MOMENTUM AND ENERGY PRESERVING INTEGRATORS FOR NONHOLONOMIC DYNAMICS

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Abstract. In this paper, we propose a geometric integrator for nonholonomic mechanical systems. It can be applied to discrete Lagrangian systems specified through a discrete Lagrangian $L_d\colon Q\times Q\to\mathbb{R}$, where Q is the configuration manifold, and a (generally nonintegrable) distribution $\mathcal{D}\subset TQ$. In the proposed method, a discretization of the constraints is not required. We show that the method preserves the discrete nonholonomic momentum map, and also that the nonholonomic constraints are preserved in average. We study in particular the case where Q has a Lie group structure and the discrete Lagrangian and/or nonholonomic constraints have various invariance properties, and show that the method is also energy-preserving in some important cases.

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1. Introduction

During the last years, there has been an increasing interest in nonholonomic mechanical systems, in part motivated by some open questions in the subject, such as those concerning reduction, integrability, stabilization or controllability; and also for their applicability in engineering, specially in robotics, mainly since it describes the motion of wheeled devices (see [2, 3, 11] and the expository paper [4]).

When a mechanical system is subjected to some external constraints, the latter may be expressed in terms of relations imposing restrictions on the allowable positions and velocities. The constraints are then called *nonholonomic* if the velocity dependence is essential, in the sense that the constraint relations can not be reduced, by integration, to relations depending on the position coordinates only. Geometrically, nonholonomic constraints are globally described by a submanifold $\mathcal{D}$ of the velocity phase space TQ. In most of the known examples $\mathcal{D}$ is a vector subbundle of TQ, i.e., the constraints have a linear dependence on the velocities. Lagrange–d'Alembert's principle allow us to determine the set of possible values of the constraint forces from the constraint manifold $\mathcal{D}$. Then, to determine the dynamics of the nonholonomic system, it is only necessary to fix initially the pair $(L, \mathcal{D})$, where $L\colon TQ \to \mathbb{R}$ is a Lagrangian function, usually of mechanical type (see [2, 6] and [8] for an extension of the classical Lagrange–d'Alembert's principle).

Very recently, many authors [10, 12, 13, 16, 21] started the study of geometric integrators adapted to nonholonomic systems, obtaining very stable numerical integrators with some preservation properties (such as discrete nonholonomic momentum map preservation) and very good energy behavior. This problem is of considerable interest given the crucial role of nonholonomic dynamics in many applications in engineering. From the numerical point of view, in [22] it appeared as an open question: "...The problem for the more general class of non-holonomic constraints is still open, as is the question of the correct analogue of symplectic integration for non-holonomically constrained Lagrangian systems...".

The most interesting approach to nonholonomic integrators appears as an adaptation of the so-called variational integrators [20] incorporating a discrete constraint submanifold, in addition to a discretization of the Lagrangian function and the vector subbundle $\mathcal{D}$. Then, the numerical method is obtained from the so-called Discrete Lagrange–d'Alembert's principle [10], recovering many of the geometric properties of the continuous system.

Obviously, since nonholonomic mechanics is not symplectic-preserving, it seems interesting to try to preserve another geometric invariance property of the continuous nonholonomic system, as for instance, the energy function in the autonomous case. This is precisely the starting point of view of our paper. Moreover, a discretization of the constraints is not required here. We show that the method preserves the discrete nonholonomic momentum map, and also that the constraints are preserved in average. We study in particular the case where the configuration space is a Lie group and the discrete Lagrangian and/or nonholonomic constraints have various invariance properties, and show that the method is also energy-preserving in many important cases. In particular, the main result of the paper, Theorem 1, states that if the configuration space is a Lie group and the Lagrangian is defined by a bi-invariant Riemannian metric, then, from a left-invariant discretization of the Lagrangian, we obtain a ***fixed time-step, energy-preserving numerical method*** for the continuous nonholonomic system, without invariance conditions on $\mathcal{D}$. See [9] for a variable time-step algorithm that preserves energy.

The paper is structured as follows. In Section 2, we introduce continuous nonholonomic mechanical systems for the case of mechanical energy Lagrangians defined by a given Riemannian metric and a potential function. In this case, the equations of motion for the constrained system are geodesic equations for an affine connection (in the kinetic case) that is not generally Levi-Civita, obtained from the induced orthogonal projection onto the nonholonomic distribution (see [7, 17]). In Section 3 we recall some definitions concerning discrete variational mechanics (discrete Lagrangian, discrete Euler–Lagrange equations, discrete flow, momentum map...). The new proposed method appears in Section 4, constructed from the discrete Lagrangian and the orthogonal projectors induced by the distribution $\mathcal{D}$ and the Riemannian metric. Then we consider the case when the configuration space is a Lie group and we obtain under adequate invariance properties the preservation of energy. In addition, we study the momentum nonholonomic map for the proposed nonholonomic integrator. In the last section we test our method in three examples (the nonholonomic particle, the snakeboard and the Chaplygin sleigh).

2. Continuous Nonholonomic mechanics

We shall start with a configuration space Q, which is an n-dimensional differentiable manifold with local coordinates (q^i), $1 \leq i \leq n = \dim Q$. Constraints linear in the velocities are given by equations of the form

$$\phi^a(q^i, \dot{q}^i) = \mu_i^a(q)\dot{q}^i = 0, \quad 1 \leq a \leq m,$$

depending, in general, on configuration coordinates and their velocities. From an intrinsic point of view, the linear constraints are defined by a distribution $\mathcal{D}$ on Q of rank $n - m$ such that the annihilator of $\mathcal{D}$ is locally given by

$$\mathcal{D}^o = \operatorname{span}\left\{\mu^a = \mu_i^a dq^i \; ; 1 \leq a \leq m\right\}$$

where the one-forms μ^a are independent.

The various kinds of constraints we are concerned with will roughly come in two types: ***holonomic*** and ***nonholonomic***, depending on whether the constraint is derived from a constraint in the configuration space or not. Therefore, the dimension of the space of configurations is reduced by holonomic constraints but not by nonholonomic constraints. Thus, holonomic constraints permit a reduction in the number of coordinates of the configuration space needed to formulate a given problem (see [23]).

We will restrict ourselves to the case of nonholonomic constraints. In this case, the constraints are given by a nonintegrable distribution $\mathcal{D}$. Fixed these constraints, we need to specify the dynamical evolution of the system, usually by fixing a Lagrangian function $L\colon TQ \to \mathbb{R}$. In mechanics, the central concepts permitting the extension of mechanics from the Newtonian point of view to the Lagrangian one are the notions of virtual displacements and virtual work; these concepts were formulated in the developments of mechanics, in their application to statics. In nonholonomic dynamics, the procedure is given by the ***Lagrange–d'Alembert principle***. This principle allows us to determine the set of possible values of the constraint forces from the set $\mathcal{D}$ of admissible kinematic states alone. The resulting equations of motion are

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right]\delta q^i = 0,$$

where δq^i denotes the virtual displacements verifying

$$\mu_i^a \delta q^i = 0$$

(for the sake of simplicity, we will assume that the system is not subject to nonconservative forces). This must be supplemented by the constraint equations. By using the Lagrange multiplier rule, we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \mu_i^a.$$

The term on the right represents the constraint force or reaction force induced by the constraints. The functions λ_a are Lagrange multipliers which, after being computed using the constraint equations, allow us to obtain a set of second order differential equations.

Now we restrict ourselves to the case of nonholonomic mechanical systems where the Lagrangian is of mechanical type

$$L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q), \quad v_q \in T_qQ,$$

where g is a Riemannian metric on the configuration space Q. Locally, the metric is determined by the matrix $M = (g_{ij})_{1\leq i,j\leq n}$ where $g_{ij} = g(\partial/\partial q^i, \partial/\partial q^j)$.

Using some basic tools of Riemannian geometry, we may write the equations of motion of the unconstrained system as

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\text{grad } V(c(t)), \tag{1}$$

where ∇ is the Levi–Civita connection associated to g. Observe that if $V \equiv 0$ then the Euler–Lagrangian equations are the equations of the geodesics for the Levi-Civita connection.

When the system is subjected to nonholonomic constraints, the equations become

$$\nabla_{\dot{c}(t)}\dot{c}(t) = -\text{grad } V(c(t)) + \lambda(t), \quad \dot{c}(t) \in \mathcal{D}_{c(t)},$$

where λ is a section of $\mathcal{D}^\perp$ along c. Here $\mathcal{D}^\perp$ stands for the orthogonal complement of $\mathcal{D}$ with respect to the metric g.

In coordinates, by defining the n^3 functions Γ^k_{ij} (Christoffel symbols for ∇) by

$$\nabla_{\frac{\partial}{\partial q^i}} \frac{\partial}{\partial q^j} = \Gamma^k_{ij}\frac{\partial}{\partial q^k},$$

we may rewrite the nonholonomic equations of motion as

$$\begin{aligned} \ddot{q}^k(t) + \Gamma^k_{ij}(c(t))\dot{q}^i(t)\dot{q}^j(t) &= -g^{ki}(c(t))\frac{\partial V}{\partial q^i} + \lambda_a(t)g^{ki}(c(t))\mu^a_i(c(t)) \\ \mu^a_i(c(t))\dot{q}^i(t) &= 0 \end{aligned}$$

where $t \mapsto (q^1(t), \dots, q^n(t))$ is the local representative of c and (g^{ij}) is the inverse matrix of M.

Since g is a Riemannian metric, the $m \times m$ matrix $(C^{ab}) = (\mu^a_i g^{ij} \mu^b_j)$ is symmetric and regular. Define now the vector fields Z^a, $1 \le a \le m$ on Q by

$$g(Z^a, Y) = \mu^a(Y), \quad \text{for all vector fields } Y,\ 1 \le a \le m;$$

that is, Z^a is the gradient vector field of the 1-form μ^a. Thus, $\mathcal{D}^\perp$ is spanned by Z^a, $1 \le a \le m$. In local coordinates, we have

$$Z^a = g^{ij}\mu^a_i \frac{\partial}{\partial q^j}.$$

We can construct two complementary projectors

$$\begin{aligned} \mathcal{P}&\colon TQ \to \mathcal{D} \\ \mathcal{Q}&\colon TQ \to \mathcal{D}^\perp, \end{aligned}$$

orthogonal with respect to the metric g. The projector $\mathcal{Q}$ is locally described by

$$\mathcal{Q} = C_{ab} Z^a \otimes \mu^b = C_{ab} g^{ij} \mu^a_i \mu^b_k \frac{\partial}{\partial q^j} \otimes dq^k.$$

Using these projectors we may rewrite the equations of motion as follows. A curve $c(t)$ is a motion for the nonholonomic system if it satisfies the constraints, i.e., $\dot{c}(t) \in \mathcal{D}_{c(t)}$, and, in addition, the "projected equation of motion"

$$\mathcal{P}(\nabla_{\dot{c}(t)}\dot{c}(t)) = -\mathcal{P}(\text{grad } V(c(t))) \tag{2}$$

is fulfilled.

Summarizing, we have obtained the dynamics of the nonholonomic system (2) applying the projector $\mathcal{P}$ to the dynamics of the free system (1). In Section 4, we will use $\mathcal{P}$ and $\mathcal{Q}$ to obtain a geometric integrator for nonholonomic systems.

3. Variational Integrators

The equations of motion for a Lagrangian system given by a Lagrangian function $L\colon TQ \to \mathbb{R}$ are the well-known Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad 1 \le i \le n.$$

It is well known that the origin of these equations is variational (see [1]). Now, variational integrators retain this variational character and also some of the geometric properties of the continuous system, such as symplecticity and momentum conservation (see [14, 20] and references therein).

In the following we will summarize the main features of this type of numerical integrators. A ***discrete Lagrangian*** is a map $L_d\colon Q \times Q \to \mathbb{R}$, which may be considered as an approximation of a continuous Lagrangian $L\colon TQ \to \mathbb{R}$. Define the ***action sum*** $S_d\colon Q^{N+1} \to \mathbb{R}$ corresponding to the Lagrangian L_d by

$$S_d = \sum_{k=1}^{N} L_d(q_{k-1}, q_k),$$

where $q_k \in Q$ for $0 \le k \le N$. The discrete variational principle states that the solutions of the discrete system determined by L_d must extremize the action sum given fixed endpoints q_0 and q_N. By extremizing S_d over q_k, $1 \le k \le N-1$, we obtain the system of difference equations

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0. \tag{3}$$

or, in coordinates,

$$\frac{\partial L_d}{\partial q_1^i}(q_k, q_{k+1}) + \frac{\partial L_d}{\partial q_2^i}(q_{k-1}, q_k) = 0, \ 1 \le i \le n, \ 1 \le k \le N-1.$$

These equations are usually called the ***discrete Euler–Lagrange equations***. Under some regularity hypotheses (the matrix $(D_{12}L_d(q_k, q_{k+1}))$ is regular), it is possible to define a (local) discrete flow $\Upsilon\colon Q \times Q \to Q \times Q$, by $\Upsilon(q_{k-1}, q_k) = (q_k, q_{k+1})$ from (3). Define the discrete Legendre transformations associated to L_d as

$$\begin{aligned} \mathbb{F}^- L_d\colon Q \times Q &\to T^*Q \\ (q_0, q_1) &\longmapsto (q_0, -D_1 L_d(q_0, q_1)) \\ \mathbb{F}^+ L_d\colon Q \times Q &\to T^*Q \\ (q_0, q_1) &\longmapsto (q_1, D_2 L_d(q_0, q_1)) \ , \end{aligned}$$

and the discrete Poincaré–Cartan 2-form $\omega_d = (\mathbb{F}^+ L_d)^* \omega_Q = (\mathbb{F}^- L_d)^* \omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. The discrete algorithm determined by Υ preserves the symplectic form ω_d, i.e., $\Upsilon^* \omega_d = \omega_d$. Moreover, if the discrete Lagrangian is invariant under the diagonal action of a Lie group G, then the discrete momentum map $J_d\colon Q \times Q \to \mathfrak{g}^*$ defined by

$$\langle J_d(q_k, q_{k+1}), \xi \rangle = \langle D_2 L_d(q_k, q_{k+1}), \xi_Q(q_{k+1}) \rangle$$

is preserved by the discrete flow. Therefore, these integrators are symplectic-momentum preserving. Here, ξ_Q denotes the fundamental vector field determined by $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.

4. A geometric nonholonomic integrator

This work proposes a numerical method for the integration of nonholonomic systems. It is not truly variational; however, it is geometric in nature and we show in Corollary 3 that it preserves the discrete nonholonomic momentum map in the presence of horizontal symmetries. Moreover, we prove in Theorem 1 that under certain symmetry conditions, the energy of the system is preserved.

Consider a discrete Lagrangian $L_d \colon Q \times Q \to \mathbb{R}$. The proposed discrete nonholonomic equations are

$$\mathcal{P}^*_{q_k}(D_1 L_d(q_k, q_{k+1})) + \mathcal{P}^*_{q_k}(D_2 L_d(q_{k-1}, q_k)) = 0 \tag{4a}$$

$$\mathcal{Q}^*_{q_k}(D_1 L_d(q_k, q_{k+1})) - \mathcal{Q}^*_{q_k}(D_2 L_d(q_{k-1}, q_k)) = 0, \tag{4b}$$

where the subscript q_k emphasizes the fact that the projections take place in the fiber over q_k. The first equation is the projection of the discrete Euler–Lagrange equations to the constraint distribution $\mathcal{D}$, while the second one can be interpreted as an elastic impact of the system against $\mathcal{D}$ (see [15]). This is what will provide the preservation of energy. Note that we can combine both equations into

$$D_1 L_d(q_k, q_{k+1}) + (\mathcal{P}^* - \mathcal{Q}^*) D_2 L_d(q_{k-1}, q_k) = 0,$$

from which we see that the system defines a unique discrete evolution operator if and only if the matrix $(D_{12} L_d)$ is regular, that is, if the discrete Lagrangian is regular. Locally, the method can be written as

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = \lambda_b \mu^b \tag{5a}$$

$$g^{ij}(q_k)\mu^a_i(q_k)\left(\frac{\partial L_d}{\partial q^j_1}(q_k, q_{k+1}) - \frac{\partial L_d}{\partial q^j_2}(q_{k-1}, q_k)\right) = 0. \tag{5b}$$

Using the discrete Legendre transformations defined above, define the pre- and post-momenta, which are covectors at q_k, by

$$p^+_{k-1,k} = p^+(q_{k-1}, q_k) = \mathbb{F}^+ L_d(q_{k-1}, q_k) = D_2 L_d(q_{k-1}, q_k)$$

$$p^-_{k,k+1} = p^-(q_k, q_{k+1}) = \mathbb{F}^- L_d(q_k, q_{k+1}) = -D_1 L_d(q_k, q_{k+1}).$$

In these terms, equation (5b) can be rewritten as

$$g^{ij}(q_k)\mu^a_i(q_k)\left(\frac{(p^-_{k,k+1})_j + (p^+_{k-1,k})_j}{2}\right) = 0$$

which means that the average of post- and pre-momenta satisfies the constraints. In this sense the proposed numerical method also preserves the nonholonomic constraints.

We may rewrite the discrete nonholonomic equations as

$$p^-_{k,k+1} = (\mathcal{P} - \mathcal{Q})^*_{q_k}(p^+_{k-1,k}). \tag{6}$$

We interpret this equation as a jump of momenta during the nonholonomic evolution. Compare this with the condition $p^-_{k,k+1} = p^+_{k-1,k}$ imposed by the discrete Euler–Lagrange equations (that is, for unconstrained systems). In our method, the momenta are related by a reflection with respect to the image of the projector $\mathcal{P}^* \colon T^*Q \to (\mathcal{D}^\perp)^o$. This is illustrated, in the context of Section 4.1, in figure 1.

4.1. **Left-invariant discrete Lagrangians on Lie groups.** Consider a discrete nonholonomic Lagrangian system on a Lie group G, with a discrete Lagrangian $L_d \colon G \times G \to \mathbb{R}$ that is invariant with respect to the left diagonal action of G on $G \times G$ (see [5, 19]). We do not impose yet any invariance conditions on the distribution $\mathcal{D}$. If we write $W_k = g_k^{-1} g_{k+1}$, then we can define the reduced discrete Lagrangian $l_d \colon G \to \mathbb{R}$ as $l_d(W_k) = L_d(g_k, g_{k+1})$. Note that $Dl_d(W_k) \in T^*_{W_k} G$.

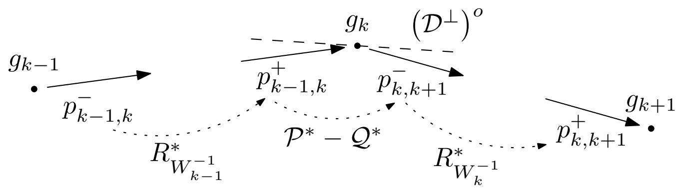

FIGURE 1. Evolution of momenta, depicted here as solid arrows. The right translations are a consequence of the left-invariance of L_d, and the reflection at g_k is the proposed method.

Computing the derivative, we obtain

$$p^-_{k,k+1} = -D_1 L_d(g_k, g_{k+1}) = L^*_{g_k^{-1}} R^*_{W_k} Dl_d(W_k),$$

where L^* and R^* are the mappings on T^*G induced by left and right multiplication on the group, respectively (this should not be confused with the Lagrangian L). We use this to write

$$p^+_{k,k+1} = D_2 L_d(g_k, g_{k+1}) = L^*_{g_k^{-1}} Dl_d(W_k) = L^*_{g_k^{-1}} R^*_{W_k^{-1}} L^*_{g_k} p^-_{k,k+1} = R^*_{W_k^{-1}} p^-_{k,k+1}.$$

Therefore, the discrete nonholonomic equations (6) become

$$p^-_{k,k+1} = (\mathcal{P} - \mathcal{Q})^* \left(R^*_{W^{-1}_{k-1}} p^-_{k-1,k} \right). \tag{7}$$

The relationships between the pre- and post-momenta are depicted in figure 1.

Note that we do not need here that the metric used to build the projectors is the metric giving the kinetic energy in the Lagrangian.

4.2. **Left-invariant Lagrangian and projectors.** Take a left-invariant discrete Lagrangian $L_d \colon G \times G \to \mathbb{R}$ as in the previous section, and assume that $\mathcal{D}$ and $\mathcal{D}^\perp$ are left-invariant. This is typically a consequence of $\mathcal{D}$ and the metric on G being left-invariant, although it can be assured by weaker conditions on the metric (preserving the orthogonality of $\mathcal{D}_e$ and $\mathcal{D}_e^\perp$ by left translations). This is equivalent to the left-invariance of the projectors $\mathcal{P}$ and $\mathcal{Q}$, which in turn is equivalent to the left-invariance of $\mathcal{P} - \mathcal{Q}$, as a straightforward verification shows.

Since our goal is to rewrite equation (7) on the dual $\mathfrak{g}^*$ of the Lie algebra, we define the discrete body momentum $p_k \colon G \times G \to \mathfrak{g}^*$ as

$$p_k = L^*_{g_k} p^-_{k,k+1},$$

which agrees with the definition in [13]. Then (7) reads

$$L^*_{g_k^{-1}} p_k = (\mathcal{P} - \mathcal{Q})^* \left(R^*_{W^{-1}_{k-1}} L^*_{g^{-1}_{k-1}} p_{k-1} \right).$$

Since $(\mathcal{P} - \mathcal{Q})^*$ is left-invariant, we obtain

$$p_k = (\mathcal{P} - \mathcal{Q})^* \left(L^*_{g_k} R^*_{W^{-1}_{k-1}} L^*_{g^{-1}_{k-1}} p_{k-1} \right) = (\mathcal{P} - \mathcal{Q})^* \left(L^*_{g^{-1}_{k-1} g_k} R^*_{W^{-1}_{k-1}} p_{k-1} \right),$$

that is,

$$p_k = (\mathcal{P} - \mathcal{Q})^* \left(\mathrm{Ad}^*_{W_{k-1}} p_{k-1} \right).$$

4.3. **Preserving energy on Lie groups.** Let us now consider the case where Q is a Lie group G, the nonholonomic distribution $\mathcal{D}$ is not necessarily G-invariant, and L is regular and bi-invariant.

Since we are restricting ourselves to Lagrangians of mechanical type, the potential energy is necessarily zero. The left-invariance of L implies that it must be of the form

$$L(v_g) = \frac{1}{2} \left\langle \mathbb{I} g^{-1} v_g, g^{-1} v_g \right\rangle, \tag{8}$$

where $\mathbb{I}\colon \mathfrak{g} \to \mathfrak{g}^*$ is a symmetric non-singular inertia tensor[1]. The bi-invariance, however, imposes the equivariance condition $\mathrm{Ad}^*_{g^{-1}} \circ \mathbb{I} = \mathbb{I} \circ \mathrm{Ad}_g$ for all $g \in G$, as is straightforward to check. We remark that in this section, the metric used to build the projectors will be the same that defines the Lagrangian. If we take a discretization $L_d\colon G \times G \to \mathbb{R}$ (which needs to be *left*-invariant only), the equations of motion (7) hold. Then we can prove the following result.

Theorem 1. *Consider a nonholonomic system on a Lie group with a regular, bi-invariant Lagrangian and with an arbitrary distribution $\mathcal{D}$, and take a discrete Lagrangian that is left-invariant. Then the proposed discrete nonholonomic method* (4) *is energy-preserving.*

Proof. The equivariant inertia tensor $\mathbb{I}$ induces an Ad-invariant scalar product on $\mathfrak{g}$ and a bi-invariant metric on G. It also defines an inner product $\langle\, , \rangle_{\mathbb{I}}$ and a corresponding norm $\|\cdot\|_{\mathbb{I}}$ on each fiber of T^*G that inherit this bi-invariance. If $p \mapsto p^\sharp$ is the index-raising operation associated to the kinetic energy metric, then

$$\|p_g\|^2_{\mathbb{I}} = \left\langle p_g, p_g^\sharp \right\rangle = \left\langle p_g, L_g \mathbb{I}^{-1} L_g^* p_g \right\rangle = \left\langle p_g, R_g \mathbb{I}^{-1} R_g^* p_g \right\rangle .$$

The dual applications of the projectors $\mathcal{P}$ and $\mathcal{Q}$ are orthogonal complementary projectors with respect to this inner product, and thus for $p \in T^*G$,

$$\|(\mathcal{P} - \mathcal{Q})^* p\|^2_{\mathbb{I}} = \langle \mathcal{P}^* p, \mathcal{P}^* p\rangle_{\mathbb{I}} + \langle \mathcal{Q}^* p, \mathcal{Q}^* p\rangle_{\mathbb{I}} = \|(\mathcal{P} + \mathcal{Q})^* p\|^2_{\mathbb{I}} = \|p\|^2_{\mathbb{I}}.$$

The energy function is given in the continuous setting by $H = \langle \partial L/\partial \dot g, \dot g\rangle - L$ as a function of the position g and momentum $p = \partial L/\partial \dot g$. For L given by (8) we have

$$H(g, p) = \frac{1}{2}\left\langle L_g^* p, \mathbb{I}^{-1} L_g^* p \right\rangle = \frac{1}{2}\|p\|^2_{\mathbb{I}}.$$

Proving that the energy is preserved amounts to showing that equation (7) preserves $\|\cdot\|_{\mathbb{I}}$. Since $\|\cdot\|_{\mathbb{I}}$ is in particular right-invariant, then $R^*_{W_{k-1}^{-1}} : T^*_{g_{k-1}} G \to T^*_{g_k} G$ is an isometry. In addition, we have shown above that $(\mathcal{P} - \mathcal{Q})^*$ is also norm-preserving, so we obtain

$$H(g_k, p^-_{k,k+1}) = H(g_{k-1}, p^-_{k-1,k}).$$ □

Remark. While the proof above shows that the norm of the post-momenta is preserved, the norm of the pre-momenta is also preserved since they are related by a reflection (equation (6)).

4.4. **The average momentum.** Take a discrete nonholonomic system on G as in the previous section, but add the condition that $\mathcal{D}$ is right-invariant. Since the metric on the group is right-invariant, so is the projector $\mathcal{P}$. Take a trajectory of the system and define at each g_k the average momentum

$$\widetilde{p}_k = \frac{1}{2}\left(p^+_{k-1,k} + p^-_{k,k+1}\right).$$

Using (6), (7) and the fact that $(\mathcal{P} - \mathcal{Q})^*$ is its own inverse, we have

$$\begin{aligned}\widetilde{p}_k &= \frac{1}{2}\left((\mathcal{P} - \mathcal{Q})^*(p^-_{k,k+1}) + p^-_{k,k+1}\right) = \mathcal{P}^*(p^-_{k,k+1}) = \mathcal{P}^*(R^*_{W_{k-1}^{-1}} p^-_{k-1,k}) \\ &= R^*_{W_{k-1}^{-1}} \mathcal{P}^*(p^-_{k-1,k}) = R^*_{W_{k-1}^{-1}} \widetilde{p}_{k-1}.\end{aligned}$$

Since the norm $\|\cdot\|_{\mathbb{I}}$ on each fiber of T^*G defined in the proof of Theorem 1 is right-invariant, we obtain $\|\widetilde{p}_k\|_{\mathbb{I}} = \|\widetilde{p}_{k-1}\|_{\mathbb{I}}$, so

$$H(g_k, \widetilde{p}_k) = H(g_{k-1}, \widetilde{p}_{k-1}).$$

In addition, by equation (6), we have that $\mathcal{Q}^*(\widetilde{p}_k) = 0$, so $\widetilde{p}_k$ satisfies the constraints.

[1] In the context of Lie groups, g will denote an element of G instead of the metric.

4.5. **Preservation of the nonholonomic momentum map.** Let us recall some concepts regarding symmetries of nonholonomic systems. Suppose that a Lie group G acts on the configuration manifold Q. Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ consisting of those elements of $\mathfrak{g}$ whose infinitesimal generators at q satisfy the nonholonomic constraints, i.e.,

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} \mid \xi_Q(q) \in \mathcal{D}_q\}\,.$$

The (generalized) bundle over Q whose fiber at q is $\mathfrak{g}^q$ is denoted by $\mathfrak{g}^{\mathcal{D}}$.

A horizontal symmetry is an element $\xi \in \mathfrak{g}$ such that $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$. Note that a horizontal symmetry is related naturally to a constant section of $\mathfrak{g}^{\mathcal{D}}$.

Now consider a discrete Lagrangian $L_d \colon Q \times Q \to \mathbb{R}$, and define the discrete nonholonomic momentum map $J_d^{\mathrm{nh}} \colon Q \times Q \to (\mathfrak{g}^{\mathcal{D}})^*$ as in [10] by

$$\begin{aligned} J_d^{\mathrm{nh}}(q_{k-1}, q_k) \colon \mathfrak{g}^{q_k} &\to \mathbb{R} \\ \xi &\mapsto \langle D_2 L_d(q_{k-1}, q_k), \xi_Q(q_k)\rangle\,. \end{aligned}$$

For any smooth section $\widetilde{\xi}$ of $\mathfrak{g}^{\mathcal{D}}$ we have a function $(J_d^{\mathrm{nh}})_{\widetilde{\xi}} \colon Q \times Q \to \mathbb{R}$, defined as $(J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_{k-1}, q_k) = J_d^{\mathrm{nh}}(q_{k-1}, q_k)\left(\widetilde{\xi}(q_k)\right)$. We can now prove the following result.

Theorem 2. *Assume that L_d is G-invariant, and let $\widetilde{\xi}$ be a smooth section of $\mathfrak{g}^{\mathcal{D}}$. Then, under the proposed nonholonomic integrator, $(J_d^{\mathrm{nh}})_{\widetilde{\xi}}$ evolves according to the equation*

$$(J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_k, q_{k+1}) - (J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_{k-1}, q_k) = \Big\langle D_2 L_d(q_k, q_{k+1}), (\xi_{k+1} - \xi_k)_Q\,(q_{k+1})\Big\rangle$$

where $\xi_k, \xi_{k+1} \in \mathfrak{g}$ are the result of dropping the base points of $\widetilde{\xi}(q_k)$ and $\widetilde{\xi}(q_{k+1})$ respectively.

Proof. By the invariance of L_d we have

$$L_d(\exp(s\xi_k)q_k, \exp(s\xi_k)q_{k+1}) = L_d(q_k, q_{k+1}),$$

and differentiating at $s = 0$ we get

$$\langle D_1 L_d(q_k, q_{k+1}), (\xi_k)_Q(q_k)\rangle + \langle D_2 L_d(q_k, q_{k+1}), (\xi_k)_Q(q_{k+1})\rangle = 0.$$

On the other hand, the proposed integrator implies

$$(\mathcal{P} - \mathcal{Q})^*(D_1 L_d(q_k, q_{k+1})) + D_2 L_d(q_{k-1}, q_k) = 0.$$

From this, and using the fact that $(\xi_k)_Q(q_k) \in \mathcal{D}$, we have

$$\begin{aligned} (J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_{k-1}, q_k) &= \langle D_2 L_d(q_{k-1}, q_k), (\xi_k)_Q(q_k)\rangle \\ &= -\langle D_1 L_d(q_k, q_{k+1}), (\mathcal{P} - \mathcal{Q})\left((\xi_k)_Q(q_k)\right)\rangle = -\langle D_1 L_d(q_k, q_{k+1}), (\xi_k)_Q(q_k)\rangle \\ &= \langle D_2 L_d(q_k, q_{k+1}), (\xi_k)_Q(q_{k+1})\rangle\,. \end{aligned}$$

Then

$$\begin{aligned} &(J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_k, q_{k+1}) - (J_d^{\mathrm{nh}})_{\widetilde{\xi}}(q_{k-1}, q_k) = \\ &\qquad = \langle D_2 L_d(q_k, q_{k+1}), (\xi_{k+1})_Q(q_{k+1})\rangle - \langle D_2 L_d(q_k, q_{k+1}), (\xi_k)_Q(q_{k+1})\rangle \\ &\qquad = \langle D_2 L_d(q_k, q_{k+1}), (\xi_{k+1} - \xi_k)_Q(q_{k+1})\rangle\,. \end{aligned}$$

□

Corollary 3. *If L_d is G-invariant and ξ is a horizontal symmetry, then the proposed nonholonomic integrator preserves $(J_d^{\mathrm{nh}})_\xi$.*

5. Examples

Example 1. The following typical example will illustrate some of the constructions of previous sections. It corresponds to a discretization of the nonholonomic particle in $\mathbb{R}^3$ described by

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$$

and the nonholonomic constraint $\varphi = \dot{z} - y\dot{x} = 0$, which is represented by the distribution

$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\}.$$

Lagrange–d'Alembert's principle gives the equations of motion

$$\begin{aligned} \ddot{x} + y\ddot{z} &= 0 \\ \ddot{y} &= 0 \\ \dot{z} - y\dot{x} &= 0. \end{aligned}$$

Discretize the system by defining the discrete Lagrangian $L_d\colon \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ as

$$L_d(x_0,y_0,z_0,x_1,y_1,z_1) = \frac{1}{2}\left[\left(\frac{x_1-x_0}{h}\right)^2 + \left(\frac{y_1-y_0}{h}\right)^2 + \left(\frac{z_1-z_0}{h}\right)^2\right].$$

Then the discrete nonholonomic equations are

$$\left(\frac{x_2-2x_1+x_0}{h^2}\right) + y_1\left(\frac{z_2-2z_1+z_0}{h^2}\right) = 0 \tag{9a}$$

$$\frac{y_2-2y_1+y_0}{h^2} = 0 \tag{9b}$$

$$\frac{z_2-z_0}{2h} - y_1\frac{x_2-x_0}{2h} = 0. \tag{9c}$$

Regarding $\mathbb{R}^3$ as a Lie group under translations, the Euclidean metric is bi-invariant. Since L is induced by this metric and L_d is left-invariant, we have preservation of energy by Theorem 1. Figure 2 compares the energy behavior for our method against the DLA (discrete Lagrange–d'Alembert) algorithm in [10].

In order to write the discrete nonholonomic momentum equation in Theorem 2 with respect to this group action, take two linearly independent sections of $\mathfrak{g}^{\mathcal{D}}$ given by $\widetilde{\xi}_1(x,y,z) = (1,0,y)$ and $\widetilde{\xi}_2(x,y,z) = (0,1,0)$. The equation for $\widetilde{\xi}_1$ reads

$$\left(\frac{x_2-x_1}{h^2} + y_2\frac{z_2-z_1}{h^2}\right) - \left(\frac{x_1-x_0}{h^2} + y_1\frac{z_1-z_0}{h^2}\right) = (y_2-y_1)\left(\frac{z_2-z_1}{h^2}\right),$$

which turns out to be (9a). Similarly, if we consider $\widetilde{\xi}_2$ we reobtain (9b).

The DLA method proposed in [10] also yields equations (9a) and (9b), which is reasonable since both methods fulfill the discrete nonholonomic momentum equation. However, the DLA method replaces (9c) by a discretization of the constraints that does not involve (x_0,y_0,z_0), such as

$$\frac{z_2-z_1}{h} - \left(\frac{y_2+y_1}{2}\right)\frac{x_2-x_1}{h} = 0.$$

Example 2. The snakeboard is a modified version of the traditional skateboard, where the rider uses his own momentum, coupled with the constraints, to move the system. The configuration manifold is $Q = \mathrm{SE}(2)\times\mathbb{T}^2$ with coordinates (x,y,θ,ψ,ϕ) as in figure 3. The center of the board, which is also the center of mass, is located at (x,y). We are considering here the case where the angles of the front and rear wheel axles are equal and opposite, as in [6, 18]. However, we measure these angles

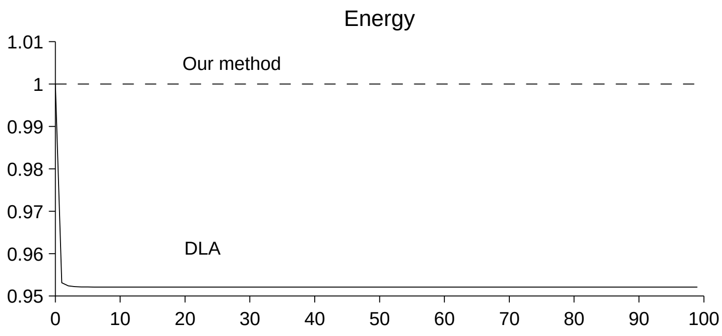

FIGURE 2. Energy behavior for the nonholonomic particle using our method and the DLA method in [10].

with respect to the board instead of the x-axis. Figure 3 shows a configuration with all the angles positive.

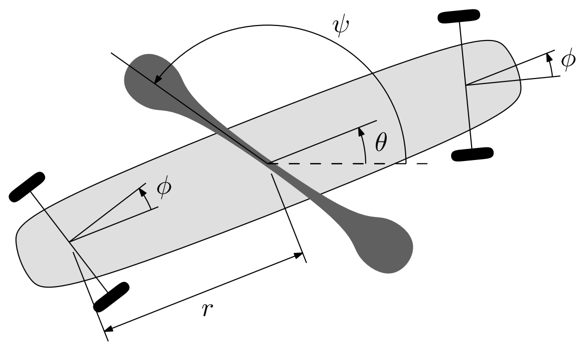

FIGURE 3. The snakeboard. The dashed line is aligned with the x-axis (not depicted).

The continuous system is described by the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(J + 2J_1)\dot{\theta}^2 + \frac{1}{2}J_0\dot{\psi}^2 + J_1\dot{\phi}^2$$

where m is the total mass of the system, J is the moment of inertia of the board about its center, J_0 is the moment of inertia of the rotor mounted on the board and J_1 is the moment of inertia of each wheel axle about its center. We assume the moments of inertia of the axles about the center of the board to be included in J. The distance between the center of the board and the wheels is denoted by r.

The wheels are not allowed to slide sideways, so the constraints turn out to be

$$\dot{x}\sin(\theta + \phi) - \dot{y}\cos(\theta + \phi) + r\dot{\theta}\cos(\phi) = 0$$
$$\dot{x}\sin(\theta - \phi) - \dot{y}\cos(\theta - \phi) - r\dot{\theta}\cos(\phi) = 0.$$

If we define the functions $a = r\cos\theta\cos\phi$, $b = r\sin\theta\cos\phi$ and $c = -\sin\phi$, then the constraint distribution is

$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi}, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial\theta}\right\}.$$

Endow Q with the Riemannian metric associated to the Lagrangian. This is represented in coordinates by the diagonal matrix

$$\mathbb{I} = \operatorname{diag}(m, m, J', J_0, 2J_1),$$

where $J' = J + 2J_1$. The orthogonal complement to $\mathcal{D}$ is then

$$\mathcal{D}^\perp = \operatorname{span}\left\{ J'c\frac{\partial}{\partial x} - ma\frac{\partial}{\partial \theta}, b\frac{\partial}{\partial x} - a\frac{\partial}{\partial y} \right\}.$$

The projection $\mathcal{Q}\colon TQ \to \mathcal{D}^\perp$ is given in coordinates by the matrix

$$\mathcal{Q} = \frac{1}{J'c^2 + m(a^2+b^2)} \begin{bmatrix} J'c^2 + mb^2 & -mab & -J'ac & 0 & 0 \\ -mab & J'c^2 + ma^2 & -J'bc & 0 & 0 \\ -mac & -mbc & m(a^2+b^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which depends on (θ, ϕ), and its dual $\mathcal{Q}^*$ is represented by the transpose.

Consider the discretization of this system determined by the discrete Lagrangian

$$\begin{aligned} L_d(q_k, q_{k+1}) &= \frac{1}{h^2}\left(\frac{1}{2}m(\Delta x_k^2 + \Delta y_k^2) + \frac{1}{2}(J + 2J_1)\Delta\theta_k^2 + \frac{1}{2}J_0\Delta\psi_k^2 + J_1\Delta\phi_k^2\right) \\ &= \frac{1}{2h^2}\Delta q_k^T \mathbb{I} \Delta q_k \end{aligned}$$

where $q_k = (x_k, y_k, \theta_k, \psi_k, \phi_k)$ (a column vector) and $\Delta z_k = z_{k+1} - z_k$.

The discrete nonholonomic equations (4) can be written as

$$D_1 L_d(q_k, q_{k+1}) + (\mathrm{Id} - 2\mathcal{Q})^*_{q_k}(D_2 L_d(q_{k-1}, q_k)) = 0,$$

so in matricial form we get

$$\frac{1}{h^2}\left(\mathbb{I}\Delta q_k + (\mathrm{Id} - 2\mathcal{Q}^T_{q_k})(-\mathbb{I}\Delta q_{k-1})\right) = 0,$$

that is,

$$q_{k+1} = (\mathrm{Id} - 2\mathbb{I}^{-1}\mathcal{Q}^T_{q_k}\mathbb{I})\Delta q_{k-1} + q_k.$$

Regarding the configuration space $\mathrm{SE}(2) \times \mathbb{T}^2$ as a Lie group, L is left-invariant. However, it cannot be right-invariant, because there are no bi-invariant metrics in $\mathrm{SE}(2)$. If one changes the group structure for the variables (x, y, θ) from $\mathrm{SE}(2)$ to $\mathbb{R}^2 \times \mathrm{S}^1$, then both the continuous and discrete Lagrangians are bi-invariant. The numerical method itself does not depend on which symmetry group one takes, but considering this last group structure allows us to apply Theorem 1 to show that there is preservation of energy.

On the other hand, we can still use the non-abelian group structure to write the discrete nonholonomic momentum equations, since only the left-invariance of L_d is required. Let us consider the action of the subgroup $\mathrm{SE}(2)$ on $\mathrm{SE}(2) \times \mathbb{T}^2$, and take the typical basis of $\mathfrak{se}(2)$: $e_1 = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right)$ and $e_3 = \left(\begin{smallmatrix} 0&-1&0\\1&0&0\\0&0&0 \end{smallmatrix}\right)$. Consider the section $\tilde{\xi}\colon Q \to \mathfrak{se}(2)$ defined by $\tilde{\xi}(x, y, \theta, \psi, \phi) = (a(\theta,\phi) + c(\theta,\phi)y)e_1 + (b(\theta,\phi) - c(\theta,\phi)x)e_2 + c(\theta,\phi)e_3$, so we have $\tilde{\xi}_Q = a(\theta,\phi)\frac{\partial}{\partial x} + b(\theta,\phi)\frac{\partial}{\partial y} + c(\theta,\phi)\frac{\partial}{\partial \theta}$. Therefore, the discrete nonholonomic momentum equation in this case is

$$\begin{aligned} (J_d^{\mathrm{nh}})_{\tilde{\xi}}(q_k, q_{k+1}) - (J_d^{\mathrm{nh}})_{\tilde{\xi}}(q_{k-1}, q_k) = \\ m(a(\theta_{k+1}, \phi_{k+1}) - a(\theta_k, \phi_k))\frac{x_{k+1} - x_k}{h^2} \\ + m(b(\theta_{k+1}, \phi_{k+1}) - b(\theta_k, \phi_k))\frac{y_{k+1} - y_k}{h^2} \\ + (J + 2J_1)(c(\theta_{k+1}, \phi_{k+1}) - c(\theta_k, \phi_k))\frac{\theta_{k+1} - \theta_k}{h^2}. \end{aligned}$$

Example 3. The Chaplygin sleigh consists in a rigid body that moves on a plane and is supported at three points. One of them is a knife edge and cannot slide sideways, and the other two can slide freely. Assume that the sleigh is symmetric, meaning that the center of mass is located on the line determined by the knife edge, at a distance a of the point of contact (x, y) (see figure 4).

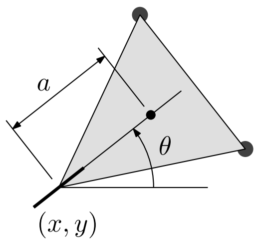

FIGURE 4. The Chaplygin sleigh.

The position of the sleigh is determined by $q = (x, y, \theta) \in \mathbb{R}^2 \times \mathrm{S}^1$, and the nonholonomic constraint is $\dot{x}\sin\theta - \dot{y}\cos\theta = 0$. If m is the mass of the sleigh, I is its moment of inertia and (x_C, y_C) denotes the position of the center of mass, then the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}m\left(\dot{x}^2 - 2a\dot{\theta}\dot{x}\sin\theta + \dot{y}^2 + 2a\dot{\theta}\dot{y}\cos\theta + a^2\dot{\theta}^2\right) + \frac{1}{2}I\dot{\theta}^2.$$

The kinetic energy metric is represented by the matrix

$$\begin{bmatrix} m & 0 & -am\sin\theta \\ 0 & m & am\cos\theta \\ -am\sin\theta & am\cos\theta & I + ma^2 \end{bmatrix}$$

so the constraint distribution and its orthogonal complement are

$$\begin{aligned} \mathcal{D} &= \operatorname{span}\left\{\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \frac{\partial}{\partial\theta}\right\} \\ \mathcal{D}^\perp &= \operatorname{span}\left\{-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} - \frac{am}{I + ma^2}\frac{\partial}{\partial\theta}\right\}. \end{aligned}$$

The dual of the projector onto $\mathcal{D}^\perp$ is then given by

$$\mathcal{Q}^* = \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta & \dfrac{am\sin\theta}{I + ma^2} \\ -\sin\theta\cos\theta & \cos^2\theta & -\dfrac{am\cos\theta}{I + ma^2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Discretize the Lagrangian by replacing $\dot{x}$ by $(x_1 - x_0)/h$ (analogously for $\dot{y}$ and $\dot{\theta}$), and θ by $(\theta_0 + \theta_1)/2$. We have applied the DLA algorithm, discretizing the constraints by $(x_2 - x_1)\sin((\theta_1 + \theta_2)/2) - (y_2 - y_1)\cos((\theta_1 + \theta_2)/2) = 0$, and compared the results with the trajectory of the continuous system. This trajectory was obtained by applying standard numerical methods to the Lagrange–d'Alembert differential equations (see for example [2, p. 25]). Figure 5 shows the evolution of $((x_k - \bar{x}_k)^2 + (y_k - \bar{y}_k)^2 + (\theta_k - \bar{\theta}_k)^2)^{1/2}$ for both DLA and our method, where $(\bar{x}_k, \bar{y}_k, \bar{\theta}_k)$ are the values at $t = hk$ of the trajectory of the continuous system. The results shown correspond to a particular trajectory with the initial points extracted from the continuous solution, but in general the errors are similar for the two methods. We used $m = J = 1$, $a = .2$, $q_0 = (0, 0, 0)$ and $q_1 = (-.2395, -.0070, .0589)$, which produces the heart-shaped loop typically described by the sleigh.

It is worth mentioning that if we take a different discretization of the constraints for the DLA algorithm, such as $(x_2 - x_1)\sin\theta_1 - (y_2 - y_1)\cos\theta_1 = 0$, the error becomes larger by one to two orders of magnitude. Taking the right discretization

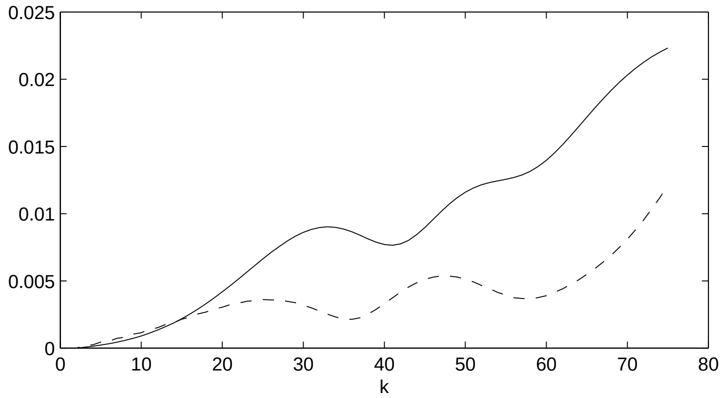

FIGURE 5. Error in $\mathbb{R}^3$ of the trajectories computed with our method (dashed line) and DLA (solid).

is crucial in the DLA algorithm; in contrast, the accuracy of our method is close to that of DLA without the need of such a choice.

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