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CONTINUITY OF TRANSLATION AND SEPARABLE INVARIANT SUBSPACES OF BANACH SPACES ASSOCIATED TO LOCALLY COMPACT GROUPS

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1. Introduction.

Let G be a locally compact group. How big can a separable ideal of the algebra of regular Borel measures on G be? More generally, let Φ be a normed linear space. Assume that G acts on Φ as linear isometries. Suppose that Ψ is a G–invariant subspace of Φ. How big is Ψ? A fairly elementary argument shows that under a very mild continuity assumption on the action of G, if Ψ is separable, then $\Psi \subseteq \Phi_c$, where Φ_c denotes elements of Φ on which the operation of G is continuous.

It is the purpose of this note to report on results we have obtained on separability of Ψ, the dimension of Φ_c, and characterizations of the set Φ_c for various Ψ and Φ. Details and related results will appear in [GLL1] and [GLL2].

This work was motivated by a question raised by Professor Ryll–Nardewski and communicated to us by Professor Hartman: "Must a translation–invariant subspace of $M(G)$ that is not contained in $L^1(G)$ have dimension c?" We are grateful to them both.

2. Lower semicontinuous representations.

Let G be a locally compact group. By a *representation* T of G on a normed linear space Φ we mean a mapping $x \to T(x)$ from G into the group of linear isometries from Φ into Φ such that $T(x_1x_2) = T(x_1) \circ T(x_2)$, $x_1, x_2 \in G$. T is said to be *lower semicontinuous* if for each $\mu \in \Phi$, and

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each $\varepsilon > 0$, the set

$$\{x \in G : \|T(x)\mu - \mu\| > \varepsilon\}$$

is open in G. Let Φ_c denote the set of all $\mu \in \Phi$ such that $x \to T(x)\mu$ is continuous from G into Φ when Φ has the norm topology. The principal tool in this section is the following observation:

LEMMA 2.1. *Let G be a locally compact group and T be a lower semicontinuous representation of G on the normed space Φ. Let $\mu \notin \Phi_c$. Then there exists $\varepsilon > 0$ such that $H = \{x \in G : \|T(x)\mu - \mu\| > \varepsilon\}$ is an open dense subset of G.*

Lemma 2.1 yields the following generalization and improvement of the result of Larsen [L] and Tam [T]:

THEOREM 2.2. *Let G be a non–discrete locally compact group, and T be a lower semi–continuous representation of G as linear isometries on the normed space Φ, and Ψ a G–invariant subspace of Φ. If Ψ is separable, then $\Psi \subseteq \Phi_c$.*

THEOREM 2.3. *Let G be a non–discrete locally compact group, and T be a lower semi–continuous representation of G as linear isometries on the normed space Φ, and Ψ a G–invariant subspace of Φ. If $\Psi \not\subset \Phi_c$, then Ψ has dimension at least c in the norm topology.*

Let $M(G)$ denote the space of regular Borel measures on G and $M_a(G)$ denote the closed ideal of measures in $M(G)$ absolutely continuous with respect to Haar measure.

COROLLARY 2.4. *Let G be a non–discrete locally compact group and Ψ a left translation–invariant subspace of $M(G)$. If $\Psi \not\subset M_a(G)$, then Ψ has dimension at least c.*

If G is a locally compact abelian group, let $M_0(G)$ denote all measures in $M(G)$ whose Fourier transform vanishes at infinity. Let $\|\mu\|_0 = \sup\{\,|\hat{\mu}(\chi)| : \chi \in \hat{G}\}$ where $\hat{G}$ denotes the dual group of G, $\mu \in M(G)$.

COROLLARY 2.5. *Let G be a non–discrete locally compact group and Ψ a left translation–invariant subspace of $M(G)$. If $\Psi \not\subset M_0(G)$, then Ψ has dimension at least c in the $\|\cdot\|_0$–norm topology.*

For each $x \in G$, let δ_x denote the Dirac measure at x. By Rad M_a(G) we shall mean the intersection of all maximal ideals of $M(G)$ that are not contained in the set of ideals identified with the dual group $\hat{G}$ of G. Since we do not know whether or not the action of G on $M(G)$ is lower semicontinuous when G is abelian and $M(G)$ has the spectral radius norm (see problem 1 in section 6), we cannot apply Theorem 2.3 to prove:

THEOREM 2.6. *Let G be a non–discrete locally compact abelian group and Ψ be a translation–invariant subspace of $M(G)$. If $\Psi \not\subset$ Rad $M_a(G)$, then Ψ has dimension at least c in the spectral radius norm topology.*

The proof of Theorem 2.6 depends on the following observation:

LEMMA 2.7. *Let ν be a singular measure on a locally compact group G and let $H = \{x \in G : \delta_x * \nu \perp \nu\}$. Then there exists a set C of cardinality at least c such that $x^{-1}y \notin H$ whenever x and y are distinct elements of C.*

3. Separable subspaces in $B(G)$.

Let G be a locally compact group and $P(G)$ be the set of continuous positive definite functions on G. Let $B(G)$ be the linear span of $P(G)$. Then $B(G)$ is an algebra under pointwise mulitiplication and invariant under left and right translations by elements of G. $B(G)$ can be identified as the continuous dual of $C^*(G)$, the enveloping C^*–algebra of $L^1(G)$, i.e. $\langle \varphi, f \rangle = \int \varphi(t) f(t)\, dt$ for any $\varphi \in B(G), f \in L^1(G)$. Then $B(G)$, with the dual norm on $C^*(G)^*$, is a commutative Banach algebra, called the *Fourier–Stieltjes algebra* of G. Furthermore, if G is abelian, then $B(G)$ is isometrically isomorphic to $M(\hat{G})$ by the Bochner's Theorem. Let $A(G)$ denote all elements in $B(G)$ of the form:

$$\varphi(x) = \langle \ell_x h, k \rangle, \quad h, k \in L^2(G)$$

$\ell_x h(y) = h(x^{-1}y)$, $x, y \in G$. Then $A(G)$ is a closed ideal in $B(G)$ (called the *Fourier algebra* of G) isometrically isomorphic to $L^1(\hat{G})$ when G is abelian. We refer the readers to [E] for more details about $A(G)$ and $B(G)$.

Let E denote the weak*–closure of the extreme points of $P_0(G) = \{\varphi \in P(G) : \varphi(e) \leq 1\}$. Then, when G is abelian, $E \setminus \{0\}$ corresponds exactly to the characters on G. A subset Φ of $B(G)$ is *invariant* if $\varphi f \in \Phi$ for all $\varphi \in E$ and all $f \in \Phi$. Clearly, every ideal is invariant.

THEOREM 3.1. (*i*) *Let G be an amenable locally compact grup. Let Φ be an invariant separable subspace of $B(G)$. If $\phi \in \Phi$ and $\phi \neq 0$, then there exists $f \notin E$ such that $\phi f \in$ and $\phi f \neq 0$. In particular, if G is abelian, then $\Phi \subseteq A(G)$.*

(ii) The "ax + b" group contains a separable ideal $\psi \subseteq B(G)$ *and* $\psi \not\subset A(G)$.

THEOREM 3.2. *(i) Let* Φ *be a separable invariant subspace of* $(B(G), \|\cdot\|_\infty)$ *where* $\|\cdot\|_\infty$ *denotes the supremum norm on* $B(G)$. *If G is amenable, then for each* $\phi \in \Phi$, $\phi \neq 0$, *there exists* $\gamma \in E$ *such that* $\gamma\phi \neq 0$ *and* $\gamma\phi \in C_0(G)$. *In case that G is abelian, that implies* $\Phi \subseteq C_0(G)$.

(ii) If G is either the Euclidean motion group or the SL(2,R), then $(B(G), \|\cdot\|_\infty)$ *is separable.*

(iii) If G is a [Moore]–group (i.e. each of its irreducible unitary representations is finite dimensional), and if $(B(G), \|\cdot\|_\infty)$ *is separable, then G is compact.*

4. Continuity of translation in $L^\infty(G)^*$ and related subspaces.

Let G be a locally compact group and $L^\infty(G)$ be the space of essentially bounded complex measurable functions on G with the essential sup norm. Let W be a C^*–subalgebra of $L^\infty(G)$ containing constants and invariant under left translation ℓ_a, $a \in G$, where $(\ell_a f)(x) = f(a^{-1}x)$, $x \in G$. A linear functional m on W is called *left invariant mean* if $m \geq 0$, $\|m\| = 1$ and $m(\ell_a f) = m(f)$ for all $a \in G, f \in W$. G is *amenable* if $L^\infty(G)$ has a left invariant mean. As well known, all compact and all abelian groups are amenable. But any locally compact group G containing the free group on two generators as a closed subgroup (i.e. $SL(2,\mathbf{R})$) is not amenable. We refer the interested readers to the classic of Greenleaf [Gr] and the recent books of Pier [P] and Paterson [Pa].

Let $(W^*)_c$ denote all elements φ in W^* such that the map $G \to W^*$, $x \to \ell_x^*\varphi$ is continuous when W^* has the norm topology. Then, obviously, $(W^*)_c$ contains the linear span of the set of left invariant means on W.

THEOREM 4.1. *Let G be a locally compact group. Then G is compact if and only if for each* $\mu \in (L^\infty(G)^*)_c$ *there exists a left invariant mean* ν *such that* $\mu << \nu$.

THEOREM 4.2. *Let G be a noncompact locally compact group. Let W be a left translation–invariant* C^**–subalgebra of* $L^\infty(G)$ *such that* $W \cap CB(G)$ *separates points and contains constant functions. Then there exists* $\mu \in (W^*)_c$ *such that* μ *is singular with respect to every left translation–invariant mean on W.*

THEOREM 4.3. *Let G be a unimodular locally compact group with an infinite closed discrete subgroup H. Then there exists an element* $\mu \in (L^\infty(G)^*)_c$ *that is singular with respect to every* *translation–invariant mean on G and with respect to* $L^1(G)$.

THEOREM 4.4. *Let G be a locally compact group.*

(*i*) $(L^\infty(G)^*)_c = L^\infty(G)^*$ *if and only if G is dsicrete.*

(*ii*) $(L^\infty(G)^*)_c = L^1(G)$ *if and only if* $L^\infty(G)$ *has a unique left invariant mean.*

REMARK. If G is amenable as a discrete group, then $L^\infty(G)$ has more than one left invariant mean (see [Gn] and [R]). However, for $n \geq 3$, and for $G = SO(n,\mathbf{R})$, the situation is different: $L^\infty(G)$ has a unique left invariant mean (see [M] and [D]).

Let $LUC(G)$ denote the space of bounded complex–valued left uniformly continuous function on G. Then, as well known, $LUC(G) = L^\infty(G)_c$ when G acts on $L^\infty(G)$ by translation. It can be shown that $(LUC(G)^*)_c$ is an L–subspace of $LUC(G)^*$. However, $(L^\infty(G)^*)_c$ is not an L–space in general (e.g. when $G = \mathbf{R}$).

THEOREM 4.5. *Let G be a locally compact group. Then*

(*i*) $(LUC(G)^*)_c = LUC(G)^*$ *if and only if G is discrete.*

(*ii*) $(LUC(G)^*)_c = L^1(G)$ *if and only if G is compact.*

(*iii*) $(LUC(G)^*)_c$ *contains a measure on* $\Delta(LUC(G))$ *with a non–zero part if and only if G is discrete.*

5. Continuity of translation in $VN(G)$ and $VN(G)^*$.

Let $VN(G)$ denote the von Neumann algebra generated by left translations on $L^2(G)$. Then, as well known, $VN(G)$ coincides with the closure of $\{\rho(f) : f \in L^1(G)\}$ in $\mathcal{B}(L^2(G))$ in the weak operator topology where $\rho(f)(h) = f * h$, $h \in L^2(G)$. If G is a locally compact abelian group, then $VN(G) \cong L^\infty(\hat{G})$. Furthermore, $A(G)$ can be identified as the unique predual of $VN(G)$ with $\langle \varphi, \rho(f) \rangle = \int \varphi(t) f(t)\, dt$, $f \in L^1(G)$, and $\|\varphi\| = \sup\{ | \int \varphi(t) f(t)\, dt | : \|\rho(f)\| \leq 1\}$.

G acts naturally on $VN(G)$ by the map $(x,T) \to \ell_x \circ T$, $x \in G$, $T \in VN(G)$ and G acts on $VN(G)^*$ via the adjoint action.

THEOREM 5.1. *Let G be a locally compcat group.*

(*i*) $VN(G)_c = VN(G)$ *if and only if G is discrete.*

(*ii*) $VN(G)_c = C^*_\lambda(G)$ (*the C**–*algebra generated by* $\rho(f), f \in L^1(G)$) *if and only if G is compact.*

Let G_d denote the group G with the discrete topology.

THEOREM 5.2. *Let G be a locally compact group. Suppose that* $(VN(G)^*)_c = A(G)$. *Then the following hold*:

(*i*) *If G is amenable, then G is compact.*

(*ii*) *If* G_d *is amenable, then G is finite.*

Note that in Theorems 4.4, 4.5, and 5.1, the first part is rather easy.

6. Some open problems.

1. Let G be a locally compact abelian group. Is the action of G on $M(G)$ defined by $(x,\mu) \to \delta_x * \mu$, $x \in G$, $\mu \in M(G)$ lower semicontinuous when $M(G)$ has the spectral radius norm?

2. Can the amenability assumption be dropped in Theorem 3.1(*i*) and Theorem 3.2(i) ? Note that "amenability" in both Theorems 3.1(*i*) and 3.2(*i*) can be replaced by the following weaker condition which holds for all amenable groups, all free groups, and the group $SL(2,\mathrm{R})$ (see [CH]): there is a net in $A(G)$ with multiplier norm bounded by 1 and which converges to the constant one function in the weak*–topology.

3. Let G be a non–discrete locally compact group. Does there always exist an element $\mu \in (L^\infty(G)^*)_c$ which is singular with respect to every translation–invariant mean on G and with respect to $L^1(G)$ (see Theorem 4.3).

4. Let G be a locally compact group such that $(VN(G^*))_c = A(G)$. Is G necessarily finite? (see Theorem 5.2).

5. If $(L^\infty(G)^*)_c$ is an L–space, is G necessarily discrete?

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