A MAXIMAL DOMAIN OF PREFERENCES FOR TOPS-ONLY RULES IN THE DIVISION PROBLEM*

by

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Abstract: The division problem consists of allocating an amount M of a perfectly divisible good among a group of n agents. Sprumont (1991) showed that if agents have single-peaked preferences over their shares, the uniform rule is the unique strategy-proof, efficient, and anonymous rule. Ching and Serizawa (1998) extended this result by showing that the set of single-plateaued preferences is the largest domain, for all possible values of M, admitting a rule (the extended uniform rule) satisfying strategy-proofness, efficiency and symmetry. We identify, for each M and n, a maximal domain of preferences under which the extended uniform rule also satisfies the properties of strategy-proofness, efficiency, continuity, and "tops-onlyness". These domains (called weakly single-plateaued) are strictly larger than the set of single-plateaued preferences. However, their intersection, when M varies from zero to infinity, coincides with the set of single-plateaued preferences.

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1 Introduction

The division problem consists of allocating an amount M of a perfectly divisible good among a group of n agents. A rule maps preference profiles into n shares of the amount M. Sprumont (1991) shows that, given M, if agents have single-peaked preferences over their shares, the uniform rule is the unique strategy-proof, efficient, and anonymous rule. This is a nice example of a large literature that, by restricting the domain of preferences, investigates the possibility of designing strategy-proof rules.[1] Moreover, in this case, single-peakedness does not only allow strategy-proof rules but also efficient ones.

Whether or not nontrivial strategy-proof rules exist depends on the domain of preferences where we want them to operate. However, by restricting sufficiently the set of preferences it would always be possible to design non-dictatorial strategy-proof rules for any environment. Therefore, and once a domain restriction has been identified (as one under which there are non-dictatorial strategy-proof rules), it is natural and meaningful to ask how much this domain can be enlarged to still allow for non-dictatorial strategy-proof rules. The Gibbard-Satterthwaite Theorem says that this maximal domain is strictly smaller than the universal domain of preferences.

Ching and Serizawa (1998) show that, when the rule depends not only on preferences but also on the amount M to be allocated, the maximal domain under which there exists at least one rule (the extended uniform rule) satisfying strategy-proofness, efficiency, and symmetry coincides with the set of single-plateaued preferences.

In Massó and Neme (2001) we show that, given M, the set of feebly single-plateaued preferences is the unique maximal domain of preferences that includes the set of single-peaked ones for which there exists at least one rule satisfying strategy-proofness, efficiency, and strong symmetry. This set is strictly larger than the set of single-plateaued preferences; in particular, preferences might have special intervals of indifference away from the set of best shares. However, the rule that we exhibit when showing our maximality result is very complex and hence difficult to be implemented because it is not "tops-only" (it does not exclusively depend on the n sets of best shares). Efficiency and strong symmetry force the rule to be sensitive to intervals of indifference away from the "top".

In this paper we ask how much the set of single-peaked preferences can be en-

[1] See Sprumont (1995), Barberà (1996), and Barberà (2001) for three comprehensive surveys of this literature as well as for three exhaustive bibliographies.

larged to still allow for strategy-proof, efficient, and simple rules. In particular, we identify a maximal domain of preferences (that includes the set of single-peaked ones) for which there exists at least one rule (the extended uniform rule) satisfying strategy-proofness, efficiency, tops-onlyness, and continuity. We refer to this domain as the set of weakly single-plateaued preferences. It turns out that this maximal domain depends crucially on both M and n, and contains (for each value of M) the set of single-plateaued preferences. Moreover, the intersection of all sets of weakly single-plateaued preferences, when M varies from zero to infinity, coincides with the set of single-plateaued preferences. Notice that in Ching and Serizawa (1998) M is treated as a variable of the problem rather than one of its data. We want to emphasize though that, in spite of their result, our analysis with a fixed amount M is meaningful since there are many allocation problems where to assume the contrary would be senseless. Furthermore, we do not claim that the domain identified here has economic relevance *per se*; rather, we understand our result as giving a precise and definite answer to an interesting and economically relevant question.

The result here differs from our previous one in Massó and Neme (2001) on at least three grounds. The first difference has to do with the list of properties a rule is required to satisfy on the maximal domain. Instead of requiring strong symmetry we demand now that the rule be tops-only and continuous. Any property related with anonymity is inappropriate when there are asymmetries among the agents that one wishes to respect (due to repeated relationships, seniority, etc.).[2] On the other hand, tops-onlyness and continuity are two natural properties to ask for if we want to exclude complex rules that would be difficult to use. Second, while in Massó and Neme (2001) we identify *the unique* maximal domain containing the set of single-peaked preferences here we only identify *a* maximal domain. The reason why this difference arises is that on the domain of single-peaked preferences there is only one rule (the uniform one) satisfying strategy-proofness, efficiency, and strong symmetry while there are many that satisfy strategy-proofness, efficiency, tops-onlyness, and continuity; the extension may now be dependent upon the rule, yielding different maximal domains for different rules (in the last section of the paper we elaborate on this point by means of an example). Third, the two maximal domains are different; in particular, the set of feebly single-plateaued preferences strictly contains the set of weakly single-plateaued preferences.

A number of papers have also identified maximal domains of preferences allowing

[2] See Barberà, Jackson, and Neme (1997) for many justifications of the use of non-anonymous rules.

for strategy-proof social choice functions in voting environments. Barberà, Sonnenschein, and Zhou (1991) show that the set of separable preferences is the maximal domain that preserves strategy-proofness of voting by committees without dummies and vetoers. Serizawa (1995), Berga (1997), Barberà, Massó, and Neme (1999), Berga and Serizawa (2000), and Berga (2002) improve upon this result in several directions; for instance, either by looking at a more general voting model and/or by admitting larger classes of social choice functions.

In all papers mentioned above either the assumption of anonymity (or any other related property like symmetry, strong symmetry, or equal treatment of equals) and/or the assumption that the domain contains the set of single-peaked preferences guarantee that, on a subdomain of preferences, the outcome of the rule is unambiguously determined (by the uniform rule in the division problem and by a generalized median voter scheme in voting environments).[3] And this supports very much the process of identifying the maximal domain of preferences. In contrast, we do not require here that the rule satisfy any property related to anonymity;[4] therefore, even for single-peaked preference profiles, we do not know their associated vectors of shares since there are many rules satisfying the required properties yielding different outcomes. This is precisely one of our main difficulties here, which we overcome using an anonymity like property established by Lemma 2 in the proof of our Theorem. At the end of the paper we will comment on this apparent anonymity implied by Lemma 2.

The paper is organized as follows. Section 2, which closely follows Massó and Neme (2001), contains preliminary notation and definitions. The set of weakly single-plateaued preferences and the result are presented in Section 3. Section 4 contains the proof of the result. Section 5 concludes with some final remarks.

2 Preliminary Notation and Definitions

Agents are indexed by the elements of a finite set $N = \{1, ..., n\}$ where $n \geq 2$. They have to share the amount $M \in \mathbb{R}_{++}$ of a perfectly divisible good. An *allocation* is a vector $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n_+$ such that $\sum x_i = M$. We denote by Z the set

[3] Theorem 1 in Berga and Serizawa (2000) assumes only that the domain of preferences contains a "minimally rich domain" in the sense that for each alternative a, there is at least one preference whose top is a.

[4] See Barberà, Jackson, and Neme (1997) to understand the difficulties of characterizing, in the division problem, non-anonymous rules on the domain of single-peaked preferences.

of allocations. Each agent has a complete preorder R over $[0, M]$, his *preference relation*. Let P be the strict preference relation associated with R and let I be the corresponding indifference relation. We assume that preferences are continuous in the sense that for each $x \in [0, M]$ the sets $\{y \in [0, M] \mid xRy\}$ and $\{y \in [0, M] \mid yRx\}$ are closed. We denote by $\mathcal{R}$ the set of continuous preferences on $[0, M]$ and by $\mathcal{V}$ a generic subset of $\mathcal{R}$. Given a preference relation $R \in \mathcal{R}$ we denote the set of preferred shares according to R as $t(R) = \{x \in [0, M] \mid xRy \text{ for all } y \in [0, M]\}$. Let $\underline{t}(R) = \min\{x \in [0, M] \mid x \in t(R)\}$ and $\overline{t}(R) = \max\{x \in [0, M] \mid x \in t(R)\}$. Abusing notation, we also denote by $t(R)$ the unique element of the set $t(R)$ whenever $\underline{t}(R) = \overline{t}(R)$. We call it the *top* of the preference relation R.

Preference profiles are n-tuples of continuous preference relations over $[0, M]$ and they are denoted by $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{R}^n$. When we want to stress the role of agent i's preference we will represent a preference profile $\mathbf{R}$ by $(R_i, \mathbf{R}_{-i})$. Given a preference profile $\mathbf{R}$ and a preference relation $R \neq R_i$ we will write $(R, \mathbf{R}_{-i})$ to represent the preference profile where R_i in $\mathbf{R}$ is replaced by R.

A *rule on* $\mathcal{V}^n \subseteq \mathcal{R}^n$ is a function $\Phi : \mathcal{V}^n \longrightarrow Z$; that is, $\sum \Phi_i(\mathbf{R}) = M$ for all $\mathbf{R} \in \mathcal{V}^n$.

Rules require each agent to report a preference. A rule is strategy-proof if it is always in the best interest of an agent to reveal his preferences truthfully. Formally,

Definition 1 *A rule on* $\mathcal{V}^n$, Φ, *is* strategy-proof *if for all* $\mathbf{R} = (R_1, ..., R_n) \in \mathcal{V}^n$, *all* $i \in N$, *and all* $R \in \mathcal{V}$, *we have* $\Phi_i(R_i, \mathbf{R}_{-i}) R_i \Phi_i(R, \mathbf{R}_{-i})$.

Given a preference profile $\mathbf{R} \in \mathcal{V}^n$, an allocation $x \in Z$ is *efficient* if there is no $z \in Z$ such that for all $i \in N$, $z_i R_i x_i$, and for at least one $j \in N$ we have $z_j P_j x_j$. Denote by $E(\mathbf{R})$ the set of efficient allocations. A rule is efficient if it selects an efficient allocation. Formally,

Definition 2 *A rule on* $\mathcal{V}^n$, Φ, *is* efficient *if for all* $\mathbf{R} \in \mathcal{V}^n$, *we have* $\Phi(\mathbf{R}) \in E(\mathbf{R})$.

Here, we are specially interested in simple rules satisfying the following two properties. The first one says that the rule has the informationally nice feature that it only requires to each agent to reveal his set of best-shares since it depends only on their top-sets. This is a very common feature of all rules used in the study of the division problem. In fact, except Massó and Neme (2001) all papers in this literature have confined to study exclusively tops-only rules, although (a) tops-onlyness is not explicitly imposed on the rule but derived as a consequence of other properties (strategy-proofness and efficiency) and (b) the rule is supposed to operate on a fixed domain of preferences (single-peaked or single-plateaued ones). However, as our result in Massó and Neme (2001) points out, when the domain of the rules is not fixed (rather, the objective is to identify their maximal admissible domain) non tops-only rules arise because they may still be strategy-proof and efficient on these domains. Therefore, and again because we find it to be the most unquestionable property if we want to restrict ourselves to use simple rules, here we do explicitly require tops-onlyness but we dispense the rule to satisfy any anonymity like condition. Formally,

Definition 3 *A rule on* $\mathcal{V}^n$*,* Φ*, is* tops-only *if for all* $\mathbf{R}, \mathbf{R}' \in \mathcal{V}^n$ *such that* $t(R_i) = t(R_i')$ *for all* $i \in N$*, we have* $\Phi(\mathbf{R}) = \Phi(\mathbf{R}')$*.*

From now on, we shall abuse notation and identify with a tops-only rule Φ a function mapping vectors of best-shares to allocations; namely, given $(R_1, ..., R_n) \in \mathcal{V}^n$, we write $\Phi(x_1, ..., x_n)$ for $\Phi(R_1, ..., R_n)$ where $x_i = t(R_i)$ for all $i \in N$. Therefore, a tops-only rule Φ can be seen as a mapping $\Phi : \mathcal{T}^n \rightarrow Z$, where $\mathcal{T}$ is the family of non-empty subsets of $[0, M]$. Since best-shares sets will be closed intervals (Lemma 3 will establish this fact) we define $\mathcal{I} = \{[a, b] \subseteq [0, M] \mid 0 \leq a \leq b \leq M\}$ as the family of all closed intervals contained in $[0, M]$.

The second property related to simple rules refers to a weak requirement of continuity.

Definition 4 *A tops-only rule on* $\mathcal{V}^n$*,* Φ*, is* continuous *if it is continuous with the Hausdorff topology.*[5]

We will identify as a maximal domain of preferences a set of preferences closely related to the set of single-plateaued preferences.[6] A preference is said to be single-plateaued if the set of best-shares is an interval and at each of its sides the preference is strictly monotonic. Formally,

Definition 5 *A preference* $R \in \mathcal{R}$ *is* single-plateaued *if* $t(R) = \left[\underline{t}(R), \overline{t}(R)\right]$ *and for all* $x, y \in [0, M]$ *we have* xPy *whenever* $y < x < \underline{t}(R)$ *or* $\overline{t}(R) < x < y$*.*

[5]The Hausdorff topology on $\mathcal{V}^n$ is the induced topology from the Hausdorff topology on $\mathcal{T}^n$. See page 16 in Hildenbrand (1974) for the definition of the Hausdorff distance between sets.

[6]See Moulin (1984) and Berga (1998) for characterizations of strategy-proof rules under the domain of single-plateaued preferences in a public-good context.

Let $\mathcal{R}_{sp}$ be the set of single-plateaued preferences. The subset of single-plateaued preferences whose top is a singleton is called the set of *single-peaked* preferences and it will be denoted by $\mathcal{R}_s$.

The following rule on $\mathcal{V}^n$ constitutes a natural extension of the uniform rule (defined on the set of single-peaked preferences) to a generic domain of preferences. Ching and Serizawa (1998) used this rule to establish their maximal domain result in the context of a variable M.

Definition 6 *The* extended uniform rule on $\mathcal{V}^n$*,* U*, is defined as follows: for all* $\mathbf{R} \in \mathcal{V}^n$ *and all* $i \in N$*,*

$$U_i(\mathbf{R}) = \begin{cases} \min\{\underline{t}(R_i), \lambda(\mathbf{R})\} & \text{if } M \leq \sum \underline{t}(R_j), \\ \min\{\overline{t}(R_i), \underline{t}(R_i) + \lambda(\mathbf{R})\} & \text{if } \sum \underline{t}(R_j) \leq M \leq \sum \overline{t}(R_j), \\ \max\{\overline{t}(R_i), \lambda(\mathbf{R})\} & \text{if } \sum \overline{t}(R_j) \leq M, \end{cases}$$

where $\lambda(\mathbf{R})$ *solves* $\sum U_j(\mathbf{R}) = M$*.*

3 Weakly Single-Plateaued Preferences

Our result identifies the set of weakly single-plateaued preferences as a maximal domain of preferences admitting strategy-proof, efficient, tops-only, and continuous rules. This set, which depends on M and n, is strictly larger than the set of single-plateaued preferences.

Before stating the formal definition, it seems useful to give a verbal explanation of the set of weakly single-plateaued preferences. A preference relation $R \in \mathcal{R}$ is *weakly single-plateaued* if its set of best shares is a closed interval and the following additional properties are satisfied: (a) If $\frac{M}{n} < \underline{t}(R)$, then the preference has to be "increasing" between M/n and its smallest best share $\underline{t}(R)$, although it may have intervals of indifference provided these intervals are above $M/2$. Moreover, the egalitarian share M/n has to be strictly preferred to all smaller shares, but all orderings are possible among them. (b) If $\overline{t}(R) < \frac{M}{n}$, then the preference has to be "decreasing" between its largest best share $\overline{t}(R)$ and M/n, although it may have intervals of indifference provided that $n = 2$ and these intervals are below $M/2$. Moreover, the egalitarian share M/n has to be strictly preferred to all larger shares, but also all orderings are possible among them. Finally, if $\underline{t}(R) \leq M/n \leq \overline{t}(R)$, then, no additional requirement is imposed.[7] Formally,

[7] The set of *feebly single-plateaued* preferences identified in Massó and Neme (2001) satisfy the

Definition 7 *A preference relation* $R \in \mathcal{R}$ *is* weakly single-plateaued *if:*
(a) $t(R) = [\underline{t}(R), \overline{t}(R)]$.
(b) Given $x < y \leq \underline{t}(R)$ *such that* $\frac{M}{n} \leq y$*, then:*
(b.1) yRx*, and*
(b.2) if xIy *then* $\frac{M}{2} \leq x < y < \underline{t}(R)$.
(c) Given $\overline{t}(R) \leq y < x$ *such that* $y \leq \frac{M}{n}$*, then:*
(c.1) yRx*, and*
(c.2) if xIy *then* $n = 2$ *and* $x \leq \frac{M}{n}$.

We denote by $\mathcal{R}_{wsp}(n)$ the set of weakly single-plateaued preferences. Note that $\mathcal{R}_{sp} \subsetneq \mathcal{R}_{wsp}(n)$ for all $n \geq 2$. Figure 1 illustrates three possible types of weakly single-plateaued preferences depending on whether $\frac{M}{n} \in \left[\underline{t}(R), \overline{t}(R)\right]$, $\frac{M}{n} < \underline{t}(R)$ (as well as $\frac{M}{2} < \underline{t}(R)$), and $\overline{t}(R) < \frac{M}{n}$.

Insert Figure 1 about here

Figure 2 illustrates a preference relation with an indifference interval below $M/2$ that is weakly single-plateaued for $n = 2$ and $n = 4$ but it is not for $n = 3$.[8]

Insert Figure 2 about here

Following Ching and Serizawa (1998) we can define, given a list of properties that a rule may satisfy, the concept of "a maximal domain of preferences for this list".

Definition 8 *A set* $\mathcal{R}_m$ *of preferences is a* maximal domain for a list of properties *if: (1)* $\mathcal{R}_m \subseteq \mathcal{R}$*; (2) there exists a rule on* $\mathcal{R}_m^n$ *satisfying the properties; and (3) there is no rule on* $\mathcal{Q}^n$ *satisfying the same properties such that* $\mathcal{R}_m \subsetneq \mathcal{Q} \subseteq \mathcal{R}$.

same properties but in (a) the intervals of indifference have to be sufficiently large in relation to M (the sum of the extremes has to be larger than M), although they do not have to be necessarily above $M/2$; in (b) the intervals of indifference have to be sufficiently small in relation to M (the sum of the extremes has to be smaller than M) although no condition on n is imposed and the intervals of indifference do not have to be necessarily below $M/2$; and moreover M/n, instead of strictly preferred, has to be at least as good as all smaller shares (in (a)) and all larger shares (in (b)).

[8] See Examples 1, 2 and 3 in the last section of the paper to understand first, why the set of weakly single-plateaued preferences contains preference relations with indifference intervals away from the top of only a very special kind and second, the special role played by the share $\frac{M}{2}$ in the definition of these intervals.

Now, we can state our result.

Theorem 1 *The set of weakly single-plateaued preferences, $\mathcal{R}_{wsp}(n)$, is a maximal domain for the properties of strategy-proofness, efficiency, tops-onlyness, and continuity.*

Before moving to the proof of Theorem 1, a comment about its content is appropriate since the reader should not be surprised by the fact that, on the one hand, a rule responds only to the information about the top ranking shares of preferences and, on the other hand, the maximal domain of preferences crucially depends on the information about other non-top ranking shares. Strategy-proofness and efficiency excludes only some preferences while others are admitted. To better understand that this is not contradictory consider the uniform rule defined on any domain of preferences having a unique top; it is a tops-only rule, and strategy-proof and efficient on the domain of single-peaked preferences, as well. Yet, if the domain of single-peaked preferences is enlarged by admitting a non-monotonic ordering (at one side of the top) then the uniform rule fails to be either strategy-proof or efficient (or both). That is, the rule does not pay attention to information other than the top ranked shares and yet its strategy-proofness and efficiency depend on whether or not the domain admits preferences which order in a particular way non-top ranked shares.

4 The Proof of Theorem 1

It is easy to check that the extended uniform rule U on $(\mathcal{R}_{wsp}(n))^n$ satisfies the properties of strategy-proofness, efficiency, tops-onlyness, and continuity. Assume that $\mathcal{V} \subseteq \mathcal{R}$ is a maximal domain for these properties and $\mathcal{R}_{wsp}(n) \subseteq \mathcal{V}$. We will show that $\mathcal{V} = \mathcal{R}_{wsp}(n)$, indeed.

Lemma 1 *Let $\Phi : \mathcal{V}^n \to Z$ be a tops-only and efficient rule. Then, for all $\mathbf{R} \in \mathcal{V}^n$ and all $i \in N$:*
(a) If $\sum \overline{t}(R_j) \leq M$ then $\overline{t}(R_i) \leq \Phi_i(\mathbf{R})$.
(b) If $M \leq \sum \underline{t}(R_j)$ then $\Phi_i(\mathbf{R}) \leq \underline{t}(R_i)$.
(c) If $\sum \underline{t}(R_j) = \sum \overline{t}(R_j) = M$ then $\Phi_i(\mathbf{R}) = \underline{t}(R_i) = \overline{t}(R_i)$.
(d) If $\sum \underline{t}(R_j) \leq M \leq \sum \overline{t}(R_j)$ and $t(R_j) = [\underline{t}(R_j), \overline{t}(R_j)]$ for all $j \in N$ then $\underline{t}(R_i) I_i \Phi_i(\mathbf{R}) I_i \overline{t}(R_i)$.

Proof. (a) Suppose otherwise; that is, there exist $\mathbf{R} \in \mathcal{V}^n$ and $i \in N$ such that $\sum \bar{t}(R_j) \leq M$ and $\bar{t}(R_i) > \Phi_i(\mathbf{R})$. Since $\mathcal{R}_{wsp}(n) \subseteq \mathcal{V}$ and Φ is tops-only we can assume that $\mathbf{R}$ has the property that for all $j \in N$ and all pairs x, y such that $\bar{t}(R_j) < x < y$ the condition

$$\bar{t}(R_j)\, P_j x P_j y \tag{1}$$

holds. Define $N_- = \{j \in N \mid \bar{t}(R_j) > \Phi_j(\mathbf{R})\}$, $N_+ = \{j \in N \mid \bar{t}(R_j) \leq \Phi_j(\mathbf{R})\}$, and $\omega = \sum_{j \in N_-} \left(\bar{t}(R_j) - \Phi_j(\mathbf{R})\right)$. Note that by assumption both sets are non-empty and ω is strictly positive. Since $\sum \bar{t}(R_j) \leq M$ it is possible to find a vector of non-negative numbers $(z_j)_{j \in N_+}$ such that $\sum_{j \in N_+} z_j = \omega$ and $\Phi_j(\mathbf{R}) - z_j \geq \bar{t}(R_j)$ for all $j \in N_+$. Therefore, the existence of the vector $(\alpha_1, ..., \alpha_n) \in Z$, defined by

$$\alpha_j = \begin{cases} \bar{t}(R_j) & \text{if } j \in N_- \\ \Phi_j(\mathbf{R}) - z_j & \text{if } j \in N_+ \end{cases},$$

and condition (1) imply $\Phi(\mathbf{R}) \notin E(\mathbf{R})$, contradicting the efficiency of Φ.

(b) Its proof is omitted since it follows an argument which is symmetric to the one used to prove case (a).

(c) It follows immediately from (a) and (b).

(d) Since $\sum \underline{t}(R_j) \leq M \leq \sum \bar{t}(R_j)$, there exists a vector $(\alpha_1, ..., \alpha_n) \in Z$ such that $\alpha_j \in [\underline{t}(R_j), \bar{t}(R_j)]$ for all $j \in N$. Then, by efficiency of Φ, we have $\underline{t}(R_i) I_i \Phi_i(\mathbf{R}) I_i \bar{t}(R_i)$ for all $i \in N$. ■

To prove the next lemma we need the following notation. Given $[a, b] \subseteq [0, M]$, let $R^{[a,b]}$ be any preference relation in $\mathcal{R}_{wsp}(n)$ with the property that $t(R^{[a,b]}) = [a, b]$. Again, to stress the role of agent i's preference relation (or the preference relations of agents in S) we will represent a profile of preferences, given a vector $[\mathbf{a}, \mathbf{b}] = ([a_1, b_1], ..., [a_n, b_n])$, as $\mathbf{R}^{[\mathbf{a},\mathbf{b}]} \equiv \left(R_i^{[a_i,b_i]}, \mathbf{R}_{-i}^{[\mathbf{a},\mathbf{b}]}\right) = \left(\mathbf{R}_S^{[\mathbf{a}_S,\mathbf{b}_S]}, \mathbf{R}_{-S}^{[\mathbf{a},\mathbf{b}]}\right)$. Finally, given $[a, b] \subseteq [0, M]$, we will represent by $\mathbf{R}^{[a,b]}$ any preference profile $\left(R_1^{[a,b]}, ..., R_n^{[a,b]}\right)$.

Lemma 2 *Let $\Phi : \mathcal{V}^n \to Z$ be a strategy-proof, efficient, tops-only, and continuous rule.*

(a) Let $(x_1, ..., x_n)$ be a vector such that $x_i \leq \frac{M}{n}$ for all i. Then, $\Phi_i(x_1, ..., x_n) = \frac{M}{n}$ for all i.

(b) Let $(x_1, ..., x_n)$ be a vector such that $\sum x_j \leq M$ and assume that $x_i \geq \frac{M}{n}$ and $x_j \leq \frac{M}{n}$ for every $j \neq i$. Then $\Phi_i(x_1, ..., x_n) = x_i$.

(c) Let $(x_1, ..., x_n)$ be a vector such that $x_i \geq \frac{M}{n}$ for all i. Then, $\Phi_i(x_1, ..., x_n) = \frac{M}{n}$

for all i.

(d) Let $(x_1, ..., x_n)$ be a vector such that $\sum x_j \geq M$ and assume that $x_i \leq \frac{M}{n}$ and $x_j \geq \frac{M}{n}$ for every $j \neq i$. Then $\Phi_i(x_1, ..., x_n) = x_i$.

(e) Let $([a_1, b_1], ..., [a_n, b_n])$ be a vector of intervals such that $\sum_{j \neq i} a_j \leq M \leq \sum_{j \neq i} b_j$ and $a_i \geq \frac{M}{n}$ for at least one i. Then either $\Phi_i([a_1, b_1], ..., [a_n, b_n]) = 0$ or $\Phi_i([a_1, b_1], ..., [a_n, b_n]) \geq \frac{M}{n}$.

Proof. (a) Suppose otherwise; that is, there exist $\mathbf{x} = (x_1, ..., x_n) \in [0, M]^n$ and $i \in N$ with the properties that $x_j \leq \frac{M}{n}$ for all j and $\Phi_i(x_1, ..., x_n) \neq \frac{M}{n}$. By Lemma 1, part (a), we may assume (since $\Phi(x_1, ...x_n) \in Z$) that $\frac{M}{n} < \Phi_i(x_1, ..., x_n) \leq X$ holds, where $X = M - \sum_{j \neq i} x_j$. We distinguish between the following two cases:

<u>Case 1</u>: $\frac{M}{n} < \Phi_i(x_1, ..., x_n) < X$. Let $R' \in \mathcal{R}_{wsp}(n)$ be any preference with the property that $t(R') = x_i$ and

$$XP'\Phi_i(x_1, ..., x_n). \tag{2}$$

Observe that such preference exists since both X and $\Phi_i(x_1, ..., x_n)$ are strictly larger than $\frac{M}{n}$. Since Φ is tops-only,

$$\Phi_i(x_1, ...x_n) = \Phi_i(R', \mathbf{R}^{\mathbf{x}}_{-i}). \tag{3}$$

On the other hand, by Lemma 1, part (c), $\Phi_i(R^X, \mathbf{R}^{\mathbf{x}}_{-i}) = X$, implying that Φ is not strategy-proof because, by conditions (2) and (3),

$$\Phi_i(R^X, \mathbf{R}^{\mathbf{x}}_{-i})P'\Phi_i(R', \mathbf{R}^{\mathbf{x}}_{-i}).$$

<u>Case 2</u>: $\frac{M}{n} < \Phi_i(x_1, ..., x_n) = X$. Without loss of generality assume $i \neq 1$. By Lemma 1, part (d), $\Phi\left(R_1^{[x_1,X]}, \mathbf{R}^{\mathbf{x}}_{-1}\right) = (X, x_2, ..., x_n)$. By continuity of Φ, and since $\Phi_i\left(R_1^{[x_1,X]}, \mathbf{R}^{\mathbf{x}}_{-1}\right) = x_i \leq \frac{M}{n} < X = \Phi_i(R_1^{x_1}, \mathbf{R}^{\mathbf{x}}_{-1})$, there exist $r \in \left(\frac{M}{n}, X\right)$ and $y \in [x_1, X]$ such that $\Phi_i\left(R_1^{[x_1,y]}, \mathbf{R}^{\mathbf{x}}_{-1}\right) = r$. Then, by Lemma 1, part (d),

$$\Phi_i\left(R_1^{[x_1,y]}, R_i^X, \mathbf{R}^{\mathbf{x}}_{-\{1,i\}}\right) = X, \tag{4}$$

for any $R_i^X \in \mathcal{R}_{wsp}(n)$. Consider now any preference $R_i' \in \mathcal{R}_{wsp}(n)$ with the properties that $t(R_i') = x_i$ and

$$XP_i'r. \tag{5}$$

Since Φ is tops-only,

$$\Phi_i\left(R_1^{[x_1,y]}, R_i^{x_i}, \mathbf{R}^{\mathbf{x}}_{-\{1,i\}}\right) = r = \Phi_i\left(R_1^{[x_1,y]}, R_i', \mathbf{R}^{\mathbf{x}}_{-\{1,i\}}\right). \tag{6}$$

But then, conditions (4), (5), and (6) imply

$$\Phi_i\left(R_1^{[x_1,y]}, R_i^X, \mathbf{R}^{\mathbf{x}}_{-\{1,i\}}\right) P_i' \Phi_i\left(R_1^{[x_1,y]}, R_i', \mathbf{R}^{\mathbf{x}}_{-\{1,i\}}\right),$$

which means that Φ is not strategy-proof.

(b) Let $\mathbf{x} = (x_1, ..., x_n)$ be a vector such that $\sum x_j \leq M$, $x_i \geq \frac{M}{n}$ for i, and $x_j \leq \frac{M}{n}$ for all $j \neq i$. By Lemma 1, part (a), we may assume that $\Phi_i(x_1, ..., x_n) \geq x_i$. To get a contradiction, assume $\Phi_i(x_1, ..., x_n) > x_i$. Consider any preference $R' \in \mathcal{R}_{wsp}(n)$ such that $t(R') = x_i$ and $\frac{M}{n} P' \Phi_i(x_1, ..., x_n)$. Since Φ is tops-only,

$$\Phi_i(R', \mathbf{R}^{\mathbf{x}}_{-i}) = \Phi_i(R^{x_i}, \mathbf{R}^{\mathbf{x}}_{-i}). \tag{7}$$

On the other hand, by Lemma 2, part (a), $\Phi_i(R^0, \mathbf{R}^{\mathbf{x}}_{-i}) = \frac{M}{n}$, which implies, together with condition (7), that Φ is not strategy-proof because

$$\Phi_i(R^0, \mathbf{R}^{\mathbf{x}}_{-i}) P' \Phi_i(R', \mathbf{R}^{\mathbf{x}}_{-i}).$$

The proofs of parts (c) and (d) are omitted since they follow arguments which are symmetric to the ones used to prove parts (a) and (b), respectively.

(e) Assume otherwise; that is, there exist $[\mathbf{a}, \mathbf{b}] = ([a_1, b_1], ..., [a_n, b_n])$ and i such that $\sum_{j \neq i} a_j \leq M \leq \sum_{j \neq i} b_j$, $a_i \geq \frac{M}{n}$, and $0 < \Phi_i\left(\mathbf{R}^{[\mathbf{a},\mathbf{b}]}\right) < \frac{M}{n}$. Because $\sum_{j \neq i} a_j \leq M \leq \sum_{j \neq i} b_j$, Lemma 1, part (b), implies $\Phi_i\left(R^0, \mathbf{R}^{[\mathbf{a},\mathbf{b}]}_{-i}\right) = 0$. Let $R' \in \mathcal{R}_{wsp}(n)$ be any preference such that $t(R') = [a_i, b_i]$ and $0 P' \Phi_i\left(\mathbf{R}^{[\mathbf{a},\mathbf{b}]}\right)$. Observe that the hypothesis that $a_i \geq \frac{M}{n}$ guarantees the existence of such a preference. Since Φ is tops-only, $\Phi_i\left(R', \mathbf{R}^{[\mathbf{a},\mathbf{b}]}_{-i}\right) = \Phi_i\left(\mathbf{R}^{[\mathbf{a},\mathbf{b}]}\right)$, which implies that Φ is not strategy-proof because $\Phi_i\left(R^0, \mathbf{R}^{[\mathbf{a},\mathbf{b}]}_{-i}\right) P' \Phi_i\left(R', \mathbf{R}^{[\mathbf{a},\mathbf{b}]}_{-i}\right)$. ■

Lemma 3 *Let $\Phi : \mathcal{V}^n \to Z$ be a strategy-proof, efficient, tops-only, and continuous rule. Then, $t(R) = [\underline{t}(R), \overline{t}(R)]$ for all $R \in \mathcal{V}$.*

Proof. Assume otherwise; that is, there exist $R \in \mathcal{V}$ and $y \in \left(\underline{t}(R), \overline{t}(R)\right)$ such that $\underline{t}(R) P y$. Define

$$\underline{x} = \frac{M - \underline{t}(R)}{n-1}$$

and

$$\overline{x} = \frac{M - \overline{t}(R)}{n-1}.$$

Let i be arbitrary. Since Φ is efficient,

$$\Phi_j\left(R, \mathbf{R}^{\underline{x}}_{-i}\right) = \begin{cases} \underline{t}(R) & \text{if } j = i \\ \underline{x} & \text{if } j \neq i \end{cases}$$

and

$$\Phi_j\left(R, \mathbf{R}_{-i}^{\overline{x}}\right) = \begin{cases} \overline{t}(R) & \text{if } j = i \\ \overline{x} & \text{if } j \neq i \end{cases}.$$

<u>Claim</u>: Either there exists $z \in [\overline{x}, \underline{x}]$ such that $\Phi_i\left(R, \mathbf{R}_{-i}^{[z,\underline{x}]}\right) > y$ or there exists $\widehat{z} \in [\overline{x}, \underline{x}]$ such that $\Phi_i\left(R, \mathbf{R}_{-i}^{[\overline{x},\widehat{z}]}\right) < y$.

<u>Proof of the Claim</u>: Assume otherwise; that is, $\Phi_i\left(R, \mathbf{R}_{-i}^{[z,\underline{x}]}\right) \leq y$ for all $z \in [\overline{x}, \underline{x}]$ and $\Phi_i\left(R, \mathbf{R}_{-i}^{[\overline{x},\widehat{z}]}\right) \geq y$ for all $\widehat{z} \in [\overline{x}, \underline{x}]$. Therefore, taking $z = \overline{x}$ and $\widehat{z} = \underline{x}$, $\Phi_i\left(R, \mathbf{R}_{-i}^{[\overline{x},\underline{x}]}\right) = y$. However, the vector $(\alpha_1, ..., \alpha_n)$, where

$$\alpha_j = \begin{cases} \underline{t}(R) & \text{if } j = i \\ \underline{x} & \text{if } j \neq i \end{cases},$$

is feasible and has the property that $\underline{t}(R)Py$ and $\underline{x}R_j^{[\overline{x},\underline{x}]}\Phi_j\left(R, \mathbf{R}_{-i}^{[\overline{x},\underline{x}]}\right)$ for all $j \neq i$. Therefore, $\Phi\left(R, \mathbf{R}_{-i}^{[\overline{x},\underline{x}]}\right) \notin E\left(R, \mathbf{R}_{-i}^{[\overline{x},\underline{x}]}\right)$, contradicting the efficiency of Φ. Therefore, the claim is proved.

Without loss of generality, assume there exists $z \in [\overline{x}, \underline{x}]$ such that $\Phi_i\left(R, \mathbf{R}_{-i}^{[z,\underline{x}]}\right) > y$. By continuity of Φ, and since Lemma 1, part (c), implies $\Phi_i\left(R, \mathbf{R}_{-i}^{\underline{x}}\right) = \underline{t}(R)$, there exists $z' \in [z, \underline{x}]$ such that $\Phi_i\left(R, \mathbf{R}_{-i}^{[z',\underline{x}]}\right) = y$. But again, the vector $(\alpha_1, ..., \alpha_n)$, where

$$\alpha_j = \begin{cases} \underline{t}(R) & \text{if } j = i \\ \underline{x} & \text{if } j \neq i \end{cases},$$

is feasible and has the property that $\alpha_i Py$ and $\alpha_j R_j^{[z',\underline{x}]}\Phi_j\left(R, \mathbf{R}_{-i}^{[z',\underline{x}]}\right)$ for all $j \neq i$. Therefore, $\Phi\left(R, \mathbf{R}_{-i}^{[z',\underline{x}]}\right) \notin E\left(R, \mathbf{R}_{-i}^{[z',\underline{x}]}\right)$, contradicting the efficiency of Φ. ■

By the above lemma we can look at any strategy-proof, efficient, continuous, and tops-only rule Φ on $\mathcal{V}^n$ as just a function $\Phi : \mathcal{I}^n \rightarrow Z$ defined on vectors of n intervals.

Lemma 4 *Let $R \in \mathcal{V}$ and x and y be such that $\frac{M}{n} \leq x < y \leq \underline{t}(R)$. Then, yRx.*

Proof. Assume otherwise; that is, there exist x and y such that $\frac{M}{n} \leq x < y \leq \underline{t}(R)$ and xPy. Consider any $R^M \in \mathcal{R}_s$. Because $\Phi_i(R^M, \mathbf{R}_{-i}^0) = M$, $\Phi_i(R^M, \mathbf{R}_{-i}^{M/n}) = \frac{M}{n}$, by Lemma 2, part (d), and continuity of Φ there exists $z \leq M/n$ such that

$$\Phi_i(R^M, \mathbf{R}_{-i}^z) = y. \tag{8}$$

By strategy-proofness of Φ,

$$\Phi_i(R, \mathbf{R}^z_{-i}) \leq y, \tag{9}$$

otherwise $\Phi_i(R, \mathbf{R}^z_{-i}) P^M \Phi_i(R^M, \mathbf{R}^z_{-i})$.

Because $z \leq \frac{M}{n}$ and Lemma 2, part (a), $\Phi_i(R^0, \mathbf{R}^z_{-i}) = \frac{M}{n}$. By continuity of Φ there exists z' such that $\Phi_i(R^{z'}, \mathbf{R}^z_{-i}) = x$. By strategy-proofness of Φ, $\Phi_i(R, \mathbf{R}^z_{-i}) RxPy$, which imply, by condition (9), $\Phi_i(R, \mathbf{R}^z_{-i}) < y$. Let $R' \in \mathcal{R}_{wsp}(n)$ be such that $t(R') = t(R)$ and $\underline{t}(R') P' \widehat{y} P' \widehat{x}$ for every $\widehat{x} < \widehat{y} < \underline{t}(R')$. Because Φ is tops-only, $\Phi_i(R', \mathbf{R}^z_{-i}) = \Phi_i(R, \mathbf{R}^z_{-i})$. Therefore, by condition (8)

$$\Phi_i\left(R^M, \mathbf{R}^y_{-i}\right) P' \Phi_i\left(R', \mathbf{R}^z_{-i}\right),$$

which contradicts strategy-proofness of Φ. ■

Lemma 5 *Let $R \in \mathcal{V}$, x and y be such that $\frac{M}{n} \leq x < y \leq \min\{\frac{M}{2}, \underline{t}(R)\}$. Then, yPx.*

Proof. First notice that the hypothesis of Lemma 5 imply that $n \geq 3$. Assume $\frac{M}{n} \leq x < y \leq \min\{\frac{M}{2}, \underline{t}(R)\}$ and xRy. By Lemma 4, we may assume that there exist a and b such that $\frac{M}{n} \leq a < b \leq \min\{\frac{M}{2}, \underline{t}(R)\}$ and $aI\widehat{x}$ for all $\widehat{x} \in [a, b]$. By part (b) of Lemma 1, for all $j \geq 3$,

$$\Phi_j\left(R_1^M, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) = 0.$$

Hence, there exists $i \in \{1, 2\}$ such that $\Phi_i\left(R_1^M, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) \geq \frac{M}{2}$. Assume that $i = 1$. Since Φ is strategy-proof,

$$\Phi_1\left(R_1^{\underline{t}(R)}, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) \geq \min\left\{\frac{M}{2}, \underline{t}(R)\right\}. \tag{10}$$

To see it, assume $z' = \Phi_1\left(R_1^{\underline{t}(R)}, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) < \min\left\{\frac{M}{2}, \underline{t}(R)\right\}$ and consider any $R^{\underline{t}(R)} \in \mathcal{R}_s$ such that $\widehat{z} P^{\underline{t}(R)} z'$ for all $\widehat{z} \in \left[\min\left\{\frac{M}{2}, \underline{t}(R)\right\}, M\right]$; then agent 1 would manipulate Φ at profile $\left(R_1^{\underline{t}(R)}, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right)$ with R_1^M. Using again strategy-proofness of Φ, condition (10) implies

$$\Phi_1\left(R_1^{[\underline{t}(R), \bar{t}(R)]}, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) \geq \min\left\{\frac{M}{2}, \underline{t}(R)\right\}.$$

Hence, by tops-onlyness of Φ,

$$\Phi_1\left(R, R_2^M, \mathbf{R}^0_{-\{1,2\}}\right) \geq \min\left\{\frac{M}{2}, \underline{t}(R)\right\}. \tag{11}$$

By Lemma 2, part (c),

$$\Phi\left(R_1^{\underline{t}(R)}, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) = \left(\frac{M}{n}, ..., \frac{M}{n}\right).$$

Since Φ is strategy-proof, $\Phi_1\left(R_1^{[\underline{t}(R),\bar{t}(R)]}, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) = M/n$. Since Φ is tops-only, we may assume that $R^M \in \mathcal{R}_s$ and

$$\Phi_1\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) = \frac{M}{n}. \tag{12}$$

Let $x' \in (a,b)$ be arbitrary. Conditions (11) and (12) and continuity of Φ imply that there exists $z \in [0,M]$ such that $\Phi_1\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^z\right) = x'$; and thus, $\Phi_2\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^z\right) < M$. Therefore, there exists $\epsilon > 0$ sufficiently small such that

$$\left[\Phi_1\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) - \epsilon\right] I \Phi_1\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) \tag{13}$$

and

$$\left[\Phi_2\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) + \epsilon\right] P_2^M \Phi_2\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right). \tag{14}$$

The existence of the feasible vector $(\alpha_1, ..., \alpha_n)$, where

$$\begin{aligned} \alpha_1 &= \left[\Phi_1\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) - \epsilon\right], \\ \alpha_2 &= \left[\Phi_2\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) + \epsilon\right], \text{ and} \\ \alpha_j &= \Phi_j\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) \text{ for all } j \geq 3, \end{aligned}$$

and conditions (13) and (14) imply that $\Phi\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right) \notin E\left(R, R_2^M, \mathbf{R}_{-\{1,2\}}^M\right)$, contradicting the efficiency of Φ. ■

Lemma 6 *Let $R \in \mathcal{V}$ and x be such that $x < \frac{M}{n} < \underline{t}(R)$. Then, $\frac{M}{n} P x$.*

Proof. Assume $x < \frac{M}{n} < \underline{t}(R)$ and $x R \frac{M}{n}$. By Lemma 2, part (c), $\Phi\left(R^{\underline{t}(R)}, \mathbf{R}_{-i}^M\right) = \left(\frac{M}{n}, ..., \frac{M}{n}\right)$. Consider any preference profile $\left(R^{\underline{t}(R)}, \mathbf{R}_{-i}^M\right) \in \mathcal{R}_s^n$ and any preference relation $R^{[\underline{t}(R),\bar{t}(R)]} \in \mathcal{R}_{sp.}$ Since Φ is strategy-proof, $\Phi_i\left(R^{[\underline{t}(R),\bar{t}(R)]}, \mathbf{R}_{-i}^M\right) = \frac{M}{n}$, and by tops-onlyness of Φ, $\Phi_i(R, \mathbf{R}_{-i}^M) = \frac{M}{n}$. Now, for each $j \neq i$ there exists $\alpha_j > 0$ such that $\sum_{j \neq i} \alpha_j = \frac{M}{n} - x$ and $\Phi_j(R, \mathbf{R}_{-i}^M) + \alpha_j \leq M$. Then, the vector of feasible shares $(z_1, ..., z_n)$, where

$$z_j = \begin{cases} x & \text{if } j = i \\ \Phi_j\left(R, \mathbf{R}_{-i}^M\right) + \alpha_j & \text{if } j \neq i \end{cases},$$

has the property that $z_i R \Phi_i\left(R, \mathbf{R}_{-i}^M\right)$ and $z_j P_j^M \Phi_j\left(R, \mathbf{R}_{-i}^M\right)$ for every $j \neq i$, implying that $\Phi\left(R, \mathbf{R}_{-i}^M\right) \notin E\left(R, \mathbf{R}_{-i}^M\right)$, which contradicts the efficiency of Φ. ■

Lemma 7 *Let $R \in \mathcal{V}$, x, and y be such that $\bar{t}(R) < y < x \leq \frac{M}{n}$. Then, yRx.*

Proof. Assume otherwise; that is, there exist x and y such that $\bar{t}(R) < y < x \leq \frac{M}{n}$ and xPy. Since R is continuous we may assume that there exist a and b such that $\bar{t}(R) < a < b \leq \frac{M}{n}$, $aIbP\widehat{x}$ for all $\widehat{x} \in (a,b)$, and $bR\widehat{z}$ for all $\widehat{z} \in (b, \frac{M}{n})$. Let $x' \in (a,b)$ be such that bPx'. By Lemma 2, $\Phi_i(R^0, \mathbf{R}_{-i}^{M/n}) = \frac{M}{n}$, by part (a), and $\Phi_i(R^0, \mathbf{R}_{-i}^{M}) = 0$, by part (d). Therefore, by continuity of Φ there exists $z > \frac{M}{n}$ such that

$$\Phi_i \left(R^0, \mathbf{R}_{-i}^{z}\right) = x'. \tag{15}$$

Consider any preference relation $R^{t(R)} \in \mathcal{R}_{sp}$. Since Φ is strategy-proof,

$$\Phi_i \left(R^{t(R)}, \mathbf{R}_{-i}^{z}\right) \geq x', \tag{16}$$

otherwise, agent i would manipulate Φ at profile $\left(R^{t(R)}, \mathbf{R}_{-i}^{z}\right)$ by declaring R^0. By part (c) of Lemma 2,

$$\Phi_i \left(R^{M}, \mathbf{R}_{-i}^{z}\right) = \frac{M}{n}. \tag{17}$$

By strategy-proofness of Φ,

$$\Phi_i \left(R^{t(R)}, \mathbf{R}_{-i}^{z}\right) \leq \frac{M}{n}. \tag{18}$$

Since Φ is continuous, conditions (15) and (17) imply that there exists $z' \in [0, M]$ such that $\Phi_i \left(R^{z'}, \mathbf{R}_{-i}^{z}\right) = b$. Conditions (16), (18), and tops-onlyness of Φ imply $x' \leq \Phi_i \left(R, \mathbf{R}_{-i}^{z}\right) \leq \frac{M}{n}$. By strategy-proofness of Φ, $\Phi_i \left(R, \mathbf{R}_{-i}^{z}\right) Ib$, implying that $x' < \Phi_i \left(R, \mathbf{R}_{-i}^{z}\right)$. Let $R' \in \mathcal{R}_{sp}$ be any preference relation such that $t(R') = t(R)$. Thus, for every $\bar{t}(R') < \widehat{y} < \widehat{x}$ we have $\bar{t}(R')P'\widehat{y}P'\widehat{x}$. Because Φ is tops-only, $\Phi_i \left(R', \mathbf{R}_{-i}^{z}\right) = \Phi_i \left(R, \mathbf{R}_{-i}^{z}\right)$. But $\Phi_i \left(R^0, \mathbf{R}_{-i}^{z}\right) P' \Phi_i \left(R', \mathbf{R}_{-i}^{z}\right)$ contradicts strategy-proofness of Φ. ■

Lemma 8 *Let $R \in \mathcal{V}$, x, and y be such that $\bar{t}(R) < y < x \leq \frac{M}{n}$ and assume $n \geq 3$ Then, yPx.*

Proof. Assume otherwise; that is, $n \geq 3$ and there exist x and y such that $\bar{t}(R) < y < x \leq \frac{M}{n}$ and xRy. By Lemma 7, we may assume that xIy and there exist a and b such that $\bar{t}(R) < a < b \leq \frac{M}{n}$, $aI\widehat{x}$ for all $\widehat{x} \in [a,b]$, $\widehat{z}Pa$ for all $\bar{t}(R) < \widehat{x} < a$, and $bP\widehat{y}$ for all $b < \widehat{y} < \frac{M}{n}$. Consider any preference relation $R^{t(R)} \in \mathcal{R}_{sp}$. By Lemma 1, part (b), $\Phi_i \left(R^{\bar{t}(R)}, R^{\bar{t}(R)}, \mathbf{R}_{-\{1,2\}}^{M}\right) \leq \underline{t}(R)$ for $i \in \{1,2\}$. By Lemma 2, part (a), $\Phi \left(R^{\bar{t}(R)}, R^{\bar{t}(R)}, \mathbf{R}_{-\{1,2\}}^{0}\right) = \left(\frac{M}{n}, ..., \frac{M}{n}\right)$. By continuity of Φ (and without loss of generality in the roles of agents 1 and 2), there exist z, x', and y' such that $\bar{t}(R) < x' \leq y' < \frac{M}{n}$,

$$\Phi_1\left(R^{\bar{t}(R)}, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) = x'$$

and

$$\Phi_2\left(R^{\bar{t}(R)}, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) = y'.$$

Since $\bar{t}(R) < x'$, Lemma 1, part (a), implies that $2\bar{t}(R)+(n-2)z < M$ and therefore,

$$x' + y' + (n-2)z \leq M. \tag{19}$$

Since Φ is strategy-proof, $\Phi_1\left(R^{[\underline{t}(R),\bar{t}(R]}, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) = x'$, and by tops-onlyness of Φ, $\Phi_1\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) = x'$. Now, if $\Phi_2\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) \neq \bar{t}(R)$ then Φ is not efficient since there exists ϵ sufficiently small (positive if $\Phi_2\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) < \bar{t}(R)$ and negative otherwise) such that the feasible vector $(\alpha_1, ..., \alpha_n)$, where

$$\alpha_j = \begin{cases} x' - \epsilon & \text{if } j = 1 \\ \Phi_j\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) + \epsilon & \text{if } j = 2 \\ \mathbf{\Phi}_j\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) & \text{if } j \geq 3 \end{cases},$$

has the properties that $\alpha_2 P^{\bar{t}(R)} \Phi_2\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right)$, $\alpha_1 I \Phi_1\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right)$, and $\alpha_j I^z_j \Phi_j\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right)$ for all $j \geq 3$; thus, the rule Φ would not be efficient. If $\Phi_2\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) = \bar{t}(R)$, condition (19) implies that there exists an agent $j \geq 3$ such that $\Phi_j\left(R, R^{\bar{t}(R)}, \mathbf{R}^z_{-\{1,2\}}\right) \neq z$, in which case, using a similar argument, Φ would not be efficient. ■

Lemma 9 *Let $R \in \mathcal{V}$ and assume that x is such that $\bar{t}(R) < \frac{M}{n} < x$. Then, $\frac{M}{n} P x$.*

Proof. Let $R \in \mathcal{V}$ and assume otherwise; that is, there exists $x \in \left(\frac{M}{n}, M\right]$ such that $x R \frac{M}{n}$. Consider any preference profile $\mathbf{R}^0_{-i} \in \mathcal{R}_s$. By part (a) of Lemma 2, $\Phi\left(R^{\bar{t}(R)}, \mathbf{R}^0_{-i}\right) = \left(\frac{M}{n}, ..., \frac{M}{n}\right)$. Since the rule Φ is strategy-proof and tops-only, $\Phi_i\left(R^{[\underline{t}(R),\bar{t}(R)]}, \mathbf{R}^0_{-i}\right) = \frac{M}{n}$. Again, by tops-onlyness of Φ, $\Phi_i\left(R, \mathbf{R}^0_{-i}\right) = \frac{M}{n}$. Therefore,

$$\textstyle\sum_{j\neq i} \Phi_j\left(R, \mathbf{R}^0_{-i}\right) = M - \dfrac{M}{n} \geq x - \dfrac{M}{n}.$$

For each $j \neq i$ there exist $\alpha_j \geq 0$ such that $\Phi_j\left(R, \mathbf{R}^0_{-i}\right) - \alpha_j \geq 0$ and $x + \sum_{j\neq i}\left[\Phi_j\left(R, \mathbf{R}^0_{-i}\right) - \alpha_j\right] = M$. Therefore, the vector $\mathbf{z} = (x, \mathbf{z}_{-i})$, where $z_j = \Phi_j\left(R, \mathbf{R}^0_{-i}\right) - \alpha_j$, is feasible and has the property that $z_i I \Phi_i\left(R, \mathbf{R}^0_{-i}\right)$, $z_j R^0_j \Phi_j\left(R, \mathbf{R}^0_{-i}\right)$ for all $j \neq i$, and there exists at least an agent $j' \neq i$ such that $z_{j'} P^0_{j'} \Phi_{j'}\left(R, \mathbf{R}^0_{-i}\right)$. Hence, $\Phi\left(R, \mathbf{R}^0_{-i}\right) \notin E\left(R, \mathbf{R}^0_{-i}\right)$, which contradicts the efficiency of Φ. ■

5 Final Remarks

We close with five remarks. First, we want to illustrate why our domain only admits preference relations with a very special kind of indifference intervals away from the top. Most of these intervals are excluded because our properties impose on the rule the following feature: it is not possible that an agent receives a share on an indifference interval away from the top while another agent is rationed (namely, he is receiving a non-maximal share). Examples 1 and 2 illustrate why this feature excludes some preferences while Example 3 exhibits a preference relation which is not excluded by this feature.

Example 1 Let $M = 10$ and $N = \{1, 2, 3\}$. Assume Φ is a strategy-proof, efficient, tops-only, and continuous rule on $\mathcal{V}^3$. Consider the preference profile $\mathbf{R} = (R_1, R_2, R_3)$ where, for $i = 1, 2$,

$$yP_ix \text{ for all } 0 \leq x < y \leq 10,$$

and

$$yP_3x \text{ for all } 0 \leq x < y \leq 3 \text{ and all } 4 \leq x < y \leq 10,$$

and

$$yI_3x \text{ for all } x, y \in [3, 4].$$

By Lemma 2, part (c), $\Phi(10, 10, 10) = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right)$. Since Φ is tops-only, $\Phi(R_1, R_2, R_3) = \left(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}\right)$. The vector $(3.5, 3.5, 3)$ is feasible and $3.5P_i\Phi_i(R_1, R_2, R_3)$ for $i = 1, 2$ and $3I_3\Phi_3(R_1, R_2, R_3)$. Therefore, $\Phi(R_1, R_2, R_3) \notin E(R_1, R_2, R_3)$, which contradicts the efficiency of Φ. Hence, R_3 can not belong to $\mathcal{V}$. Note that R_3 does not satisfy condition (b.2) of Definition 7 and therefore, $R_3 \notin \mathcal{R}_{wsp}(3)$.

Example 2 Let $M = 10$ and $N = \{1, 2\}$. Assume Φ is a strategy-proof, efficient, tops-only, and continuous rule on $\mathcal{V}^2$. Consider the preference profile $\mathbf{R} = (R_1, R_2)$, where,

$$yP_1x \text{ for all } 0 \leq y < x \leq 10,$$

$$yP_2x \text{ for all } 0 \leq y < x \leq 3 \text{ and all } 7 \leq y < x \leq 10,$$

and

$$yI_2x \text{ for all } x, y \in [3, 7].$$

By Lemma 2, part (a), $\Phi(0, 0) = (5, 5)$. Since Φ is tops-only, $\Phi(R_1, R_2) = (5, 5)$. The vector $(3, 7)$ is feasible and $3P_1\Phi_1(R_1, R_2)$ and $7I_2\Phi_2(R_1, R_2)$. Therefore, $\Phi(R_1, R_2) \notin E(R_1, R_2)$, which contradicts the efficiency of Φ. Hence, R_2 can not belong to

$\mathcal{V}$. Note that R_2 does not satisfy condition (c.2) of Definition 7 and therefore, $R_2 \notin \mathcal{R}_{wsp}(2)$.

Example 3 Let $M = 10$ and $N = \{1, 2\}$. Consider the extended uniform rule U on $\mathcal{V}^2$. Consider the preference relation R_1, where

$$xP_1y \text{ for all } 0 \leq y < x \leq 6 \text{ and all } 7 \leq y < x \leq 10$$

and

$$yI_1x \text{ for all } x, y \in [6, 7].$$

Let $R_2 \in \mathcal{V}$ be any preference relation such that $U_1(R_1, R_2) \in (6, 7)$. We want to show that $U_2(R_1, R_2) \in t(R_2)$. To see it, observe that $U_1(R_1, R_2) < \underline{t}(R_1)$ implies, by part (b) of Lemma 1, $U_2(R_1, R_2) \leq \underline{t}(R_2)$. To obtain a contradiction, assume $U_2(R_1, R_2) < \underline{t}(R_2)$. Take any $\overline{R}_2 \in \mathcal{R}_{sp}$ with the properties that $t\left(\overline{R}_2\right) = t(R_2)$ and $5\overline{P}_2U_2(R_1, R_2)$. Since $U(10, 10) = (5, 5)$ and U is tops-only, $U(R_1, \widehat{R}_2) = (5, 5)$ whenever $t(\widehat{R}_2) = 10$. But then, agent 2 manipulates U at $\left(R_1, \overline{R}_2\right)$ by declaring any $\widehat{R}_2$ with $t(\widehat{R}_2) = 10$.

Second, parts (a) and (c) of Lemma 2 suggest that the properties of strategy-proofness, efficiency, tops-onlyness, and continuity imply (surprisingly) some partial anonymity condition. But this is only apparent. It is a consequence of the following two unrelated reasons: (i) An indirect anonymity axiom is assumed when we impose the natural condition that the domain of the rule be the Cartesian product of the same set of preferences for all agents. Otherwise, given an arbitrary feasible vector $(y_1, ..., y_n)$, we could define agent i's specific maximal set of preferences by replacing in Definition 7 the share $\frac{M}{n}$ by y_i. Then, the statement of Lemma 2 would also hold after replacing the role of the vector of equal shares $\left(\frac{M}{n}, ..., \frac{M}{n}\right)$ by $(y_1, ..., y_n)$. (ii) We are looking for a maximal domain of preferences. There are other domains of preferences (for instance, the set of preferences satisfying Definition 7 after replacing $\frac{M}{n}$ by $\frac{M}{2n}$ in condition (b.1)) under which we could have a vector $(x_1, ..., x_n)$ with the property that $x_i \geq \frac{M}{n}$ for all i, but $\frac{M}{2n} \leq \Phi_j(x_1, ..., x_n) < \frac{M}{n}$ for some j; hence, Φ does not satisfy the partial anonymity condition of part (c) of Lemma 2. However, this domain of preferences would not be maximal since it is smaller than $\mathcal{R}_{wsp}(n)$ because these preferences are all strictly monotonic between $\frac{M}{2n}$ and $\frac{M}{n}$,[9] while preferences in $\mathcal{R}_{wsp}(n)$ do not have to. This lack of anonymity (or symmetry) suggests, though, the possibility that our list of properties admits, for the domain of weakly single-

[9]The same argument would apply if we replace $\frac{M}{n}$ in Definition 7 by any smaller share.

plateaued preferences, a large class of functions (and not only the extended uniform rule) satisfying our list of properties.[10]

Third, the maximal domain of preferences identified in Ching and Serizawa (1998) and Massó and Neme (2001) is, in each case, the unique one containing the set of single-peaked preferences. In contrast (and as another consequence of not requiring any property related to symmetry or anonymity), the set of weakly single-plateaued preferences is not the unique one *containing* the set of single-peaked preferences for our list of properties. To see it, consider the domain of preferences $\mathcal{R}_L \subseteq \mathcal{R}$ where $R_i \in \mathcal{R}_L$ if and only if:

$$yP_ix \text{ for all } 0 \leq x < y \leq \underline{t}(R_i).$$

Let $N = \{1, 2, 3\}$ and consider the rule ψ on $(\mathcal{R}_L)^3$ defined as follows: for all $\mathbf{R} \in (\mathcal{R}_L)^3$,

$$\begin{aligned}
\psi_1(\mathbf{R}) &= \underline{t}(R_1), \\
\psi_2(\mathbf{R}) &= \max\{M - \underline{t}(R_1), \underline{t}(R_2)\}, \text{ and} \\
\psi_3(\mathbf{R}) &= M - \psi_1(\mathbf{R}) - \psi_2(\mathbf{R}).
\end{aligned}$$

It is easy to check that $\psi : (\mathcal{R}_L)^3 \to Z$ is strategy-proof, efficient, tops-only, and continuous. Moreover, it is possible to show that $\mathcal{R}_L$ is a maximal domain of preferences for this list of properties.

Fourth, the intersection of all our maximal domains of preferences, fixed M and when n varies from two to infinity, is strictly larger than the set of single-plateaued preferences; namely, $\bigcap_{n \geq 2} \mathcal{R}_{wsp}(n) \supsetneqq \mathcal{R}_{sp}$. To see that, observe that there are many preference relations that are weakly single-plateaued for all $n \geq 2$ but they are not single-plateaued.

Fifth, the intersection of all our maximal domains of preferences, fixed $n \geq 2$ and when M varies from zero to infinity, coincides with the single-plateaued domain. This implies that, when the rule depends not only on preferences but also on the amount M to be allocated (as in Ching and Serizawa, 1998), the maximal domain coincides with the set of single-plateaued preferences as already shown by Ching and Serizawa (1998) for the properties of strategy-proofness, efficiency, and symmetry.

[10] Barberà, Jackson, and Neme (1997) shows that, on the single-peaked domain, the class is larger even if we add replacement monotonicity to our list of properties.

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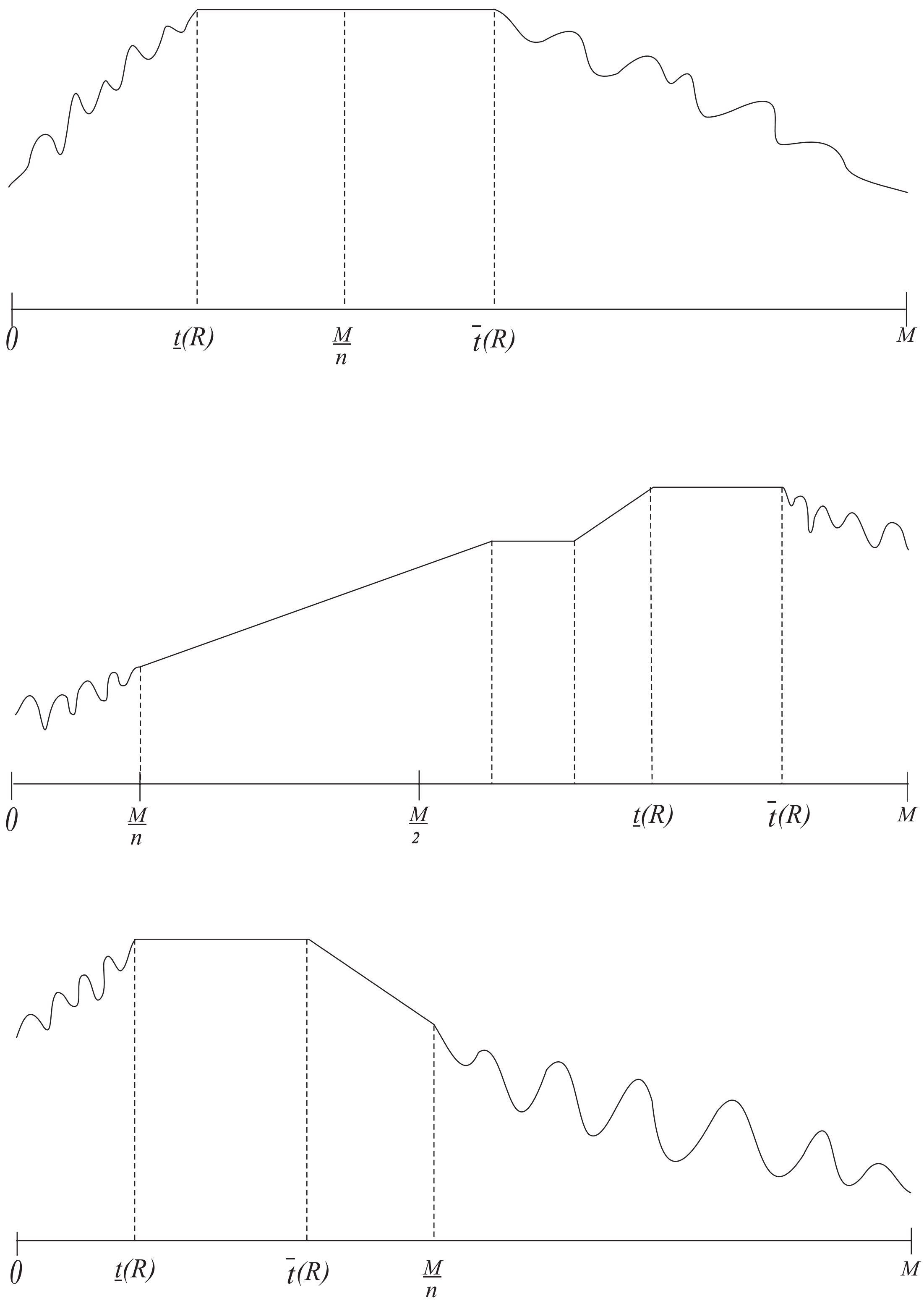
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Figure 1

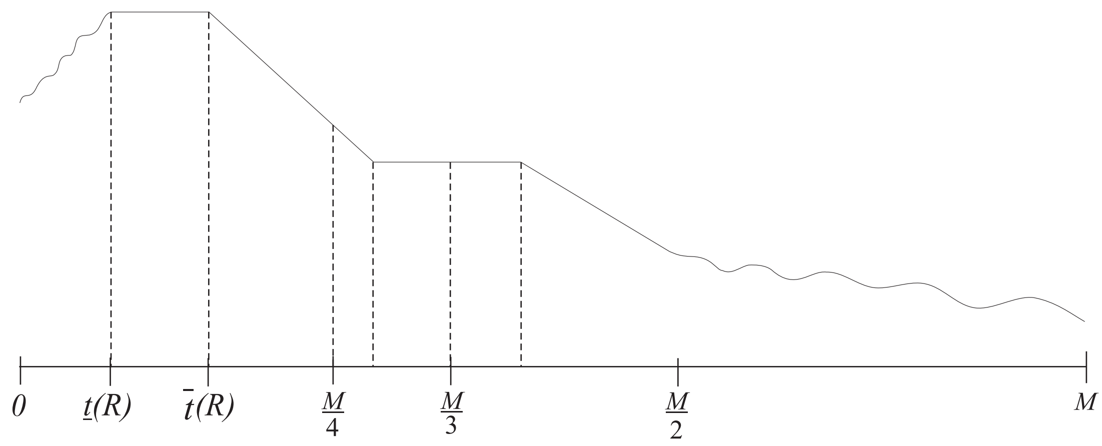

Figure 2