1(X)$ in G we understand a homomorphism $\rho: \pi_1(X) \rightarrow G$. The set of all such homomorphisms, $\text{Hom}(\pi_1(X), G)$, can be naturally identified with the subset of G^{2g} consisting of $2g$ -tuples $(A_1, B_1, \dots, A_g, B_g)$ satisfying the algebraic equation $\prod_{i=1}^g [A_i, B_i] = 1$. This shows that $\text{Hom}(\pi_1(X), G)$ is a real analytic variety, which is algebraic if G is algebraic.

The group G acts on $\text{Hom}(\pi_1(X), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \text{Hom}(\pi_1(X), G)$ and $\gamma \in \pi_1(X)$. If we restrict the action to the subspace $\text{Hom}^+(\pi_1(X), G)$ consisting of *reductive representations*, the orbit space is Hausdorff. By a reductive representation we mean one that composed with the adjoint representation in the Lie algebra of G decomposes as a sum of irreducible representations (when G is compact every representation is reductive). Define the *moduli space of representations* of $\pi_1(X)$ in G to be the orbit space

$$\mathcal{R}(G) = \text{Hom}^+(\pi_1(X), G)/G$$

with the quotient topology.

Given a representation $\rho: \pi_1(X) \rightarrow G$, there is an associated flat G -bundle on X , defined as $E_\rho = \tilde{X} \times_\rho G$, where $\tilde{X} \rightarrow X$ is the universal cover and $\pi_1(X)$ acts on G via ρ . This gives in fact an identification between the set of equivalence classes of representations $\text{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat G -bundles, which in turn is parameterized by the cohomology set $H^1(X, G)$. We can then assign a topological invariant to a representation ρ given by the characteristic class $c(\rho) := c(E_\rho) \in \pi_1(G)$ corresponding to E_ρ . To define this, let \tilde{G} be the universal covering group of G . We have an exact sequence

$$1 \longrightarrow \pi_1(G) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

which gives rise to the (pointed sets) cohomology sequence

$$(2.1) \quad H^1(X, \tilde{G}) \longrightarrow H^1(X, G) \xrightarrow{c} H^2(X, \pi_1(G)).$$

Since $\pi_1(G)$ is abelian, we have

$$H^2(X, \pi_1(G)) \cong \pi_1(G),$$

and $c(E_\rho)$ is defined as the image of E under the last map in (2.1). Thus the class $c(E_\rho)$ measures the obstruction to lifting E_ρ to a flat \tilde{G} -bundle, and hence to lifting ρ to a representation of $\pi_1(X)$ in \tilde{G} . For a fixed $d \in \pi_1(G)$, the *moduli space of reductive representations* $\mathcal{R}_d(G)$ with topological invariant d is defined as the subvariety

$$\mathcal{R}_d(G) := \{\rho \in \mathcal{R}(G) : c(\rho) = d\}.$$

2.2. G -Higgs bundles. Let now X be a compact Riemann surface and let K be its canonical line bundle. Let G be a connected reductive real Lie group. Let $H \subset G$ be a maximal compact subgroup, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition of \mathfrak{g} . A G -Higgs bundle over X is a pair (E, φ) consisting of a principal holomorphic $H^{\mathbb{C}}$ -bundle E over X and a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes K$, i.e. $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to E via the isotropy representation of $H^{\mathbb{C}}$ in $\mathfrak{m}^{\mathbb{C}}$.

If $G = \mathrm{GL}(n, \mathbb{C})$ (with its underlying real structure), we recover the original notion of Higgs bundle introduced by Hitchin [25], consisting of a holomorphic vector bundle $\mathbb{E} := E(\mathbb{C}^n)$ — associated to a principal $\mathrm{GL}(n, \mathbb{C})$ -bundle E via the standard representation — and a homomorphism

$$\Phi : \mathbb{E} \longrightarrow \mathbb{E} \otimes K.$$

The Higgs bundle (\mathbb{E}, Φ) is said to be *stable* if

$$(2.2) \quad \frac{\deg \mathbb{E}'}{\mathrm{rank} \mathbb{E}'} < \frac{\deg \mathbb{E}}{\mathrm{rank} \mathbb{E}}$$

for every proper subbundle $\mathbb{E}' \subset \mathbb{E}$ such that $\Phi(\mathbb{E}') \subset \mathbb{E}' \otimes K$. The Higgs bundle (\mathbb{E}, Φ) is *polystable* if $(\mathbb{E}, \Phi) = \bigoplus_i (\mathbb{E}_i, \Phi_i)$ where (\mathbb{E}_i, Φ_i) is a stable Higgs bundles and $\deg \mathbb{E}_i / \mathrm{rank} \mathbb{E}_i = \deg \mathbb{E} / \mathrm{rank} \mathbb{E}$. The *moduli space of polystable Higgs bundles* $\mathcal{M}(n, d)$ is defined as the set of isomorphism classes of polystable Higgs bundles (\mathbb{E}, Φ) with $\mathrm{rank} \mathbb{E} = n$ and $\deg \mathbb{E} = d$. Another important concept is that of *semistability* which is defined by replacing the strict inequality in (2.2) by the weaker inequality. It is immediate that polystability implies semistability.

Similarly, there is a notion of stability, semistability and polystability for G -Higgs bundles. If $G \subset \mathrm{GL}(n, \mathbb{C})$ is a classical group, to a G -Higgs bundle we can naturally associate a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle. The polystability of a G -Higgs bundle is in fact equivalent to the polystability of the corresponding $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle. However, a G -Higgs bundle can be stable as a G -Higgs bundle but not as a $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle ([4]). Now, topologically, $H^{\mathbb{C}}$ -bundles E on X are classified by a characteristic class $d = c(E) \in \pi_1(H^{\mathbb{C}}) = \pi_1(H) = \pi_1(G)$, and for a fixed such class d , the *moduli space of polystable G -Higgs bundles* $\mathcal{M}_d(G)$ is defined as the set of isomorphism classes of polystable G -Higgs bundles (E, φ) such that $c(E) = d$.

2.3. Correspondence of moduli spaces. We assume now that G is semisimple. With the notation of the previous sections, we have the following.

Theorem 2.1. *Let G be a connected non-compact semisimple real Lie group. There is a homeomorphism $\mathcal{R}_d(G) \cong \mathcal{M}_d(G)$.*

Remark 2.2. This correspondence is in fact an isomorphism of real analytic varieties.

Remark 2.3. There is a similar correspondence when G is reductive, replacing the fundamental group of X by its universal central extension.

The proof of Theorem 2.1 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let \mathbf{E}_G be a C^∞ principal G -bundle over X with fixed characteristic class $d \in \pi_1(G) = \pi_1(H)$. Let D be a G -connection on \mathbf{E}_G and let F_D be its curvature. If D is flat, i.e. $F_D = 0$, then the holonomy of D around a closed loop in X only depends on the homotopy class of the loop and thus defines a representation of $\pi_1(X)$ in G . This gives an identification²,

$$\mathcal{R}_d(G) \cong \{\text{Reductive } G\text{-connections } D : F_D = 0\} / \mathcal{G},$$

²even when G is complex algebraic, this is merely a real *analytic* isomorphism, see Simpson [36, 37, 38]

where, by definition, a flat connection is reductive if the corresponding representation of $\pi_1(X)$ in G is reductive, and \mathcal{G} is the group of automorphisms of \mathbf{E}_G — the *gauge group*. Let h_0 be a fixed reduction of \mathbf{E}_G to a C^∞ H -bundle \mathbf{E}_H . Every G -connection D on \mathbf{E}_G decomposes uniquely as

$$D = d_A + \psi,$$

where d_A is an H -connection on \mathbf{E}_H and $\psi \in \Omega^1(X, \mathbf{E}_H(\mathfrak{m}))$. Let F_A be the curvature of d_A . We consider the following set of equations for the pair (d_A, ψ) :

$$(2.3) \quad \begin{aligned} F_A + \frac{1}{2}[\psi, \psi] &= 0 \\ d_A \psi &= 0 \\ d_A^* \psi &= 0. \end{aligned}$$

These equations are invariant under the action of \mathcal{H} , the gauge group of \mathbf{E}_H . A theorem of Corlette [13], and Donaldson [15] for $G = \mathrm{SL}(2, \mathbb{C})$, says the following.

Theorem 2.4. *There is a homeomorphism*

$$\{\text{Reductive } G\text{-connections } D : F_D = 0\} / \mathcal{G} \cong \{(d_A, \psi) \text{ satisfying (2.3)}\} / \mathcal{H}.$$

The first two equations in (2.3) are equivalent to the flatness of $D = d_A + \psi$, and Theorem 2.4 simply says that in the \mathcal{G} -orbit of a reductive flat G -connection D_0 we can find a flat G -connection $D = g(D_0)$ such that if we write $D = d_A + \psi$, the additional condition $d_A^* \psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h = g(h_0)$ of \mathbf{E}_G to an H -bundle obtained by the action of $g \in \mathcal{G}$ on h_0 . Equation $d_A^* \psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$ -equivariant map $\tilde{X} \rightarrow G/H$ corresponding to the new reduction of structure group h .

To establish the link with Higgs bundles, we consider the H -bundle \mathbf{E}_H and the moduli space of solutions to the *Hitchin's equations* for a pair (d_A, φ) consisting of an H -connection d_A and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_H(\mathfrak{m}^{\mathbb{C}}))$:

$$(2.4) \quad \begin{aligned} F_A - [\varphi, \tau(\varphi)] &= 0 \\ \bar{\partial}_A \varphi &= 0. \end{aligned}$$

Here $\bar{\partial}_A$ is the $(0, 1)$ part of d_A , which defines a holomorphic structure on \mathbf{E}_H , and τ is the conjugation on $\mathfrak{g}^{\mathbb{C}}$ defining its compact form. The gauge group of \mathbf{E}_H acts on the space of solutions defining the moduli space of solutions. A theorem of Hitchin [25] for $G = \mathrm{SL}(2, \mathbb{C})$ and Simpson [36] for an arbitrary semisimple complex Lie group G can be adapted to a semisimple real Lie group G [7] to give the following.

Theorem 2.5. *There is a homeomorphism*

$$\mathcal{M}_d(G) \cong \{(d_A, \varphi) \text{ satisfying (2.4)}\} / \mathcal{H}.$$

To explain this correspondence we interpret the moduli space of G -Higgs bundles in terms of pairs $(\bar{\partial}_E, \varphi)$ consisting of a $\bar{\partial}$ -operator on the $H^{\mathbb{C}}$ -bundle $\mathbf{E}_{H^{\mathbb{C}}}$ obtained from \mathbf{E}_H by the extension of structure group $H \subset H^{\mathbb{C}}$, and $\varphi \in \Omega^{1,0}(X, \mathbf{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ satisfying $\bar{\partial}_E \varphi = 0$. Such pairs are in correspondence with G -Higgs bundles (E, φ) , where E is the holomorphic $H^{\mathbb{C}}$ -bundle defined by the operator $\bar{\partial}_E$ on $\mathbf{E}_{H^{\mathbb{C}}}$ and $\bar{\partial}_E \varphi = 0$ is equivalent

to $\varphi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. The moduli space of polystable G -Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{\text{Polystable } (\bar{\partial}_E, \varphi) : \bar{\partial}_E \varphi = 0\} / \mathcal{H}^{\mathbb{C}},$$

where $\mathcal{H}^{\mathbb{C}}$ is the gauge group of $\mathbf{E}_{H^{\mathbb{C}}}$, which is in fact the complexification of \mathcal{H} . Since there is a one-to-one correspondence between H -connections on \mathbf{E}_H and $\bar{\partial}$ -operators on $\mathbf{E}_{H^{\mathbb{C}}}$, the correspondence given in Theorem 2.5 can be interpreted by saying that in the $\mathcal{H}^{\mathbb{C}}$ -orbit of a polystable G -Higgs bundle $(\bar{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\bar{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies $F_A - [\varphi, \tau(\varphi)] = 0$, and this is unique up to H -gauge transformations.

To complete the circle, leading to Theorem 2.1, we just need the following.

Proposition 2.6. *The correspondence $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ defines a homeomorphism*

$$\{(d_A, \varphi) \text{ satisfying (2.4)}\} / \mathcal{H} \cong \{(d_A, \psi) \text{ satisfying (2.3)}\} / \mathcal{H}.$$

For the benefit of the reader we have outlined the correspondences between the various moduli spaces explained in this Section in a diagram (2.5) below. The notation used is as follows:

- $\mathbf{E}_{H^{\mathbb{C}}} \rightarrow X$ is a smooth $H^{\mathbb{C}}$ -bundle,
 - $\bar{\partial}_E$ is a holomorphic structure on $\mathbf{E}_{H^{\mathbb{C}}}$,
 - $\varphi \in \Omega^{1,0}(\mathbf{E}_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$, and
 - $\mathcal{H}^{\mathbb{C}}$ is the gauge group of $\mathbf{E}_{H^{\mathbb{C}}}$;
- $\mathbf{E}_G \rightarrow X$ is a smooth G -bundle,
 - D is a connection on \mathbf{E}_G , and
 - \mathcal{G} is the gauge group of \mathbf{E}_G ;
- $\mathbf{E}_H \rightarrow X$ is a smooth H -bundle (a reduction of $\mathbf{E}_{H^{\mathbb{C}}}$ and \mathbf{E}_G),
 - A is a connection on \mathbf{E}_H ,
 - $\psi \in \Omega^1(\mathbf{E}_H(\mathfrak{m}))$, and
 - \mathcal{H} is the gauge group of \mathbf{E}_H ;

and the maps (1), (2) and (3) are homeomorphisms given by

- (1) $(d_A, \varphi) \mapsto (\bar{\partial}_A, \varphi)$ [25, 35, 7];
- (2) $(d_A, \varphi) \mapsto (d_A, \psi = \varphi - \tau(\varphi))$;
- (3) $(d_A, \Psi) \mapsto D = d_A + \psi$ [25, 13].

The following diagram outlines the correspondences between the various moduli spaces explained in this Section (cf. Theorems 2.1, 2.4 and 2.5 and Proposition 2.6).

(2.5)

$$\begin{array}{ccc}
 \mathcal{M}_d(G) \cong \left\{ (\bar{\partial}_E, \varphi) : \begin{array}{l} \bar{\partial}_E \varphi = 0 \\ \text{polystable} \end{array} \right\} / \mathcal{H}^{\mathbb{C}} & \xleftarrow{(1)} & \left\{ (d_A, \varphi) : \begin{array}{l} \bar{\partial}_A \varphi = 0 \\ F_A - [\varphi, \tau(\varphi)] = 0 \end{array} \right\} / \mathcal{H} \\
 \Downarrow & & \downarrow (2) \\
 \mathcal{R}_d(G) \cong \left\{ D : \begin{array}{l} F_D = 0 \\ \text{reductive} \end{array} \right\} / \mathcal{G} & \xleftarrow{(3)} & \left\{ (d_A, \psi) : \begin{array}{l} F_A + \frac{1}{2}[\psi, \psi] = 0 \\ d_A \psi = 0 \\ d_A^* \psi = 0 \end{array} \right\} / \mathcal{H}.
 \end{array}$$

3. ISOMETRY GROUPS OF HERMITIAN SYMMETRIC SPACES

3.1. G -Higgs bundles for the classical Hermitian symmetric spaces. Our goal in this paper is to study representations of the fundamental group in the case in which G/H is a Hermitian symmetric space. This means that G/H admits a complex structure compatible with the Riemannian structure of G/H , making G/H a Kähler manifold. If G/H is irreducible, the centre of \mathfrak{h} is one-dimensional and the almost complex structure on G/H is defined by a generating element in $J \in Z(\mathfrak{h})$ (acting through the isotropy representation on $\mathfrak{m}^{\mathbb{C}}$). This complex structure defines a decomposition

$$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_+ + \mathfrak{m}_-,$$

where \mathfrak{m}_+ and \mathfrak{m}_- are the $(1, 0)$ and the $(0, 1)$ part of $\mathfrak{m}^{\mathbb{C}}$ respectively. Table 1 (see Sec. 7) shows the main ingredients for the irreducible classical Hermitian symmetric spaces.

Let now (E, φ) be a G -Higgs bundle over a compact Riemann surface X . The decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_+ + \mathfrak{m}_-$ gives a vector bundle decomposition $E(\mathfrak{m}^{\mathbb{C}}) = E(\mathfrak{m}_+) \oplus E(\mathfrak{m}_-)$ and hence

$$\varphi = (\beta, \gamma) \in H^0(X, E(\mathfrak{m}_+) \otimes K) \oplus H^0(X, E(\mathfrak{m}_-) \otimes K) = H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K).$$

In Table 3, we describe the G -Higgs bundles for the various groups appearing in Table 1. It is sometimes convenient to replace the $H^{\mathbb{C}}$ -bundle E by a vector bundle associated to the standard representation of $H^{\mathbb{C}}$. For example, for $G = \text{SU}(p, q)$,

$$H^{\mathbb{C}} = S(\text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{C})) := \{(A, B) \in \text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{C}) : \det B = (\det A)^{-1}\},$$

and the $H^{\mathbb{C}}$ -bundle E is replaced by two holomorphic vector bundles V and W of rank p and q , respectively such that $\det W = (\det V)^{-1}$.

As mentioned in Sec. 2.2, via the natural inclusion $G \subset G^{\mathbb{C}} \subset \text{SL}(N, \mathbb{C})$ for G in Table 3, to a G -Higgs bundle (E, φ) we can naturally associate an $\text{SL}(N, \mathbb{C})$ -Higgs bundle (\mathbb{E}, Φ) , where \mathbb{E} is a holomorphic vector bundle with trivial determinant. This is a very useful correspondence that we also describe in Table 3.

3.2. Toledo invariant and Milnor–Wood inequalities. Let G be a semisimple Lie group such that G/H is an irreducible Hermitian symmetric space. Then the torsion-free part of $\pi_1(H)$ is isomorphic to \mathbb{Z} and hence the topological invariant of either a

representation of $\pi_1(X)$ in G , or of a G -Higgs bundle, is measured by an integer $d \in \mathbb{Z}$, known as the *Toledo invariant*. In the classical cases described in Table 1 this coincides with the degree of a certain vector bundle. In fact, besides $G = \mathrm{SO}_0(2, n)$ with $n \geq 3$, for which $\pi_1(H) \cong \mathbb{Z} \oplus \mathbb{Z}_2$, for all the other groups in Table 1 $\pi_1(H) \cong \mathbb{Z}$.

Let $G = \mathrm{SU}(p, q)$. As we can see from Table 3, an $\mathrm{SU}(p, q)$ -Higgs bundle over X is defined by a 4-tuple (V, W, β, γ) consisting of holomorphic vector bundles V and W of rank p and q , respectively, such that $\det W = (\det V)^{-1}$, and homomorphisms

$$\beta : W \longrightarrow V \otimes K \quad \text{and} \quad \gamma : V \longrightarrow W \otimes K.$$

If (V, W, β, γ) is polystable then the associated Higgs bundle (\mathbb{E}, Φ) with

$$\mathbb{E} = V \oplus W \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

is also polystable and in particular semistable. Applying the semistability numerical criterion to special Higgs subbundles defined by the kernel and image of Φ (see [4]) we obtain

$$(3.1) \quad d \leq \mathrm{rank}(\gamma)(g - 1)$$

$$(3.2) \quad -d \leq \mathrm{rank}(\beta)(g - 1),$$

which gives the inequality

$$(3.3) \quad |d| \leq \min\{p, q\}(g - 1),$$

that generalizes the inequality of Milnor–Wood for $G = \mathrm{SU}(1, 1)$. Using similar arguments one can show the various *Milnor–Wood type inequalities* for G in Table 3. In fact, the inequalities for $G = \mathrm{Sp}(2n, \mathbb{R})$ and $G = \mathrm{SO}^*(2n)$ can be obtained from that of $\mathrm{SU}(n, n)$, via the natural inclusion of these two groups in $\mathrm{SU}(n, n)$. These inequalities for representations of $\pi_1(X)$ in G have been proved by Domic and Toledo [14] using other methods. One can observe that the bound for the Toledo invariant can uniformly written as $\mathrm{rank}(G/H)(g - 1)$.

Remark 3.1. Duality gives an isomorphism $\mathcal{M}_d(G) \cong \mathcal{M}_{-d}(G)$ for every G in Table 3. For example, for $\mathrm{SU}(p, q)$ this isomorphism is defined by the map

$$(V, W, \beta, \gamma) \mapsto (V^*, W^*, \gamma^t, \beta^t).$$

There is hence no loss of generality in considering only the case with positive Toledo invariant, which we will do from now on.

Our main interest in this paper is the case when the Toledo invariant d is maximal, that is $|d| = d_{\max}$. Let

$$\mathcal{M}_{\max}(G) := \mathcal{M}_{d_{\max}}(G)$$

be the moduli space of G -Higgs bundles with maximal Toledo invariant, and similarly $\mathcal{R}_{\max}(G) := \mathcal{R}_{d_{\max}}(G)$ for the corresponding moduli space of representations.

Our goal in the following sections is to study the geometry of maximal Higgs bundles and to count the number of connected components of $\mathcal{M}_{\max}(G)$ for G in Table 3. The case of $G = \mathrm{SU}(1, 1) \cong \mathrm{SL}(2, \mathbb{R})$ and $G = \mathrm{SO}_0(2, 1) \cong \mathrm{PSL}(2, \mathbb{R})$ was studied by Goldman [20], who showed that the moduli space of maximal representations in

$\mathrm{SL}(2, \mathbb{R})$ has 2^{2g} connected components isomorphic to Teichmüller space, all of which get identified when we consider representations in $\mathrm{PSL}(2, \mathbb{R})$.

3.3. Low rank phenomena. There are certain low rank coincidences between the Higgs bundles given in Table 3, coming from the special low dimensional isomorphisms between Lie groups.

The most basic instance of this is isomorphism $\mathrm{SU}(1, 1) \cong \mathrm{Sp}(2, \mathbb{R})$ which clearly gives rise to equivalent Higgs vector bundle data (\mathbb{E}, Φ) , namely a line bundle M (which equals the V of Table 3) and a pair of sections $\beta \in H^0(M^2K)$ and $\gamma \in H^0(M^{-2}K)$ — we note that this is also the data corresponding to a $\mathrm{SL}(2, \mathbb{R})$ -Higgs bundle (cf. Hitchin [25]), as is to be expected from the isomorphism $\mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$.

In a similar way, the covering $\mathrm{SU}(1, 1) \cong \mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{Spin}_0(2, 1) \rightarrow \mathrm{SO}_0(2, 1)$ shows that a Higgs bundle for one of the former groups gives rise to one for the latter. Explicitly, if (M, β, γ) is as above, then the associated $\mathrm{SO}_0(2, 1)$ -Higgs bundle $(\mathbb{E} = V \oplus W, \Phi)$ has $V = M^2 \oplus M^{-2}$ and $W = \mathcal{O}$; thus the L of Table 3 is $L = M^2$. Note that the Toledo invariants are related by

$$d_{\mathrm{Spin}_0(2,1)} = \deg(M) = \frac{1}{2} \deg(L) = \frac{1}{2} d_{\mathrm{SO}_0(2,1)},$$

and that a $\mathrm{SO}_0(2, 1)$ -Higgs bundle lifts to a $\mathrm{Spin}_0(2, 1)$ -Higgs bundle if and only if it has even Toledo invariant.

Analogous phenomena occur for various other (local) isomorphisms, here we shall describe explicitly just one more such situation of particular interest, corresponding to the covering $\mathrm{Sp}(4, \mathbb{R}) \cong \mathrm{Spin}_0(2, 3) \rightarrow \mathrm{SO}_0(2, 3)$. Let (V, β, γ) be the vector bundle data corresponding to a $\mathrm{Sp}(4, \mathbb{R})$ -Higgs bundle as in Table 3 and let $L = \Lambda^2 V$ be the determinant bundle of the rank 2 bundle V . Then $S^2 V$ has a non-degenerate quadratic form with values in L^2 defined by

$$Q(x \otimes y, x' \otimes y') = (x \wedge x') \otimes (y \wedge y')$$

and thus we obtain an orthogonal bundle (W, Q_W) letting

$$W = S^2 V \otimes L^{-1}$$

with the induced quadratic form Q_W . Thus we have the required bundle data (W, Q_W) and L to define a $\mathrm{SO}_0(2, 3)$ -Higgs bundle. Since $S^2 V = W \otimes L \cong W^* \otimes L$, the section $\beta \in H^0(S^2 V \otimes K)$ can be viewed as a section of $\mathrm{Hom}(W, L) \otimes K$ and, similarly, γ can be viewed as a section of $\mathrm{Hom}(W, L^{-1}) \otimes K$. Note that, since $\deg(V) = \deg(\Lambda^2 V) = \deg(L)$, in this case the Toledo invariants are the same. However, the topological classification of $\mathrm{SO}_0(2, 3)$ bundle involves *two* classes, namely the degree of L (i.e. the Toledo invariant) *and* the second Stiefel–Whitney class $w_2(W, Q_w) \in \mathbb{Z}/2$ of the $\mathrm{SO}(3, \mathbb{C})$ bundle (W, Q_W) . One can show without too much difficulty that an $\mathrm{SO}_0(2, 3)$ -Higgs bundle given by the data $(L, W, Q_W, \beta, \gamma)$ lifts to a $\mathrm{Sp}(4, \mathbb{R})$ -Higgs bundle if and only if

$$\deg(L) = w_2(W, Q_W).$$

Remark 3.2. The group $\mathrm{SO}_0(2, 2)$ is special because the associated Hermitian symmetric space is not irreducible; in fact $\mathrm{SO}_0(2, 2)$ is isogenous to $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \cong \mathrm{Spin}_0(2, 2)$. Of course, the results for irreducible Hermitian symmetric spaces given

here, can be applied to obtain results for isometry groups of all Hermitian symmetric spaces. For this reason we shall exclude this group from our considerations in this paper.

4. MAXIMAL TOLEDO INVARIANT

4.1. Tube type condition. We refer to [16, 23, 27, 34] for details regarding this section.

It is well-known that a Hermitian symmetric space of non-compact type G/H can be realized as a bounded symmetric domain. For the classical groups this is due to Cartan, while the general case is given by the Harish-Chandra embedding $G/H \rightarrow \mathfrak{m}_+$ which defines a biholomorphism between G/H and the bounded symmetric domain \mathcal{D} given by the image of G/H in the complex vector space \mathfrak{m}_+ . Now, for any bounded domain \mathcal{D} there is the *Shilov boundary* of \mathcal{D} which is defined as the smallest closed subset \check{S} of the topological boundary $\partial\mathcal{D}$ for which every function f continuous on $\overline{\mathcal{D}}$ and holomorphic on \mathcal{D} satisfies that

$$|f(z)| \leq \max_{w \in \check{S}} |f(w)| \text{ for every } z \in \mathcal{D}.$$

The Shilov boundary \check{S} is the unique closed G -orbit in $\partial\mathcal{D}$.

The simplest situation to consider is that of the hyperbolic plane. The Poincaré disc is its realization as a bounded symmetric domain. However, we know that the hyperbolic plane can also be realized as the upper-half plane. There are other Hermitian symmetric spaces that, like the hyperbolic plane, admit a realization similar to the upper-half plane. These are the tube type symmetric spaces.

Let V be a real vector space and let $\Omega \subset V$ be an open cone in V . A tube over the cone Ω is a domain of the form

$$T_\Omega = \{u + iv \in V^\mathbb{C}, u \in V, v \in \Omega\}.$$

A domain \mathcal{D} is said to be of *tube type* if it is biholomorphic to a tube T_Ω . In the case of a symmetric domain the cone Ω is also symmetric. An important characterization of the tube type symmetric domains is given by the following.

Proposition 4.1. *Let \mathcal{D} be a bounded symmetric domain. The following are equivalent:*

- (i) \mathcal{D} is of tube type.
- (ii) $\dim_{\mathbb{R}} \check{S} = \dim_{\mathbb{C}} \mathcal{D}$.
- (iii) \check{S} is a symmetric space of compact type.

There is a generalization of the Cayley map that sends the unit disc biholomorphically to the upper-half plane. Let \mathcal{D} be the bounded domain associated to a Hermitian symmetric space G/H . Acting by a particular element in G , known as the *Cayley element*, one obtains a map

$$c : \mathcal{D} \longrightarrow \mathfrak{m}_+$$

which is called the *Cayley transform*. A relevant fact for us is the following.

Proposition 4.2. *Let \mathcal{D} be the symmetric domain corresponding to the Hermitian symmetric space G/H . Let \mathcal{D} be of tube type. Then the image by the Cayley transform*

$c(\mathcal{D})$ is biholomorphic to a tube domain T_Ω where the symmetric cone Ω is the non-compact dual of the Shilov boundary of \mathcal{D} . In fact the Shilov boundary is a symmetric space isomorphic to H/H' for a certain subgroup $H' \subset H$, and $\Omega = G'/H'$ is its non-compact dual symmetric space.

Proposition 4.3. (1) *The symmetric spaces defined by $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}_0(2, n)$ are of tube type.*

(2) *The symmetric space defined by $\mathrm{SU}(p, q)$ is of tube type if and only if $p = q$.*

(3) *The symmetric space defined by $\mathrm{SO}^*(2n)$ is of tube type if and only if n is even.*

For a tube type classical irreducible symmetric space G/H , Table 4 indicates the Shilov boundary $\check{S} = H/H'$, its non-compact dual $\Omega = G'/H'$, the isotropy representation space \mathfrak{m}' and its complexification $\mathfrak{m}'^{\mathbb{C}}$, corresponding to the Cartan decomposition of the Lie algebra $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ of G' . The vector space \mathfrak{m}' has the structure of a Euclidean Jordan algebra, where the cone Ω is realized.

The study of certain problems in non-tube type domains can be reduced to the tube type thanks to the following.

Proposition 4.4. *Let G/H be a Hermitian symmetric space of non-compact type. There exists a subgroup $\tilde{G} \subset G$ such that $\tilde{G}/\tilde{H} \subset G/H$ is a maximal isometrically embedded symmetric space of tube type, where $\tilde{H} \subset \tilde{G}$ is a maximal compact subgroup.*

Table 5 gives the maximal symmetric space of tube type isometrically embedded in the two series of irreducible classical symmetric spaces of non-tube type. We describe also the Shilov boundaries of G/H and \tilde{G}/\tilde{H} which are of the form $\check{S} = H/H'$, and $\check{\tilde{S}} = \tilde{H}/\tilde{H}'$, respectively. Notice that in the non-tube case the Shilov boundary \check{S} is a homogeneous space H/H' but it is not symmetric.

4.2. Tube type domains and Cayley correspondence. It turns out that the behaviour of maximal representations and G -Higgs bundles is governed by the tube type nature of G/H and the geometry of its Shilov boundary. To explain this, let L be a holomorphic line bundle over X . Let G be a real reductive Lie group. Let $H \subset G$ be a maximal compact subgroup, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be the Cartan decomposition of \mathfrak{g} . An L -twisted G -Higgs pair is a pair (E, φ) consisting of a principal holomorphic $H^{\mathbb{C}}$ -bundle E and a holomorphic section of $E(\mathfrak{m}^{\mathbb{C}}) \otimes L$, where $E(\mathfrak{m}^{\mathbb{C}})$ is the bundle associated to E via the isotropy representation of $H^{\mathbb{C}}$ in $\mathfrak{m}^{\mathbb{C}}$. Note that a G -Higgs bundle is simply a K -twisted G -Higgs pair. Let $\mathcal{M}_L(G)$ be the moduli space of polystable L -twisted G -Higgs pairs.

Theorem 4.5. *Let G be a connected semisimple classical Lie group such that G/H is a Hermitian symmetric space of tube type, and let $\Omega = G'/H'$ be the non-compact dual of the Shilov boundary $\check{S} = H/H'$ of G/H . Then*

$$(4.1) \quad \mathcal{M}_{\max}(G) \cong \mathcal{M}_{K^2}(G').$$

In analogy with the Cayley transform of the previous section, we call the isomorphism given in Theorem 4.5 *Cayley correspondence*. To prove this correspondence, it suffices to do it for all G in Table 4. We will sketch the main arguments case by case. See [22, 3, 4, 19, 18, 6] for details.

- $G = \mathrm{SU}(n, n)$:

An $\mathrm{SU}(n, n)$ -Higgs bundle over X is defined by a 4-tuple (V, W, β, γ) consisting of two holomorphic vector bundles V and W of rank n such that $\det W = (\det V)^{-1}$, and homomorphisms

$$\beta : W \longrightarrow V \otimes K \quad \text{and} \quad \gamma : V \longrightarrow W \otimes K.$$

Suppose that the Toledo invariant $d = \deg V$ is maximal and positive, that is, $d = n(g-1)$. From (3.1) we deduce that γ must be an isomorphism. Let $\theta : W \rightarrow W \otimes K^2$ be defined as $\theta = (\gamma \otimes I_K) \circ \beta$, where $I_K : K \rightarrow K$ is the identity map.

The condition $\det W = (\det V)^{-1}$, together with the isomorphism γ imply that $(\det W)^2 \cong K^{-n}$. Now, if we choose a square root of the canonical bundle, $L_0 = K^{1/2}$, and define $\widetilde{W} = W \otimes L_0$, we have that $(\det \widetilde{W})^2 = \mathcal{O}$ and hence the structure group of \widetilde{W} is the kernel of the group homomorphism $\mathrm{GL}(n, \mathbb{C}) \rightarrow \mathbb{C}^*$ given by $A \mapsto (\det A)^2$. This kernel is isomorphic to the semidirect product $\mathrm{SL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{\pm I\}$ and has then two connected components. The choice of a 2-torsion element in the Jacobian of X for $\det \widetilde{W}$ defines an invariant that takes 2^{2g} values.

Let $\widetilde{\theta} : \widetilde{W} \rightarrow \widetilde{W} \otimes K^2$ be defined as $\widetilde{\theta} = \theta \otimes I_{L_0}$. The map

$$(4.2) \quad (V, W, \beta, \gamma) \mapsto (\widetilde{W}, \widetilde{\theta})$$

gives the bijection (4.1) with $G' = \mathrm{SL}(n, \mathbb{C}) \rtimes \mathbb{Z}_2$ (see Table 4).

- $G = \mathrm{Sp}(2n, \mathbb{R})$:

A $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle over X is defined by a triple (V, β, γ) consisting of a rank n holomorphic vector bundles V and *symmetric* homomorphisms

$$\beta : V^* \longrightarrow V \otimes K \quad \text{and} \quad \gamma : V \longrightarrow V^* \otimes K.$$

If the Toledo invariant $d = \deg V$ is maximal and positive, that is, $d = n(g-1)$, again from (3.1) we deduce that γ is an isomorphism. Let $L_0 = K^{1/2}$ be a fixed square root of K , and define $W = V^* \otimes L_0$. Then $Q := \gamma \otimes I_{L_0^{-1}} : W^* \rightarrow W$ is a symmetric isomorphism defining an orthogonal structure on W , in other words, (W, Q) is an $\mathrm{O}(n, \mathbb{C})$ -holomorphic bundle. The K^2 -twisted endomorphism $\theta : W \rightarrow W \otimes K^2$ defined by $\theta = (\gamma \otimes I_{K \otimes L_0}) \circ \beta \otimes I_{L_0}$ is Q -symmetric and hence (W, θ) defines a K^2 -twisted $\mathrm{GL}(n, \mathbb{R})$ -Higgs pair, from which we can recover the original $\mathrm{Sp}(2n, \mathbb{R})$ -Higgs bundle, giving the bijection (4.1) in this case.

- $G = \mathrm{SO}^*(2n)$, with $n = 2m$:

A $\mathrm{SO}^*(2n)$ -Higgs bundle is over X is defined by a triple (V, β, γ) consisting of a rank n holomorphic vector bundles V and skew-symmetric homomorphisms

$$\beta : V^* \longrightarrow V \otimes K \quad \text{and} \quad \gamma : V \longrightarrow V^* \otimes K.$$

Since $n = 2m$ is even, the maximal value of the Toledo invariant (see Table 3) is $d_{\max} = n(g-1)$. If $d = n(g-1)$ then, as in the previous cases, γ is an isomorphism, and if $L_0 = K^{1/2}$ is a fixed square root of K , and we define $W = V^* \otimes L_0$, the homomorphism $\omega := \gamma \otimes I_{L_0^{-1}} : W^* \rightarrow W$ is a skew-symmetric isomorphism defining a symplectic structure on W , that is, (W, ω) is a $\mathrm{Sp}(2m, \mathbb{C})$ -holomorphic bundle. The

K^2 -twisted endomorphism $\theta : W \longrightarrow W \otimes K^2$ defined by $\theta = (\gamma \otimes I_{K \otimes L_0}) \circ \beta \otimes I_{L_0}$ is in this case skew-symmetric with respect to ω and hence (W, θ) defines a K^2 -twisted G' -Higgs pair for $G' = \mathrm{U}^*(2m)$. The map $(V, \beta, \gamma) \mapsto (W, \theta)$ gives the bijection (4.1) in this case.

- $G = \mathrm{SO}_0(2, n)$:

A $\mathrm{SO}_0(2, n)$ -Higgs bundle is defined by a $\mathrm{SO}(2, \mathbb{C})$ -bundle

$$(V = L \oplus L^{-1}, Q_V = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})$$

where L is a holomorphic line bundle, and a $\mathrm{SO}(n, \mathbb{C})$ -bundle (W, Q_W) , together with homomorphisms

$$\beta : W \longrightarrow L \otimes K \quad \text{and} \quad \gamma : W \longrightarrow L^{-1} \otimes K.$$

The maximal case corresponds to $d = \deg L = 2g - 2$. In this situation one can show that γ has (maximal) rank one at all points and hence it is surjective. If we define $F := \ker \gamma$, we have a sequence

$$(4.3) \quad 0 \longrightarrow F \longrightarrow W \longrightarrow L^{-1} \otimes K \longrightarrow 0.$$

One can show that this sequence splits and F inherits a $\mathrm{O}(n - 1, \mathbb{C})$ -structure. Let $L_0 := L^{-1} \otimes K$. From the exact sequence we deduce that $L_0 \otimes \det F \cong \mathcal{O}$ and hence $L_0^2 = \mathcal{O}$. In other words, $L^2 \cong K^2$. Now, according to the decomposition $W \cong F \oplus L_0$, we can decompose $\beta = \beta' + \beta''$ with $\beta' : F \longrightarrow L \otimes K$ and $\beta'' : L_0 \longrightarrow L \otimes K$. Tensoring these homomorphisms by L_0 , we obtain $\theta' : F \otimes L_0 \rightarrow K^2$ and $\theta'' : \mathcal{O} \rightarrow K^2$. The map

$$(L, W, Q_W, \beta, \gamma) \mapsto (F, \theta', \theta'')$$

defines the correspondence (4.1) now.

In all the cases above we have to show of course that the corresponding (poly)stability conditions in $\mathcal{M}_{\max}(G)$ and $\mathcal{M}_{K^2}(G')$ are equivalent.

The correspondence (4.1) brings to the surface new topological invariants of a maximal representation — the invariants of the $H^{\mathbb{C}}$ -bundle — which are not *a priori* “visible”. The new invariants will account to a certain extent for the abundance of connected components in most cases.

Remark 4.6. We believe that Theorem 4.5 is true also in the non-classical case. To show this it would suffice to check the case of the rank 3 irreducible exceptional domain, which is obtained from a real form of E_7 . It would be very interesting, however, to find a proof independent of classification theory.

4.3. Non-tube type domains and rigidity of maximal representations. We study now maximal G -Higgs bundles and representations when G/H is not of tube type (see Table 5). Let us start with $G = \mathrm{SU}(p, q)$ with $p \neq q$. Without loss of generality we assume that $p < q$. The maximal value of the Toledo invariant is then $d_{\max} = p(g - 1)$. As shown in [22, 4], it turns out that there are no stable $\mathrm{SU}(p, q)$ -Higgs bundles. In fact, every polystable $\mathrm{SU}(p, q)$ -Higgs bundle (V, W, β, γ) is strictly

semistable and decomposes as a direct sum

$$(4.4) \quad (V, W, \beta, \gamma) \cong (V, W', \beta, \gamma) \oplus (0, W'', 0, 0),$$

of a maximal polystable $U(p, p)$ -Higgs bundle and a polystable $GL(q-p, \mathbb{C})$ -bundle with zero Chern class, where $\gamma: V \xrightarrow{\cong} W' \otimes K$, with $W' = \text{im } \gamma \otimes K^{-1}$ and $W'' = W/W'$. Since

$$\det(V) \otimes \det(W') \otimes (W'') \cong \mathcal{O},$$

this means that the $SU(p, q)$ -Higgs bundle reduces to an $S(U(p, p) \times U(q-p))$ -Higgs bundle.

We have the exact sequence

$$\begin{aligned} 1 \rightarrow SU(p, p) \rightarrow S(U(p, p) \times U(q-p)) \rightarrow S(U(1) \times U(q-p)) \rightarrow 1 \\ (A, B) \mapsto (\det(A), B), \end{aligned}$$

from which we conclude the following.

Theorem 4.7. *Let $p < q$. Then the moduli space $\mathcal{M}_{\max}(SU(p, q))$ fibres over $M(GL(q-p, \mathbb{C}))$ with fibre isomorphic to $\mathcal{M}_{\max}(SU(p, p))$, where $M(GL(q-p, \mathbb{C}))$ is the moduli space of polystable vector bundles of rank $q-p$ and zero Chern class.*

Similarly, for $G = SO^*(2n)$ with $n = 2m+1$, the maximal value of the Toledo invariant is $d_{\max} = (n-1)(g-1)$, and if $d = d_{\max}$ there are no stable $SO^*(4m+2)$ -Higgs bundles and every polystable $SO^*(4m+2)$ -Higgs bundle (V, β, γ) decomposes as

$$(V, \beta, \gamma) = (V', \beta, \gamma) \oplus (L, 0, 0),$$

where (V', β, γ) is a maximal polystable $SO^*(4m)$ -Higgs bundle, with $V' = (\text{im } \gamma)^* \otimes K$ and $L = V/V'$ is a line bundle of 0 degree (see [6] for details).

We thus have the following.

Theorem 4.8.

$$\mathcal{M}_{\max}(SO^*(4m+2)) \cong \mathcal{M}_{\max}(SO^*(4m)) \times J(X),$$

where $J(X)$ is the Jacobian of X .

Since, as we know from Table 5, the two cases discussed above are the only ones defining classical irreducible Hermitian symmetric spaces of non-tube type, we conclude the following.

Theorem 4.9. *Let G be a connected semisimple classical Lie group such that G/H is Hermitian symmetric space of non-compact type. Let $\tilde{G} \subset G$ be a subgroup defining the largest isometrically embedded subspace $\tilde{G}/\tilde{H} \subset G/H$, and let $H'' = H'/\tilde{H}'$ where $\tilde{S} = H/H'$ and $\tilde{S} = \tilde{H}/\tilde{H}'$ are the Shilov boundaries of G/H and \tilde{G}/\tilde{H} , respectively. Then the following holds:*

(1) *Every G -Higgs bundle in $\mathcal{M}_{\max}(G)$ is strictly polystable and reduces to a $N_G(\tilde{G})$ -Higgs bundle, where the normalizer of \tilde{G} in G , $N_G(\tilde{G})$, fits in the exact sequence*

$$1 \rightarrow \tilde{G} \rightarrow N_G(\tilde{G}) \rightarrow H'' \rightarrow 1.$$

(2) The moduli space $\mathcal{M}_{\max}(G)$ fibres over $M(H''^{\mathbb{C}})$, with fibre $\mathcal{M}_{\max}(\tilde{G})$, where $M(H''^{\mathbb{C}})$ is the moduli space of polystable holomorphic $H''^{\mathbb{C}}$ -bundles with zero characteristic class.

From Theorem 2.1 and the theorems of Narasimhan–Seshadri [31] and Ramanathan [33], which identify the moduli space of polystable holomorphic $H''^{\mathbb{C}}$ -bundles with trivial characteristic class with the moduli space of representations of $\pi_1(X)$ in H'' , we obtain the following.

Theorem 4.10. *With the same hypotheses and notation as in Theorem 4.9 we have the following:*

(1) Every representation in $\mathcal{R}_{\max}(G)$ is reducible and factors through a representation in $N_G(\tilde{G})$.

(2) The moduli space $\mathcal{R}_{\max}(G)$ fibres over $\mathcal{R}(H'')$, with fibre $\mathcal{R}_{\max}(\tilde{G})$, where $\mathcal{R}(H'')$ is the moduli space of representations of $\pi_1(X)$ in H'' .

Theorem 4.10 had been proved by Toledo [39] for $G = \mathrm{SU}(1, q)$. For general p the result had been proved by Hernández [24], under a certain non-degeneracy condition on the representation, which he was able to show is always satisfied for $p = 2$. The result was then proved in [3, 4] for any reductive representation in $\mathrm{SU}(p, q)$. Finally Burger, Iozzi and Wienhard [9] showed that any maximal representation stabilizes a maximal tube type subdomain of $\tilde{G}/\tilde{H} \subset G/H$. From this, and their results in [10], Theorem 4.10 should follow directly for general G . On the other hand, to generalize Theorems 4.9 and 4.10 from the Higgs bundle point of view, it suffices to prove corresponding results for the only non-tube rank 2 irreducible exceptional domain which is obtained from a real form of E_6 .

Remark 4.11. Theorems 4.9 and 4.10 establish a certain kind of *rigidity* for maximal G -Higgs bundles, and hence for surface group representations in G , when G/H is not of tube type. Namely, since the expected complex dimension of $\mathcal{M}_d(G) = \dim_{\mathbb{R}} \mathfrak{g}(g - 1)$ (as can be computed using the deformation theory in Sec. 5.1), in the non tube situation $\dim_{\mathbb{R}} \tilde{\mathfrak{g}} + \dim_{\mathbb{R}} \mathfrak{h}'' < \dim \mathfrak{g}$ and hence the dimension of \mathcal{M}_{\max} is smaller than expected (here \mathfrak{h}'' and $\tilde{\mathfrak{g}}$ are the Lie algebras of \tilde{G} and \tilde{H}'' in Theorem 4.9).

The fact that $M(H''^{\mathbb{C}})$ is connected ([31, 33]) leads to the following Corollary to Theorem 4.9.

Corollary 4.12. *If G , \tilde{G} and H'' are as in Theorem 4.9, then the number of connected components of $\mathcal{M}_{\max}(G)$ is bounded by the number of connected components of $\mathcal{M}_{\max}(\tilde{G})$. In particular, if $\mathcal{M}_{\max}(\tilde{G})$ is connected so is $\mathcal{M}_{\max}(G)$.*

5. MORSE THEORY ON THE MODULI SPACE OF G -HIGGS BUNDLES

5.1. Deformation theory. Below we shall be doing analysis, in the form of Morse theory, on the moduli spaces of G -Higgs bundles and therefore we need a description of their tangent spaces. This can be conveniently done using hypercohomology of certain

complexes of sheaves. This idea probably has its origin in Welters [42]. A convenient reference is Biswas and Ramanan [2].

Let (E, φ) be a G -Higgs bundle. The *deformation complex* of (E, φ) is the following complex of sheaves:

$$(5.1) \quad C^\bullet(E, \varphi): E(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{\text{ad}(\varphi)} E(\mathfrak{m}^{\mathbb{C}}) \otimes K.$$

Note that this makes sense because $[\mathfrak{m}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}] \subseteq \mathfrak{m}^{\mathbb{C}}$.

The following result generalizes the fact that the infinitesimal deformation space of a holomorphic vector bundle V is isomorphic to $H^1(\text{End } V)$.

Proposition 5.1. *The space of infinitesimal deformations of a G -Higgs bundle (E, φ) is isomorphic to the hypercohomology group $\mathbb{H}^1(C^\bullet(E, \varphi))$.*

In particular, if (E, φ) represents a non-singular point of the moduli space $\mathcal{M}_d(G)$ then the tangent space at this point is canonically isomorphic to $\mathbb{H}^1(C^\bullet(E, \varphi))$.

For usual holomorphic vector bundles, the analogue of the following result is the fact that the only endomorphisms of a stable bundle are the constant multiples of the identity.

Proposition 5.2. *Let (E, φ) be a stable G -Higgs bundle which represents a smooth point of $\mathcal{M}_d(G)$. Then*

$$\mathbb{H}^0(C^\bullet(E, \varphi)) = \mathbb{H}^2(C^\bullet(E, \varphi)) = 0.$$

5.2. Morse theory. The idea of applying Morse theory to the study of moduli of holomorphic vector bundles has its origin in the fundamental work of Atiyah and Bott [1]. Here the moduli space of stable bundles was studied using equivariant Morse theory on the infinite dimensional space of unitary connections. The use of Morse theory in moduli spaces of Higgs bundles was introduced by Hitchin [25]. In this section we explain how to apply these methods in moduli spaces of G -Higgs bundles. In particular, we give a criterion (Corollary 5.4) for finding the local minima of the Morse function, which is extremely useful in the context of problem of counting connected components of the moduli space.

In order to define the Morse function, we shall consider the moduli space $\mathcal{M}_d(G)$ of G -Higgs bundles from the gauge theory point of view, as explained in Sec. 2.3. Thus we identify $\mathcal{M}_d(G)$ with the moduli space of solutions to Hitchin's equations (2.4). From this point of view it makes sense to define

$$(5.2) \quad \begin{aligned} f: \mathcal{M}_d(G) &\longrightarrow \mathbb{R}, \\ (d_A, \varphi) &\mapsto \|\varphi\|^2, \end{aligned}$$

where $\|\varphi\|^2 = \int_X |\varphi|^2 d\text{vol}$ is the L^2 -norm of φ . Note that this norm is well defined because $|\varphi|^2$ is invariant under H -gauge transformations.

The maps f has its origin in symplectic geometry: away from the singular locus of $\mathcal{M}_d(G)$ it is a moment map for the hamiltonian S^1 -action given by

$$e^{i\theta}: (d_A, \varphi) \mapsto (d_A, e^{i\theta} \varphi).$$

This fact is important for two reasons. Firstly, a theorem of Frankel [17] guarantees that, when $\mathcal{M}_d(G)$ is smooth, f is a perfect Bott–Morse function. Secondly, the critical points of f are exactly the fixed points of the S^1 -action.

The function f can be used to obtain information about connected components even when $\mathcal{M}_d(G)$ has singularities due to the following result, proved by Hitchin [25], using Uhlenbeck’s weak compactness theorem [41].

Proposition 5.3. *The function $f: \mathcal{M}_d(G) \rightarrow \mathbb{R}$ is a proper map.*

Corollary 5.4. *Let $\mathcal{M} \subseteq \mathcal{M}_d(G)$ be a closed subspace and let $\mathcal{N} \subseteq \mathcal{M}$ be the subspace of local minima of f on \mathcal{M} . If \mathcal{N} is connected, then so is \mathcal{M} .*

5.3. A criterion for minima. In view of Corollary 5.4 it is clearly of fundamental importance to have a criterion which allows one to identify the local minima of the function f . As has already been pointed out, the critical points of f are just the fixed points of the S^1 -action on $\mathcal{M}_d(G)$. The G -Higgs bundles corresponding to fixed points are the so-called *Hodge bundles* (see Hitchin [25, 26] and Simpson [36]), described in the following proposition.

Proposition 5.5. *A polystable G -Higgs bundle (E, φ) corresponds to a fixed point of the action of S^1 on $\mathcal{M}_d(G)$ if and only if (E, φ) is a Hodge bundle, i.e., there is a semi-simple element $\psi \in H^0(E(\mathfrak{h}))$ and decompositions*

$$E(\mathfrak{h}^{\mathbb{C}}) = \bigoplus_k E(\mathfrak{h}^{\mathbb{C}})_k,$$

$$E(\mathfrak{m}^{\mathbb{C}}) = \bigoplus_k E(\mathfrak{m}^{\mathbb{C}})_k$$

in eigen-bundles for ψ such that

$$\psi|_{E(\mathfrak{h}^{\mathbb{C}})_k} = ik \quad \text{and} \quad \psi|_{E(\mathfrak{m}^{\mathbb{C}})_k} = ik,$$

and, moreover, $[\psi, \varphi] = i\varphi$.

Notice that the condition $[\psi, \varphi] = i\varphi$ means that

$$\varphi \in H^0(E(\mathfrak{m}^{\mathbb{C}})_1 \otimes K).$$

Hence, if (E, φ) is a Hodge bundle as described in the preceding proposition, there is an induced decomposition of the deformation complex $C^\bullet(E, \varphi)$ defined in (5.1), as follows:

$$C^\bullet(E, \varphi) = \bigoplus_k C_k^\bullet(E, \varphi),$$

where for each k we define the complex

$$C_k^\bullet(E, \varphi): E(\mathfrak{h}^{\mathbb{C}})_k \xrightarrow{\text{ad}(\varphi)} E(\mathfrak{m}^{\mathbb{C}})_{k+1} \otimes K.$$

This decomposition gives us a corresponding decomposition of the infinitesimal deformation space of (E, φ) :

$$\mathbb{H}^1(C^\bullet(E, \varphi)) = \bigoplus_k \mathbb{H}^1(C_k^\bullet(E, \varphi)).$$

This decomposition is important because of the following result.

Proposition 5.6. *Let (E, φ) be a stable G -Higgs bundle which represents a non-singular point of $\mathcal{M}_d(G)$. If (E, φ) represents a fixed point of the S^1 -action, then the eigenspace $-k$ of the tangent space for the Hessian of f is isomorphic to $\mathbb{H}^1(C_k^\bullet(E, \varphi))$. In particular, (E, φ) corresponds to a local minimum of f if and only if*

$$\mathbb{H}^1(C_k^\bullet(E, \varphi)) = 0 \quad \forall k > 0.$$

A key result proved in [4] gives a very useful criterion for deciding when the hypercohomology $\mathbb{H}^1(C_k^\bullet(E, \varphi))$ vanishes. In order to state this result, it is convenient to use the Euler characteristic of the complex $C_k^\bullet(E, \varphi)$ defined by

$$\chi(C_k^\bullet(E, \varphi)) = \dim \mathbb{H}^0(C_k^\bullet(E, \varphi)) - \dim \mathbb{H}^1(C_k^\bullet(E, \varphi)) + \dim \mathbb{H}^2(C_k^\bullet(E, \varphi)).$$

Theorem 5.7 ([4, Proposition 4.14]). *Let (E, φ) be a semi-stable G -Higgs bundle, fixed under the action of S^1 . Then*

$$\chi(C_k^\bullet(E, \varphi)) \leq 0$$

and equality holds if and only if $\text{ad}(\varphi): E(\mathfrak{h}^{\mathbb{C}})_k \rightarrow E(\mathfrak{m}^{\mathbb{C}})_{k+1} \otimes K$ is an isomorphism.

Together with Proposition 5.2, Theorem 5.7 gives the promised criterion for finding the local minima of f .

Corollary 5.8. *Let (E, φ) be a stable G -Higgs bundle which represents a non-singular point of $\mathcal{M}_d(G)$. Then (E, φ) represents a local minimum of f if and only if*

$$\text{ad}(\varphi): E(\mathfrak{h}^{\mathbb{C}})_k \longrightarrow E(\mathfrak{m}^{\mathbb{C}})_{k+1} \otimes K$$

is an isomorphism for all $k > 0$.

6. MINIMA AND COUNTING OF COMPONENTS

6.1. Minima. It is the purpose of this section to describe the local minima of the function $f: \mathcal{M}_d(G) \rightarrow \mathbb{R}$ defined in (5.2) for these groups.

Recall from Remark 3.1 that there is no loss of generality in assuming that the Toledo invariant is positive. Thus, to all the results stated here for positive Toledo invariant, there are parallel results for negative Toledo invariant.

Since it creates no extra difficulties, we state the following Theorem for arbitrary (positive) Toledo invariant, even though we are presently only interested in the case of maximal Toledo invariant. The Theorem is proved by using the criterion given in Corollary 5.8, together with an extra argument to deal with strictly polystable G -Higgs bundles.

Theorem 6.1. *Let $(E, \varphi = \beta + \gamma)$ be a polystable G -Higgs bundle with positive Toledo invariant.*

- (1) *If G is one of the groups $\text{SU}(p, q)$, $\text{SO}^*(2n)$ and $\text{SO}_0(2, n)$ (with $n \neq 2, 3$) then $(E, \varphi = \beta + \gamma)$ represents a local minimum on $\mathcal{M}_d(G)$ if and only if $\beta = 0$.*
- (2) *Let $(E, \varphi = \beta + \gamma)$ be a polystable $\text{Sp}(2n, \mathbb{R})$ -Higgs bundle with $n \neq 2$ and let $(\mathbb{E}, \Phi) = (V \oplus V^*, \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})$ be the associated Higgs vector bundle. Then (E, φ) represents a local minimum of f if and only if one of the following situations occurs:*

- (a) *The vanishing $\beta = 0$ holds.*
 (b) *The number n is odd and there is a square root L of the canonical bundle K and a decomposition $V = LK^{-2[n/2]} \oplus LK^{-2[n/2]+2} \oplus \dots \oplus LK^{2[n/2]}$ with respect to which*

$$(6.1) \quad \gamma = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}.$$

In this case, necessarily the Toledo invariant is maximal, i.e. $\deg(V) = n(g-1)$.

- (c) *The number n is even and there is a square root L of the canonical bundle K and a decomposition $V = L^{-1}K^{2-n} \oplus L^{-1}K^{4-n} \oplus \dots \oplus L^{-1}K^n$ with respect to which β and γ are given by (6.1). Also in this case, necessarily we are in the situation of maximal Toledo invariant, $\deg(V) = n(g-1)$.*

Remark 6.2. The case of $G = \mathrm{SU}(1,1) \cong \mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SL}(2, \mathbb{R})$ is covered by both (1) and (2) of Theorem 6.1. This case (together with $G = \mathrm{SO}_0(2,1) \cong \mathrm{PSL}(2, \mathbb{R})$) was studied by Hitchin [25].

Remark 6.3. Recall from Remark 3.2 that we have excluded the group $\mathrm{SO}_0(2,2)$ from our considerations — in fact, the results for this group do not fit into the general statement given in Theorem 6.1.

Remark 6.4. The minima for the split real group $\mathrm{Sp}(2n, \mathbb{R})$ described in (2b) and (2c) of Theorem 6.1 are exactly the ones that belong to the Teichmüller components defined by Hitchin [26].

It remains to deal with the special cases $\mathrm{Sp}(4, \mathbb{R})$ and $\mathrm{SO}_0(2,3)$ (cf. Sec. 3.3).

Theorem 6.5. *Let G be one of the groups $\mathrm{Sp}(4, \mathbb{R})$ or $\mathrm{SO}_0(2,3)$ and let $(E, \varphi = \beta + \gamma)$ be a polystable G -Higgs bundle with positive Toledo invariant. Then (E, φ) represents a local minimum of f if and only if one of the following situations occurs:*

- (1) *The vanishing $\beta = 0$ holds.*
 (2) *If $G = \mathrm{SO}_0(2,3)$ and the associated Higgs vector bundle is $(V \oplus W, \Phi)$, then there are decompositions in line bundles*

$$V = K \oplus K^{-1} \quad \text{and} \quad W = M \oplus \mathcal{O} \oplus M^{-1},$$

with $0 < \deg(M) \leq 4g-4$. With respect to these decompositions,

$$Q_V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q_W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

γ is the canonical section 1 of $\mathrm{Hom}(\mathcal{O}, K^{-1}) \otimes K \cong \mathcal{O}$ while β is a non-zero section of $\mathrm{Hom}(M, K) \otimes K$, i.e.

$$\beta \in H^0(M^{-1}K^2).$$

In this case, necessarily the Toledo invariant is maximal, i.e., $d = 2g-2$.

- (3) *If $G = \mathrm{Sp}(4, \mathbb{R})$ and the associated Higgs vector bundle is $(V \oplus V^*, \Phi)$, then there is a decomposition in line bundles*

$$V = N \oplus N^{-1}K,$$

with $g-1 < \deg(N) \leq 3g-3$. With respect to this decomposition, $\gamma \in H^0(S^2V^* \otimes K)$ is given by the tautological section

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\beta \in H^0(S^2V \otimes K)$ is a non-vanishing section of the form

$$\beta = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\beta} \end{pmatrix} \quad \text{with} \quad \tilde{\beta} \in H^0(N^{-2}K^3)$$

Also in this case, necessarily the Toledo invariant is maximal, i.e., $d = 2g - 2$.

Remark 6.6. Following through the correspondence between $\mathrm{Sp}(4, \mathbb{R})$ -Higgs bundles and $\mathrm{SO}_0(2, 3)$ -Higgs bundles described in Sec. 3.3, one sees that a minimum for $\mathrm{Sp}(4, \mathbb{R})$ of the type given in (3) of Theorem 6.5 gives rise to a minimum of the type given in (2) of the theorem with $M = N^2K^{-1}$. Thus, in particular, $\deg(M)$ is even and, in fact, one can see that the second Stiefel–Whitney class $w_2(W, Q_W)$ is exactly the modulo 2 reduction of the degree of M , thus confirming (in the case of these minima) that this is the obstruction to lifting to $\mathrm{Sp}(4, \mathbb{R})$.

Remark 6.7. In the case $G = \mathrm{SO}_0(2, 3)$, the Cayley correspondence of Theorem 4.5 becomes particularly simple to describe for the $\mathrm{SO}_0(2, 3)$ -Higgs bundles which are local minima (of maximal Toledo invariant, of course): the kernel of γ is $F = M \oplus M^{-1}$ (cf. (4.3)) and the restriction of Q_W to F is clearly non-degenerate. Furthermore, $L_0 = L^{-1}K = K^{-1}K$ is trivial, $\theta'' = 0$ and $\theta': F \rightarrow K^2$ is given by $\beta \in H^0(M^{-1}K^2) = H^0(\mathrm{Hom}(M, K^2))$.

Remark 6.8. The minima for $\mathrm{SO}_0(2, 3)$ which belong to a Teichmüller component are of the type described in (2) of Theorem 6.5 with $\deg(M) = 4g - 4$. Note that, since $\beta \neq 0$, this forces $M = K^{-2}$. Thus there is a unique such minimum and this lifts to a minimum for $\mathrm{Sp}(4, \mathbb{R})$ because $\deg(M)$ is even (cf. Remark 6.6).

The minima for $\mathrm{Sp}(2n, \mathbb{R})$ which belong to a Teichmüller component are of the type described in (3) of the Theorem with $\deg(N) = 3g - 3$. Note that, since $\beta \neq 0$, this means that $N = K^{3/2}$. Hence we see that there are 2^{2g} such minima, corresponding to the choices of the square root of the canonical bundle K . Clearly each of these minima are lifts of the unique minimum for $\mathrm{SO}_0(2, 3)$.

6.2. The counting of components. In this section we give the count of the number of components of $\mathcal{M}_{\max}(G)$. Using Theorems 4.7 and 4.8, we can reduce the problem to the case where G/H is of tube type, i.e., when G is one of the groups $\mathrm{SU}(n, n)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(2, n)$ or $\mathrm{SO}^*(2n)$ with n even. The number of connected components of $\mathcal{M}_{\max}(G)$ is given in Table 2.

The general strategy for counting the components of $\mathcal{M}_{\max}(G)$ when G/H is of tube type is as follows:

- (1) Use the Cayley correspondence of Theorem 4.5 to obtain extra topological invariants, via the identification of $\mathcal{M}_{\max}(G)$ with the moduli space of K^2 -twisted G' -Higgs pairs. The relevant topological invariants can be read off Table 4 as those of bundles whose structure group is the maximal compact $H' \subseteq G'$. This

provides a subdivision

$$\mathcal{M}_{\max}(G) \cong \mathcal{M}_{K^2}(G') = \bigcup_c \mathcal{M}_{K^2,c}(G'),$$

according to the values of these topological invariants.

- (2) Use the results of Sec. 6.1 to identify the local minima of f on each of the subspaces $\mathcal{M}_{K^2,c}(G')$ defined in (1).
- (3) For each subspace $\mathcal{M}_{K^2,c}(G')$, determine whether the space of local minima is connected and non-empty: if this is the case, then $\mathcal{M}_{K^2,c}(G')$ is a connected component of \mathcal{M}_{\max} .
- (4) When the subspace of local minima of f on $\mathcal{M}_{K^2,c}(G')$ is not connected, find its connected components. It turns out that this only happens due to the presence of one or more Teichmüller components — in this case, non-Teichmüller components with the same invariants may or may not exist.

In the following we outline how this strategy is carried out for each of the groups mentioned above.

- $G = \mathrm{SU}(n, n)$:

Consulting Table 4 we see that the relevant topological invariants are those of $H' = \mathrm{SU}(n) \times \mathbb{Z}_2$ -bundles. Since $\pi_0(H') = \mathbb{Z}_2$ and $\pi_1(H') = \{1\}$, it follows that the invariant takes values in $H^1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. The corresponding minima are given by (1) of Theorem 6.1 as having $\beta = 0$. In terms of the Cayley correspondence (4.2) this means that the pair $(\widetilde{W}, \widetilde{\theta})$ has $\widetilde{\theta} = 0$. Hence the subspace of local minima is isomorphic to the moduli space³ of vector bundles \widetilde{W} such that $(\det \widetilde{W})^2 = \mathcal{O}$. This moduli space has a connected component for each of the 2^{2g} choices of square root of the trivial line bundle and it is not difficult to see that this choice corresponds exactly to the value of the topological invariant in $H^1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. This gives a total of 2^{2g} connected components of $\mathcal{M}_{\max}(G)$, as stated in Table 2. It is interesting to notice that this analysis is also valid for the case $\mathrm{SU}(1, 1) \cong \mathrm{Sp}(2, \mathbb{R})$; however it is somewhat special, in that each of the 2^{2g} components is a Teichmüller component (in fact isomorphic to Teichmüller space). This is obviously not the case when $n \neq 1$.

- $G = \mathrm{Sp}(2n, \mathbb{R})$:

In this case Table 4 shows that the relevant group is $G' = \mathrm{GL}(n, \mathbb{R})$. Thus the invariants are the first and second Stiefel–Whitney classes $w_1 \in H^1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ and $w_2 \in H^2(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$. For each of the possible values of (w_1, w_2) there are minima of the type given in (2a) of Theorem 6.1, i.e., with $\beta = 0$. Under the Cayley correspondence (4.1) these correspond to pairs (W, θ) with $\theta = 0$ (cf. Sec. 4.2) where W is an orthogonal bundle. Thus, for given (w_1, w_2) , the space of minima of this type can be identified with the moduli space of $\mathrm{O}(n, \mathbb{C})$ -bundles with these invariants. The moduli space of principal bundles for a *connected* group and fixed topological type is known to be connected by Ramanathan [33, Proposition 4.2]. Now, since $\mathrm{O}(n, \mathbb{C})$ is not connected the result of Ramanathan cannot be applied directly, however, all that is really required

³Here, and in the following, one must check that the various stability conditions involved in defining the moduli spaces agree. This is rather technical and we shall ignore this question in the present paper.

for his argument is that semistability is an open condition and thus one obtains the desired result (cf. [32]). Hence there are $2 \cdot 2^{2g} = 2^{2g+1}$ connected components of $\mathcal{M}_{\max}(\mathrm{Sp}(2n, \mathbb{R}))$ corresponding to minima with $\beta = 0$.

The case $n \geq 3$. In this case there are additional minima of the type described in (2b) and (2c) of Theorem 6.1. Each of these minima are easily seen to be the unique minimum of f on a Teichmüller component (it can be seen that they all have $w_2 = 0$, whereas $w_1 = 0$ when n is even and arbitrary for n odd). Since there is such a minimum for each square root of the canonical bundle, there are 2^{2g} Teichmüller components. Hence the total number of components is $3 \cdot 2^{2g}$.

The case $n = 2$. Here $G' = \mathrm{GL}(2, \mathbb{R})$ and the maximal compact subgroup is $H' = \mathrm{O}(2)$. One easily sees that the minima with $\beta \neq 0$, described in (3) of Theorem 6.5 all have $w_1 = 0$. Excluding the case $w_1 = 0$ we thus have $(2^{2g} - 1) \cdot 2 = 2^{2g+1} - 2$ possible values for the invariants, giving rise to the same number of components (the subspaces of minima are connected, as above). When the first Stiefel–Whitney class vanishes, there is a reduction of structure group to $\mathrm{SO}(2) \cong S^1$ and then the second Stiefel–Whitney class w_2 lifts to an integer invariant, namely the first Chern class c_1 of the S^1 -bundle. For the case of the minima with $\beta = 0$, we have $c_1 = 0$ and again the space is connected. Hence, corresponding to minima with $\beta = 0$, there are in total $2^{2g+1} - 1$ components. For the minima with $\beta \neq 0$ described in (3) of Theorem 6.5 the value is $c_1 = \deg(N) - (g - 1)$. It follows from the bound $g - 1 < \deg(N) \leq 3g - 3$ that the corresponding subspace is non-empty only for $c_1 \leq 2g - 2$. For each c_1 satisfying $0 < c_1 < 2g - 2$ it can be proved that the subspace of minima is connected, thus showing that there is a unique (non-Teichmüller, in fact) component for this value. When $c_1 = 2g - 2$ the minima belong to a Teichmüller component and there are 2^{2g} of these (again depending on the choice of a square root of the canonical bundle). The total number of components is thus $2^{2g+1} - 1 + (2g - 3) + 2^{2g} = 3 \cdot 2^{2g} + 2g - 4$.

The case $n = 1$. Since $\mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$ this case has been treated above but, it can of course also be seen directly that $\mathcal{M}_{\max}(\mathrm{Sp}(2, \mathbb{R}))$ has 2^{2g} Teichmüller-components.

- $G = \mathrm{SO}_0(2, n)$:

The case $n \geq 4$. In this case the group giving rise to the extra invariants is $H' = \mathrm{O}(n - 1)$. Thus the new invariants are again Stiefel–Whitney classes w_1 and w_2 . From (1) of Theorem 6.1 we know that the only minima are the ones with $\beta = 0$ so, exactly as explained for $\mathrm{Sp}(2n, \mathbb{R})$ above, the subspace of minima with given (w_1, w_2) can be identified with the moduli space of $\mathrm{O}(n - 1, \mathbb{C})$ -bundles, which is connected. This gives the total of $2 \cdot 2^{2g} = 2^{2g+1}$ components of $\mathcal{M}_{\max}(G)$.

The case $n = 3$. Here we have $G' = \mathrm{SO}(1, 1) \times \mathrm{SO}(1, 2)$ with maximal compact subgroup $H' = \mathrm{O}(2)$ and the analysis is parallel to the one given above for $G = \mathrm{Sp}(4, \mathbb{R})$. The invariants are (w_1, w_2) and all minima with $w_1 \neq 0$ have $\beta = 0$. When $w_1 = 0$, the class w_2 lifts to an integer class c_1 and for $c_1 = 0$ there is one component whose minima have $\beta = 0$. The remaining components have minima with $\beta \neq 0$ of the type described in (2) of Theorem 6.5 and the invariant is $c_1 = \deg(M)$. For each allowed value $0 < \deg(M) \leq 4g - 4$ there is one connected component (which lifts to $\mathrm{Sp}(4, \mathbb{R})$ when $\deg(M)$ is even). The Teichmüller component occurs for $\deg(M) = 4g - 4$; in this

case there is just one, which is covered by the 2^{2g} projectively equivalent components for $\mathrm{Sp}(4, \mathbb{R})$. Thus the total number of components is $2^{2g+1} - 1 + 4g - 4 = 2^{2g+1} + 4g - 5$

Remark 6.9. The maximal compact subgroup of $\mathrm{SO}_0(2, n)$ is $H = \mathrm{SO}(2) \times \mathrm{SO}(n)$. Hence $\pi_1(H) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ if $n \geq 3$. There is thus a second invariant attached to an $\mathrm{SO}_0(2, n)$ -Higgs bundle (an element in \mathbb{Z}_2) in addition to the Toledo invariant. This can be identified as the second Stiefel–Whitney class of the $\mathrm{SO}(n, \mathbb{C})$ -bundle (W, Q_W) in Table 3. We could have used this invariant to distinguish two disjoint closed subspaces in $\mathcal{M}_{\max}(\mathrm{SO}_0(2, n))$, but this is taken into account by the Cayley correspondence.

The case $n = 1$. In this case $G = \mathrm{SO}_0(2, 1) \cong \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{PSp}(2, \mathbb{R})$ and there is a unique minimum with $\beta = 0$. This belongs to the unique Teichmüller component and lifts to the 2^{2g} projectively equivalent Teichmüller components for $G = \mathrm{SL}(2, \mathbb{R})$.

- $G = \mathrm{SO}^*(2n)$, with $n = 2m$:

In this case Theorem 6.1 shows that all minima have $\beta = 0$ and so from the Cayley correspondence (cf. Sec. 4.2) we deduce that the subspace of minima on $\mathcal{M}_{\max}(G)$ can be identified with the moduli space of principal $H^{\mathbb{C}}$ -bundles. Since H' is the simply connected group $\mathrm{Sp}(n)$ (see Table 4), it follows from Ramanathan [33] that the space of minima is connected, thus showing that $\mathcal{M}_{\max}(G)$ is also connected.

We now study the case when G/H is not of tube type.

- $G = \mathrm{SU}(p, q)$ with $p \neq q$:

Without loss of generality we assume that $p < q$. From Theorem 4.7 we see that $\mathcal{M}_{\max}(G)$ is isomorphic to the moduli space \mathcal{M}' of G -Higgs bundles of the form (4.4) with $\deg(V) = p(g - 1)$ and

$$\det(V) \otimes \det(W') \otimes (W'') \cong \mathcal{O},$$

which means, as already noted, that every maximal $\mathrm{SU}(p, q)$ -Higgs bundle reduces to an $\mathrm{S}(\mathrm{U}(p, p) \times \mathrm{U}(q - p))$ -Higgs bundle.

Define a map from \mathcal{M}' to the Jacobian parameterizing line bundles of degree $p(g - 1)$ by

$$\begin{aligned} \mathcal{M}' &\rightarrow J^{p(g-1)}(X) \\ (V, W', W'', \beta, \gamma) &\mapsto \det(V) \end{aligned}$$

The fibre of this map over a line bundle L is a product of the moduli space

$$M_{L^{-2}K^p}(\mathrm{GL}(q - p, \mathbb{C}))$$

of polystable bundles W'' of rank $q - p$ and fixed determinant $L^{-2}K^p$ and the moduli space

$$\widetilde{\mathcal{M}} = \{(V, W', \beta, \gamma) : \det(V) = L\}$$

which is a subspace of the moduli space $\mathcal{M}_{L^2K^{-p}}(\mathrm{U}(p, p))$ of (maximal) $\mathrm{U}(p, p)$ -Higgs bundles with fixed determinant L^2K^{-p} . Clearly $\mathcal{M}_{L^2K^{-p}}(\mathrm{U}(p, p))$ is isomorphic to the moduli space $\mathcal{M}_{\max}(\mathrm{SU}(p, p))$ and, similarly to this case, $\mathcal{M}_{L^2K^{-p}}(\mathrm{U}(p, p))$ has 2^{2g} connected components, indexed by the square roots of L^2K^{-p} . Fixing $\det(V)$ obviously amounts to fixing one of these square roots and hence $\widetilde{\mathcal{M}}$ is just one of these connected

components. The map to the Jacobian defined above is surjective since, if we are given another line bundle $L \otimes L_0$, the map

$$(V, W', W'', \beta, \gamma) \mapsto (V \otimes L_0^{1/p}, W' \otimes L_0^{1/p}, W'' \otimes L_0^{-2/(q-p)}, \beta, \gamma)$$

gives an isomorphism between the fibre over L and the fibre over $L \otimes L_0$. Hence, the connectedness of the Jacobian, of $M_{L^{-2}K^p}(\mathrm{GL}(q-p, \mathbb{C}))$ and of $\widetilde{\mathcal{M}}$ imply that \mathcal{M}' is connected and, consequently, $\mathcal{M}_{\max}(G)$ is connected.

- $G = \mathrm{SO}^*(2n)$, with $n = 2m + 1$:

From Theorem 4.8 and the fact that the Jacobian $J(X)$ is connected, we have that $\mathcal{M}(\mathrm{SO}^*(4m+2))$ is connected since, as we have seen, $\mathcal{M}(\mathrm{SO}^*(4m))$ is connected.

7. TABLES

G	H	$H^{\mathbb{C}}$	$\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}_+ + \mathfrak{m}_-$
$SU(p, q)$	$S(U(p) \times U(q))$	$S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$	$\text{Hom}(\mathbb{C}^q, \mathbb{C}^p) + \text{Hom}(\mathbb{C}^p, \mathbb{C}^q)$
$Sp(2n, \mathbb{R})$	$U(n)$	$GL(n, \mathbb{C})$	$S^2(\mathbb{C}^n) + S^2(\mathbb{C}^{n*})$
$SO^*(2n)$	$U(n)$	$GL(n, \mathbb{C})$	$\Lambda^2(\mathbb{C}^n) + \Lambda^2(\mathbb{C}^{n*})$
$SO_0(2, n)$	$SO(2) \times SO(n)$	$SO(2, \mathbb{C}) \times SO(n, \mathbb{C})$	$\text{Hom}(\mathbb{C}^n, \mathbb{C}) + \text{Hom}(\mathbb{C}, \mathbb{C}^n)$

 TABLE 1. Irreducible classical Hermitian symmetric spaces G/H

G	$\#\pi_0(\mathcal{M}_{\max}(G))$	Teichmüller components	Reference
$SU(n, n)$	2^{2g}	– (2^{2g} if $n = 1$)	[4, 30]
$SU(p, q) \quad (p \neq q)$	1	–	[4, 44, 45]
$Sp(2n, \mathbb{R}) \quad (n \geq 3)$	$3 \cdot 2^{2g}$	2^{2g}	[18]
$SO_0(2, n) \quad (n \geq 4)$	2^{2g+1}	–	[5]
$SO^*(2n)$	1	–	[5]
$Sp(4, \mathbb{R}) \cong Spin_0(2, 3)$	$3 \cdot 2^{2g} + 2g - 4$	2^{2g}	[22]
$SO_0(2, 3)$	$2^{2g+1} + 4g - 5$	1	[5]
$Sp(2, \mathbb{R}) \cong SL(2, \mathbb{R})$	2^{2g}	2^{2g}	[21, 25]
$SO_0(2, 1) \cong PSL(2, \mathbb{R})$	1	1	[21, 25]

 TABLE 2. Components of $\mathcal{M}_{\max}(G)$

G	$SU(p, q)$	$Sp(2n, \mathbb{R})$	$SO^*(2n)$	$SO_0(2, n)$
(E, φ) $\varphi = \beta + \gamma$	V : rank p bundle W : rank q bundle $\det V \otimes \det W = \mathcal{O}$ $\beta \in H^0(\text{Hom}(W, V) \otimes K)$ $\gamma \in H^0(\text{Hom}(V, W) \otimes K)$	V : rank n bundle $\beta \in H^0(S^2V \otimes K)$ $\gamma \in H^0(S^2V^* \otimes K)$	V : rank n bundle $\beta \in H^0(\Lambda^2V \otimes K)$ $\gamma \in H^0(\Lambda^2V^* \otimes K)$	$(V = L \oplus L^{-1}, Q_V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ (W, Q_W) : rank n orthogonal bundle L : line bundle; $\det W = \mathcal{O}$ $\beta \in H^0(\text{Hom}(W, L) \otimes K)$ $\gamma \in H^0(\text{Hom}(W, L^{-1}) \otimes K)$
$G^{\mathbb{C}} \subset SL(N, \mathbb{C})$ $\mathbb{E} = E(\mathbb{C}^N)$ $\Phi \in H^0(\text{End } \mathbb{E} \otimes K)$	$\mathbb{E} = V \oplus W$ $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$	$\mathbb{E} = V \oplus V^*$ $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$	$\mathbb{E} = V \oplus V^*$ $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$	$\mathbb{E} = V \oplus W$ $\Phi = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \gamma \\ -\gamma^t & -\beta^t & 0 \end{pmatrix}$
Toledo invariant	$d = \deg V = -\deg W$	$d = \deg V$	$d = \deg V$	$d = \deg L$
Milnor–Wood inequality $ d \leq d_{\max}$	$ d \leq \min\{p, q\}(g - 1)$	$ d \leq n(g - 1)$	$ d \leq [\frac{n}{2}](2g - 2)$	$ d \leq 2g - 2$

TABLE 3. Higgs bundles for irreducible classical symmetric spaces G/H

G	H	G'	H'	$\check{S} = H/H'$	\mathfrak{m}'	$\mathfrak{m}'^{\mathbb{C}}$
$SU(n, n)$	$S(U(n) \times U(n))$	$\{A \in GL(n, \mathbb{C}) : \det(A)^2 = 1\}$ $\cong SL(n, \mathbb{C}) \rtimes \mathbb{Z}_2$	$\{A \in U(n) : \det(A)^2 = 1\}$ $\cong SU(n) \rtimes \mathbb{Z}_2$	$U(n)$	$\text{Herm}(n, \mathbb{C})$	$\text{Mat}(n, \mathbb{C})$
$Sp(2n, \mathbb{R})$	$U(n)$	$GL(n, \mathbb{R})$	$O(n)$	$U(n)/O(n)$	$\text{Sym}(n, \mathbb{R})$	$\text{Sym}(n, \mathbb{C})$
$SO^*(2n), n = 2m$	$U(n)$	$U^*(n)$	$Sp(n)$	$U(n)/Sp(n)$	$\text{Herm}(m, \mathbb{H})$	$\text{Skew}(n, \mathbb{C})$
$SO_0(2, n)$	$SO(2) \times SO(n)$	$SO_0(1, 1) \times SO(1, n-1)$	$O(n-1)$	$\frac{U(1) \times S^{n-1}}{\mathbb{Z}_2}$	$\mathbb{R} \times \mathbb{R}^{n-1}$	$\mathbb{C} \times \mathbb{C}^{n-1}$

 TABLE 4. Irreducible classical tube type Hermitian symmetric spaces G/H

G	H	H'	\tilde{G}	\tilde{H}	\tilde{H}'	$H'' = H'/\tilde{H}'$
$SU(p, q), p < q$	$S(U(p) \times U(q))$	$\{(A, B) \in U(p) \times U(q-p) : \det(A)^2 \det(B) = 1\}$ $\cong S(U(p) \times U(q-p)) \rtimes \mathbb{Z}_2$	$SU(p, p)$	$S(U(p) \times U(p))$	$SU(p) \rtimes \mathbb{Z}_2$	$S(U(1) \times U(q-p))$
$SO^*(4m+2)$	$U(2m+1)$	$Sp(2m) \times U(1)$	$SO^*(4m)$	$U(2m)$	$Sp(2m)$	$U(1)$

 TABLE 5. Irreducible classical non-tube type Hermitian symmetric spaces G/H

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