Pythagorean Fuzzy Linguistic Muirhead Mean Operators and Their Applications to Multiattribute Decision Making

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Abstract

Pythagorean fuzzy set (PFS), as an extension of intuitionistic fuzzy set (IFS) to deal with uncertainty, has attracted much attention since its introduction, in both theory and application aspects. In this paper, we investigate the multiple attribute decision making (MADM) problems with the Pythagorean linguistic information based on some new aggregation operators. To begin with, we present some new Pythagorean fuzzy linguistic Muirhead mean operators to deal with MADM problems with Pythagorean fuzzy linguistic information, including the Pythagorean fuzzy linguistic Muirhead Mean (PFLMM) operator, the Pythagorean fuzzy linguistic weighted Muirhead Mean (PFLWMM) operator, the Pythagorean fuzzy linguistic dual Muirhead Mean (PFLDMM) operator and the Pythagorean fuzzy linguistic dual weighted Muirhead Mean (PFLDWMM) operator, the main advantages of these aggregation operators are that they can capture interrelationships of multiple attributes among any number of attributes by a parameter vector P and make information aggregation process more flexible by the parameter vector P. In addition, the some properties of these new aggregation operators are proved and some special cases are discussed where the parameter vector takes some different values. Moreover, we present two new methods to solve the MADM problems with Pythagorean fuzzy linguistic information. Finally, an illustrative example is provided to show the feasibility and validity of the new methods, investigate the influences of parameter vector P on the decision making results and also analyze the advantages of proposed methods by comparing with the other existing methods.

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1. INTRODUCTION

Multiple attribute decision making (MADM), as an effective framework for comparison, has always been used to find the most desirable one from a finite set of alternatives on the predefined attributes. An important problem of decision process is to express the attribute value. However, due to the intrinsic complexity of natural objects, there exists much uncertain information in many real-world problems. So, it is difficult for experts or decision makers (DMs) to give their assessments on attributes by crisp numbers. Intuitionistic fuzzy set (IFS) [1], is an effective tool to express the complex fuzzy information due to it is characterized by three parameters, namely, a membership degree, a nonmembership degree and an indeterminacy degree. That is, an IFS A in a finite universe of discourse X has such a structure $A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | x \in X\}$, where μ_A represents the membership degree and ν_A is the nonmembership degree with the condition that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. Since IFS's appearance, it becomes a powerful tool to deal with some information with imprecision, uncertainty and vagueness. However, Yager [43, 44] pointed out that there exists such a kind of useful extension of IFS $A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | x \in X\}$ which satisfies the condition $0 \leq \mu_A^2(x) + \nu_A^2(x) \leq 1$. Such a useful extension of IFSs is called Pythagorean fuzzy set (PFS). The main difference between the IFSs and PFSs focuses on the membership degrees and the nonmembership degrees of them. Therefore, it follows from the above analysis of IFSs and PFSs that PFS has more powerful ability than IFS to deal with uncertain information in MADM problems. Since PFS was proposed, a lot of research achievements about theory and methods have been made, and it has three aspects: (A) the basic theory, such as the operational laws [30, 31], comparison method [28], distance [13, 21], similarity degree [49], correlation measure [5], information measure [33], and other properites [9]; (B) the extended traditional MADM or MAGDM methods for PFS, such as Stochastic MCDM method [27], MABAC method [29], TODIM method [37, 40], Mathematical programming method [38], QUALIFLEX [48], TOPSIS method [50] and so on; (C) the MADM or MAGDM methods [4, 6, 7, 8, 14, 15, 23, 36, 41, 45] based on Pythagorean fuzzy aggregation operators.

In the field of information fusion, information aggregation is an important research topic as it is a critical process of gathering relevant information from multiple sources. However, aggregation operator as a tool to aggregate relevant information has been focused and also used in many decision making problems. The main advantage of decision methods based on aggregation operator is that these methods can not only give the ranking information but also provide the comprehensive values of the alternatives. Due to the increasing complexity of the real worlds, numerical numbers may not always be adequate to solve the uncertain and fuzzy information in practical decision making problems, especially for qualitative aspects, while it is easy to provide the assessment values taking the form of linguistic variables. Therefore, some linguistic decision making methods are developed [10, 11, 24, 25, 42]. Based on the idea of intuitionistic fuzzy set, Wang [39] proposed the intuitionistic linguistic set (ILS), which uses an intuition-

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istic fuzzy number (IFN) to describe the membership and non-membership degree of a linguistic variable simultaneously. Zhang [46] proposed MAGDM method based on linguistic intuitionistic fuzzy numbers. Many corresponding decision making methods were proposed based on some intuitionistic fuzzy linguistic aggregation operator, such as, Chen [3] proposed a MADM methods based on linguistic inituitionistic fuzzy numbers, Jun [12] proposed MADM method based on intuitionistic linguistic Maclaurin symmetric mean aggregation (ILMSM) operators, Liu [16, 17, 19, 20] proposed intuitionistic linguistic geometric aggregation (ILGA)operators, intuitionistic linguistic weighted Bonferroni mean (ILWBM) operator, improved intuitionistic linguistic fuzzy aggregation operator and applied to MADM or MAGDM problems, Zhang [47] proposed extended outranking approach under linguistic intuitionistic fuzzy environment. It is obvious that the ILS is an efficient approximate technique to deal with the uncertain and fuzzy information by integrating the advantages of IFS and linguistic variables. Motivated by the idea of linguistic variables and PFSs, Peng [32] proposed Pythagorean fuzzy linguistic term (PFLT) and applied to MADM problems. Some novel linguistic decision making methods based on Pythagorean fuzzy set have been developed, such as, Liu [18] proposed MCDM decision making based on Pythagorean fuzzy uncertain linguistic aggregation operators, Du [4] proposed novel MADM method with interval-value Pythagorean fuzzy linguistic information. In addition, some decision methods based on 2-tuple linguistic [2, 35] and 2-dimension linguistic aggregation operator [22] are also developed.

Muirhead mean (MM) [26] is a well-known aggregation operator for it can consider the interrelationships among any number of aggregation arguments and it also a universal operator since it contain other general operators by assessing different parameter vectors. when the parameter vector is assess different values, MM reduced to some existing operators, such as arithmetic and geometric operators which do not consider the interrelationships of aggregation arguments, Maclaurin symmetric mean [34, 41], are the special cases of MM operator. So, some extended MM operators [18, 35] have been developed and applied to solve the MAGDM problems. Because PFNs have stronger abilities than IFSs in describing the information, linguistic variables are more suitable to describe practical problems that are ill-defined by using quantitative information and the MM can capture interrelationships among multi-input arguments assigned by a variable vector. Therefore, it is necessary and significant to develop some new linguistic aggregation operators based on MM that not only accommodate Pythagorean linguistic information but also can capture the interrelationships among multi-input arguments.

The goal of this paper is to develop some methods for MADM problems with Pythagorean fuzzy linguistic information based on some new Pythagorean fuzzy linguistic MM (PFLMM) operators by combining MM and Pythagorean fuzzy linguistic information. To begin with, some new Pythagorean fuzzy linguistic Muirhead mean operators to deal with MADM problems with Pythagorean fuzzy linguistic information, included the Pythagorean fuzzy linguistic Muirhead Mean (PFLMM) operator, the Pythagorean fuzzy linguistic weighted Muirhead Mean (PFLWMM) operator, the Pythagorean fuzzy linguistic dual Muirhead Mean (PFLDMM) operator, the Pythagorean fuzzy linguistic dual weighted Muirhead Mean (PFLDWMM) operator, are presented. In addition, some properties of these new aggregation operators are proved and some special cases are discussed. Finally, two new methods are presented to solve an MADM problem with Pythagorean fuzzy linguistic information. To do so, the rest of the paper is organized as follows. In Section 2, we review some definitions on PFSs, PFLNs and Muirhead mean, which are used in the analysis throughout this paper. Section 3 is devoted to the main results concerning PFLMM operator and PFLWMM operator along with their properties. Section 4 is focused on PFLDMM operator and PLDWMM operator along with their properties. In Section 5, we construct MADM approaches based on PFLWMM operator and PFLDWMM operator proposed in Section 3 and Section 4. Consequently, a practical example is provided in Section 6 to verify the validity of the proposed methods and to show their advantages. In Section 7, we give some conclusions of this study.

2. PRELIMINARIES

In this section, some basic concepts related to PFS, Pythagorean fuzzy linguistic set and Muirhead mean are recapped, which are the basis of this work.

2.1. Pythagorean fuzzy set

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of discourse, an intuitionistic fuzzy set (IFS) [1] A in X characterized by a membership function $\mu_A : X \to [0, 1]$ and a nonmembership function $\nu_A : X \to [0, 1]$, which satisfy the condition $0 \le \mu_A(x) + \nu_A(x) \le 1$. An IFS A can be expressed as

$$A = \{\langle x, (\mu_A(x), \nu_A(x))\rangle | x \in X\}.$$

$\pi_A(x) = 1-\mu_A(x)-\nu_A(x)$ is called the degree of indeterminacy. For convenience, called $(\mu_A(x), \nu_A(x))$ is an intuitionistic fuzzy number (IFN) and denoted by (μ_A, ν_A).

However, there are some decision-making problems in which the DMs or the experts' attitudes are possibly not suitable to be described by applying an IFS. Under such situations, Pythagorean fuzzy set (PFS), introduced by Yager[44], which is a novel concept to deal with this situation and also an extension of IFS:

In a finite universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$, a PFS P with the structure

$$P = \{\langle x, (\mu_P(x), \nu_P(x))\rangle | x \in X\}.$$

where $\mu_P : X \to [0, 1]$ denotes the membership degree and $\nu_P : X \to [0, 1]$ denotes the nonmembership degree of the element $x \in X$ to the set P, respectively, with the condition that $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1$. $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is called the degree of indeterminacy. For the convenience, Zhang and Xu[50] called $p = (\mu_p(x), \nu_p(x))$ a Pythagorean fuzzy number (PFN) denoted by $p = (\mu_p, \nu_p)$.

2.2. The Pythagorean Fuzzy Linguistic Set

Let $S = \{s_0, s_1, \cdots, s_g\}$ be a finite linguistic term set with odd cardinality, where s_i represents a possible value for linguistic term, $g + 1$ is the cardinality of S. For example, $S = \{s_0 = \text{e}xtremely\ poor, s_1 = \text{v}ery\ poor, s_2 = \text{p}oor, s_3 = \text{f}air, s_4 = \text{g}ood, s_5 = \text{v}ery\ good, s_6 = \text{e}xtremely\ good\}$. Obviously, the mid linguistic term represents an assessment of "indifference", and the rest of other linguistic labels are placed by symmetrically around it.

Let s_i and s_j be any two linguistic numbers in linguistic set S, they must satisfy the following properties[10, 11]:

(1) If $i > j$, then $s_i \succ s_j$;

(2) There exists negative operator: $Neg(s_i) = s_j$, such that $j = gCi$;

(3) If $s_i \succ s_j$, $max(s_i, s_j) = s_i$ and $min(s_i, s_j) = s_j$.

To preserve all the given information, the discrete linguistic term set S can be extended to a continuous linguistic term set$\bar{S} = \{s_\alpha | \alpha \in [0, g]\}$. If $s_\alpha \in S$, then we call s_α the original linguistic term; $s_\alpha \notin S$, we call s_α the virtual linguistic term. In general, the decision makers use the original linguistic terms to evaluate alternatives, and the virtual linguistic terms can only appear in calculation[42].

Now, we recall some definitions of Pythagorean fuzzy linguistic term set.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite nonempty universe of discourse and $\bar{S}$ be a continuous linguistic term set of $s = \{s_0, s_1, \cdots, s_g\}$, a Pythagorean fuzzy linguistic set (PFLS) P on X with the structure

$$P = \{\langle x, s_{\vartheta(x)}, (\mu_P(x), \nu_P(x))\rangle | x \in X\}.$$

where $s_{\vartheta(x)} \in \bar{S}$, $\mu_P : X \rightarrow [0, 1]$ denotes the membership degree and $\nu_P : X \rightarrow [0, 1]$ denotes the nonmembership degree of the element $x \in X$ to the linguistic term $s_{\vartheta(x)}$, respectively, with the condition that $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$. $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is called the degree of indeterminacy to linguistic term $s_{\vartheta(x)}$. For the convenience, we note a Pythagorean fuzzy linguistic number (PFLN) as $a = \langle s_{\vartheta(a)}, (\mu(a), \nu(a))\rangle$.

Obviously, if $0 \leq \mu_P(x)) + (\nu_P(x) \leq 1$, a PFLN is reduced to an intuitionistic fuzzy linguistic number (IFLN).

Let $a_1 = \langle s_{\vartheta_1}, (\mu_1, \nu_1)\rangle$ and $a_2 = \langle s_{\vartheta_2}, (\mu_2, \nu_2)\rangle$ be any two PFLNs and $\lambda \geq 0$, the

$$(1) a_1 \oplus a_2 = \langle s_{\vartheta_1+\vartheta_2}, (\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2);$$

$$(2) a_1 \otimes a_2 = \langle s_{\vartheta_1\times\vartheta_2}, (\mu_1\mu_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2});$$

$$(3) \lambda a_1 = \langle s_{\lambda\times\vartheta_1}, (\sqrt{1 - (1 - \mu_1^2)^\lambda}, \nu_1^\lambda);$$

$$(4) a_1^\lambda = \langle s_{\vartheta_1^\lambda}, (\mu_1^\lambda, \sqrt{1 - (1 - \nu_1^2)^\lambda}).$$

Let $a = \langle s_{\vartheta(a)}, (\mu(a), \nu(a))\rangle$ be a PFLN, the score of a can be evaluated by a new score function $S(a)$, which is shown as

$$S(a) = \frac{1}{2}(\mu(a)^2 + 1 - \nu(a)^2) \times s_{\vartheta(a)}. \tag{1}$$

The larger the score value of $S(a)$, the greater the PFLN a.

Let $a = \langle s_{\vartheta(a)}, (\mu(a), \nu(a))\rangle$ be a PFLN, the degree of accuracy of a can be evaluated by a new accuracy function $H(a)$, which is shown as

$$H(a) = \frac{1}{2}(\mu(a)^2 + \nu(a)^2) \times s_{\vartheta(a)}. \tag{2}$$

The larger the degree of accuracy of $S(a)$, the greater the PFLN a.

Based on the score function S and accuracy function H, the comparison rules between two PFLNs are given as follows:

Let $a_1 = \langle s_{\vartheta_1}, (\mu_1, \nu_1)\rangle$ and $a_2 = \langle s_{\vartheta_2}, (\mu_2, \nu_2)\rangle$ be any two PFLNs, then

(1) If $S(a_1) \leq S(a_2)$, then $a_1 \prec a_2$;

(2) If $S(a_1) = S(a_2)$, then

(2.1) If $H(a_1) \leq H(a_2)$, then $a_1 \prec a_2$;

(2.2) If $H(a_1) = H(a_2)$, then $a_1 = a_2$.

2.3. Muirhead Mean Operator

The Muirhead mean (MM) operator [26] is a general aggregation function and firstly proposed by Muirhead in 1902, it is defined as follows:

DEFINITION 1. Let $a_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative real numbers, $A = \{a_1, a_2, \cdots, a_n\}$ and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector, if

$$MM^P(a_1, \cdots, a_n) = \Big(\frac{1}{n!}\Big(\sum_{\theta\in S_n}\Big(\prod_{j=1}^{n} a_{\theta(j)}^{p_j}\Big)\Big)\Big)^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{3}$$

The we call MM^P the Muirhead mean (MM), where $\theta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$ and S_n is the collection of all permutation of $\theta(j)(j = 1, 2, \cdots, n)$. There are some special cases when the parameter vector assessed different values.

(1) If $P = (1, 0, \cdots, 0)$, MM operator will reduces to arithmetic averaging operator

$$MM^{(1,0,\cdots,0)}(a_1, \cdots, a_n) = \frac{1}{n}\sum_{j=1}^{n} a_j. \tag{4}$$

(2) If $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})$, PFLMM operator will reduces to Maclaurin symmetric mean (MSM) operator

$$PFLMM^{(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})}(a_1, \cdots, a_n) = \Big(\frac{\sum_{1\leq i_1\leq\cdots\leq i_k\leq n}\prod_{j=1}^{k} a_j}{C_n^k}\Big)^{\frac{1}{k}}; \tag{5}$$

(3) If $P = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, MM operator will reduces to geometric averaging operator

$$MM^{(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})}(a_1, \cdots, a_n) = \prod_{j=1}^{n} a_j^{\frac{1}{n}}. \tag{6}$$

From the above discussion we can see that the advantage of the MM operator is that it can capture the interrelationships among the multiple aggregated arguments and it is a generalization of most existing aggregation operators.

3. PYTHAGOREAN FUZZY LINGUISTIC WEIGHTED MUIRHEAD MEAN OPERATORS

Because the traditional MM can only process the crisp number, and PFLNs can easily express the fuzzy information, it is necessary and significant to extend MM to process PFLNs. In this section, we propose the Pythagorean fuzzy linguistic Muirhead mean (PFLMM) operator and the Pythagorean fuzzy linguistic weighted Muirhead mean (PFLWMM) operator, and discuss the properties of these operators.

3.1. Pythagorean Fuzzy Linguistic Muirhead Mean Operators

DEFINITION 2. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $A = \{a_1, a_2, \cdots, a_n\}$ and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a Pythagorean fuzzy linguistic Muirhead mean operator is a function $\text{PFLMM}^P\text{:}A^n \to A$, and

$$PFLMM^P(a_1, \cdots, a_n) = (\frac{1}{n!}(\oplus_{\theta \in S_n}(\otimes_{j=1}^n a_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}, \tag{7}$$

where $\theta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$ and S_n is the collection of all permutation of $\theta(j)(j = 1, 2, \cdots, n)$.

THEOREM 1. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $PFLMM^P(a_1, \cdots, a_n)$ is still a PFLN and

$$\begin{aligned} PFLMM^P(a_1, \cdots, a_n) \quad &= \langle s_{(\frac{1}{n!}(\sum_{\theta \in S_n}(\prod_{j=1}^n \vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}}, \\ &((\sqrt{1 - (\prod_{\theta \in S_n}(1 - (\prod_{j=1}^n \mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}}, \\ &\sqrt{1 - (1 - (\prod_{\theta \in S_n}(1 - \prod_{j=1}^n (1 - \nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}})\rangle. \end{aligned} \tag{8}$$

Proof. Firstly, we prove Eq. (8). According to the operational law of PFLNs, we obtain

$$(a_{\theta(j)})^{p_j} = \langle s_{\vartheta_{\theta(j)}^{p_j}}, (\mu_{\theta(j)}^{p_j}, \sqrt{1 - (1 - \nu_{\theta(j)}^2)^{p_j}})\rangle, \quad and$$

$$\otimes_{j=1}^n a_{\theta(j)^{p_j}} = \langle s_{\prod_{j=1}^n \vartheta_{\theta(j)}^{p_j}}, (\prod_{j=1}^n \mu_{\theta(j)}^{p_j}, \sqrt{1 - \prod_{j=1}^n (1 - \nu_{\theta(j)}^2)^{p_j}})\rangle,$$

then we get

$$\begin{aligned} \oplus_{\theta \in S_n} \otimes_{j=1}^n a_{\theta(j)^{p_j}} \quad &= \langle s_{\sum_{\theta \in S_n} \prod_{j=1}^n \vartheta_{\theta(j)}^{p_j}}, \\ &(\sqrt{1 - \prod_{\theta \in S_n}(1 - (\prod_{j=1}^n \mu_{\theta(j)}^{p_j})^2)}, \prod_{\theta \in S_n} \sqrt{1 - \prod_{j=1}^n (1 - \nu_{\theta(j)}^2)^{p_j}})\rangle, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{n!} \oplus_{\theta \in S_n} \otimes_{j=1}^n a_{\theta(j)^{p_j}} \quad &= \langle s_{\frac{1}{n!} \sum_{\theta \in S_n} \prod_{j=1}^n \vartheta_{\theta(j)}^{p_j}}, \\ &(\sqrt{1 - (\prod_{\theta \in S_n}(1 - (\prod_{j=1}^n \mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}}, (\prod_{\theta \in S_n} \sqrt{1 - \prod_{j=1}^n (1 - \nu_{\theta(j)}^2)^{p_j}})^{\frac{1}{n!}})\rangle. \end{aligned}$$

Therefore,

$$
\begin{aligned}
(\frac{1}{n!}\oplus_{\theta\in S_n}\otimes_{j=1}^{n} a_{\theta(j)^{p_j}})^{\frac{1}{\sum_{j=1}^{n} p_j}} &= \langle s_{(\frac{1}{n!}\sum_{\theta\in S_n}\prod_{j=1}^{n}\vartheta_{\theta(j)}^{p_j})^{\frac{1}{\sum_{j=1}^{n}p_j}}}, \\
&((\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^{n}p_j}}, \\
&\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}})\rangle.
\end{aligned}
$$

In addition, we need to prove $PFLMM^P(a_1,\cdots,a_n)$ is also a PFLN.

Since $\mu_{\theta(j)}\in[0,1]$, we have $\mu_{\theta(j)}^{p_j}\in[0,1]$ and $(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2\in[0,1]$. And then

$$
1-(\prod_{j=1}^{n}\mu_{a_{\theta(j)}}^{p_j})^2\in[0,1] \ \ and \prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2)^{\frac{1}{n!}}\in[0,1]
$$

And so,

$$
\sqrt{(1-\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}})}^{\frac{1}{\sum_{j=1}^{n}p_j}}\in[0,1]
$$

Similarly,

$$
\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}}\in[0,1]
$$

Let

$$
\mu=\sqrt{(1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}})}^{\frac{1}{\sum_{j=1}^{n}p_j}},
$$

$$
\nu=\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}};
$$

that is, $\mu,\nu\in[0,1]$.

Now we need to prove $\mu^2+\nu^2\in[0,1]$.

Since $\mu_{\theta(j)}^2 + \nu_{\theta(j)}^2 \leq 1$, then $\mu_{\theta(j)}^2 \leq 1 - \nu_{\theta(j)}^2$. Furthermore, we have

$$
\begin{aligned}
\mu^2 + \nu^2 \quad & = (1 - (\prod_{\theta \in S_n} (1 - (\prod_{j=1}^{n} \mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}} \\
& +1 - (1 - (\prod_{\theta \in S_n} (1 - \prod_{j=1}^{n} (1 - \nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}} \\
& \leq (1 - (\prod_{\theta \in S_n} (1 - (\prod_{j=1}^{n} (1 - \nu_{\theta(j)}^2)^{p_j})))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}} \\
& +1 - (1 - (\prod_{\theta \in S_n} (1 - \prod_{j=1}^{n} (1 - \nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}} = 1.
\end{aligned}
$$

That is, $\mu^2 + \nu^2 \in [0, 1]$. Obviously, $s_{(\frac{1}{n!}(\sum_{\theta \in S_n}(\prod_{j=1}^{n} \vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^{n} p_j}}} \in S$. Hence, $PFLMM^P(a_1, \cdots, a_n)$ is also a PFLN.

EXAMPLE 1. Let $a_1 = \langle s_2, (0.5, 0.3) \rangle$, $a_2 = \langle s_4, (0.7, 0.5) \rangle$, $a_3 = \langle s_3, (0.8, 0.2) \rangle$ and $P = (1, 0.5, 0.4)$. Let

$$
s_{(\frac{1}{3!}(\sum_{\theta \in S_3}(\prod_{j=1}^{3} \vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^{3} p_j}}} = s_b
$$

where

$$
\begin{aligned}
b \quad & = (\frac{1}{6} \times (2 \times 4^{0.5} \times 3^{0.4} + 2 \times 3^{0.5} \times 4^{0.4} + 4 \times 2^{0.5} \times 3^{0.4} \\
& +4 \times 3^{0.5} \times 2^{0.4} + 3 \times 4^{0.5} \times 2^{0.4} + 3 \times 2^{0.5} \times 4^{0.4}))^{\frac{1}{1+0.5+0.4}} \\
& = 2.9033.
\end{aligned}
$$

Therefore,

$$
s_{(\frac{1}{3!}(\sum_{\theta \in S_3}(\prod_{j=1}^{3} \vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^{3} p_j}}} = s_{2.9033}.
$$

Since

$$
\begin{aligned}
& (\sqrt{1 - (\prod_{\theta \in S_3} (1 - (\prod_{j=1}^{3} \mu_{\theta(j)}^{p_j})^2))^{\frac{1}{3!}}})^{\frac{1}{\sum_{j=1}^{3} p_j}} = ((1 - ((1 - (0.5 \times 0.7^{0.5} \times 0.8^{0.4})^2) \\
& \times (1 - (0.5 \times 0.8^{0.5} \times 0.7^{0.4})^2) \times (1 - (0.7 \times 0.5^{0.5} \times 0.8^{0.4})^2) \\
& \times (1 - (0.7 \times 0.8^{0.5} \times 0.5^{0.4})^2) \times (1 - (0.8 \times 0.5^{0.5} \times 0.7^{0.4})^2) \\
& \times (1 - (0.8 \times 0.7^{0.5} \times 0.5^{0.4})^2))^{\frac{1}{6}})^{\frac{1}{2}})^{\frac{1}{1+0.5+0.4}} = 0.6592
\end{aligned}
$$

and

$$\begin{aligned}&\sqrt{1-(1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{3!}})^{\frac{1}{\sum_{j=1}^{3}p_j}}} = (1-(1-((1-(1-0.3^2)\\
&\times(1-0.5^2)^{0.5}\times(1-0.2^2)^{0.4})\times(1-(1-0.3^2)\times(1-0.2^2)^{0.5}\times(1-0.5^2)^{0.4})\\
&\times(1-(1-0.5^2)\times(1-0.3^2)^{0.5}\times(1-0.2^2)^{0.4})\times(1-(1-0.5^2)\times(1-0.2^2)^{0.5}\\
&\times(1-0.3^2)^{0.4})\times(1-(1-0.2^2)\times(1-0.3^2)^{0.5}\times(1-0.5^2)^{0.4})\times(1-(1-0.2^2)\\
&\times(1-0.5^2)^{0.5}\times(1-0.3^2)^{0.4}))^{\frac{1}{3!}})^{\frac{1}{1+0.5+0.4}})^{\frac{1}{2}} = 0.3581.\end{aligned}$$

So, $PFLMM^P(a_1, a_2, a_3) = \langle s_{2.9033}, (0.6592, 0.3581)\rangle$.

In the process of decision making, the aggregation results would be more reliable if the selected operator is monotonic, the lack of monotonicity may debase the reliability and dependability of the final decision-making results. However, we can prove $PFLMM^P(a_1, \cdots, a_n)$ are idempotent, bounded, and monotonic.

PROPERTY 1 (IDEMPOTENCY). Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector and all $a_i (i = 1, 2, \cdots, n)$ are equal, i.e., $a_i = a = \langle s_\vartheta, (\mu, \nu)\rangle (i = 1, 2, \cdots, n)$, then

$$PFLMM^P(a_1, \cdots, a_n) = a.$$

Proof. Since

$$\begin{aligned}PFLMM^P(a_1, \cdots, a_n) &= \langle s_{(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}\vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}}},\\
&((\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^{n}p_j}},\\
&\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}})\rangle.\end{aligned}$$

and $\vartheta_i = \vartheta, \mu_i = \mu, \nu_i = \nu$, we have

$$\begin{aligned}(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}\vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}} &= (\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}\vartheta^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}} = (\frac{1}{n!}(\sum_{\theta\in S_n}(\vartheta^{\sum_{j=1}^{n}p_j})^{\frac{1}{\sum_{j=1}^{n}p_j}}\\
&= (\frac{1}{n!}\cdot n!\cdot(\vartheta^{\sum_{j=1}^{n}p_j}))^{\frac{1}{\sum_{j=1}^{n}p_j}} = \vartheta.\end{aligned}$$

$$
\begin{aligned}
(\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}} &= (\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\mu^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}}\\
&= (\sqrt{1-(\prod_{\theta\in S_n}(1-\mu^{2\sum_{j=1}^n p_j}))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}}\\
&= (\sqrt{1-(((1-\mu^{2\sum_{j=1}^n p_j})n!)^{\frac{1}{n!}}})})^{\frac{1}{\sum_{j=1}^n p_j}}\\
&= (\sqrt{1-(1-\mu^{2\sum_{j=1}^n p_j})})^{\frac{1}{\sum_{j=1}^n p_j}}\\
&= (\sqrt{\mu^{2\sum_{j=1}^n p_j}})^{\frac{1}{\sum_{j=1}^n p_j}} = \mu.\\
\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu_{\theta(j)}^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}} &= \sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu^2)^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}}\\
&= \sqrt{1-(1-(\prod_{\theta\in S_n}(1-(1-\nu^2)^{\sum_{j=1}^n p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}}\\
&= \sqrt{1-(1-((1-(1-\nu^2)^{\sum_{j=1}^n p_j})n!)^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}}\\
&= \sqrt{1-(1-(1-(1-\nu^2)^{\sum_{j=1}^n p_j}))^{\frac{1}{\sum_{j=1}^n p_j}}}\\
&= \sqrt{1-((1-\nu^2)^{\sum_{j=1}^n p_j})^{\frac{1}{\sum_{j=1}^n p_j}}}\\
&= \sqrt{1-(1-\nu^2)} = \nu.
\end{aligned}
$$

therefore $PFLMM^P(a_1,\cdots,a_n)=\langle s_\vartheta,(\mu,\nu)\rangle = a$.

PROPERTY 2 (MONOTONICITY). Let $a_i=\langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle(i=1,2,\cdots,n)$ and $a_i^{'}=\langle s_{\vartheta_i^{'}},(\mu_i^{'},\nu_i^{'})\rangle(i=1,2,\cdots,n)$ be two collections of PLFNs, $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. If $s_{\vartheta_i}\le s_{\vartheta_i^{'}},\mu_i\le\mu_i^{'}$ and $\nu_i\ge\nu_i^{'}$, then

$$PFLMM^P(a_1,\cdots,a_n)\le PFLMM^P(a_1^{'},\cdots,a_n^{'}).$$

Proof. Let $PFLMM^P(a_1,\cdots,a_n)=\langle s_\vartheta,(\mu,\nu)\rangle, PFLMM^P(a_1^{'},\cdots,a_n^{'})=\langle s_{\vartheta^{'}},(\mu^{'},\nu^{'})\rangle$. Since $s_\vartheta = s_{(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^n\vartheta_{\theta(j)}^{p_j})))^{\frac{1}{\sum_{j=1}^n p_j}}}$ and $s_{\vartheta_j}\le s_{\vartheta_j^{'}}$, we have

$$
\begin{aligned}
s_{\vartheta_j}\le s_{\vartheta_j^{'}} &\Rightarrow s_{\vartheta_j^{p_j}}\le s_{(\vartheta_j^{'})^{p_j}} \Rightarrow s_{\prod_{j=1}^n(\vartheta_j^{p_j})}\le s_{\prod_{j=1}^n((\vartheta_j^{'})^{p_j})}\\
&\Rightarrow s_{\sum_{j=1}^n(\prod_{j=1}^n(\vartheta_j^{p_j}))}\le s_{\sum_{j=1}^n(\prod_{j=1}^n((\vartheta_j^{'})^{p_j}))}\\
&\Rightarrow s_{\frac{1}{n!}(\sum_{j=1}^n(\prod_{j=1}^n(\vartheta_j^{p_j})))}\le s_{\frac{1}{n!}(\sum_{j=1}^n(\prod_{j=1}^n((\vartheta_j^{'})^{p_j})))}\\
&\Rightarrow s_{(\frac{1}{n!}(\sum_{j=1}^n(\prod_{j=1}^n(\vartheta_j^{p_j}))))^{\frac{1}{\sum_{j=1}^n p_j}}}\le s_{(\frac{1}{n!}(\sum_{j=1}^n(\prod_{j=1}^n((\vartheta_j^{'})^{p_j}))))^{\frac{1}{\sum_{j=1}^n p_j}}}.
\end{aligned}
$$

That is, $s_\vartheta\le s_{\vartheta^{'}}$.

Since $\mu_i \le \mu_i^{'}$, then we have $\mu_{\theta(j)}^{p_j} \le (\mu_{\theta(j)}^{'})^{p_j}$ and $(\prod_{j=1}^n(\mu_{\theta(j)}^{p_j}))^2 \le (\prod_{j=1}^n(\mu_{\theta(j)}^{'})^{p_j})^2$. Furthermore, $1-(\prod_{j=1}^n(\mu_{\theta(j)}^{'})^{p_j})^2 \le 1-(\prod_{j=1}^n(\mu_{\theta(j)}^{p_j}))^2$ and $\prod_{\theta\in S_n}(1-(\prod_{j=1}^n(\mu_{\theta(j)}^{'})^{p_j})^2) \le \prod_{\theta\in S_n}(1-(\prod_{j=1}^n(\mu_{\theta(j)}^{p_j}))^2)$, and so $(\prod_{\theta\in S_n}(1-(\prod_{j=1}^n(\mu_{\theta(j)}^{'})^{p_j}))^2)^{\frac{1}{n!}} \le (\prod_{\theta\in S_n}(1-(\prod_{j=1}^n(\mu_{\theta(j)}^{p_j}))^2))^{\frac{1}{n!}}$. So we have

$$(\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^n \mu_{\theta(j)}^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}} \le (\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^n(\mu_{\theta(j)}^{'})^{p_j})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}},$$

that is, $\mu \le \mu^{'}$. Similarly, we also get $\nu \ge \nu^{'}$.

In order to prove the $\langle s_\vartheta, (\mu,\nu)\rangle \le \langle s_{\vartheta'}, (\mu^{'},\nu^{'})\rangle$. There are the following cases need to be discussed.

(**A.**) If $\mu < \mu^{'}$ and $\nu \ge \nu^{'}$, then $\mu^2+1-\nu^2 < (\mu^{'})^2+1-(\nu^{'})^2$. Furthermore, $\frac{1}{2}(\mu^2+1-\nu^2)\times s_\vartheta < \frac{1}{2}((\mu^{'})^2+1-(\nu^{'})^2)\times s_{\vartheta'}$. That is, $S(\langle s_\vartheta,(\mu,\nu)\rangle) \le S(\langle s_{\vartheta'},(\mu^{'},\nu^{'})\rangle)$. That is, $PFLMM^P(a_1,\cdots,a_n) \le PFLMM^P(a_1^{'},\cdots,a_n^{'})$.

(**B.**) If $\mu = \mu^{'}$ and $\nu > \nu^{'}$, then $\mu^2+1-\nu^2 < (\mu^{'})^2+1-(\nu^{'})^2$. Furthermore, $\frac{1}{2}(\mu^2+1-\nu^2)\times s_\vartheta < \frac{1}{2}((\mu^{'})^2+1-(\nu^{'})^2)\times s_{\vartheta'}$. That is, $S(\langle s_\vartheta,(\mu,\nu)\rangle) \le S(\langle s_{\vartheta'},(\mu^{'},\nu^{'})\rangle)$. That is, $PFLMM^P(a_1,\cdots,a_n) \le PFLMM^P(a_1^{'},\cdots,a_n^{'})$.

(**C.**) If $\mu = \mu^{'}$ and $\nu = \nu^{'}$, then $\mu^2+1-\nu^2 = (\mu^{'})^2+1-(\nu^{'})^2$. Furthermore, $\frac{\mu^2+1-\nu^2}{2} = \frac{(\mu^{'})^2+1-(\nu^{'})^2}{2}$.

(**C1.**) If $s_\vartheta < s_{\vartheta'}$, then $\frac{1}{2}(\mu^2+1-\nu^2)\times s_\vartheta < \frac{1}{2}((\mu^{'})^2+1-(\nu^{'})^2)\times s_{\vartheta'}$. That is, $S(\langle s_\vartheta,(\mu,\nu)\rangle) \le S(\langle s_{\vartheta'},(\mu^{'},\nu^{'})\rangle)$. That is,

$$PFLMM^P(a_1,\cdots,a_n) \le PFLMM^P(a_1^{'},\cdots,a_n^{'}).$$

(**C2.**) If $s_\vartheta = s_{\vartheta'}$, then $\frac{1}{2}(\mu^2+1-\nu^2)\times s_\vartheta = \frac{1}{2}((\mu^{'})^2+1-(\nu^{'})^2)\times s_{\vartheta'}$. That is, $S(\langle s_\vartheta,(\mu,\nu)\rangle) = S(\langle s_{\vartheta'},(\mu^{'},\nu^{'})\rangle)$. Furthermore, $H(\langle s_\vartheta,(\mu,\nu)\rangle) = \frac{1}{2}(\mu^2+\nu^2)\times s_\vartheta = \frac{1}{2}((\mu^{'})^2+(\nu^{'})^2)\times s_{\vartheta'} = H(\langle s_{\vartheta'},(\mu^{'},\nu^{'})\rangle)$, therefore That is,

$$PFLMM^P(a_1,\cdots,a_n) = PFLMM^P(a_1^{'},\cdots,a_n^{'}).$$

From above discussion, we have $PFLMM^P(a_1,\cdots,a_n) \le PFLMM^P(a_1^{'},\cdots,a_n^{'})$.

From the idempotency and monotonicity of PFLMM operator, it is easy to obtain that PFLMM operator is bounded, that is,

PROPERTY 3 (BOUNDEDNESS). Let $a_i = \langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle (i=1,2,\cdots,n)$ be a collection of PLFNs, $P=(p_1,p_2,\cdots,p_n)\in \mathbf{R}^n$ be a parameter vector,

$$a^- = \langle min_{1\le i\le n}\{s_{\vartheta_i}\}, min_{1\le i\le n}\{\mu_i\}, max_{1\le i\le n}\{\nu_i\}\rangle,$$
$$a^+ = \langle max_{1\le i\le n}\{s_{\vartheta_i}\}, max_{1\le i\le n}\{\mu_i\}, min_{1\le i\le n}\{\nu_i\}\rangle,$$

then

$$a^- \le PFLMM^P(a_1,\cdots,a_n) \le PFLMM^P(a_1^{'},\cdots,a_n^{'}) \le a^+.$$

Now, we will develop some special cases of PFLMM operator with respect to the parameter vector. Let $a_i = \langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle (i=1,2,\cdots,n)$ be a collection of PLFNs and $P=(p_1,p_2,\cdots,p_n)\in \mathbf{R}^n$ be a parameter vector.

(1) If $P = (1, 0, \cdots, 0)$, PFLMM operator will reduces to Pythagorean fuzzy linguistic arithmetic averaging (PFLAA) operator

$$PFLMM^{(1,0,\cdots,0)}(a_1, \cdots, a_n) = \langle s_{\frac{\sum_{j=1}^{n} \vartheta_i}{n}}, (\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_j^2)^{\frac{1}{n}}}, \prod_{j=1}^{n} \nu_j^{\frac{1}{n}} \rangle. \tag{9}$$

(2) If $P = (\lambda, 0, \cdots, 0)$, PFLMM operator will reduces to generalized Pythagorean fuzzy linguistic arithmetic averaging (GPFLAA) operator

$$\begin{aligned} PFLMM^{(1,0,\cdots,0)}(a_1, \cdots, a_n) \quad &= \langle s_{(\frac{\sum_{j=1}^{n} \vartheta_i^{\lambda}}{n})^{\frac{1}{\lambda}}}, (\sqrt{1 - \prod_{j=1}^{n} (1 - \mu_j^{2\lambda})^{\frac{1}{n}})^{\frac{1}{\lambda}}}, \\ &\sqrt{1 - (1 - \prod_{j=1}^{n} (1 - (1 - \nu_j^2)^{\lambda})^{\frac{1}{n}})^{\frac{1}{\lambda}}} \rangle. \end{aligned} \tag{10}$$

(3) If $P = (\underbrace{1, 1, \cdots, 1}_{k}, \underbrace{0, \cdots, 0}_{n-k})$, PFLMM operator will reduces to Pythagorean fuzzy linguistic Maclaurin symmetric mean (PFLMSM) operator

$$\begin{aligned} PFLMM^{(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})}(a_1, \cdots, a_n) = \langle s_{(\frac{1}{C_n^k} (\sum_{1 \leq i_1 \leq \cdots \leq i_k \leq 1} (\prod_{j=1}^{k} \vartheta_{i_j})))^{\frac{1}{k}}}, \\ (\sqrt{(1 - \prod_{1 \leq i_1 \cdots \leq i_k \leq n} (1 - \prod_{j=1}^{k} \mu_{i_j}^2)^{\frac{1}{C_n^k}})^{\frac{1}{k}}}, \\ \sqrt{1 - (1 - (\prod_{1 \leq i_1 \leq \cdots \leq i_k \leq n} (1 - \prod_{j=1}^{k} (1 - \nu_{i_j}^2))^{\frac{1}{C_n^k}}))^{\frac{1}{k}}} \rangle. \end{aligned} \tag{11}$$

(4) If $P = (1, 1, \cdots, 1)$, PFLMM operator will reduces to Pythagorean fuzzy linguistic geometric averaging (PFLGA) operator

$$PFLMM^{(1,1,\cdots,1)}(a_1, \cdots, a_n) = \langle s_{\prod_{j=1}^{n} \vartheta_j^{\frac{1}{n}}}, ((\prod_{j=1}^{n} \mu_j)^{\frac{1}{n}}, \sqrt{1 - (\prod_{j=1}^{n} (1 - \nu_j^2))^{\frac{1}{n}}}) \rangle. \tag{12}$$

(5) If $P = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, PFLMM operator will reduces to Pythagorean fuzzy linguistic geometric averaging (PFLGA) operator

$$PFLMM^{(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})}(a_1, \cdots, a_n) = \langle s_{\prod_{j=1}^{n} \vartheta_j^{\frac{1}{n}}}, ((\prod_{j=1}^{n} \mu_j)^{\frac{1}{n}}, \sqrt{1 - (\prod_{j=1}^{n} (1 - \nu_j^2))^{\frac{1}{n}}}) \rangle. \tag{13}$$

3.2. Pythagorean Fuzzy Linguistic Weighted Muirhead Mean Operators

Weights of attributes play a vital role in decision making and will directly the results of decision making results. In the Section 3.1, we proposed the PFLMM aggregation operators which can not consider the weights of attributes, so it is very important to consider to weights of attributes in the process of information aggregation.

DEFINITION 3. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $A = \{a_1, a_2, \cdots, a_n\}$, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $a_i (i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a Pythagorean fuzzy linguistic weighted Muirhead mean operator is a function PFLWMM^P: $A^n \to A$, and

$$PFLWMM^P(a_1, \cdots, a_n) = (\frac{1}{n!}(\oplus_{\theta \in S_n}(\otimes_{j=1}^{n}(w_{\theta(j)} a_{\theta(j)})^{p_j})))^{\frac{1}{\sum_{j=1}^{n} p_j}}, \tag{14}$$

where $\theta(j) (j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$ and S_n is the collection of all permutation of $\theta(j) (j = 1, 2, \cdots, n)$.

THEOREM 2. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $a_i (i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $PFLWMM^P(a_1, \cdots, a_n)$ is still a PFLN and

$$\begin{aligned} PFLWMM^P(a_1, \cdots, a_n) = \langle & s_{(\frac{1}{n!}(\sum_{\theta \in S_n}(\prod_{j=1}^{n}(w_{\theta(j)} \vartheta_{\theta(j)})^{p_j})))^{\frac{1}{\sum_{j=1}^{n} p_j}}}, \\ & ((\sqrt{1 - (\prod_{\theta \in S_n}(1 - (\prod_{j=1}^{n}(1 - (1 - \mu_{\theta(j)}^2)^{w_{\theta(j)}})^{p_j})))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^{n} p_j}}, \\ & \sqrt{1 - (1 - (\prod_{\theta \in S_n}(1 - \prod_{j=1}^{n}(1 - (\nu_{\theta(j)}^2)^{w_{\theta(j)}})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}}})\rangle. \end{aligned} \tag{15}$$

Proof. Since $a_{\theta(j)}$ is a PFLN, we have $w_{\theta(j)} a_{\theta(j)}$ is also a PFLN. By the operation of PFLNs, we have $w_{\theta(j)} a_{\theta(j)} = (\sqrt{1 - (1 - \mu_{\theta(j)}^2)^{w_{\theta(j)}}}, \nu_{\theta(j)}^{w_{\theta(j)}})$. Therefore, we can directly obtain the result according to Theorem 1.

EXAMPLE 2. Let $a_1 = \langle s_2, (0.3, 0.5)\rangle$, $a_2 = \langle s_4, (0.2, 0.4)\rangle$, $a_3 = \langle s_3, (0.6, 0.2)\rangle$, $w = (0.25, 0.4, 0.35)$ and $P = (1, 1, 0)$. According to Eq. (6), since

$$\begin{aligned} (\frac{1}{3!}(\sum_{\theta \in S_3}(\prod_{j=1}^{3}(w_{\theta(j)} \vartheta_{\theta(j)}))))^{\frac{1}{\sum_{j=1}^{3} p_j}} &= (\frac{1}{6} \times ((2 \times 0.25) \times (4 \times 0.4) + (2 \times 0.25) \\ &\times (3 \times 0.35) + (4 \times 0.4) \times (2 \times 0.25) + (4 \times 0.3) \times (3 \times 0.35) + (3 \times 0.35) \\ &\times (4 \times 0.4) + (3 \times 0.35) \times (2 \times 0.25)))^{\frac{1}{1+1+0}} = 1.0008. \end{aligned}$$

and

$$(\sqrt{1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}(1-(1-\mu^2_{\theta(j)})^{w_{\theta(j)}}))))^{\frac{1}{3!}})^{\frac{1}{1+1}}} = ((1-((1-((1-0.3^2)^{0.25}$$
$$\times(1-0.2^2)^{0.4}))\times(1-((1-0.3^2)^{0.25}\times(1-0.6^2)^{0.35}))\times(1-((1-0.2^2)^{0.4}$$
$$\times(1-0.2^2)^{0.25}))\times(1-((1-0.2^2)^{0.4}\times(1-0.6^2)^{0.35}))\times(1-((1-0.6^2)^{0.35}$$
$$\times(1-0.3^2)^{0.25}))\times(1-((1-0.6^2)^{0.35}\times(1-0.2^2)^{0.4})))^{\frac{1}{6}})^{\frac{1}{2}})^{\frac{1}{1+1+0}} = 0.1973;$$
$$\sqrt{1-(1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}(1-\nu^2_{\theta(j)})^{p_j}))^{\frac{1}{3!}})^{\frac{1}{\sum_{j=1}^{3}p_j}}} = (1-(1-((1-(1-0.5^{0.5})$$
$$\times(1-0.4^{0.8}))\times(1-(1-0.5^{0.5})\times(1-0.2^{0.7}))\times(1-(1-0.4^{0.8})\times(1-0.5^{0.5}))$$
$$\times(1-(1-0.4^{0.8})\times(1-0.2^{0.7}))\times(1-(1-0.2^{0.7})\times(1-0.5^{0.5})\times(1-(1-0.2^{0.7})$$
$$\times(1-0.4^{0.8}))))^{\frac{1}{3!}})^{\frac{1}{1+1+0}})^{\frac{1}{2}} = 0.7151.$$

Therefore, $PFLMM^P(a_1,a_2,a_3) = \langle s_{1.0008},(0.1973,0.7151)\rangle$.

In the process of decision making, the aggregation results would be more reliable if the selected operator is monotonic, the lack of monotonicity may debase the reliability and dependability of the final decision-making results. Similar to Property 2 and Property 3, we can prove $PFLWMM^P(a_1,\cdots,a_n)$ are bounded, and monotonic.

PROPERTY 4 (MONOTONICITY). Let $a_i = \langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle (i=1,2,\cdots,n)$ and $a_i^{'} = \langle s_{\vartheta_i^{'}},(\mu_i^{'},\nu_i^{'})\rangle (i=1,2,\cdots,n)$ be two collections of PLFNs, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i (i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i = 1$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector. If $s_{\vartheta_i}\le s_{\vartheta_i^{'}}, \mu_i\le\mu_i^{'}$ and $\nu_i\ge\nu_i^{'}$, then

$$PFLWMM^P(a_1,\cdots,a_n)\le PFLWMM^P(a_1^{'},\cdots,a_n^{'}).$$

PROPERTY 5 (BOUNDEDNESS). Let $a_i = \langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle (i=1,2,\cdots,n)$ be a collection of PLFNs, $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i (i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i = 1$,

$$a^- = \langle min_{1\le i\le n}\{s_{\vartheta_i}\},(min_{1\le i\le n}\{\mu_i\},max_{1\le i\le n}\{\nu_i\})\rangle,$$
$$a^+ = \langle max_{1\le i\le n}\{s_{\vartheta_i}\},(max_{1\le i\le n}\{\mu_i\},min_{1\le i\le n}\{\nu_i\})\rangle,$$

then

$$a^-\le PFLWMM^P(a_1,\cdots,a_n)\le PFLWMM^P(a_1^{'},\cdots,a_n^{'})\le a^+.$$

Now, we will develop some special cases of PFLWMM operator with respect to the parameter vector. Let $a_i = \langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle (i=1,2,\cdots,n)$ be a collection of PLFNs, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i (i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i = 1$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector.

(1) If $P=(1,0,\cdots,0)$, PFLWMM operator will reduces to

$$PFLWMM^{(1,0,\cdots,0)}(a_1,\cdots,a_n) = \langle s_{\sum_{j=1}^n \frac{w_j}{n}\vartheta_j},(\sqrt{1-\prod_{j=1}^n(1-\mu_j^2)^{\frac{w_j}{n}}},\prod_{j=1}^n \nu_j^{\frac{w_j}{n}})\rangle. \quad (16)$$

(2) If $P = (\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, PFLWMM operator will reduces to Pythagorean fuzzy linguistic weighted Maclaurin symmetric mean (PFLWMSM) operator

$$
\begin{aligned}
&PFLWMM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(a_1,\cdots,a_n) = \langle s_{(\frac{1}{C_n^k}(\sum_{1\le i_1\le\cdots\le i_k\le n}(\prod_{j=1}^{k}(w_j\vartheta_{i_j}))))^{\frac{1}{k}}},\\
&(\sqrt{(1-(\prod_{1\le i_1\le\cdots\le i_k\le n}(1-\prod_{j=1}^{k}(1-\mu_{i_j}^{2})^{w_j})^{\frac{1}{C_n^k}}))^{\frac{1}{k}}},\\
&\sqrt{1-(1-(\prod_{1\le i_1\le\cdots\le i_k\le n}(1-\prod_{j=1}^{k}(1-\nu_{i_j}^{2w_j}))^{\frac{1}{C_n^k}}))^{\frac{1}{k}}})\rangle.
\end{aligned}
\tag{17}
$$

4. PYTHAGOREAN FUZZY LINGUISTIC DUAL WEIGHTED MUIRHEAD MEAN OPERATORS

It is well-known that geometric average operator is the dual operator of arithmetic average operator. Similarly, we study the Pythagorean fuzzy linguistic dual weighted Muirhead mean operators in this section.

4.1. Pythagorean Fuzzy Linguistic Dual Muirhead Mean Operators

DEFINITION 4. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $A = \{a_1, a_2, \cdots, a_n\}$ and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a Pythagorean fuzzy linguistic Dual Muirhead mean operator is a function PFLDMM^P: $A^n \to A$, and

$$
PFLDMM^{P}(a_1,\cdots,a_n) = \frac{1}{\sum_{j=1}^{n} p_j}(\otimes_{\theta\in S_n}(\oplus_{j=1}^{n} p_j a_{\theta(j)}))^{\frac{1}{n!}},
\tag{18}
$$

where $\theta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$ and S_n is the collection of all permutation of $\theta(j)(j = 1, 2, \cdots, n)$.

THEOREM 3. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $PFLDMM^P(a_1, \cdots, a_n)$ is still a PFLN and

$$
\begin{aligned}
PFLDMM^{P}(a_1,\cdots,a_n) \quad &= \langle s_{\frac{1}{\sum_{j=1}^{n} p_j}(\prod_{\theta\in S_n}(\sum_{j=1}^{n} p_j\vartheta_{\theta(j)}))^{\frac{1}{n!}}},\\
&(\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\mu_{\theta(j)}^{2})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}}},\\
&(\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu_{\theta(j)}^{p_j})^{2}))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^{n} p_j}})\rangle.
\end{aligned}
\tag{19}
$$

Proof. Firstly, we prove Eq. (19). According to the operational law of PFLNs, we obtain

$$p_j a_{\theta(j)} = \langle s_{p_j \vartheta_{\theta(j)}}, (\sqrt{1-(1-\nu^2_{\theta(j)})^{p_j}}, \nu^{p_j}_{\theta(j)})\rangle, \quad and$$
$$\sum_{j=1}^{n} p_j a_{\theta(j)} = \langle s_{\sum_{j=1}^{n} p_j \vartheta_{\theta(j)}}, (\sqrt{1-\prod_{j=1}^{n}(1-\nu^2_{\theta(j)})^{p_j}}, \prod_{j=1}^{n} \nu^{p_j}_{\theta(j)})\rangle,$$

then we get

$$\begin{aligned}\otimes_{\theta\in S_n} \oplus_{j=1}^{n} p_j a_{\theta(j)} = \langle s_{\prod_{\theta\in S_n}\sum_{j=1}^{n} p_j \vartheta_{\theta(j)}}, \quad & (\prod_{\theta\in S_n} \sqrt{1-\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}}, \\ & \sqrt{1-\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2)})\rangle,\end{aligned}$$

and

$$\begin{aligned}(\otimes_{\theta\in S_n} \oplus_{j=1}^{n} p_j a_{\theta(j)})^{\frac{1}{n!}} \quad & = \langle s_{(\prod_{\theta\in S_n}\sum_{j=1}^{n} p_j \vartheta_{\theta(j)})^{\frac{1}{n!}}}, ((\prod_{\theta\in S_n} \sqrt{1-\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}})^{\frac{1}{n!}}, \\ & \sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{n!}}})\rangle.\end{aligned}$$

330 Therefore,

$$\begin{aligned}\frac{1}{\sum_{j=1}^{n} p_j}(\oplus_{\theta\in S_n} \otimes_{j=1}^{n} a_{\theta(j)^{p_j}})^{\frac{1}{n!}} \quad & = \langle s_{\frac{1}{\sum_{j=1}^{n} p_j}(\sum_{\theta\in S_n}\prod_{j=1}^{n}\vartheta^{p_j}_{\theta(j)})^{\frac{1}{n!}}}, \\ & (\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n} p_j}}}, \\ & (\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^{n} p_j}})\rangle.\end{aligned}$$

In addition, we need to prove $PFLDMM^P(a_1, \cdots, a_n)$ is also a PFLN.

Since $\mu_{\theta(j)} \in [0,1]$, we have $(1-\mu^2_{\theta(j)})^{p_j} \in [0,1]$ and $\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j} \in [0,1]$. And then

$$1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}) \in [0,1] \;\; and \;\; (1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}} \in [0,1].$$

And so,

$$\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}} \in [0,1].$$

Further,

$$1-\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}}\in[0,1]$$

and

$$(1-\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}\in[0,1].$$

And so

$$\sqrt{1-(1-\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}}\in[0,1].$$

Similarly, we have

$$\sqrt{(1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}}\in[0,1].$$

Let

$$\begin{aligned}\mu &= \sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}},\\ \nu &= \sqrt{(1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}},\end{aligned}$$

that is, $\mu,\nu\in[0,1]$.

Now we need to prove $\mu^2+\nu^2\in[0,1]$.

Since $\mu^2_{\theta(j)}+\nu^2_{\theta(j)}\le 1$, then $\nu^2_{\theta(j)}\le 1-\mu^2_{\theta(j)}$. Furthermore, we have

$$\begin{aligned}\mu^2+\nu^2 \quad &= 1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\nu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}\\ &+(1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}\\ &\le 1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}\\ &+(1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-\mu^2_{\theta(j)})^{p_j})))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}=1.\end{aligned}$$

That is, $\mu^2+\nu^2\in[0,1]$. In addition, obviously, we have $s_{\frac{1}{\sum_{j=1}^{n}p_j}(\prod_{\theta\in S_n}(\sum_{j=1}^{n}p_j\vartheta_{\theta(j)}))^{\frac{1}{n!}}}\in S$.

Hence, $PFLDMM^P(a_1,\cdots,a_n)$ is also a PFLN.

EXAMPLE 3. Let $a_1 = \langle s_2, (0.5, 0.3)\rangle$, $a_2 = \langle s_4, (0.7, 0.5)\rangle$, $a_3 = \langle s_3, (0.8, 0.2)\rangle$ and $P = (1, 0.5, 0.4)$. According to Eq. (6), we have

$$s_{\frac{1}{\sum_{j=1}^{3} p_j}((\prod_{\theta\in S_3}(\sum_{j=1}^{3} p_j\vartheta_{\theta(j)}))^{\frac{1}{3!}})} = s_a,$$

where

$a = \frac{1}{1+0.5+0.4} \times ((2\times1+4\times0.5+3\times0.4)\times(2\times1+3\times0.5+4\times0.4)\times(4\times1+2\times0.5+3\times 0.4)\times(4\times1+3\times0.5+2\times0.4)\times(3\times1+4\times0.5+2\times0.4)\times(3\times1+2\times0.5+4\times0.4))^{\frac{1}{6}} = 2.9904$

and

$$\sqrt{1-(1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}(1-\mu^2_{\theta(j)})^{p_j}))^{\frac{1}{3!}})^{\frac{1}{\sum_{j=1}^{3} p_j}})}$$

$$\begin{aligned}
&= (1-(1-((1-(1-0.5^2)\times(1-0.7^2)^{0.5}\times(1-0.8^2)^{0.4})\\
&\times(1-(1-0.5^2)\times(1-0.8^2)^{0.5}\times(1-0.7^2)^{0.4})\times(1-(1-0.7^2)\times(1-0.5^2)^{0.5}\\
&\times(1-0.8^2)^{0.4})\times(1-(1-0.7^2)\times(1-0.8^2)^{0.5}\times(1-0.5^2)^{0.4})\times(1-(1-0.8^2)\\
&\times(1-0.5^2)^{0.5}\times(1-0.7^2)^{0.4})\times(1-(1-0.8^2)\times(1-0.7^2)^{0.5}\\
&\times(1-0.5^2)^{0.4}))^{\frac{1}{3!}})^{\frac{1}{1+0.5+0.4}})^{\frac{1}{2}} = 0.6790.
\end{aligned}$$

$$\begin{aligned}
&(\sqrt{1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}\nu^{p_j}_{\theta(j)})^2))^{\frac{1}{3!}}})^{\frac{1}{\sum_{j=1}^{3} p_j}} = ((1-((1-(0.3\times0.5^{0.5}\times0.2^{0.4})^2)\\
&\times(1-(0.3\times0.2^{0.5}\times0.5^{0.4})^2)\times(1-(0.5\times0.3^{0.5}\times0.2^{0.4})^2)\\
&\times(1-(0.5\times0.2^{0.5}\times0.3^{0.4})^2)\times(1-(0.2\times0.3^{0.5}\times0.5^{0.4})^2)\\
&\times(1-(0.2\times0.5^{0.5}\times0.3^{0.4})^2))^{\frac{1}{6}})^{\frac{1}{2}})^{\frac{1}{1+0.5+0.4}} = 0.3178;
\end{aligned}$$

Therefore, $PFLDMM^P(a_1, a_2, a_3) = \langle s_{2.9904}, (0.6790, 0.3178)\rangle$.

Similar to Property 1, 2, 3, it is easy to prove $PFLDMM^P(a_1, \cdots, a_n)$ are idempotent, bounded, and monotonic, the details of their proofs are omitted.

PROPERTY 6 (IDEMPOTENCY). Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector and all $a_i (i = 1, 2, \cdots, n)$ are equal, i.e., $a_i = a = \langle s_\vartheta, (\mu, \nu)\rangle (i = 1, 2, \cdots, n)$, then

$$PFLDMM^P(a_1, \cdots, a_n) = a.$$

PROPERTY 7 (MONOTONICITY). Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ and $a_i^{'} = \langle s_{\vartheta_i^{'}}, (\mu_i^{'}, \nu_i^{'})\rangle (i = 1, 2, \cdots, n)$ be two collections of PLFNs, $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. If $s_{\vartheta_i} \leq s_{\vartheta_i^{'}}, \mu_i \leq \mu_i^{'}$ and $\nu_i \geq \nu_i^{'}$, then

$$PFLDMM^P(a_1, \cdots, a_n) \leq PFLMM^P(a_1^{'}, \cdots, a_n^{'}).$$

PROPERTY 8 (BOUNDEDNESS). Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector,

$$a^- = \langle min_{1\leq i\leq n}\{s_{\vartheta_i}\}, min_{1\leq i\leq n}\{\mu_i\}, max_{1\leq i\leq n}\{\nu_i\}\rangle,$$
$$a^+ = \langle max_{1\leq i\leq n}\{s_{\vartheta_i}\}, max_{1\leq i\leq n}\{\mu_i\}, min_{1\leq i\leq n}\{\nu_i\}\rangle,$$

then

$$a^{-} \leq PFLDMM^{P}(a_1, \cdots, a_n) \leq PFLMM^{P}(a_1^{'}, \cdots, a_n^{'}) \leq a^{+}.$$

Now, we will develop some special cases of PFLDMM operator with respect to the parameter vector. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector.

(1) If $P = (1, 0, \cdots, 0)$, PFLDMM operator will reduces to Pythagorean fuzzy linguistic geometric averaging (PFLGA) operator

$$PFLDMM^{(1,0,\cdots,0)}(a_1, \cdots, a_n) = \langle s_{\prod_{j=1}^{n} \vartheta_i^{\frac{1}{n}}}, (\prod_{j=1}^{n} \mu_j^{\frac{1}{n}}, \sqrt{1 - \prod_{j=1}^{n}(1-\nu_j^2)^{\frac{1}{n}}}\rangle. \quad (20)$$

(2) If $P = (\lambda, 0, \cdots, 0)$, PFLDMM operator will reduces to generalized Pythagorean fuzzy linguistic geometric (GPFLG) operator

$$\begin{aligned} &PFLDMM^{(1,0,\cdots,0)}(a_1, \cdots, a_n) \\ &= \langle s_{\frac{1}{\lambda}(\prod_{j=1}^{n}(\lambda\vartheta_i)^{\lambda})^{\frac{1}{n}}}, (\sqrt{1-(1-\prod_{j=1}^{n}(1-(1-\mu_j^2)^{\lambda})^{\frac{1}{n}})^{\frac{1}{\lambda}}}, (\sqrt{1-\prod_{j=1}^{n}(1-\nu_j^{2\lambda})^{\frac{1}{n}})^{\frac{1}{\lambda}}})\rangle. \end{aligned} \quad (21)$$

(3) If $P = (\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})$, PFLDMM operator will reduces to Pythagorean fuzzy linguistic geometric Maclaurin symmetric mean (PFLGMSM) operator

$$\begin{aligned} &PFLDMM^{(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{n-k})}(a_1, \cdots, a_n) = \langle s_{\frac{1}{k}(\prod_{1 \leq i_1 \leq \cdots \leq i_k \leq 1}(\sum_{j=1}^{k} \vartheta_{i_j}))^{\frac{1}{C_n^k}}}, \\ &(\sqrt{1-(1-(\prod_{1 \leq i_1 \leq \cdots \leq i_k \leq n}(1-\prod_{j=1}^{k}(1-\mu_{i_j}^2))^{\frac{1}{C_n^k}}))^{\frac{1}{k}}}, \\ &(\sqrt{1-\prod_{1 \leq i_1 \leq \cdots \leq i_k \leq n}(1-(\prod_{j=1}^{k}\nu_{i_j})^2)^{\frac{1}{C_n^k}})^{\frac{1}{k}}})\rangle. \end{aligned} \quad (22)$$

(4) If $P = (1, 1, \cdots, 1)$, PFLDMM operator will reduces to Pythagorean fuzzy linguistic arithmetic averaging (PFLMA) operator

$$PFLDMM^{(1,1,\cdots,1)}(a_1, \cdots, a_n) = \langle s_{\frac{1}{n}(\sum_{j=1}^{n} \vartheta_j)}, (\sqrt{1-(\prod_{j=1}^{n}(1-\mu_j^2))^{\frac{1}{n}}}, (\prod_{j=1}^{n} \nu_j)^{\frac{1}{n}})\rangle. \quad (23)$$

(5) If $P = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$, PFLMM operator will reduces to Pythagorean fuzzy linguistic arithmetic averaging (PFLAA) operator

$$PFLDMM^{(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})}(a_1, \cdots, a_n) = \langle s_{\frac{1}{n}(\sum_{j=1}^{n} \vartheta_j)}, (\sqrt{1-(\prod_{j=1}^{n}(1-\mu_j^2))^{\frac{1}{n}}}, (\prod_{j=1}^{n} \nu_j)^{\frac{1}{n}})\rangle. \quad (24)$$

4.2. **Pythagorean Fuzzy Linguistic Dual Weighted Muirhead Mean Operators**

Similar to PFLWMM operators. In this Section, we proposed the PFLDWMM aggregation operators which consider the weights of attributes in the process of information aggregation.

DEFINITION 5. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $A = \{a_1, a_2, \cdots, a_n\}$, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $a_i (i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then a Pythagorean fuzzy linguistic dual weighted Muirhead mean operator is a function PFLDWMM^P: $A^n \to A$, and

$$PFLDWMM^P(a_1, \cdots, a_n) = \frac{1}{\sum_{j=1}^n p_j}(\otimes_{\theta \in S_n}(\oplus_{j=1}^n p_j (a_{\theta(j)})^{w_{\theta(j)}}))^{\frac{1}{n!}}, \tag{25}$$

where $\theta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$ and S_n is the collection of all permutation of $\theta(j)(j = 1, 2, \cdots, n)$.

THEOREM 4. Let $a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle (i = 1, 2, \cdots, n)$ be a collection of PLFNs, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $a_i (i = 1, 2, \cdots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^n w_i = 1$, and $P = (p_1, p_2, \cdots, p_n) \in \mathbf{R}^n$ be a parameter vector. Then $PFLDWMM^P(a_1, \cdots, a_n)$ is still a PFLN and

$$\begin{aligned} PFLDWMM^P(a_1, \cdots, a_n) = \langle s_{\frac{1}{\sum_{j=1}^n p_j}(\prod_{\theta \in S_n}(\sum_{j=1}^n p_j (\vartheta_{\theta(j)})^{w_{\theta(j)}}))^{\frac{1}{n!}}}, \\ (\sqrt{1 - (1 - (\prod_{\theta \in S_n}(1 - \prod_{j=1}^n (1 - (\mu_{\theta(j)}^2)^{w_{\theta(j)}})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}}, \\ (\sqrt{1 - (\prod_{\theta \in S_n}(1 - (\prod_{j=1}^n (1 - (1 - \nu_{\theta(j)}^2)^{w_{\theta(j)}})^{p_j})))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}})\rangle. \end{aligned} \tag{26}$$

Proof. Since $a_{\theta(j)}$ is a PFLN, we have $a_{\theta(j)}^{w_{\theta(j)}}$ is also a PFLN. By the operation of PFLNs, we have $a_{\theta(j)}^{w_{\theta(j)}} = (\mu_{\theta(j)}^{w_{\theta(j)}}, \sqrt{1 - (1 - \nu_{\theta(j)}^2)^{w_{\theta(j)}}})$. Therefore, we can directly obtain the result according to Theorem 2.

EXAMPLE 4. Let $a_1 = \langle s_2, (0.3, 0.5)\rangle$, $a_2 = \langle s_4, (0.2, 0.4)\rangle$, $a_3 = \langle s_3, (0.6, 0.2)\rangle$, $w = (0.25, 0.4, 0.35)$ and $P = (1, 1, 0)$. According to Eq. (6), since

$$\begin{aligned} &\frac{1}{\sum_{j=1}^3 p_j}((\prod_{\theta \in S_3}(\sum_{j=1}^3 p_j (\vartheta_{\theta(j)})^{w_{\theta(j)}})))^{\frac{1}{3!}} = \frac{1}{1+1+0} \times ((2^{0.25} + 4^{0.4}) \times (2^{0.25} + 3^{0.35}) \\ &\times (4^{0.4} + 2^{0.25}) \times (4^{0.4} + 3^{0.35}) \times (3^{0.35} + 4^{0.4}) \times (3^{0.35} + 2^{0.25}))^{\frac{1}{3!}} = 1.5373 \end{aligned}$$

and

$$\sqrt{1-(1-(\prod_{\theta\in S_3}(1-(\prod_{j=1}^{3}(1-(\mu^2_{\theta(j)})^{w_{\theta(j)}})^{p_j}))^{\frac{1}{3!}})^{\frac{1}{\sum_{j=1}^{3}p_j}}}$$
$$=(1-(1-((1-(1-0.3^{0.5})\times(1-0.2^{0.8}))\times(1-(1-0.3^{0.5})\times(1-0.6^{0.7}))$$
$$\times(1-(1-0.2^{0.8})\times(1-0.3^{0.5}))\times(1-(1-0.2^{0.8})\times(1-0.6^{0.7}))$$
$$\times(1-(1-0.6^{0.7})\times(1-0.3^{0.5})\times(1-(1-0.6^{0.7})\times(1-0.2^{0.8}))))^{\frac{1}{3!}})^{\frac{1}{1+1+0}})^{\frac{1}{2}}=0.7206.$$
$$(\sqrt{1-(\prod_{\theta\in S_3}((1-(\prod_{j=1}^{3}(1-(1-\mu^2_{\theta(j)})^{w_{\theta(j)}})))^{\frac{1}{3!}})^{\frac{1}{1+1}}}$$
$$=((1-(1-((1-(1-0.5^2)^{0.25})\times(1-(1-0.4^2)^{0.4}))\times(1-(1-(1-0.5^2)^{0.25})$$
$$\times(1-(1-0.2^2)^{0.35}))\times(1-(1-(1-0.4^2)^{0.4})\times(1-(1-0.5^2)^{0.25}))\times(1-(1-(1-0.4^2)^{0.4})$$
$$\times(1-(1-0.2^2)^{0.35}))\times(1-(1-(1-0.2^2)^{0.35})\times(1-(1-0.5^2)^{0.25}))\times(1-(1-(1-0.2^2)^{0.35})$$
$$\times(1-(1-0.4^2)^{0.4})))^{\frac{1}{6}})^{\frac{1}{2}})^{\frac{1}{1+1+0}}=0.2167;$$

Therefore, $PFLDWMM^P(a_1,a_2,a_3)=\langle s_{1.5373},(0.7206,0.2167)\rangle$.

Similar to Property 7 and Property 8, we can prove $PFLDWMM^P(a_1,\cdots,a_n)$ are bounded, and monotonic.

PROPERTY 9 (MONOTONICITY). Let $a_i=\langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle(i=1,2,\cdots,n)$ and $a_i^{'}=\langle s_{\vartheta_i^{'}},(\mu_i^{'},\nu_i^{'})\rangle(i=1,2,\cdots,n)$ be two collections of PLFNs, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i(i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector.If $s_{\vartheta_i}\leq s_{\vartheta_i^{'}},\mu_i\leq\mu_i^{'}$ and $\nu_i\geq\nu_i^{'}$, then

$$PFLDWMM^P(a_1,\cdots,a_n)\leq PFLDWMM^P(a_1^{'},\cdots,a_n^{'}).$$

PROPERTY 10 (BOUNDEDNESS). Let $a_i=\langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle(i=1,2,\cdots,n)$ be a collection of PLFNs, $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i(i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$,

$$a^-=\langle min_{1\leq i\leq n}\{s_{\vartheta_i}\},(min_{1\leq i\leq n}\{\mu_i\},max_{1\leq i\leq n}\{\nu_i\})\rangle,$$
$$a^+=\langle max_{1\leq i\leq n}\{s_{\vartheta_i}\},(max_{1\leq i\leq n}\{\mu_i\},min_{1\leq i\leq n}\{\nu_i\})\rangle,$$

then

$$a^-\leq PFLWMM^P(a_1,\cdots,a_n)\leq PFLDWMM^P(a_1^{'},\cdots,a_n^{'})\leq a^+.$$

Now, we will develop some special cases of PFLDWMM operator with respect to the parameter vector. Let $a_i=\langle s_{\vartheta_i},(\mu_i,\nu_i)\rangle(i=1,2,\cdots,n)$ be a collection of PLFNs, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $a_i(i=1,2,\cdots,n)$ with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$, and $P=(p_1,p_2,\cdots,p_n)\in\mathbf{R}^n$ be a parameter vector.

(1) If $P=(1,0,\cdots,0)$, we have

$$PFLDWMM^{(1,0,\cdots,0)}(a_1,\cdots,a_n)=\langle s_{\prod_{j=1}^n\vartheta_j^{\frac{w_j}{n}}},(\prod_j^n\mu_j^{\frac{w_j}{n}},\sqrt{1-\prod_j^n(1-\nu_j^2)^{\frac{w_j}{n}}})\rangle. \quad (27)$$

(2) If $P = (\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})$, PFLDWMM operator will reduces to Pythagorean fuzzy linguistic weighted geometric Maclaurin symmetric mean (PFLWGMSM) operator

$$
\begin{aligned}
&PFLDWMM^{(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,\cdots,0}^{n-k})}(a_1,\cdots,a_n) = \langle s_{\frac{1}{k}(\prod_{1\le i_1\le\cdots\le i_k\le n}(\sum_{j=1}^{k}\vartheta_{i_j}^{w_{i_j}}))^{\frac{1}{C_n^k}}},\\
&(\sqrt{1-(1-(\prod_{1\le i_1\le\cdots\le i_k\le n}(1-\prod_{j=1}^{k}(1-\mu_{i_j}^{2w_j}))^{\frac{1}{C_n^k}}))^{\frac{1}{k}}},\\
&(\sqrt{1-\prod_{1\le i_1\le\cdots\le i_k\le n}(1-(\prod_{j=1}^{k}(1-(1-\nu_{i_j}^{2})^{w_{i_j}}))^{\frac{1}{C_n^k}})^{\frac{1}{k}}})\rangle. \qquad (28)
\end{aligned}
$$

5. MODEL FOR MULTIPLE ATTRIBUTE DECISION MAKING WITH PYTHAGOREAN FUZZY LINGUISTIC INFORMATION

In this section, we develop a MADM method with Pythagorean fuzzy linguistic information based on the proposed PFLWMM operator or PFLDWMM operator. The following assumptions or notations are used to represent the MADM problems for potential evaluation of emerging technology commercialization with Pythagorean fuzzy information.

Based on the given linguistic term set $S = \{s_0, s_1, \cdots, s_g\}$, let $A = \{A_1, A_2, \cdots, A_m\}$ be a set of m alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be the set of attributes, and $w = \{w_1, \cdots, w_n\}$ be the weight vector of attributes with $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$. Suppose that $A = (a_{ij})_{m\times n}$ is the decision making matrix, where $a_{ij} = \langle s_{\vartheta_{ij}}, (\mu_{ij}, \nu_{ij})\rangle$, $s_{\vartheta_{ij}} \in S$, μ_{ij} indicates the indicates the degree that the alternative A_i satisfies the attribute G_j given by the decision maker, ν_{ij} indicates the degree that the alternative Ai does not satisfy the attribute G_j given by the decision maker, $\mu_{ij}, \nu_{ij} \in [0,1]$ and $\mu_{ij}^2 + \nu_{ij}^2 \in [0,1]$, $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$.

In the following, two novel MADM methods are developed with Pythagorean fuzzy linguistic information based on PFLWMM operator or PFLDWMM operator, which are shown in the following:

Step 1. Aggregate all assessment values $a_{ij} = \langle s_{\theta_{ij}}, (\mu_{ij}, \nu_{ij})\rangle$ of the alternative $A_i (i = 1,2,\cdots,m)$ on all attributes $G_j (j = 1,2,\cdots,n)$ into the overall assessment $a_i = \langle s_{\theta_i}, (\mu_i, \nu_i)\rangle$ based on the

$$
\begin{aligned}
&a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i)\rangle = PFLMM^{P}(a_{i1},\cdots,a_{in}) = \langle s_{(\frac{1}{n!}(\sum_{\theta\in S_n}(\prod_{j=1}^{n}(w_{\theta(j)}\vartheta_{\theta(ij)})^{p_j})))^{\frac{1}{\sum_{j=1}^{n}p_j}}},\\
&((\sqrt{1-(\prod_{\theta\in S_n}(1-(\prod_{j=1}^{n}(1-(1-\mu_{\theta(ij)}^{2})^{w_{\theta(ij)}})^{p_j})))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}},\\
&\sqrt{1-(1-(\prod_{\theta\in S_n}(1-\prod_{j=1}^{n}(1-(\nu_{\theta(ij)}^{2})^{w_{\theta(j)}})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^{n}p_j}}})\rangle. \qquad (29)
\end{aligned}
$$

or

$$a_i = \langle s_{\vartheta_i}, (\mu_i, \nu_i) \rangle = PFLDWMM^P(a_{i1}, \cdots, a_{in}) = \langle s_{\frac{1}{\sum_{j=1}^n p_j}(\prod_{\theta \in S_n}(\sum_{j=1}^n p_j(\vartheta_{\theta(ij)})^{w_{\theta(j)}}))^{\frac{1}{n!}}},$$
$$(\sqrt{1-(1-(\prod_{\theta \in S_n}(1-\prod_{j=1}^n(1-(\mu_{\theta(ij)}^2)^{w_{\theta(j)}})^{p_j}))^{\frac{1}{n!}})^{\frac{1}{\sum_{j=1}^n p_j}}},$$
$$(\sqrt{1-(\prod_{\theta \in S_n}(1-(\prod_{j=1}^n(1-(1-\nu_{\theta(ij)}^2)^{w_{\theta(j)}})^{p_j})))^{\frac{1}{n!}}})^{\frac{1}{\sum_{j=1}^n p_j}})\rangle. \tag{30}$$

Step 2. Calculate the score values $S(a_i)$ of all collective overall values to rank the all alternatives $A_i(i = 1, 2, \cdots, m)$, the bigger the $S(a_i)$, the better the A_i, where

$$S(a_i) = \frac{1}{2}(\mu_i^2 + 1 - \nu_i^2) \times s_{\vartheta_i} = s_{\frac{1}{2}(\mu_i^2+1-\nu_i^2)\times\vartheta_i}. \tag{31}$$

If there is no difference between two scores a_i and a_j, then we need to calculate the accuracy degree $H(a_i)$ and $H(a_j)$ by the following equation:

$$H(a_i) = \frac{1}{2}(\mu_i^2 + \nu_i^2) \times s_{\vartheta_i} = s_{\frac{1}{2}(\mu_i^2+\nu_i^2)\times\vartheta_i}. \tag{32}$$

and then rank the alternatives A_i and A_j accordance with degrees $H(a_i)$ and $H(a_j)$.

Step 3. Rank all alternatives $A_i(i = 1, 2, \cdots, m)$ and determine the desirable alternative according to $S(a_i)$ and $H(a_i)(i = 1, 2, \cdots, m)$.

Step 4. End.

6. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

6.1. Numerical Example

In order to show the application of the proposed approach in this paper, an illustrative example was cited and adapted from [4], which an evaluation on the emergency response capabilities of relevant department when some disasters occurred. There is a panel with four emerging departments $A_i(i = 1, 2, 3, 4)$ should be considered that have taken part in the rescue work. A_1 is the transportation department, A_2 is the health departments, A_3 is the telecommunications department, and A_4 is the supplies department. The government needs to give an evaluation according to four attributes: (1) G_1 is the emergency forecasting capability; (2) G_2 is the emergency process capability; (3) G_3 is the after-disaster loss evaluation capability; and (4) G_4 is the after-disaster reconstruction capability, $w = (0.1, 0.4, 0.2, 0.3)$ is the weight vector of them. Several experts are invited to evaluate the four departments in anonymity with the linguistic term set $S = \{s_0 = \text{e}xtremely\ low, s_1 = \text{v}ery\ low, s_2 = \text{l}ow, s_3 = \text{m}edium, s_4 = \text{h}igh, s_5 = \text{v}ery\ high, s_6 = \text{e}xtremely\ high\}$. The four possible alternatives $\{A_1, A_2, A_3, A_4\}$ are evaluated by using the Pythagorean fuzzy linguistic information, and the Pythagorean fuzzy linguistic decision matrix $A = (a_{ij})_{4\times5}$ is shown in Table 1.

(1) Based on PFLWMM operator to drive the collective overall value when parameter $P = (1, 1, 1, 1)$, we obtain following:

Table 1: Pythagorean fuzzy linguistic decision matrix

	G_1	G_2	G_3	G_4
A_1	$\langle s_2,(0.6,0.4)\rangle$	$\langle s_3,(0.6,0.4))\rangle$	$\langle s_4,(0.8,0.2)\rangle$	$\langle s_4,(0.7,0.4)\rangle$
A_2	$\langle s_3,(0.7,0.5)\rangle$	$\langle s_5,(0.7,0.5))\rangle$	$\langle s_5,(0.7,0.4)\rangle$	$\langle s_4,(0.7,0.3)\rangle$
A_3	$\langle s_2,(0.6,0.3)\rangle$	$\langle s_4,(0.6,0.5))\rangle$	$\langle s_5,(0.7,0.3)\rangle$	$\langle s_4,(0.6,0.4)\rangle$
A_4	$\langle s_3,(0.8,0.2)\rangle$	$\langle s_4,(0.8,0.3))\rangle$	$\langle s_4,(0.6,0.4)\rangle$	$\langle s_5,(0.8,0.3)\rangle$

Step 1. Based on Eq.(29), we have

$$a_1 = \langle s_{0.6928},(0.3530,0.7978)\rangle; a_2 = \langle s_{0.8712},(0.3704,0.8334)\rangle;$$
$$a_3 = \langle s_{0.7872},(0.3210,0.8073)\rangle; a_4 = \langle s_{0.8712},(0.4057,0.7715)\rangle.$$

Step 2. Based on Eq.(31), we utilize the score function to calculate the score values of collective overall assessment values $a_i(i = 1,2,3,4)$,

$$S(a_1) = s_{0.1691}; S(a_2) = s_{0.1928}; S(a_3) = s_{0.1777}; S(a_4) = s_{0.2481}.$$

Step 3. According the score values of $a_i(i = 1,2,3,4)$ calculated in Step 2, all feasible alternative $A_i(i = 1,2,3,4)$ are ranked as follows:

$$A_1 \prec A_3 \prec A_2 \prec A_4,$$

Therefore, the desirable alternative is A_4.

(2) Based on PFLDWMM operator to drive the collective overall value when parameter $P = (1,1,1,1)$, we obtain following:

Step 1. Based on Eq.(30), we have

$$a_1 = \langle s_{1.3647},(0.9208,0.1623)\rangle; a_2 = \langle s_{1.4382},(0.9249,0.2036)\rangle;$$
$$a_3 = \langle s_{1.4271},(0.9034,0.1772)\rangle; a_4 = \langle s_{1.4493},(0.9416,0.1396)\rangle.$$

Step 2. Based on Eq.(31), we utilize the score function to calculate the score values of collective overall assessment values $a_i(i = 1,2,3,4)$,

$$S(a_1) = s_{1.2429}; S(a_2) = s_{1.3044}; S(a_3) = s_{1.2735}; S(a_4) = s_{1.3531}.$$

Step 3. According the score values of $a_i(i = 1,2,3,4)$ calculated in Step 2, all feasible alternative $A_i(i = 1,2,3,4)$ are ranked as follows:

$$A_1 \prec A_3 \prec A_2 \prec A_4,$$

Therefore, the desirable alternative is A_4.

6.2. The Influence of the Parameter Vector P on the Decision Making Results

In order to show the influence of the parameter vectors P on the decision making results, we use different parameter vectors P in our proposed methods based on

Table 2: Ranking results by using different parameter vector P in PFLWMM operator

Parameter Vector P	The score values of $A_i (i = 1, 2, 3, 4)$	Ranking Results
$(1, 0, 0, 0)$	$S(a_1) = s_{0.6807}, S(a_2) = s_{0.7052}, S(a_3) = s_{0.6806}, S(a_4) = s_{0.7536}$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$(1, 1, 0, 0)$	$S(a_1) = s_{0.4520}, S(a_2) = s_{0.4801}, S(a_3) = s_{0.4564}, S(a_4) = s_{0.5427}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(1, 1, 1, 0)$	$S(a_1) = s_{0.2895}, S(a_2) = s_{0.3154}, S(a_3) = s_{0.2978}, S(a_4) = s_{0.3764}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(1, 1, 1, 1)$	$S(a_1) = s_{0.1691}, S(a_2) = s_{0.1928}, S(a_3) = s_{0.1777}, S(a_4) = s_{0.2481}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$S(a_1) = s_{0.1691}, S(a_2) = s_{0.1928}, S(a_3) = s_{0.1777}, S(a_4) = s_{0.2481}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(2, 0, 0, 0)$	$S(a_1) = s_{0.5079}, S(a_2) = s_{0.5436}, S(a_3) = s_{0.5095}, S(a_4) = s_{0.6267}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(3, 0, 0, 0)$	$S(a_1) = s_{0.3954}, S(a_2) = s_{0.4410}, S(a_3) = s_{0.3992}, S(a_4) = s_{0.5491}$	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 3: Ranking results by using different parameter vector P in PFLDWMM operator

Parameter Vector P	The score values of $A_i (i = 1, 2, 3, 4)$	Ranking Results
$(1, 0, 0, 0)$	$S(a_1) = s_{0.1217}, S(a_2) = s_{0.1242}, S(a_3) = s_{0.1143}, S(a_4) = s_{0.1391}$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$(1, 1, 0, 0)$	$S(a_1) = s_{0.4616}, S(a_2) = s_{0.4967}, S(a_3) = s_{0.4743}, S(a_4) = s_{0.5315}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(1, 1, 1, 0)$	$S(a_1) = s_{0.8262}, S(a_2) = s_{0.8809}, S(a_3) = s_{0.8532}, S(a_4) = s_{0.9065}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(1, 1, 1, 1)$	$S(a_1) = s_{1.2429}, S(a_2) = s_{1.3044}, S(a_3) = s_{1.2735}, S(a_4) = s_{1.3531}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$S(a_1) = s_{1.2429}, S(a_2) = s_{1.3044}, S(a_3) = s_{1.2735}, S(a_4) = s_{1.3531}$	$A_4 \succ A_2 \succ A_3 \succ A_1$
$(2, 0, 0, 0)$	$S(a_1) = s_{0.3584}, S(a_2) = s_{0.3728}, S(a_3) = s_{0.3409}, S(a_4) = s_{0.4060}$	$A_4 \succ A_2 \succ A_1 \succ A_3$
$(3, 0, 0, 0)$	$S(a_1) = s_{0.6303}, S(a_2) = s_{0.6622}, S(a_3) = s_{0.6080}, S(a_4) = s_{0.7072}$	$A_4 \succ A_2 \succ A_1 \succ A_3$

PFLWMM and PFLDWMM operators to rank the alternatives. The ranking results are shown in Table 2 and Table 3.

We explain the following aspects to illustrate the influence of parameter vector P on the decision making results:

(1) We see from the Section 3 and Section 4 that our methods are more general. Specially, when $P = (\overbrace{1, 1, \underset{k}{\cdots}, 1}, \overbrace{0, 0, \underset{k}{\cdots}, 0})$, the PFLWMM operator will become Pythagorean fuzzy linguistic weighted Maclaurin mean, which is also family aggregation operators when the parameter k takes different value.

(2) It follows from Table 2 and Table 3 that the aggregation results obtained by PFLWMM and PFLDWMM operators are almost remain unchanged in this example though the parameter vector P change, this phenomenon also illustrates PFLWMM and PFLDWMM operators have good robust property.

(3) Parameter vector P can capture interrelationship between the individual arguments that can be fully taken into account. As far as the PFLWMM operator is concerned, we can find from Table 2 that the more interrelationships of attributes which we consider, the smaller value of score functions, that is, the parameter vector P have greater control ability, the values of score function will become greater. However, for the IFDWMM operator, the result is just the opposite, the more interrelationships of attributes we consider, the greater value of score functions will become. The parameter vector P have greater control ability, the values of score function will become small. So, different parameter vector P can be regarded as the decision makers' risk preference.

6.3. **Comparisons With Other Existing Methods**

In order to verify the effectiveness of the proposed methods with PFLWMM operator and PFLDWMM operator,we compare our proposed methods with other existing methods including the PFLWA operator, PFLGA operator and PFMSM operator. The results are shown in Table 4, which indicates that five methods have the same desirable alternative, which further verifies the validity of the method proposed in this paper with PFLWMM operator and PFLDWMM operator.

Table 4: Ranking results by using different methods

Aggregation Operator	Parameter Vector	Ranking Results
PFLWA	No	$A_4 \succ A_2 \succ A_3 \succ A_1$
PFLGW	No	$A_4 \succ A_2 \succ A_3 \succ A_1$
PFLMSM	(1, 1, 1, 0)	$A_4 \succ A_2 \succ A_3 \succ A_1$
PFLWMM in this paper	(1, 1, 1, 1)	$A_4 \succ A_2 \succ A_3 \succ A_1$
PFLDWMM in this paper	(1, 1, 1, 1)	$A_4 \succ A_2 \succ A_3 \succ A_1$

In the following, we will give some comparisons of the three methods and our proposed methods with respect to some characteristic, which are listed in Table 5.

Table 5: Ranking results by using different methods

Methods	captures interrelationship of MAs	makes method flexible by PV
PFLWA	×	×
PFLGW	×	×
PFLMSM	√	√
PFLWMM in this paper	√	√
PFLDWMM in this paper	√	√

where MA means multiple attributes and PV means parameter vector.

PFLWA and PFLGA are special cases of PFLWMM and PFLDWMM operator. Compared with the method based on the PFLWA operator and PFLGA operator, in which there are two limitations: (1)the method based on PFLWA and PFLGA operator thinks that the input arguments are independent; (2) the method based on PFLWA and PFLGA operator doesn't consider the interrelationship among input arguments. However, the new proposed operators in this paper can also consider the interrelationship among all input arguments and they are also generalization of most existing aggregation operators. Therefore, the proposed methods are more general and flexible to solve MADM problems than PFLWA and PFLGA. Compared with the method in [41] based on the PFMSM operator, which consider interrelationship of multi-input arguments, but it can not deal with linguistic information. Therefore, we extend PFMSM to PFLWSMM and PFLDWMSM which are special cases of $PFLWMM$ and $PFLDWMM$ operators when parameter vector $P = (\underset{k}{\overbrace{1, 1, \cdots, 1}}, \underset{k}{\overbrace{0, 0, \cdots, 0}})$. Thus, the new methods proposed in this paper can make the linguistic information aggregation process more flexible by the parameter vector P.

7. CONCLUSIONS

In recent years, aggregation operators play a vital role in decision making and many aggregation operators under different environment have been developed. But they still have some limitations in solving some practical problems. Some traditional Maclaurin Symmetric Mean (MSM) operator and intuitionistic MSM operator are generally suitable for aggregating the information taking the form of crisp numbers and intuitionistic fuzzy numbers, but fails in dealing with the Pythagorean linguistic information. In this paper, we have investigated the MADM problems with the Pythagorean linguistic information based on some new aggregation operators which capture interrelationships of multiple attributes among any number of attributes by a parameter vector P. To begin with, we presented some new Pythagorean fuzzy linguistic MM aggregation operators to deal with MADM problems with Pythagorean fuzzy linguistic information, including the Pythagorean fuzzy linguistic Muirhead Mean (PFLMM) operator, the Pythagorean fuzzy linguistic weighted Muirhead Mean (PFLWMM) operator, the Pythagorean fuzzy linguistic dual Muirhead Mean (PFLDMM) operator, the Pythagorean fuzzy linguistic dual weighted Muirhead Mean (PFLDWMM) operator. In addition, the some properties of these new aggregation operators were proved and some special cases were discussed. Moreover, we presented two new methods to solve the MADM problems with Pythagorean fuzzy linguistic information. Finally, we used an illustrative example to show the feasibility and validity of the new methods by comparing with the other existing methods.

In further research, it is necessary to solve the real decision making problems by applying these operators. In addition, we can develop some new aggregation operators on the basis of Muirhead mean operator by considering that MM operator has the superiority of compatibility.

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