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A probabilistic analysis of the mean-variance opportunity locus.

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A PROBABILISTIC ANALYSIS OF THE MEAN-VARIANCE OPPORTUNITY LOCUS

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A PROBABILISTIC ANALYSIS OF THE MEAN-VARIANCE OPPORTUNITY LOCUS

ABSTRACT

In this dissertation it is shown that there is a high probability that the variances of portfolios selected at random from a large investment universe, and expected to yield ex ante a return E_j, will map on the mean variance plane in the immediate neighborhood of the conditional expectation $E(V|E_j)$.

The implications of this finding for the two parameter mean-variance model were analyzed with the help of ten Monte Carlo experiments. For investment universes as small as the one composed by fifteen uncertain prospects, it has been shown that $E(V|E_j)$ maps very close to the minimum variance portfolio for that level of expectation, under conditions not unlike those observed in the stock market. That is, when the correlations between pairs of investment prospects are moderate (in the neighborhood of .5 or smaller), and when E_j is not at the extremes of its feasible range.

These findings are expected to have wideranging and unsettling consequences for portfolio theory, its normative and positive implications, and even for the empirical validation of the latter. The characteristics of the conditional distribution of variances introduce strong structural patterns on the mean variance opportunity locus which may

show that a great number of the "good results" attributed to the two-parameter model are a consequence of the "structure" of the opportunity locus and not due to the explanatory power of the model

Old and new empirical research has been reexamined with the help of plausible structural models derived from the characteristics of the opportunity locus. The startling conclusion is that the structural models explain equally well the observable implications of the mean-variance model, and that they also explain observable phenomena which the two parameter model fails to predict.

A review of the pertinent literature related to normative and prescriptive portfolio models is provided in Chapter One. It begins with the concepts of admissibility and stochastic dominance, discusses moment approximation models, provides a panoramic overview of the mean-variance model and concludes with a brief presentation of popular alternative risk criteria, such as semivariance and the concept of "safety first".

The random process for the generation of portfolios x in the convex set $\{x|E_j\}$ is formulated in Chapter Two. It is based on the property of points in a closed convex set to be convex linear combinations of the extreme points of the set. The Monte Carlo methodology derived from this process is documented in Chapter Three. Chapter Five provides satisfactory evidence that the random process is ap-

propriate and that it corresponds to a process which is intuitively acceptable.

The Monte Carlo experiments were designed to observe the responsiveness of the conditional distributions of variances to changes in the size of the investment universe, in the level of correlation between returns of pairs of uncertain prospects, and to different levels of expectation E_j within its feasible range. The description of the experiments, the results and their interpretation may be found in Chapter Four.

Finally, Chapter Six develops the models for the formulation of the normative and empirical implications of the results of Chapter Four which lead to the conclusions stated at the beginning of this abstract.

INTRODUCTION

The purpose of this dissertation is to analyze the characteristics of the conditional distribution of variances resulting from a random process which selects portfolios expected to yield ex ante a fixed return E_j. The characteristics of the conditional distribution of variances define an implicit structure of the mean-variance opportunity locus which has not been previously analyzed. An analysis of the mean-variance opportunity locus has apparently never been attempted. Farrar's early study of mutual funds described the shape of the region with respect to its boundaries but did not study other characteristics.[1]

The characteristics of the conditional distribution of variances define very strong structural patterns of the opportunity locus which may be used as an alternative explanation to the normative and positive observable consequences implied by the mean-variance model. It is shown that under conditions which are also present in the stock market, the variances of random portfolios expected to yield ex ante a return E_j will exhibit an unordinately high density in the immediate neighborhood of their conditional expectation $E(V|E_j)$; that the distance between this conditional expectation and the variance of the minimum variance portfolio for that level of expectation can be explained by the size of the investment universe, the overall level of correlation between returns of pairs of uncertain prospects, and the position of E_j on its feasible range. The results of ten Monte Carlo experiments indicate that under specified conditions, $E(V\ E_j)$ is negligibly different from the minimum variance for the level of expectation E_j.

Plausible structural models have been derived from these results, and they provide a strong indication that many of the useful attributes of the two parameter mean-variance model may well be due to the structural characteristics of the opportunity locus. Since these characteristics are inherent to the region and they do not imply behavioral or theoretical assumptions about individuals and markets of uncertain prospects, the state-of-the-art implications of the mean variance model are therefore placed in jeopardy.

Chapter One reviews the development of normative and prescriptive portfolio models. It begins with the most general formulations based on the admissibility of uncertain prospects, and describes discriminators based on the concept of stochastic dominance. Through gradual addition of assumptions, several moment approximation models are examined leading to the presentation of the two parameter mean-variance model. This model and its implications are reviewed in a panoramic, if brief, historical perspective. The sporadic literature related to mean-variance prescriptive models is discussed, and used as evidence of an underlying structure in the opportunity locus which may explain the conformity of prescribed approximations with the portfolios on the mean-variance efficient set. Finally, the last section of the chapter examined popular risk criteria which are variations of the concepts of semivariance and "safety first". In this section it is also shown that one of these measures, the lower partial variance of the multivariate distribution of returns, prescribes portfolios which map in the mean variance plane at negligible distances from the mean-variance efficient frontier. The latter observations are used as preliminary evidence that a strong structure characterizes the mean-variance opportunity locus, and that it may very well explain the persistency of good results attributed to approximations. Chapter One provides the atmosphere and the motivation for the analysis of the conditional distribution of variances corresponding to random portfolios expected to yield ex ante a return E_j.

In Chapter Two a random process is defined for the generation of random portfolios x in a convex set $\{x|E_j\}$ which is defined by the target return E_j and a group of linear constraints. The random process is based on the property of the closed convex sets which allows that any point in the set can be expressed as a convex linear transformation of its extreme points. The linear coefficients constitute a random partition of the interval [0,1] and define a random portfolio in the set $\{x|E_j\}$. Based on this definition of the random process, the first two moments of the distribution of conditional variances are derived, and some of their properties established. The last sections of the chapter examine the similarity between the portfolio $E(x \varepsilon \{x|E_j\})$ and the portfolios x_* and x^* in the same set which correspond to the minimum and maximum variances for the set. This analysis of similarity provided sufficient grounds for the formulation of causality relationships which define the experimental variables of interest.

Chapter Three describes the Monte Carlo methodology and the programs used for the generation of random portfolios according to the random process formulated above.

Chapter Four describes the ten Monte Carlo experiments designed to observe the responsiveness of the conditional distributions to changes in the size of the investment universe, in the level of correlation between pairs of uncertain prospects; and to different levels of expectation E_j within its feasible range. The analysis and discussion of the results of this experiment lead to the conclusion that under specified conditions a great density of the conditional distribution can be expected in the immediate neighborhood of the minimum variance for that level of expectation. The specified conditions are loosely defined as those in which the investment universe contains fifteen or more prospects, the correlations between prospects are in the neighborhood of .5 or less; and the target level of expectation is not at either extreme of its feasible range.

The vector x in $\{x|E_j\}$ may not be uniquely determined through a convex linear transformation of the extreme points of the set by a partition of [0,1]. This introduces the suspicion that the high density observed in the neighborhood of the expectation of the conditional distribution of variances may be due to overdetermination or "double counting". A test is provided in Chapter Five which shows that "double counting" did not take place. In the same chapter, an alternative intuitively acceptable random process is also formulated, and it is shown that its results are in agreement with those of Chapter Four. This is taken as evidence of the appropriateness of the random generation process formulated in Chapter Two, and the generality of the patterns observed in the Monte Carlo experiments.

Finally, an evaluation criterion is defined in Chapter Six under which mean variance prescriptive models do not dominate ex ante two alternative selection models based on the structural properties of the opportunity locus. A brief discussion of the implications of this finding for the management of an active, high-turnover portfolio and the information systems of mutual funds constitutes the concluding remarks regarding the prescriptive implications of the results of this dissertation. The remaining sections of that chapter are devoted to the discussion of the empirical implications of the structure of the opportunity locus. The early work of Farrar[2] and Sharpe[3] is used as background for the discussion of the philosophical issues resulting from two models being empirically indistinguishable over a wide range of observations. The chapter concludes with a demonstration that the empirical observations of Fama and McBeth[4] could be explained by a structural model which is tentatively proposed. There is an indication that the structural model explains observations which they document but choose to ignore.

These results suggest that future tests of the implications of the two parameter model should include conclusive evidence that the relationships observed are not a consequence of the structural characteristics of the mean variance opportunity locus. This follows from the fact that the structural characteristics have been derived independently of the behavioral assumptions of the mean-variance model and therefore have no behavioral or theoretical content and are merely properties of the data.

In summary, this dissertation has taken a fresh look at the mean-variance paradigm and has provided new insights into the nature of the relationships underlying the model, which are useful to explain many of the implications of the two parameter model. The job is not nearly finished since this research is at best preliminary and many of the assumptions made here constitute quantum jumps that must be smoothed out. It is a novel and provocative approach to an old problem in which the direction of research seems to be to test "ad infinitum" the implciations of the behavioral assumptions implied by the mean variance model. It is hoped that by introducing structural relationships, some attention will be directed to the analysis of the characteristics of the opportunity locus which have no bearing on the elegant theoretical formulations of investment behavior, but which may constitute a critical factor rendering the theory empirically untestable.

FOOTNOTES

1. D.E. Farrar, The Investment Decision Under Uncertainty. Chicago (Markham), 1967.

2. Ibid.

3. W.F. Sharpe, "Risk Aversion in the Stock Market; Some Empirical Evidence," Journal of Finance, 20 (May 1965), pp. 416-422.

4. E.F. Fama and J.D. MacBeth, "Risk, Return and Equilibrium: Empirical Tests," Journal of Political Economy, 81 (May/June 1973), pp. 607-636.

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CHAPTER I

THE PORTFOLIO SELECTION PROBLEM

In its most general formulation, the selection of a portfolio consists of allotting proportions of a fixed initial wealth W_0 to n investments which constitute the entire investment opportunity set.

The portfolio selection problem is a two-stage process in which a suitable criterion for choice is adopted first, and then a portfolio is constructed according to this criterion.

Normative models of portfolio selection make assumptions, explicitly or implicitly, about the preference structure of individual investors, prescribe criteria for selection which are consistent with the preference assumptions, and some formulate algorithms which under specified conditions will yield optimum portfolios for the assumed preference structure.

This chapter will review the formulation, assumptions and implications of the most popular normative models for portfolio selection. Assuming a structural rather than a developmental approach, the review will begin with the most general models and through gradual imposition of assumptions and conditions, it will attempt to describe the most popular models and algorithms available for the selection of portfolios of risky assets.

A General Formulation of the Normative Portfolio Models

An investor with a preference structure conforming to a well defined group of axioms is attempting to allocate his present wealth W_0 between n investment prospects. Without loss of generality it can be assumed that each investment prospect can be described by a probability distribution function $F_i(r)$ of its single-period wealth relatives r (1 + Rate of Return). To formulate a normative portfolio selection model it is first necessary to establish an ordering over a set of stochastic variables. This ordering may be established over the distribution functions $F_i(r)$ and may be represented by a class of well behaved utility functions, the Von Neumann-Morgernstern (hereafter referred as VNM) class of utility functions, for example.[1]

For the wide class of VNM utility functions, an ordering over the distribution of wealth relatives may be defined as follows:[2]

Let $U(r)$ be a VNM utility function, and $F_i(r)$, $F_j(r)$ be the distributions of value relatives for investment prospects i and j respectively, then

(1) $$\int_{-\infty}^{\infty} U(r)\,dF_i(r) \geq \int_{-\infty}^{\infty} U(r)\,dF_j(r)$$

if and only if the i-th investment prospect is preferred or indifferent ($i \gtrsim j$) to the j-th investment prospect. Equivalently, if $E_i(U)$ and $E_j(U)$ are the expectations of $U(r)$ over $F_i(r)$ and $F_j(r)$ respectively, then

(2) $$E_i(U) - E_j(U) = \int_{-\infty}^{\infty}(F_i(r) - F_j(r))dU(r) \geq 0$$

if and only if $i \gtrsim j$.

From (1) or (2), a weak partial ordering can be established over the set of all possible portfolios[3] provided that U(r) is non-decreasing with finite values for any finite value of r. Given any two portfolios i and j, it will be said that i dominated j (iDj) if and only if

(3) $F_i(r) \leq F_j(r)$ for every r, and $F_i \neq F_j$ for at least one r.[4]

The relation described by (3) is also known as a stochastic dominance relation.

The set {P} is the set of all possible portfolios given the constraints of an individual investor, it will be called the portfolio opportunity set. If i is in {P}, i is said to be <u>inadmissible</u> if there is a portfolio j≠i in the portfolio opportunity set such that jDi. Portfolios are said to be <u>admissible</u> if and only if they are not inadmissible.[5] The efficient set {ε} is then defined as the subset of {P}containing only admissible portfolios. Efficiency depends on the definition of the dominance relation jDi, which in turn, depends on the class of utility functions for which it is defined. One is allowed to talk about an efficient set only in relation to a specific class of utilities.

For example, if the investor is assumed to be a risk averter in the Pratt[6]-Arrow[7] sense, a new weak partial ordering can be defined for utility functions in the risk averter class:

Let $U(r)$ be such that $\delta U/\delta r>0$, $\delta^2 U/\delta r^2<0$ and $\delta^3 U/\delta r^3$ exists, then a necessary and sufficient condition for dominance iDj is that

(4) $\int_{-\infty}^{r}(F_j(t)-F_i(t))dt \geq 0$ for every r, and not $F_i=F_j$ for all r's.[8a]

Alternative conditions for dominance have proliferated[8b] and will not be discussed here. Despite their elegance, criteria for dominance such as those described here are merely binary discriminators which require pairwise comparisons of all elements in $\{P\} \times \{P\}$ for the identification of efficient sets. Some attempts have been made to formulate heuristics for the identification of subsets of the efficient set,[9] but they are preliminary and at best very rudimentary.

The practical importance of a normative model must be judged on its ability to prescribe not only criteria for selection, but paths to the "optimality" implied by those criteria. From (1) or (2) above, it can be inferred that individuals acting consistently with the assumptions implied by the VNM utility functions must select an element of $\{P\}$ such that it maximizes their expected utility. The formulation of algorithms and heuristics for the accomplishment of this objective depends to a great extent on the existence of a representation of $E(U)$ which is computationally amenable for this purpose.

The following section will describe an approximation which is particularly well suited for the construction of algorithms leading to the selection of portfolios which maximize expected utility. This method will be called the "moment approximation approach," which is to be distinguished historically from the so-called state-time-preference approach developed by Arrow[10] and Debreu.[11] The latter is a very elegant framework for the analysis of the theoretical issues of economic decision making under uncertainty, but it has yet to provide empirical content to its formulations[12] and derive prescriptive paths for the attainment of maximum utility. This paper will not discuss the comparative advantages of one approach over the other, this can be found by reference to Hirschleifer[13] or Karl Borch.[14]

The Moment Approximation Approach

Assume that an investor chooses from a portfolio opportunity set such that all the elements in the set are fully described by their distributions of terminal wealth $F(W)$, where terminal wealth is the random variable

(5) $$W = rW_0$$

which can be expressed as

(6) $$W = \overline{W} + h, \text{ where } \overline{W} = E(W) \text{ so that } E(h)=0.$$

Let the utility of wealth $U(W)$ be continuous and have derivatives, then it can generally be expanded into a Taylor Series[15] about $\overline{W}$ as

$$(7) \qquad U(W) = U(\overline{W}+h) = U(\overline{W}) + U'(\overline{W})h + U''(\overline{W})(h^2/2!)+....$$
$$+ U^{(k-1)}(\overline{W})(h^{i-1}/(k-1)!) + R_k$$

where the residual R_k is defined as

$$(8) \qquad R_k = U^{(k)}(W+th)(h^k/k!), \qquad 0<t<1$$

If $m_1, m_2 \ldots\ldots\ldots, m_{k-1}$ are the first k-1 central moments of the distribution F(W), then the expected utility can be expressed as

$$(9) \qquad E(U(W)) = U(\overline{W}) + U''(\overline{W})(m_2/2!) + U^{(3)}(\overline{W})(m_3/3!) + \ldots$$
$$+ U^{(k-1)}(\overline{W})(m_{k-1}/(n-1)!) + E(R_k)$$

Tsiang,[16] to whom the previous development is owed, explicitly states that only when the series (9) can be shown to be convergent can the remainder $E(R_k)$ be neglected. Under these conditions, expected utility can be treated as a polynomial of the first k-1 central moments of F(W) with constant coefficients for a fixed $\overline{W}$. This is not to say that the utility function can be expressed as a (k-1)-th order polynomial. Borch[17] shows that this would be inconsistent with the generally accepted Arrow[18] conditions for risk aversion.[19]

A heuristic study of the conditions under which the polynomial in (9) is a reasonable approximation for expected utility can be found in the previously referenced article by Tsiang, and in a recent article by Samuelson.[20] For the purpose of this review it will suffice to state that the first two-moment approximation is appropriate when the standard deviation of the distribution of final wealth is small relative to total wealth including human capital, e.g. for "stakes" which are small compared to wealth; and in general, as Samuelson shows, when the distribution of terminal wealth is "compact."[21] Reasonably, the normal distribution is particularly well fitted since its odd moments vanish and the higher even moments can be expressed as functions of powers of m_2, generally of smaller order; so that if m_2 is small, the series will converge rapidly. How far a truncation of (9) will be permissible depends on the characteristics of its convergence which also reflects the class of utility functions which are assumed. Particularly well suited functions are the exponential utility and the constant elasticity utility functions.[22]

A serious constraint to the utilization of higher moments in the approximation is the lack of economic theory about the behavior of $U^{(i)}$ for i of order higher than the third. The state of the art for risk averters is the definition given above (n.19) from which Tsiang derives the requirement that $U^{(3)} \geq 0$. There is some indication that $U^{(4)} \leq 0$, assuming isomorphism with the exponential or logarithmic utility functions; or, as argued by Fama,[23] by observing that in comparison to the normal distribution, a highly leptokurtic distribution implies a larger probability of high losses. The latter arguments remain speculative.

Three moment approximations became popular in the finance literature[24] following the publication of empirical results by

Arditti[25,26] who presented evidence that investors are willing to trade away expected return for positive skewness in the distribution of returns. This result may now be strengthened by the demonstration that for risk averters, the third derivative must be non-negative.[27] At the danger of incurring in redundancy, it is also noticed that Alderfer and Bierman[28] report that skewness preference was a fairly prevalent pattern in the behavior of subjects selecting trivial make-believe investment prospects in a laboratory situation.

Four-moment models have not been intended as such, but have resulted from the attempts of researchers to investigate an alternative criterion for portfolio selection, the geometric mean. As an alternative for expected utility maximization, it is assumed that investors have the objective of maximizing terminal wealth in the long-run; Latané and Tuttle[29] show that this objective is consistent with the selection of portfolios which maximize ex ante expected geometric mean return, provided that returns are reinvested in a portfolio which allots in each period the same proportion of wealth to each prospect which had been allotted in the previous period. Interestingly, this policy maximizes the expected utility of terminal wealth for investors possessing logarithmic utility functions, and subject to a solvency constraint with probability one.[30] Young and Trent,[31] building upon the previous work of Latné, studied the properties of moment approximations to the geometric mean return G of a portfolio which is

$$(10) \qquad G = u_1 - (m_2/2u_1^2) + (m_3/3u_1^3) - (m_4/4u_1^4)$$

where u_1 is the expected single-period wealth relative, and m_i is the i-th central moment of the distribution of wealth relatives. It is not surprising that Young and Trent state that if the third and higher moments are small in relation to u_1, then the geometric mean can be approximated by the first two moments. These authors showed that for actual portfolios, the three and four-moment approximations did not improve significantly the accuracy of two-moment approximations, and that the error of the two-moment approximations to the geometric mean return decreases rapidly with the size of the portfolio, reaching an average error of one tenth of a percent for portfolios including 224 stocks. These results are encouraging for the proponents of the mean-variance approach since the isomorphism of expressions (9) and (10) is readibly observable, and it suggests that the first two moment approximation to expected utility may be appropriate for actual portfolios of stocks. This is a testable speculation which in the case of the popular logarithmic utility function is even tractable, as will be seen below.

Let $W=W_0r=(u_1+z)W_0$ and $U(W)=\log W$, then by (9) it can be seen that

$$(11) \qquad E(\log W)=\log(W_0u_1)-(m_2/2u_1^2)+(m_3/3u_1^3)-(m_4/4u_1^4)+\ldots.$$

and by application of (10) above, it can be seen that an accurate approximation given the results of Young and Trent is

(12) $E(\log W)=\log(W_0 u_1) + G - u_1$

when moments of order greater than four can be neglected. Since G can be accurately approximated by the first two-moments, then by examination of (12), it follows that the expected utility should be accurately approximated by (9) truncated after the first two moments. Young and Trent's empirical results are complementary to Tsiang's analysis in giving strength to a defense of the mean variance approach as an approximation. It is only fair to observe that this conclusion should not be very surprising given the previously referenced conclusion by Haakansson (see n.30) who shows the equivalence between maximization of G and E(U) when the utility U is logarithmic.

Despite the arguments presented here for the inclusion of higher moments in the approximation, it seems that the apparent power of the first two moments to approximate the most accepted normative criterion - Expected Utility Maximization - will contribute to the longevity of the mean-variance paradigm.

The approximation of expected utility by a (k-1)-th order polynomial hinges on the fact that all (k-1) first moments are assumed to exist and to be finite. Mandelbrot and Fama have presented empirical results which they have interpreted as evidence that the distributions of price changes in the stock and commodity markets are non-normal members of the Pareto-Levy family of distributions.[32] These distributions share the characteristic of possessing no finite moments, and the property of being closed under addition, so that the distributions of day-to-day changes are of the same form of those of week-to-week or year-to-year changes, thereby the name "stable" which is applied to members of this family. The normal distribution is the only member of the family which possesses all its moments.

Numerous arguments have been raised against this hypothesis, most notably by Agnew[33] who showed that other distributions such as the bilateral exponential share the stability property and possess all their moments, and who claimed that a better empirical "fit" can be achieved using other "fat-tailed" distributions. The issue is not settled yet, but the soundness of the arguments for infinite moments may be judged from the following comments of Markowitz[34] and Tsiang.[35] Markowitz denies the theoretical requirement for the stability property on the grounds that the determinants of day-by-day price changes are unrelated to the determinants of the year-to-year fortunes of an enterprise. He also claims after introspection, that since the distributions of future returns are subjectively determined, he is willing to bound his between zero and a very large finite value and expects other investors to do the same. Tsiang presents a very convincing argument that is based on the fact that downward changes in prices cannot exceed 100 percent and therefore the distributions must be truncated stable Paretian, which since they are truncated cannot have infinite moments and therefore not be Paretian. He also makes a call for the proponents of the Paretian hypothesis to define an acceptable VNM utility function which when applied to a Paretian distribution other than the normal will yield a finite positive expected utility.

The latter is a very pertinent argument since Fama[36] attempts to reconcile his claim with the "finite moment approach" by indicating that the mean variance approach provides valuable insights into diversification which remain valid even in the event of infinite variance. A suitable measure of interfractile dispersion may be used as indicated by Blume[37] as a substitute for variance for the purpose of obtaining approximate predictive descriptions of the distributions of returns for distributions which are stable Paretian. An algorithm for the identification of efficient sets under these conditions has apparently not been proposed.

This is an interesting although anticlimatic conclusion for the Paretian hypothesis since it is another indication of the willingness of the academic community to preserve the integrity of the mean-variance approach. When the method is deemed an adequate approximation for situations involving a small variance, and an adequate explanation for situations involving infinite variance, one can only expect that mean-variance approach and its close relatives, the moment approximation models, are here to stay for a much longer period than its detractors had hoped for. A closer scrutiny of the mean variance paradigm will follow.

The Mean-Variance Approach to Portfolio Selection

<u>A brief historical comment</u>. In his pioneering article[38] Harry Markowitz formulated a single period normative model of portfolio selection which is based on the assumption that individual preferences can be adequately described for the relevant range of terminal wealth values by the rising portion of a quadratic utility function. This assumption leads to the formulation of the portfolio selection problem as a quadratic program. The work of Markowitz constitutes a turning point in the development of models and theories of decision-making under uncertainty, and is the cornerstone in which a host of positive and prescriptive models rely; or the target chosen for attack when alternative prescriptive models are presented. A review of these models will follow under the arbitrary headings which attempt to chategorize them as Positive, Evaluative and Naively Prescriptive.

<u>Positive models</u>. The most important category from the point of view of economic theory is the development of "positive" models based on the assumptions of the Markowitz model. This trend was pioneered by Tobin[39] who, following the publication of the mean-variance normative model, presented a model of liquidity preference and demand for cash for investors possessing utility functions fully determined by the first two moments of the distributions of returns of a two-parameter

family, which, as it turned out,[40] must be normal for the model to hold. Tobin's model stimulated the appearance and development of what has come to be known as the Sharpe[41]-Treynor[42]-Lintner[43] theory of equilibrium in the capital markets, or colloquially, as the Capital Market Models.

These models operate on a capital market where the prices for assets fully reflect the available information, which is shared by all investors who possess the same identical beliefs about the future, and who are mean-variance expected utility maximizers. To these conditions they add the requirement that the market is in equilibrium and prognosticate the three propositions that ex ante, the expected returns of an asset are related to no other risk except the portfolio risk; that this relationship is linear ex ante; and that the risk premiums are positive.[44,45] Strictly speaking, these models are not positive since the consequences that they predict are all ex ante and therefore not observable; this was noted by Sharpe (n.41) but largely ignored thereafter. The proponents of the models (n.44) insist on labeling them as positive models, and although there is some conflicting evidence that ex ante specific risk (as measured by the residual variance) is related to ex post average returns,[46] the consensus is that until now the three implications outlined above have withstood the battery of empirical (tests using naive ex ante estimates of risk, and ex post returns) to which they have been exposed.[48]

From the point of view of the normative mean-variance model, the "positive" models are disturbing since they contain one normative implication which if confirmed would render the quadratic programming formulation totally useless. Given the assumptions and implications of capital market theory which were outlined above, and the assumption that investors may borrow and lend at the risk-free rate r_f, the normative implication is that there exists only one "efficient" portfolio of risky assets in the mean variance sense, and this is the "market portfolio" which contains all risky assets in proportion to their market value.[49]

The market portfolio can be approximated with relative ease by selecting random portfolios from the population of risky assets, with probabilities of selection proportional to the market value of the asset. The normative implication and the ease with which proxies for the market portfolio can be selected provided the foundations for the derivation of evaluative models.

Evaluative models. These models have come to be known as performance evaluation models and are due, not surprisingly, to Sharpe[50] and Treynor,[51] and most notably to Jensen.[52] They rely on the implications of capital market theory and test the assumption of market efficiency, e.g. that market prices reflect all available information. The benchmark for comparison in all three models is a naively derived portfolio. This may be a random portfolio selected as a proxy for the market portfolio, which with all possible combinations of borrowing and lending yields a linear locus of risk and return called the (ex ante) capital market line. The performance of an actual portfolio may be ascertained by comparison of the ex post differential between its returns and the returns of a portfolio which is ex post in the capital market line, with the differential between the returns of the naively derived portfolio with the same ex ante risk[53] and a portfolio in the ex post capital market line.[54] Although the methodology for comparison varies between the three authors, their philosophy is equivalent. Jensen's conclusions may be used to summarize the findings: although mutual funds (115 mutual funds over the period 1955-1964) did not outperform in general naively derived random portfolios which are proxies for the market, some funds appear to perform consistently above the market; but in general, "fund portfolios were found to be inferior after deduction of all management expenses and brokerage commissions."[55]

Given this conclusion, Sharpe and Jensen do not hesitate to issue an investment manifesto which may be properly summarized in Shapre's words: "Good managers concentrate on evaluating [portfolio] risk, spending little effort and money on the search of incorrectly priced securities."[56] But as he later admits, "if some securities are [....] mispriced, the past record will help to identify the slightly superior group. This suggests a procedure which will do little harm and may do some good: Perform a mean-variance analysis using historical data with reasonably stringent [diversification constraints]."[57]

A very recent article by Treynor and Black[58] seems to agree with the later statement. Under the assumption that "security analysis properly used can improve portfolio performance,"[59] they proceed to formulate a highly idealized model in which the decision of how much to invest in the market portfolio is independent of the decision to invest in an "active" portfolio of securities which the analyst considers wrongly priced. The selection of the active portfolio is based on the familiar tradeoff between differential return (subjective estimate based on the belief that the market price is incorrect) and residual risk (variability about the differential expected return which is not explained by the market). Behavior consistent with this model would explain Fama and MacBeth's "puzzling" results (see n.45 above) and probably the finding by Jensen that "some" mutual funds performed consistently above the market. The latter conclusions are very speculative but attractive in terms of portfolio management, particularly in view of Jensen's assertion that analysts often possess advantageous information which is not realized into differential returns due to the administrative lag between detection of the differential and implementation of an action. The market absorbs information at a very rapid rate. What this suggests is the implementation of an information system, possibly mechanized, which can rapidly convert information into action that would result in differential returns. The proposal for an active portfolio lays the grounds for the pursuit of this concept.

The comments and speculations made in the last two paragraphs breathe new life into some "dated" research directed to the implementation of the mean-variance normative model, and the implications of using historical information to obtain naively optimal mean-variance portfolios. The interest in this aspect of portfolio management had naturally disappeared in view of the strong conclusions reviewed in this section, but regains interest in light of the comments of the last paragraphs. These models have been arbitrarily labelled as naively prescriptive and will be discussed in the section that follows.

Implementation of the Markowitz Model as a naively prescriptive model. The third category of research stimulated by Markowitz has been directed to the practical implementation of the normative model for the selection of portfolios. Contributions in this area have been sporadic and probably inhibited by the host of evidence supporting the normative implications of the capital market model, and the skepticism of practitioners with respect to the utilitarian value of these models.

One argument against the implementation of the normative model is that since it requires subjective estimates of variance and expected returns for individual prospects, and of covariances between the returns of pairs of prospects, its application to any meaningful portfolio selection problem taxes disproportionately the ability of analysts to convert information into estimates of variances and covariances. For example, if the efficient set were to be selected from a universe of 1000 prospects, the analyst is required to make 501,500 estimates, not including the number of pairwise comparisons which would be required to attain a consistent positive definite covariance matrix.

To avoid this problem, Sharpe[60] suggested a simplified algorithm which requires only 3002 estimates for the same universe of prospects. His simplification is based on the assumption (as in Capital Market Theory) that covariation between securities is entirely explained by their relationship to a common underlying market factor. The resulting algorithm possesses the additional advantage of being computationally more efficient (a linear programming problem) than the quadratic programming problem proposed by Markowitz.

Cohen and Pogue[61] studied the implications of using historical data to obtain the estimates required by the Markowitz and Sharpe models, and by an approximation of their own design. Theirs was a multi-purpose study in which they compared ex ante (as defined by historical data) the positions on the mean variance plane of the efficient portfolios generated by application of Markowitz's model and the corresponding portfolios generated by the approximations; and ex post with random portfolios generated from a smaller universe of prospects and portfolios of actual mutual funds. The ex ante results of the comparison are not surprising: Markowitz portfolios were mean variance efficient in relation to those yielded by the other models; and their composition was appreciably different also, with the exception of the extremes in the range of possible returns. What is surprising is that even for the range where the composition was different, the "efficient" frontiers yielded by the approximation models were insignificantly different from their Markowitz counterpart when they were plotted on the mean-standard deviation plane. It is interesting to note that while the models do not differ in their estimation of expected returns; when compared with the "true" historical covariance matrix, the covariance matrices of the approximations exhibited errors in their corresponding correlation matrices ranging from -.6999 to .6999. Considering that correlations in the market are moderately positive, the error may be considered substantial. This is the characteristic of the mean-variance paradigm which Farrar[62] labelled robustness in relation to calibration of the inputs, and that will be labelled here "the persistency of good results."

Cohen and Pogue's ex post results have their share of surprises too as indicated by their conclusions:

> 1. The expost performance [as measured by the descriptive regression of standard deviations vs. average returns] of the efficient portfolios selected by the models and the mutual funds clearly dominates that of the random portfolios.
>
> 2. The ex post performance of [the portfolios generated by the three methods] tends to dominate the performance of mutual funds for higher levels

of returns (above 15%).

3. The performances of mutual funds with less than 15 percent return are not dominated by the efficient portfolios.

4. There is no strong evidence [....] for the absolute dominance for any of the [naively prescriptive] portfolio selection methods over the total range of returns available.[63]

It is necessary to point out, as suggested by Friend and Vickers,[64] that Cohen and Pogue unintentionally loaded their results by selecting random portfolios of a smaller number of securities and from smaller universes than the portfolios selected by the naively prescriptive models, or those of mutual funds. The reason for this comment is derived from their observation that as the number of securities in the random portfolio increases, and as the size of the universe for selection is allowed to increase, the descriptive line of the random portfolios approaches the descriptive lines of the other selection modes. Their method for the selection of random portfolios consists of selecting 20 or 40 stocks from universes respectively of 75 or 150 securities, with equal probability of selection, and distributing the wealth equally among the stocks in the portfolio. It is questionable that random portfolios selected accordingly will meaningfully represent the range of feasible points in the mean variance plane. This is made evident by the fact that the random portfolios selected from a universe of 150 securities cluster non-overlappingly with the cluster of mutual funds which group at a higher return and a higher variance.

If, as the two authors claim in their reply to these criticisms,[65] the random portfolios were selected merely as a null test, then they selected probably the weakest of all available null tests since their random selection is behaviorally isomorphic with an investor who has no objectives and utilizes no information about the market. It appears that they are willing to endow their models with nothing short of omniscence, while they refuse the alternatives (often called null tests) even the most trivial access to information. When the results show that the null test model cannot be outperformed by the model, the practice may be correctly labeled as conservative; but when the results show that the model clearly outperforms the null test model, and that there is a clear indication that this null test is not the most stringent of those available, the researcher must at the least acknowledge the possibility of alternative explanations for the conclusions.

A related paper by K.V. Smith[66] analyzed the implications of utilizing stock price indexes and economic indexes as proxis for the "underlying market factor" required by Sharpe's linear approximation. The ex ante "efficient" sets plotted on the mean-standard deviation plane and corresponding to the stock price indexes are negligibly different and often overlapping. The comparison of these mappings with the mapping of a historically derived "true" ex ante Markowitz-efficient frontier confirms the results of Cohen and Pogue. Although

Smith explains the closeness of the frontiers by the fact that the three indexes are highly correlated (of the order of .9 or greater),[67] these results are not totally what one would expect. Smith used the Standard and Poor and NYSE indexes, and the Dow Jones Average for his research. The S&P and NYSE weight their stock in proportion to the outstanding market value of the asset while the DJ Average uses only 30 stocks and weights them equally. The latter can be expected to be quite erratic in comparison to the other two, and therefore when stock returns are regressed with it, the regression coefficients may be expected to understate those of the other two indices, which represent the behavior of a market portfolio more accurately. Fisher states in reference to the Dow Jones Average that "neither in its short run nor in its long run can this index be expected to reflect the behavior of a truly diversified portfolio of common stocks in the NYSE."[68]

Covariance matrices derived from the use of all three indexes may be expected to understate the true historically derived covariance matrix since residual variances and covariances (if any) are ignored. In the case of the Dow Jones Average, the covariances may be further understated due to its erratic behavior (greater variance). As a result, the efficient set obtained from Sharpe's approximation and the Dow Jones as a market index may be expected to contain overdiversified portfolios in relation to the others. Given the previous considerations, it is again surprising that portfolios which should be different in composition map so close to each other in the mean standard deviation plane. Observe though that the criterion for rejoicement in the last two articles reviewed and, as it will be seen later, in Farrar's dissertation[69] is the "closeness" of the portfolios obtained using alternative selection modes as compared to those yielded by the Markowitz method. At the risk of being mischieviously Cartesian, one is tempted to inquire "how close is close?"; and to request an educated guess of how probable it is to be close to a Markowitz efficient portfolio under a given set of circumstances.

As part of his now dated analysis of mutual funds, Farrar (see n.69) attempted to assess the performance of mutual funds by comparing ex ante mappings of mutual fund portfolios with the mappings of historically derived Markowitz efficient portfolios, this procedure he labeled "mean-variance goodness-of-fit." He addressed himself to the question of "how close close is" by obtaining the locus of maximum variance portfolios corresponding to each level of expectation, which together with the Markowitz efficient frontier constitute the boundaries of the portfolio opportunity set in the mean-standard deviation plane. The startling result is that actual mutual fund portfolios map on the mean-standard deviation in the "immediate neighborhood" of the efficient frontier for a fixed level of expectation, and at a considerable distance from the mapping of the maximum variance portfolio for the same level of expectation. Farrar's positive conclusion was that mutual funds behaved as if they were Markowitz mean-variance optimizers. His attempt to assess the probability of mapping in the neighborhood of efficient portfolios was less successful. Although he was very willing to assess closeness by comparing the position of a portfolio between the mappings of the maximum- and minimum-variance portfolios for a given level of expectation, to obtain an indication of the probability of this observation he devised a null test similar to the one used later by Cohen and Pogue. Farrar selected portfolios with equal probability of selection between eleven classes of assets, so that a great degree of conservatism can be expected of these random portfolios since they can be expected to cluster (and they did) in the immediate neighborhood of a portfolio containing all assets, with equal weights distributed among the eleven classes of assets. This portfolio is overdiversified, but still diversified and can be expected to mapp closer to the minimum variance portfolio than to the maximum variance portfolio at an expected return which is the arithmetic average of the expected returns of the eleven classes of assets. Not surprisingly, Farrar found these results to confirm this expectation, and concluded that this may be interpreted as an indication that behaviorally, the mutual funds are isomorphisms of the mean-variance paradigm.

The mere fact that a portfolio mapps in the neighborhood of an efficient portfolio only means that the two points are in each other's neighborhood. That this is the result of a willful act, a characteristic imbedded in the process, or an act of God can only be inferred after some meaningful hypotheses have been formulated as alternatives. One of the alternative hypotheses can be used as a null test, after specification of the probability of occurrence of an event, given that the alternative hypothesis holds. The alternative hypothesis can now be rejected if it fails to satisfy some arbitrary rule of thumb such as "the probability of occurrence of the event, or of events even more extreme is less than α under the alternative hypothesis," <u>but</u> this does not justify acceptance of the proposed hypothesis.[70] The fact that Farrar must reject the hypothesis that mutual funds are isomorphs of his random portfolios does not mean that he is entitled to accept the hypothesis that they are isomorphs of the mean-variance paradigm. Such conclusion is reminiscent of the paradox of the ravens[71] in which the empirical observation "this vase is green" is accepted as confirmation of the hypothesis "all ravens are black" since the vase is a non-raven and it is also non-black.

This presentation concludes the panoramic review of the mean variance approach and its dependent models in a one-period horizon. The following section will analyze in more detail the normative-mean variance model.

The Mean Variance Normative Portfolio Selection Model

A portfolio k in {P} is said to be "mean variance efficient" if there is no portfolio l in {P} such that $u_l \geq u_k$, and $v_l \leq v_k$ where u and v represent respectively the mean and variance of the distribution of single-period wealth relatives, and both equalities do not occur simultaneously.

The normative model which results from this definition is the quadratic programming formulation of the portfolio problem. Given an investment opportunity set of n investment prospects, let u be the column vector representing the expectation of the n-variate distribution of single period wealth relatives and C its covariance matrix. If x is an n-column vector representing the proportion of total wealth invested on each one of the n prospects, the mean-variance standard portfolio selection problem consists of selecting x such that

$$x'Cx \text{ is a minimum}$$

subject to

$$x'u = E_j \tag{13}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$U \geq x \geq L$$

where E_j is the target level of expectation, and U and L are n-column vectors representing upper and lower bounds.

If the expected utility is a monotonic function of x'u - cx'Cx where c is a constant representing the absolute value of a coefficient of risk aversion, then the expected utility will reach its maximum when the argument $x'u - cx'Cx$ reaches its maximum. A quadratic programming problem can be formulated which yields parametric solutions on $\lambda=1/c$, and which consists of finding x such that

$$\lambda x'u - x'Cx \quad \text{is a maximum}$$

subject to

$$\sum_{i=1}^{n} x_i = 1 \tag{14}$$

$$U \geq x \geq L$$

The solution of (14) will yield for each value of λ in $[0,\infty]$ a unique $x(\lambda)$ which is an element of the mean-variance efficient set.

Algorithms for the solution of (13) and (14) are discussed by Markowitz[72] and Sharpe;[73] an extension for multiperiod mean-variance analysis has been developed by Hommes[74] as an extension of Sharpe's. The solution procedures are well known and will not be discussed here.

The formulation (14) implies a quadratic utility function hopefully rising in the relevant range. The criticisms against the use of quadratic utility are familiar and will not be repeated here.[75] This formulation has also been criticized in terms of its inability to yield truly admissible portfolios under conditions other than those of an n-variate normal distribution of investment prospects,[76,77] and because except under the previous condition it violates the assumption that in-terms of variance. Recently an algorithm has been presented for the solution of this problem.[84]

The definition of alternative measures of risk is motivated by the desire to provide practitioners with prescriptive models of portfolio selection which are based on concepts of risk more amenable to their beliefs. These models cannot claim to be strictly normative in the sense that they do not claim consistency with axiomatic representations of rational behavior; but rather, they claim isomorphism with behavioral assumptions which are based on observation or introspection. For this reason they will be labeled here as "behaviorally motivated models" and will be discussed in the section that follows.

Behaviorally Motivated Models

These models were pioneered by Roy[85] who also deserved to be called the precursor of diversification models since his article, which appeared simultaneously with Markowitz's, lays the ground for the same implications of the mean-variance model. Roy introduced his concept of "safety first" arguing that investors possess an absolute lower bound or disaster level for the returns of uncertain investment prospects which they are assumed to avoid at all costs. Consequently, Roy's investor will select portfolios in such a way that the probability of returns below the disaster level is a minimum. It can be shown for normal distributions of returns that the "safety first rule" is exactly equivalent to Markowitz mean-variance paradigm.

The viability of this concept has acquired modern status since it can be formulated as a chance-constrained programming problem. Machol and Lerner[86] present a very lucid argument for the adoption of such models. They formulate a model which attempts to maximize terminal wealth while requiring that the probability of returns below a "disaster level" does not exceed an appropriately chosen upper bound.

Both formulations implicitly assume utility functions which do not have to be specified, but which share the characteristic of being steeper below the disaster level, and increasing at a lower rate, or constant above this level, depending on whether wealth is to be maximized or probability of loss is to be minimized. The use of semivariance as a measure of risk implies a similar assumption about the utility functions.

Semivariance is the incomplete or partial second moment of the distribution of portfolio returns about an arbitrary point which may well be Roy's disaster level. As a risk measure it assumes a utility function which is quadratic below this level, and linearly increasing above it, so that an individual attempting to maximize expected utility will choose portfolios in such a way that he maximizes

(15) $$E(U) = \lambda x'u - S(x)_a$$

where $S(x)_a$ is the semivariance about a of the distribution vestors prefer more to less which is the cornerstone of the theory of money.[78,79] These criticisms are theoretically valid, but practically, they must be analyzed in the light of the ability of the mean variance criterion to yield adequate approximations to the expected utility as it was seen above.

Baumol[80] recognized the validity of the criticisms and proposed to eliminate from the efficient set derived from (14) those portfolios which are likely candidates for inadmissibility by means of a discriminator based on Chebyshev's inequality. His criterion has been severely criticized on the grounds that it solves the wrong problem[81] since the screening rule is only correct when the distributions are normal, and under this condition formulation (14) will yield truly admissible portfolios.

Another source of dissatisfaction with the mean variance approach, which is reflected by the practitioners indifference towards its prescriptive applications,[82,83] stems from the fact that variance as a measure of risk contradicts introspective intuitive definitions of risk. Markowitz (see n.34 above) recognized the intuitive inconsistency of equating variability and risk and formulated a measure of downward risk, the semivariance of the distribution of portfolio returns about a properly selected safety level. Although he presumably preferred this measure to variance, the computational problems which it poses led him to develop the model in of returns of the portfolio represented by the vector of weights x.

A more modern application of the semivariance concept is due to Bower and Wippern[87] who express dissatisfaction with the Sharpe-Treynor indices of portfolio performance on the grounds that they fail to differentiate downward variations, and propose a measure of the instances in which a given portfolio falls more than the market. This measure is called semideviation and may be defined as follows:

Let r_t be the return of a portfolio at the end of period t

m_t be the return of the market portfolio at the end of t, then

$$(16) \qquad d_t = \begin{cases} (r_t - r_{t-1}) - (m_t - m_{t-1}) & \text{if } d_t < 0 \\ 0 & \text{otherwise} \end{cases}$$

and the semideviation sd is

$$(17) \qquad sd = \left(\sum_{i=1}^{n} d_t^2 / N \right)^{1/2}$$

The correlation between sd and the variance of actual portfolios in the market is of the order of -.15 as reported by Bower and Wippern.

As attractively intuitive as the semivariance is, it remained an academic curiosity until the appearance of Hogan and Warren's (see n.84 above) algorithm for the solution of (15). Previous to the appearance of this algorithm, Belovicz, Hommes and Pipino[88,89] formulated an approximate measure of downward risk which they called Lower Partial Variance (LPV). LPV assumes that individuals would like to maximize expected utility as in (15), but since they do not know a priori the shape of the distributions of returns corresponding to each portfolio x, which is a requisite for the definition of $S(x)_a$, then an individual consistent with (15) will attempt to minimize the dispersion of portfolio returns which result from n-variate random vectors which do not dominate a predetermined "disaster vector" A. If R is the n-variate vector of returns of the investment prospects, and the vector

$$(18) \qquad D = \begin{cases} R-A \text{ if } R \quad A \\ \bar{0} \quad \text{otherwise} \end{cases}$$

then the conditional covariance matrix C* of observations which do not dominate A is

$$(19) \qquad C^* = E[(R-A)'(R-A) - D'D]'$$

and the lower partial variance for a disaster vector A is

$$(20) \qquad LPV(A) = x'C^*x$$

A lower partial variance efficient set will be obtained from the solution of

$$(21) \qquad \text{Maximize } Z = x'u - x'C^*x$$

a quadratic programming problem which is computationally equivalent to formulation (14). Belovicz and al. (see n.89) report that although the minimum variance and minimum lower partial variance portfolios for specified levels of expectation exhibit different compositions, they map very closely to each other when plotted on the mean variance plane. This is another surprising result to add to the "persistence of good results." Not reported in their paper was the fact that when they tested for stochastic dominance (Kolmogorov Smirnov Statistic) between pairs of portfolios yielding ex ante the same expected return, they could not reject the two-tail null hypothesis that LPV-efficient portfolios were not dominated by variance-efficient portfolios or viceversa. The results are surprising since they were using Monte Carlo simulated positively skewed distributions of returns which would be expected to yield very different covariance matrices C and C*, not being here merely the case as in Cohen and Pogue's and Smith's articles in which the covariance matrices of the approximations understated the covariances of the "true" covariance matrix.

A plausible explanation which has not been examined elsewhere in the literature is that the constraints, which are shared by all these models, excert an unordinate control over the variety of the outputs, as they map on the mean variance plane. If this is the case, the persistency of good results is doubtfully a virtue of the mean-variance paradigm since it implies that ex ante, the outcome of a portfolio choice is almost entirely determined by the definition of constraints, and thus calibration of inputs and even the definition of the model does not only allow for a wide range of errors, but makes measurement error irrelevant.

The remaining chapters will examine this speculation in more detail by analyzing the conditional distribution of portfolio variances given a set of constraints and a target expected return.

Conclusion

Mean variance analysis appears to be an inordinately robust paradigm since it withstands violations of its assumptions, demonstration of inconsistencies in its theoretical structure, gross inaccuracies in its inputs when it is used as a prescriptive model; and it is shown to be an appropriate approximation to widely acceptable normative criteria such as the expected utility maxim, to behaviorally based prescriptive models such as the lower partial variance model and even to actual investment management modes such as Farrar's mutual funds.

The persistency of good results has been found suspicious in at least one aspect, which is the robustness that the model exhibits as a prescriptive algorithm when the inputs are grossly distorted. This characteristic suggests an absence of "degrees of freedom" in the outputs. The present paper will examine the variety of the outputs corresponding to portfolio decisions in which a target expected rate of return is assumed as a constraint.

Footnotes

1. Karl Borch, "A Note on Uncertainty and Indifference Curves," Review of Economic Studies, 36 (January 1969), 1-4.

2. Ibid., p. 2.

3. J.P. Quirk and R. Saposnik, "Admissibility and Measureable Utility Functions," Review of Economic and Statistics, 29 (February 1962), 140-146.

4. G. Hanoch and H. Levy, "The Efficiency Analysis in Choices Involving Rick," Review of Economic Studies, 36 (July 1969), 335-346.

5. J.P. Quirk and R. Saposnik, op. cit.

6. J.W. Pratt, "Risk Aversion in the Small and in the Large," Econometrica, 32 (January - April 1964), 122-136.

7. K.J. Arrow, Aspects of the Theory of Risk Bearing, Helsinki (Urjö Jahnssonin Säätio), 1965.

8a. G. Hanoch and H. Levy, op. cit.

8b. G. Hanoch and H. Levy, "Efficient Portfolio Selection with Quadratic and Cubic Utility," Journal of Business, 43 (April 1970), 181-189.

9. R.B. Porter, J.W. Wart and D.L. Ferguson, "Efficient Algorithms for Conducting Dominance Tests on Large Number of Portfolios," Journal of Financial and Quantitative Analysis, 8 (January 1973), 71-81.

10. K.J. Arrow, "The Role of Securities in the Optimal Allocation of Risk Bearing," in his Essays in the Theory of Risk Bearing, (Markham Publishing Co.), 1971.

11. G. Debreu, Theory of Value: An Axiomatic Analysis of Economic Equilibrium, (John Wiley and Sons), 1959.

12. M.C. Jensen, "Capital Markets Theory and Evidence," The Bell Journal of Economics and Management Science, 3 (Autumn 72), 357-398.

13. J. Hirsleifer, "Investments Decisions Under Uncertainty: Choice-Theoretic Approaches," Quarterly Journal of Economics, 79 (November 1965), 509-536.

14. K. Borch, _op_. _cit_.

15. S.C. Tsiang, "The Rationale of the Mean-Variance Analysis, Skewness Preference and the Demand for Money," _American Economic Review_, 62 (June 1972), 354-371.

16. _Ibid_.

17. Borch, _op_. _cit_.

18. K.J. Arrow, _Aspects_ ...., _op_. _cit_.

19. The conditions for risk aversion are:

$$U'(W) > 0$$
$$U''(W) < 0$$
$$d(-U''(W)/U'(W))/dW \leq 0$$
$$d(-WU''(W)/U'(W))/dW \geq 0$$

from which Tsiang (see n.27) derives the condition $U^{(3)}(W) \geq 0$

20. P.A. Samuelson, "The Fundamental Approximation Theorem in Terms of Means, Variances and Higher Moments," _Review of Economic Studies_, 38 (October 1970), 537-542.

21. Compact distributions are those that converge to a "spike" when a parameter of the distribution tends toward zero. For example the normal distribution when variance approaches zero.

22. Tsiang, _op_. _cit_.

23. E.F. Fama, "The Behavior of Stock Market Prices," _Journal of Business_ 38 (January 1965), 94.

24a. W.H. Jean, "The Extension of Portfolio Analysis to Three or More Parameters," _Journal of Financial and Quantitative Analysis_, 6 (January 1971), 505-515.

24b. ________, "More on Multidimensional Portfolio Analysis," _Journal of Financial and Quantitative Analysis_, 8 (June 1973), 475-490.

25. F.D. Arditti, "Another Look at Mutual Fund Performance," _Journal of Financial and Quantitative Analysis_, 6 (June 1971), 909-912.

26. ___________, "Risk and the Required Return on Equity," _Journal of Finance_, 22 (March 1967), 19-36.

27. Tsiang, _op_. _cit_.

28. C.P. Alderfer and H. Bierman, "Choices with Risk: Beyond the Mean and Variance," _Journal of Business_, 43 (July 1970), 341-353.

29. H.A. Latané and D.L. Tuttle, "Criteria for Portfolio Building," Journal of Finance, 22 (September 1967), pp. 360-361.

30. N.H. Hakansson, "On Optimal Myopic Portfolio Policies With and Without Serial Correlation of Yields," Journal of Business, 44 (July 1971), 324-334.

31. R.E. Young and R.H. Trent, "Geometric Mean Approximation of Individual Security and Portfolio Performance," Journal of Financial and Quantitative Analysis, 4 (June 1969), 179-200.

32. B. Mandelbrot, "The Variation of Certain Speculative Prices," Journal of Business, 36 (October 1963), 394-419; E.F. Fama, op. cit.

33. R.A. Agnew, "Counter Examples to an Assertion Concerning the Normal Distribution and a New Stochastic Price Fluctuation Model," Review of Economic Studies, 38 (July 1971), 381-383.

34. H. Markowitz, Portfolio Selection, New York (John Wiley and Sons), 1959, xi.

35. Tsiang, op. cit.

36. E.F. Fama, "Risk, Returns and Equilibrium: Some Clarifying Comments," Journal of Finance, 23 (March 1968), n.6.

37. M.E. Blume, "Portfolio Theory: A Step Towards its Practical Application," Journal of Business, 43 (April 1970), 152-173.

38. H.M. Markowitz, "Portfolio Selection," Journal of Finance, 7 (March 1952), 77-91.

39. J. Tobin, "Liquidity Preference as Behavior Towards Risk," Review of Economic Studies, 25 (February 1958), 65-87.

40. M.S. Feldstein, "Mean Variance Analysis in the Theory of Liquidity Preference," Review of Economic Studies, 36 (January 1969), 5-12.

41. W.F. Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," Journal of Finance, 19 (September 1964), 425-442.

42. J.L. Treynor, "How to Rate Management Investment Funds," Harvard Business Review, 43 (January-February 1965), 63-75.

43. J. Lintner, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," The Review of Economics and Statistics, 47 (February 1965), 13-37.

44. M. Jensen, op. cit., p. 391.

45. E.F. Fama and J.D. MacBeth, "Risk Return and Equilibrium: Empirical Tests," Journal of Political Economy, 81 (May-June 1973), 607-636.

46. __________________________, Ibid., p. 629. Fama and MacBeth state that there are other variables in addition to portfolio risk which "systematically affect period-by-period returns......some of these variables are [non-linearly] related to [portfolio risk] and to [residual non-portfolio risk] .... [but] there is no economic rationale for their presence in [the] stochastic risk-return model." The absence of theory, or rather adherence to one theory, does not justify ignoring empirical results. A recent article by Treynor and Black may throw some light into their results.

47. J.L. Treynor and F. Black, "How to Use Security Analysis to Improve Portfolio Selection," Journal of Business, 46 (January 1973), 66-86.

48. This paper is concerned with aspects of the mean-variance model not previously examined in the literature, for this reason the treatment Capital Market Theory is superficial and has been included here to provide a full perspective. A complete analysis of the theory, its theoretical foundations and the discussion of the relevant empirical tests and results published and unpublished through 1972 may be found in the above-referenced article by Michael Jensen (see n.12). An attempt to reproduce here such thorough analysis is redundant and beyond the scope of this paper.

49. W.F. Sharpe, Portfolio Theory and Capital Markets, New York (McGraw Hill Book Co.), 1970, 82.

50. ___________, "Mutual Fund Performance," Journal of Business, 39 (January 1966), 119-138.

51. J.L. Treynor, "How to Rate ....", op. cit.

52. M.C. Jensen, "Risk, the Pricing of Assets, and Evaluation of Investment Portfolios," Journal of Business, 42 (April 1969), 167-247.

53. Risk is measured by Sharpe as the standard deviation of the portfolio, and by Jensen and Treynor as the "beta coefficient" which is the ratio of the covariance of the portfolio returns with those of a market index over the variance of the returns of the index.

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CHAPTER II

THE PORTFOLIO OPPORTUNITY SET: DISTRIBUTIONAL PROPERTIES ON THE MEAN-VARIANCE PLANE UNDER AN EXPECTATION CONSTRAINT

Introduction

In the previous chapter the portfolio opportunity set $\{P\}$ was defined as the set of all portfolios feasible under a specified set of constraints. A portfolio can be described entirely by the n-column vector of weights x representing the proportion of wealth which corresponds to each of the n investment prospects. This implies that there is a one-to-one correspondence between elements of $\{P\}$ and points of a region $\{x\}$ in a n-Euclidean space. The region $\{x\}$ is defined by a set of constraints as in (14) of the previous chapter. By introducing the additional constraint $x'u=E_j$, a subregion $\{x|E_j\}$ of $\{x\}$ is defined as a representation of the subset $\{P|E_j\}$ of $\{P\}$, where $\{P|E_j\}$ is the set of all feasible portfolios which are expected to yield ex ante a rate of return E_j.

A randomly selected portfolio from the subset $\{P|E_j\}$ will be represented on the mean-variance plane by the point (E_j,V) or, since E_j is given, simply by its variance V.

Since there is a one-to-one correspondence between $\{P\}$ and $\{x\}$, the random selection of a vector x in $\{x|E_j\}$ is the exact equivalent of the random selection of a portfolio if $\{P|E_j\}$. The distribution of variances resulting from this process of random selection will be defined as the distribution of variances given E_j, or simply $F(V|x'u=E_j)$, which is bounded from above and from below.

This chapter discusses first the definition of the subregion $\{x|E_j\}$ for portfolio selection problems consistent with the traditional formulations of Markowtiz[1] and Sharpe,[2] in which the investment opportunity set consists entirely of risky prospects and is exhaustive, e.g. it includes all possible risky investment prospects.

Following the definition of the subregion, some properties of the conditional distribution of variances will be analyzed for a specific random process formulated below.

<u>Formulation of the Random Process for the selection of points in $\{x|E_j\}$</u>. Let u and C be the expectation and covariance matrix of the n-variate distribution of wealth relatives, and let the constraints of the Markowitz-Sharpe normative model be characterized by the set of m equations on n variables $Ax=b$. The subregion $\{x|E_j\}$ is defined as the set of all x in $\{x\}$, such that

$$(1) \qquad x'u = E_j \quad \text{and} \quad Ax = b$$

where A is an (m x n) matrix of constant coefficients and b is an m-column vector of constraints.

For the standard portfolio problem defined in (13) of the previous chapter, $\{x|E_j\}$ is defined as the set of all points x in $\{x\}$, such that

(2) $$x'u = E_j$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\text{and} \quad U \geq x \geq L$$

for U and L defined as before.

The special case for which U is a vector of 1's and L a vector of 0's is the most commonly discussed. For this reason it will be used as an illustration of some of the developments that follow. For convenience and future reference, the region corresponding to this special case, is defined as

(3) $$x'u = E_j$$

$$\sum_{i=1}^{n} x_i = 1$$

$$\text{and} \quad x \geq 0$$

It will be seen in the following section that any portfolio problem with constraints as in (1), (2) or (3) can be reduced to a standard form.

<u>Representation of the elements of $\{x|E_j\}$ in terms of its extreme points</u>. In general, the region defined by (1) has a finite number k of extreme points y_i, for i=1,......,k and $k \leq \binom{n}{m}$.[3] Furthermore, y_i is the i-th basic feasible solution of the set (1),[4] and any point x in (1) can be defined as a convex linear combination of its extreme points.[5] For w, a k-column vector representing the lengths of k subintervals which form a partition of [0,1], and Y an (n x k) matrix whose columns represent all basic feasible solutions of (1),

any point x in the region (1) may be expressed as

(4) $$x = Yw$$

Since the variance of the distribution of returns of a portfolio x is $V = x'Cx$, then from (4) it follows that

(5) $$V = w'Y'CYw = w'Qw$$

where Q is a k-square positive-semidefinite[6] matrix.

From (4) and (5) it follows that the Markowitz-Sharpe portfolio selection problem may be reduced to a standard form and defined as

(6) $$\begin{aligned} &\text{Minimize} \quad V = w'Qw \\ &\text{subject to} \quad \sum_{i=1}^{k} w_i = 1 \\ &\text{and} \quad w \geq 0 \end{aligned}$$

The representation of x in (4) provides the basis for the random generation process which is described in the following section.

<u>Definition of the Random Vector $\tilde{x}$ and the Random Variable $V(\tilde{x})$</u>. The random vector $\tilde{w}$ represents a random partition of the interval [0,1] into k subintervals of length w_i ($0 \leq w_i \leq 1$, $i=1,\ldots,k$). A random partition is the (k-1) dimensional analogue of the uniform distribution. The joint density $f(w_1,w_2,\ldots,w_{k-1}) = 1/(k-1)!$, which for k=2 (the uniform distribution) reduces to $f(w_1) = 1$. Each element w_i, for $i=1,\ldots,k-1$, is independently distributed with the common distribution $P(w_i>a) = (1-a)^{k-1}$, which for k=2 reduces to $P(w_i>a) = 1-a$ as corresponds to the uniform distribution.[7]

The matrix Y is the (n x k) matrix with columns corresponding to the extreme points y_i as defined above.

The random vector x in $\{x|E_j\}$ is defined as

(7) $$\tilde{x} = Y\tilde{w}$$

which follows from (4) above.

The variance corresponding to $\tilde{x}$ is a random variable and can be defined by application of (5) and (7) as

(8) $$V(\tilde{x}) = \tilde{x}'C\tilde{x} = \tilde{w}'Y'CY\tilde{w} = \tilde{w}'Q\tilde{w}$$

If the matrix Y corresponds to the extreme points of the region defined by either (2) or (3), the random variable $V(\tilde{x})$ represents the variance of randomly selected portfolios which are expected to yield ex ante a return E_j. The distribution $F(V|x'u=E_j)$ is the distribution $F(V(\tilde{x}))$ of the random variable $V(\tilde{x})$. The underlying random process as defined in (7) corresponds to the random selection of points w in a k-dimensional Euclidean space such that its elements w_i (for i=1,....,k) conform to[8]

(9) $$\sum_{i=1}^{k-1} w_i \leq 1 \quad \text{and} \quad w_k = 1 - \sum_{i=1}^{k-1} w_i, \text{ for all } w_i \geq 0$$

and may assume with equal probability any value in the closed interval [0,1]. This is equivalent to the selection, with equal probability of being selected, of points in the simplex defined by

(10) $$\sum_{i=1}^{k} w_i = 1, \quad w \geq 0.$$

The random variable $V(\tilde{x})$ is bounded from above and from below by

(11) $$V^* = \{x^{*\prime}Cx^* : x^{*\prime}Cx^* \text{ is a maximum in (1), (2) or (3)}\}$$

and

(12) $$V_* = \{x_*'Cx_* : x_*'Cx_* \text{ is a minimum in (1), (2) or (3)}\}$$

The mean and variance of the distribution of $V(\tilde{x})$ will be derived analytically in the following sections. This derivation will be followed by a heuristic analysis of some distributional characteristics of $F(V|E_j)$ for the special case in which $\{x|E_j\}$ is defined as in (3) above.

Expectation of $V(\tilde{x})$

The expectation of $V(x)$ is

(13) $$\begin{aligned} E(V(\tilde{x})) &= E(\tilde{x}'C\tilde{x}) \\ &= E(\tilde{w}'Y'CY\tilde{w}) \\ &= E\left(\sum_{i=1}^{k} \tilde{w}_1^2\, y_i'Cy_i + \sum_{i=1}^{k} \sum_{j\neq 1}^{k} \tilde{w}_i\tilde{w}_j y_i'Cy_j\right) \end{aligned}$$

An interesting property of the random vector w is that[9]

(14) $$E(\tilde{w}_i^a\tilde{w}_j^b\tilde{w}_p^c\tilde{w}_q^d\ldots.) = (k-1)!(a!b!c!d!\ldots)/(k-1+a+b+c+d+\ldots)!$$

for $i,j,p,q = 1,\ldots\ldots,k.$

Define the average variance of the extreme points in the convex set as

(15) $$\overline{V} = (1/k) \sum_{i=1}^{k} y_i'Cy_i$$

and the variance of the conditional expectation of x as

$$(16)\qquad V(E(\tilde{x})) = E(\tilde{x})'C\,E(x)$$

$$= (1/k^2) \sum_{i=1}^{k} \sum_{j=1}^{k} y_i' C y_j$$

It follows from (13) and (14) that

$$(17)\qquad E(V(\tilde{x})) = \frac{1}{(k+1)k} \sum_{i=1}^{k} y_i' C y_i + \frac{1}{(k+1)k} \sum_{i=1}^{k} \sum_{j=1}^{k} y_i' C y_j$$

and from (15) and (16) that

$$(18)\qquad E(V(\tilde{x})) = (\bar{V} + k\,V(E(\tilde{x})))/(k+1)$$

It is interesting to notice that

$$(19)\qquad \bar{V} \geq V(E(\tilde{x}))$$

since the function $V(x)$ is convex, and therefore

$$(20)\qquad E(V(x)) \geq V(E(\tilde{x}))$$

and also that

$$(21)\qquad \lim_{k-\infty} E(V(\tilde{x})) = V(E(\tilde{x}))$$

so that for a large k,

$$(22)\qquad E(V(\tilde{x})) \doteq V(E(\tilde{x}))$$

The Variance of $V(\tilde{x})$

The variance of the random variable $V(x)$ is

$$(23)\qquad V(V(\tilde{x})) = E(\sum_{i}^{k} \sum_{j}^{k} \sum_{p}^{k} \sum_{q}^{k} w_i w_j w_p w_q\; y_i' C y_j y_p' C y_q) - (E(V(x)))^2$$

It is shown in Appendix E that $V(V(\tilde{x}))$ may be expressed as

$$(24)\qquad V(V(\tilde{x})) = [k^3/(k+3)(k+2)(k+1)]\;V(E(\tilde{x}))^2$$

$$+ [23 \sum_{i=1}^{k} Q_{ii}^2 + \sum_{i=1}^{k} \sum_{j\neq i}^{k} (20Q_{ij}Q_{ii} + 6Q_{ij}^2 + 3Q_{ii}Q_{jj})$$

$$+\sum_{i=1}^{k}\sum_{j\neq i}^{k}\sum_{p\neq j,i}^{k}(2Q_{jp}Q_{ii}+4Q_{ij}Q_{ip})][$$

$$[1/k(k+1)(k+2)(k+3)] - E(V(x))^2$$

where $Q_{ij} = y_i'Cy_j$.

Due to the order of magnitude of the terms in (24), it is also shown in Appendix E that

$$(25) \qquad \lim_{k\to\infty} V(V(\tilde{x})) = 0^+$$

which means that for large values of k, the variance of the distribution of random variances is very small, and decreases for increasing values of k.

This is a very interesting property because since k depends on the size of the n-investment prospect universe, and on the position of E_j on its feasible range;[10] for arbitrarily large values of k it will be sufficient to determine the position of $V(E(\tilde{x}))$ in the interval (V_*,V^*) to obtain an approximate description of the shape of the distribution of $V(\tilde{x})$. Notice that for n=100, and an intermediate E_j, the magnitude of k may increase to the order of 10^3 for the convex region defined by (3). For k of the order of 10^3, $V(V(\tilde{x}))$ is such that

$$(26) \quad 0 \leq V(V(\tilde{x})) \leq (10^{-3})[2\overline{Q_{jp}Q_{ii}}+4\overline{Q_{ij}Q_{ip}}-2\bar{V}V(E(\tilde{x}))]+o(10^{-6})$$

where $o(10^{-6})$ = "Terms of order of magnitude less than 10^{-6}". Since $V(V(\tilde{x}))$ must never be negative, in the event that $[2\overline{Q_{jp}Q_{ii}}+4\overline{Q_{ij}Q_{ip}}-2\bar{V}V(E(\tilde{x}))] < 0$ then the variance of the distribution of variances is even smaller, of order of magnitude less than 10^{-6}.[11]

An investment universe of 100 prospects is not uncommonly large when compared for example to the number of stocks in the New York Stock Exchange. The previous example is then a very good indication of the magnitudes that can be expected for the variance of the conditional distribution of variances. What is important to recognize is that since variance is a measure of dispersion about the mean of the distribution, and since the variance of the distribution can be expected to be very small for large values of k, then the shape of the distribution can be approximately described by the position of $E(V(\tilde{x}))$. This is particularly desirable since the derivation of the third central moment of the distribution is tractable, but, judging from the complexity of (24), not particularly amenable to derivation.

Under the assumption that the position of $E(V(\tilde{x}))$ on the closed inteval $[V_*,V^*]$ will provide a rought but good indication of the shape of the distribution, the following sections will describe a heuristic attempt to ascertain the influence on the relative position of $E(V\tilde{x}))$ of the overall level of correlation $\bar{\rho}$ between pairs of returns of investment prospects; and the effect of the level of E_j, given its feasible range, on the distribution of $V(\tilde{x})$.

Analysis of the Composition of the Maximum and Minimum Variance Portfolios

This section will be devoted to the analysis of the degree of similarity between the expected portfolio $E(\tilde{x})$ and respectively the maximum variance portfolio x^* and the minimum variance portfolio x_*. It was shown in the previous sections that for a large k, the distribution of variances of random portfolios with the same expectation can be adequately described by the position of $E(V(\tilde{x})) = V(E(\tilde{x}))$ in the interval $[V_*, V^*]$. The degree of similarity between $E(\tilde{x})$ and respectively x^* and x_* will provide an indication of the relative position of $E(V(\tilde{x}))$ in the interval.

The measure of similarity between portfolios was suggested by Farrar[12] and is defined as

$$D^2(x_i, x_j) = (x_i - x_j)'(x_i - x_j) \tag{14}$$

which is the Euclidean distance squared between x_i and x_j. $D^2(x_i, x_j)$ lies in the interval [0,2], zero denoting equality between the vectors and 2 the maximum degree of dissimilarity.

The maximum variance portfolio x^* and the minimum variance portfolio x_* can be defined by application of (6) as

$$\begin{aligned} x^* &= \{Yw : w'Qw \text{ is a maximum}\} \\ x_* &= \{Yw : w'Qw \text{ is a minimum}\} \end{aligned} \tag{27}$$

$$\text{subject to } \sum_{i=1}^{k} w_i = 1, \quad w \geq 0$$

Observe in this formulation that a necessary condition for x^* is that $x^* = y_i$, one of the extreme points in the convex region. Therefore, w^* which corresponds to $x^* = Yw^*$ must be such that its i-th element equals 1 and all others equal 0.

The failure of x^* to comply with this condition would imply that it could be expressed as a convex linear combination of two extreme points, and therefore its variance would not be a maximum for the region.

No such condition applies to the composition of x_* except for the special case of perfect correlation between extreme points which will be reviewed below. If w_* is defined as a k-column vector such that $x_* = Yw_*$, it can be observed that the "participation" t of extreme points in w_*, - e.g. the number t of non-zero elements of w_*, depends entirely on the configuration of the k-square covariance matrix Q. To illustrate this point, observe the effect of the variables E_j and $\overline{\rho}$ defined above on the "participation" t in w_*, which will be analyzed in the following paragraphs.

For the extreme case for which $\rho_{ij} = 1$ for all pairs (i,j), x^* and x_* can be redefined as

$$x^* = \{Yw : \sum_{i=1}^{k} w_i (Q_{ii})^{1/2} \text{ is a maximum}\} \tag{28}$$

$$x_* = \{Yw : \sum_{i=1}^{k} w_i (Q_{ii})^{1/2} \text{ is a minimum}\}$$

$$\text{subject to} \quad \sum_{i=1}^{k} w_i = 1 \; ; \quad w \geq 0$$

where Q_{ii} is the i-th diagonal element of Q.

Since (28) corresponds to the formulation of a linear programming problem, it follows that $x^* = y_i$, and $x_* = y_j$, - e.g. that the maximum and minimum variance portfolios are extreme points of the convex region. The "participation" t in w* and w_* is then t=1.

For configurations of the covariance matrix which differ from the case of perfect correlation presented above, the "participation" of extreme points in w* remains unaltered, while the "participation" in w_* is expected to increase, approaching t=k as the off-diagonal elements of Q approach zero, or take negative values. A rough but adequate indicator of the magnitude of t is the number of zero or negative off-diagonal elements in Q.

The analysis of the conditions for which the off-diagonal elements of Q are zero or negative will follow, after defining them as

(29) $$Q_{ij} = y_i'Cy_j, \quad i \neq j$$

the covariance between portfolios represented by the extreme points y_i and y_j.

Given that the n investment prospects are distributed independently, then $Q_{ij} = 0$ if and only if y_i and y_j are orthogonal. If the n-square covariance matrix C is strictly positive, then the k-square matrix Q is also strictly positive, - e.g. $Q_{ij} > 0$ for all pairs (i,j). When the covariance matrix C contains positive and negative elements, Q_{ij} is more likely to be negative or zero if y_i and y_j are orthogonal than if the two vectors are not.

In conclusion, the "participation" t in the vector w_* depends directly on the number of orthogonal pairs possible and the overall level of correlation between pairs of prospects. The "participation" of extreme points in w can be expected to increase as the level of correlation between pairs of prospects decreases and as the incidence of orthogonal pairs increases.

It can be demonstrated that the incidence of orthogonal pairs is directly affected by the level of the expected return E_j in the range of values it may take. For a demonstration of this argument it will be illustrative to examine the standard problem defined in (3). Let d be the number of investment prospects for which $u_i > E_j$; then the number of orthogonal pairs that can be formed such that y_i is in the pair is (n-d-1)(d-1), which will increase as d approaches n/2 and will decrease as d moves towards 1 or (n-1), - e.g. as E_j moves towards the extremes of its feasible range.

Since the composition of the extreme points y_i depends entirely on E_j (see n.10 above); other things being equal, it may be concluded that x^* will contain the minimum possible number of investment prospects allowed by the constraints; while x_* will contain a greater number of investment prospects, the number being greater for intermediate levels in the feasible range of E_j, lower levels of correlation between pairs of prospects, and a higher incidence of orthogonal extreme points in the region. These considerations will allow the analysis of the degree of similarity between $E(\tilde{x})$ and respectively x^* and x_*, which follows in the next section.

Analysis of the Degree of Similarity Between $E(\tilde{x})$ and x^*, x_*

The degree of similarity between $E(\tilde{x})$ and the portfolios corresponding to the maximum and minimum variances in the region will be a function of the overall level of correlation between pairs of investment prospects, the incidence of orthogonal pairs of extreme points and their "participation" t in the vector w_*.

In the case of perfect correlations between pairs of prospects, it can be observed that

$$(30) \qquad D^2(x^*, E(\tilde{x})) = y_j'y_j + (1/k^2)\sum_{p=1}^{k}\sum_{q=1}^{k} y_p'y_q - (2/k)\sum_{p=1}^{k} y_j'y_p$$

$$D^2(x_*, E(\tilde{x})) = y_i'y_i + (1/k^2)\sum_{p=1}^{k}\sum_{q=1}^{k} y_p'y_q - (2/k)\sum_{p=1}^{k} y_i'y_p$$

will always be of the same order of magnitude since the orthogonality of y_j and y_i with the other extreme points will affect both measurements of similarity in the same way.

This argument becomes clear for the standard problem defined in (3). Observe that for each extreme point in this region, there are (n-1) possible pairs such that $y_i'y_j \neq 0$; so that the order of magnitude of the similarity measurements can be established as follows[13]

$$(31) \qquad o(D^2(x^*, E(\tilde{x})) = (1-(n-1)/k)\,o(\overline{y_p'y_q})$$

$$o(D^2(x_*, E(\tilde{x})) = (1-(n-1)/k)\,o(\overline{y_p'y_q})$$

From (31) it can be concluded that given perfect correlations between pairs of prospects, an equivalent degree of similarity is exhibited between $E(\tilde{x})$ and respectively x^* and x_*. For this reason, it can be expected that for a high level of correlation between pairs of investment prospects, the variances of random portfolios will have their expectation at some distance of both boundaries, e.g. it can be expected that the distribution is not significantly skewed in either direction. For large values of k, given that the variance is very small, the distribution may be expected to have long and narrow tails since a high density in the neighborhood of $V(E(\tilde{x}))$ follows from the fact that the distributional variance approaches zero as k increases.

It should be noted that as the level of correlation between pairs of prospects approaches 1, the distance between the maximum and minimum variance for a level E_j diminishes accordingly, so that little or no advantages may be derived from diversification, as evidenced by the fact that at the limit (when $\rho_{ij}=1$) both the maximum and minimum variance portfolios are extreme points of the region (3), e.g. they are portfolios containing two prospects only.

For $\rho_{ij}\neq 1$ it can be observed that for the standard problem referenced above, given the measures of similarity between $E(x)$ and x^*, x_* which are

$$(32) \quad D^2(x^*,E(\tilde{x})) = y_j'y_j + (1/k^2) \sum_{p=1}^{k} \sum_{q=1}^{k} y_p'y_q - (2/k) \sum_{p=1}^{k} y_j'y_p$$

$$D^2(x_*,E(\tilde{x}))=\sum_{p=1}^{k}\sum_{q=1}^{k} w_{*p}w_{*q}y_p'y_q+(1/k^2)\sum_{p=1}^{k}\sum_{q=1}^{k} y_p'y_q$$

$$-(2/k)\sum_{p=1}^{k}\sum_{q=1}^{k} w_{*p}y_p'y_q$$

It is possible to analyze the behavior of the measures by reducing them in terms of their overall order of magnitude. Given that t is the participation of extreme points in w_*, the order of magnitude of the similarity measurements can be expressed as follows

(33) $$o(D^2(x^*,E(\tilde{x})) = (1 - (n-1)/k)o(\overline{y_p'y_q})$$

$$o(D^2(x_*,E(\tilde{x})) = (n-1)/t - (n-1)/k)o(\overline{y_p'y_q})$$

The analysis of the relations in (33) suggests that as E_j moves from the center of its feasible range to the extremes, k approaches n-1 and t becomes smaller in relation to k because of a decreased indicence of orthogonal pairs. Under these circumstances it can be observed that E(x) and x* are very similar and thus $E(V(\tilde{x}))$ is in the neighborhood of V*.

For intermediate levels of E_j, k>>n-1, with the greatest incidence of orthogonal pairs of extreme points. Under these conditions, t will approach k as the level of correlation between pairs of prospects decreases. Consequently, $E(\tilde{x})$ is then very similar to x_* and very dissimilar to x*, which for a large k will result on a high concentration of the distribution in the neighborhood of x_*. This means that the distribution of variances will be positively skewed, the degree of positive skewness depending on the level of correlation between pairs of prospects and on the level of E_j with respect to its feasible range.

The previous analysis is valuable as long as one is not concerned with ascertaining the exact shape of the distributions. It allows for the definition of tentative causality relationships regarding the factors that may or will affect the shape of the conditional distribution. For the standard problem defined by (3), it can be argued that the greater the size of the investment universe, the more dense the distribution of variances in the neighborhood of $E(V(\tilde{x})) = V(E(\tilde{x}))$. That the distribution will be highly concentrated in the neighborhood of V_*, provided that correlations between pairs of prospects are not inordinately high on the average, and that the level of E_j is not at either extreme of its feasible range; that when correlations between pairs of investment prospects uniformly approach unity, the distribution may be expected to be near symmetrical; and finally, that at the extremes of the feasible range of E_j's, the distribution will be negatively skewed, e.g. that it will be concentrated in the neighborhood of V^*.

These heuristically derived results have been supported by the results of ten Monte Carlo experiments, as it will be seen in Chapter IV. Chapter III describes the Monte Carlo methodology for the generation of random variances conditional on a fixed level of expectation E_j.

Footnotes

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4. Ibid., p. 100.

5. Ibid., p. 64.

6. This follows from the property of C of being such that it can be expressed as the product of a lower and upper triangular matrices, e.g. C=T'T, and therefore Q=Y'T'TY. If the matrix B is defined as B=TY, then Q=B'B a positive definite matrix.

7. W. Feller, An Introduction to Probability Theory and its Applications, V.II, New York (John Wiley and Sons), 1966, p. 75.

8. Ibid., p. 75.

9. W.A. Whitworth, Choice and Change, Reprint fo the Fifth Edition, New York (Hafner Publishing Co.), 1965, p. 209.

10. To illustrate, let the region $\{x|E_j\}$ be as defined by (3), and d be the number of investment prospects for which $u_i<E_j$. Then the number of basic feasible solutions is

$$k = \begin{cases} (n-d)d \text{ if no } u_i=E_j \text{ for } i-1,.....,n \\ (n-d-1)d+1 \text{ if for any prospect } u_i=E_j \end{cases}$$

11. For definition of terms see Appendix E.

12. D.E. Farrar, The Investment Decision Under Uncertainty, Chicago (Markham), 1967, p. 62.

13. The term, $\overline{y_p'y_q}$ is the average of the k dot products of p-th extreme point with itself and all other extreme points.

CHAPTER III

MONTECARLO GENERATION OF RANDOM PORTFOLIOS

Generation of the Random Vector $\tilde{w}$

The random vector w has been defined as a random partition of the interval [0,1]. The generation of w follows from the properties of the exponen tial distribution.[1] If $e_1, \ldots\ldots\ldots, e_k$ are k exponential variates from the same parent distribution, then

$$(1) \qquad w = \{w_i\} = \{e_i / \sum^k e_i\}$$

is a random partition of [0,1] in k intervals.

The generation from a set of k uniform random variates follows directly from (1): let $r_1, \ldots\ldots, r_k$ be uniform on [0,1], then

$$(2) \qquad w = \{w_i\} = \{\ln(r_i) / \ln(\pi^k r_i)\}$$

Identification of the Basic Feasible Solutions y_i

Unlike the traditional mathematical programming problems where algorithms are constructed with the purpose of avoiding inspection of all the extreme points in the region, the generation of random vectors requires the construction of heuristics to generate economically all the extreme points in the region. Two methods are described below to generate the extreme points for two special cases.

The standard problem. This problem consists of selecting random portfolios in the region defined by

(3) $$x'u = E_j$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0 \qquad \text{for } i = 1,\ldots\ldots,n$$

A strategy in which a heuristic for the generation of extreme points is based will follow:

(a) All extreme points will be basic feasible solutions of (3)

(b) Basic feasible solutions of (3) will have two elements j,l obtained from the solution of

$$x_j + x_l = 1$$

$$u_j x_j + u_l x_l = E_j$$

$$x_j, x_l \geq 0$$

while all other elements $x_i = 0$ for $i \neq j,l$

(c) The non-negativity constraints require that if u_j is greater than u_l, then $u_j \geq E_j \geq u_l$ for a basic feasible solution.

The heuristic suggests itself immediately: Order the n prospects from highest to lowest expected return; define the position of E_j in this sequence; and to obtain a basic feasible solution, select j,l according to condition (c)aabove.

Upper and lower boundaries. Since it is not unusual in portfolio selection problems to find the portfolio manager facing upper and lower boundary constraints which are either self-imposed or required by government regulation, the standard problem may be expanded to include left- and right-hand constraints on the vector x such that the region will be defined by

$$(4) \qquad x'u = E$$
$$x_i = 1$$
$$U \geq x \geq L$$

where U and L are n-column vectors.

The development of heuristics to generate extreme points in the region does not appear to be a trivial exercise, and it may require that they be tailored for the particular problem. It may be advantageous for the construction of such heuristics to utilize the property of the region (4) which stipulates that the extreme points y_i in the region are such that (n-2) of the elements of y_i are either at their upper or lower boundaries, while the remaining two are associated with a basic solution for (3).[2]

The problem will thus be reduced to a combinatorial problem where the number of extreme points to be generated will be bounded by

$$(11) \qquad k \leq \binom{n}{2} \sum_{i=0}^{n-2} \binom{n-2}{i}$$

The number of points may be reduced substantially for a given problem as it will be illustrated in the example that follows.

Assume that n=6, $\{U_q\} = .2$ and $\{L_q\} = 0$ for all q's. In this case only those combinations with n-2 elements at their upper boundaries need to be examined; and out of those, the infeasible solutions may be screened out by means of a simple filter rule. The filter rule will be constructed as follows: let y_i be a candidate such that (n-2) of its elements $y_q = .2$ for $q \neq p,1$ and

$$(12) \qquad y_p = (E_j - .2(\sum_{q=1}^{n} u_q - u_p))/(u_p - u_1)$$

$$y_1 = .2 - y_p$$

Then y_i is an extreme point of the region only if $.2 \geq y_1 \geq 0$.

Generation of Random Portfolios and Their Variances

Having obtained, as shown above, the matrix of extreme points Y in the region, and the random vector w, a random variate of the distribution of x is given by (7) of the previous chapter as

$$(13) \qquad \tilde{x} = Y\tilde{w}$$

and its corresponding variance will be

$$(14) \qquad V(\tilde{x}) = \tilde{x}'C\tilde{x}$$

Repeated generation of variates $\tilde{w}$ through the Monte Carlo method will allow for a preliminary analysis of the distributional properties of the random variable $V(\tilde{x})$ for fixed levels of expectation.

Appendix C contains the programs utilized for the generation and analysis of the random variates discussed above. All programs were written in Extended Basic and are compatible with the executive routines of the UMASS timesharing system of the University of Massachusetts at Amherst. The quadratic programming algorithms utilized to obtain the boundary values of the distribution are described elsewhere.[3]

The following chapter will describe the ten Monte Carlo experiments which were conducted to obtain a preliminary description of the conditional distribution of variances, by analyzing the properties of its sample distributions. An analysis of the results and discussion of the observations follows the description of the experiments.

Footnotes

1. W. Feller, An Introduction to Probability Theory and its Applications, New York (John Wiley and Sons), 75.

2. G. Hadley, Linear Algebra, Reading (Addison-Wesley Publishing Co.), 1963, 388.

3. R. Hommes, Documentation for Program Prtflio, Faculty Working Paper Series, No. 73-19, Center for Business and Economic Research, University of Massachusetts, Amherst, 1973.

CHAPTER IV

THE MONTE CARLO EXPERIMENTS

Description of the Experiments

Ten experiments have been selected for the analysis of the properties of the sample distributions of the variances corresponding to random portfolios expected to yield ex ante a target rate of return E_j. The convex region described in (3) of Chapter III was chosen for this analysis, so that the distributions to be analyzed correspond to the random variable

$$V(\tilde{x}) = \tilde{x}'C\tilde{x}$$

subject to $\sum_{i=1}^{n} \tilde{x}_i = 1$

and $\tilde{x}'u = E_j$

$\tilde{x} > 0$

where $\tilde{x}$ is an n-dimensional random vector generated by the Monte Carlo process described in Chapter III.

The covariance matrices C and the vectors of expectation u corresponding to each one of the experiments can be found in Appendix A.[1,2]

In Chapter II it was shown that the variables n, E_j and $\bar{\rho}$ - respectively the size of the investment opportunity set, the expected return ex ante of the portfolio and the overall level of correlation between pairs of investment prospects have an effect on the degree of similarity between $E(\tilde{x})$ and the upper and lower bounds x^*, x_*. Under the assumption that this effect will be reflected on the shape of the conditional distributions of variances, these same variables were chosen for experimental manipulation, e.g. as explanatory variables for the behavior of the conditional sample distribution of variances.

The experiments examine circumstances in which the investment universe is composed of six, ten and fifteen ($n=6, 10, 15$) investment prospects respectively. The expectation E_j is fixed at low, intermediate and high levels ($E_j = 1.1, 1.2, 1.3, 1.4, 1.5$);[3] and the covariances correspond respectively to low ($0 < \rho_{ij} < .10$), intermediate ($.25 < \rho_{ij} < .45$) and high ($.70 < \rho_{ij} < .90$) levels of correlations between the returns of investment prospects.[4] The sample distributions of conditional variances were studied under conditions corresponding to all the levels of n, E_j and $\bar{\rho}$ described above.

To allow for comparisons of the effects of changes in the explanatory variables between distributions spanning different ranges of values, the comparisons will be made between distributions of the standardized variable v which is defined as

$$(1) \qquad v = (V(\tilde{x}) - V_*)/(V^* - V_*)$$

In addition to the variable v, some new variables are introduced in this chapter and will be defined in the following section.

Definition of Variables and Nomenclature

The following notation has been used for the variables appearing in this chapter:

C	Covariance matrix of the n-variate distribution of returns of investment prospects.
u	Vector of expectations (plus 1) of the n-variate distribution of returns of investment prospects.
n	Size of the investment opportunity set.
k	Number of extreme points in the convex region defined by the constraints.
E_j	Ex ante level of expectation (plus 1) set as a constraint for the generation of random portfolios.
V_*	Minimum portfolio variance given E_j
V^*	Maximum Portfolio Variance given E_j
$E(v)$	True expectation of the standardized variable v
$\bar{v}$	Sample mean of the standardized variable v
$s^2(v)$	Sample estimate of the variance of v
$M^3(v)$	Sample estimate of the third central moment of v
Sk	$(M^3(v)/s^3(v))$, the skewness coefficient of the sample distribution of v.
M	Median of the sample distribution of v
Q	Interquartile range of the sample distribution of v
q	Upper bound of the first quartile of the sample distribution of v
v^-	Value of v for which the sample probability $P(v \le v^-) = .9$

Descriptive Variables

Given that the purpose of these experiments is to provide an approximate description of the behavior of the conditional distribution of variances corresponding to random portfolios, and that the distributional form is unknown, four sets of variables were selected to describe those characteristics of the distribution which are considered interesting. It is of interest ot determine how fast the distribution may be expected to become dense in the neighborhood of its mean, and therefore it is necessary to observe the effect of the experimental conditions on the dispersion of the distribution. It is also desirable to ascertain the location of the mean, and the behavior of the tails of the distribution since these characteristics are rough indicators of the shape of the distribution and, together with a measure of the skewness of the distribution, they provide an indication of the probability of an observation in a given interval. According to the previous considerations, the following variables were selected as descriptive variables of the characteristics of the distributions:

(1) Measures of Dispersion: The sample variance $s^2(v)$ and the interquartile range Q of the standardized distribution were selected as measures of dispersion. They are expected to be highly correlated, and in this sense redundant; but since the distributions cannot be expected to be always

symmetrical, the two measures contain distinct information.

(2) <u>Behavior of the Tails of the Distribution</u>: The upper bound q of the first quartile, and the length of the 90% probability band $\bar{v}$ were selected to analyze the response of the tails of the sample distributions to varying experimental conditions. This asymmetric choice will be better understood after casual examination of the shape of the cumulative sample distributions shown in Appendix B.

(3) <u>Measure of Location of a High Density Neighborhood</u>: Although a "high density neighborhood" is a euphemistic description of what could more accurately be labeled a modal interval, the latter carries connotations of unimodality which are to be avoided in the absence of knowledge about the distributional form. It was shown in Chapter II that $E(V(\tilde{x}))$ may be an indicator of the location of a high density interval. To this effect, E(v) will be used as a measure of location of the interval, and $s^2(v)$ as a measure of concentration.

(4) <u>Measure of Skewness</u>: The sample coefficient of skewness Sk has been selected as a proper measure, with one qualification. It is conceivable that if the true distribution has inordinately long tails, the sample distribution will not contain observations representative of those tails unless a proper method of importance sampling is designed. Unfortunately, importance or stratified sampling implies imputing a "true" distribution which in this case is un-

known. If observations on the tails are not present in the sample distribution, the sample measure of skewness may be an understatement of the true measure, or even reverse its sign. Fortunately, since the high density interval is in the neighborhood of E(v) and the distributions are bounded, the sample coefficient of skewness may be interpreted as follows: The sample distributions will be considered symmetrical if Sk is close to zero and E(v) is approximately equal to .5; if Sk is approximately equal to zero and E(v) is significantly different from .5, the location of E(v) will override the sign of Sk. That is, for Sk approximately equal to zero and E(v) to the left of .5, the distribution will be considered positively skewed; if E(v) is to the right of .5, then the distribution will be negatively skewed. This argument is supported by the observed graphical representations of the cumulative distributions in Appendix B.

A summary presentation of the results of the ten Monte Carlo experiments is given in the next section.

The Results of the Experiments

The results of the experiments for the descriptive variables defined above, and for a selected number of other variables which where considered of interest for discussion, are summarized in Tables IV-1 through IV-10. These correspond to each one of the ten experiments described in Appendix A.

A graphical summary of these results can be found in figures IV-1 through IV-7. These figures can be used for a comparative analysis of the effect of experimental conditions on the descriptive variables which characterize the sample distributions. Figures IV-1 and IV-2 depict the response of the dispersion measures Q and $s^2(v)$ to changes in the experimental variables. Figure IV-3 shows the effect of these changes on the expectation E(v). The behavior of the tails of the distributions, as described by q and v^-; and the behavior of the skewness coefficient may be found respectively in figures IV-4, IV-5 and IV-6. Finally, a composite pictorial description of the effect of the experimental conditions on the shape of the distributions has been attempted in figure IV-7.

Appendix B contains the graphical representation of the sample cumulative distributions corresponding to each one of the experiments. The Appendix consists of figures B-1 through B-50, which are properly labeled for cross reference. The graphs depict the distributions of the standardized variance v to facilitate visual inspection and comparisons.

These results will be analyzed and discussed in great detail in the remaining sections of this chapter.

TABLE IV-1

EXPERIMENT NO. 1*

Sample Size 5000

$\bar{\rho}$ = Low

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
6	5	1.13	.029	.031	.197	.197	.0174	$.18\times10^{-2}$	.794	.17	.095	.195	.375
					(.035)	(.035)	$(.169\times10^{-4})$	$(.55\times10^{-7})$					
6	8	1.2	.038	.0863	.108	.108	.0045	$.429\times10^{-4}$	1.4	.09	.055	.073	.191
					(.0463)	(.0463)	$(.26\times10^{-4})$	$(.19\times10^{-7})$					
6	9	1.3	.078	.18	.088	.089	.0039	$.45\times10^{-3}$	1.8	.07	.03	.08	.165
					(.093)	(.094)	$(.128\times10^{-3})$	$(.26\times10^{-5})$					
6	7	1.35	.114	.246	.158	.158	.01	.0015	1.42	.133	.08	.12	.29
					(.153)	(.153)	$(.64\times10^{-3})$	$(.229\times10^{-4})$					
6	8	1.4	.167	.314	.170	.168	.0106	.0016	1.46	.14	.09	.12	.295
					(.22)	(.219)	$(1.04\times10^{-3}$	$(.49\times10^{-4})$					
6	5	1.5	.436	.3455	.493	.493	.036	-.0015	-.22	.47	.33	.27	.7
					(.606)	(.606)	(.0042)	$(-.63\times10^{-4})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-2

EXPERIMENT NO. 2*

Sample Size 5000

$\bar{\rho}$ = Intermediate

n	k	E_j	V_*	V^*-V_*	$E(v)$	$\bar{v}$	$S^2(v)$	$M^3(v)$	Dk	M	q	Q	v^-
6	5	1.1	.042	.016	.378 (.048)	.38 (.048)	.035 ($.94\times10^{-5}$)	$.253\times10^{-3}$ ($.112\times10^{-8}$)	.039	.381	.225	.29	.6
6	8	1.2	.077	.0594	.094 (.0826)	.093 (.0825)	.0053 ($.186\times10^{-4}$)	$.574\times10^{-3}$ ($.12\times10^{-6}$)	1.2	.071	.021	.09	.18
6	9	1.3	.147	.1378	.109 (.162)	.109 (.162)	.0034 ($.71\times10^{-4}$)	$.373\times10^{-3}$ ($.974\times10^{-6}$)	1.63	.090	.061	.069	.185
6	8	1.4	.283	.220	.326 (.197)	.326 (.197)	.011 ($.53\times10^{-3}$)	$.14\times10^{-2}$ ($.149\times10^{-3}$)	1.23	.172	.115	.13	.33
6	5	1.5	.572	.219	.675 (.471)	.675 (.470)	.033 ($.155\times10^{-2}$)	$-.91\times10^{-3}$ ($-.96\times10^{-5}$)	-.156	.491	.335	.29	.72

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-3

EXPERIMENT NO. 3*

Sample Size 5000

$\bar{\rho}$ = High

n	k	E_j	V_*	V^*-V_*	E(v)	v^-	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
6	5	1.10	.054	.015	.591	.583	.048	-.0014	-.13	.59	.425	.325	.87
					(.0628)	(.0627)	$(.108\times10^{-4})$	$(.46\times10^{-8})$					
6	8	1.20	.11	.052	.26	.26	.012	$.86\times10^{-3}$	.64	.245	.18	.145	.41
					(.123)	(.123)	$(.32\times10^{-4})$	$(.12\times10^{-6})$					
6	9	1.3	.219	.096	.206	.207	.009	$.86\times10^{-3}$	.97	.19	.13	.125	.33
					(.238)	(.239)	$(.84\times10^{-4})$	$(.75\times10^{-6})$					
6	8	1.4	.413	.168	.194	.192	.0065	$.4\times10^{-3}$	.76	.17	.115	.11	.29
6	5	1.5	.724	.078	.366	.369	.0273	$.98\times10^{-3}$	.217	.35	.23	.245	.58
					(.752)	(.753)	$(.166\times10^{-3})$	$(.46\times10^{-6})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-4

EXPERIMENT NO. 4*

Sample Size: 3250

$\bar{\rho}$ = Low

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
10	16	1.13	.02	.06	.117	.117	.0035	$.26\times10^{-3}$	1.26	.10	.07	.07	.19
					(.0247)	(.0247)	($.56\times10^{-5}$)	($.17\times10^{-7}$)					
10	19	1.2	.027	.113	.0398	.0398	$.41\times10^{-3}$	$.98\times10^{-5}$	1.17	.03	.02	.025	.06
					(.0315)	(.0315)	($.52\times10^{-5}$)	($.14\times10^{-7}$)					
10	25	1.3	.054	.204	.0387	.0393	$.39\times10^{-3}$	$.11\times10^{-4}$	1.42	.03	.02	.025	.06
					(.0619)	(.062)	($.16\times10^{-4}$)	($.95\times10^{-7}$)					
10	19	1.4	.114	.437	.091	.092	.0014	$.5\times10^{-4}$	.99	.081	.06	.045	.138
					(.153)	(.154)	($.26\times10^{-4}$)	($.4\times10^{-5}$)					
10	9	1.5	.3	.514	.43	.43	.024	$-.23\times10^{-3}$	-.06	.42	.3	.235	.62
					(.52)	(.52)	(.0063)	($-.31\times10^{-4}$)					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-5

EXPERIMENT NO. 5*

Sample Size: 3250

$\bar{\rho}$ = Intermediate

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
10	9	1.1	.035	.0263	.35	.35	.0125	$.32\times10^{-3}$	.23	.34	.26	.165	.5
					(.0442)	(.0443)	$(.87\times10^{-5})$	$(.59\times10^{-8})$					
10	19	1.2	.064	.0755	.068	.069	.001	$.34\times10^{-4}$	1.01	.06	.04	.04	.11
					(.0691)	(.0692)	$(.59\times10^{-5})$	$(.146\times10^{-7})$					
10	25	1.3	.121	.127	.068	.068	$.8\times10^{-3}$	$.198\times10^{-4}$	.887	.06	.045	.035	.108
					(.132)	(.132)	$(.21\times10^{-5})$	$(.87\times10^{-7})$					
10	19	1.4	.215	.336	.166	.166	.0017	$.35\times10^{-4}$	.478	.16	.135	.05	.22
					(.27)	(.27)	$(.2\times10^{-3})$	$(.135\times10^{-5})$					
10	9	1.5	.458	.354	.432	.431	.0228	$-.8\times10^{-4}$	-.025	.43	.31	.23	.63
					(.612)	(.611)	(.0289)	$(-.38\times10^{-5})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-6

EXPERIMENT NO.6*

Sample Size: 3250

$\bar{\rho}$ = High

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
	9	1.1	.059	.026	.55	.55	.016	$-.28\times10^{-3}$	-.14	.548	.46	.175	.71
					(.07)	(.07)	$(.67\times10^{-5})$	$(-.25\times10^{-8})$					
	19	1.2	.13	.0565	.287	.289	.0042	$.13\times10^{-3}$	.485	.28	.235	.09	.37
					(.146)	(.146)	$(.13\times10^{-5})$	$(.236\times10^{-7})$					
	25	1.3	.258	.0862	.245	.245	.0028	$.48\times10^{-4}$	.318	.238	.205	.07	.31
					(.279)	(.279)	$(.21\times10^{-4})$	$(.3\times10^{-7})$					
	19	1.4	.443	.11	.4	.4	.00345	$-.76\times10^{-5}$	-.037	.395	.355	.08	.47
					(.487)	(.487)	$(.42\times10^{-4})$	$(-.1\times10^{-7})$					
	9	1.5	.742	.0717	.418	.418	.017	$-.7\times10^{-4}$	.033	.41	.315	.195	.58
					(.772)	(.772)	$(.86\times10^{-4})$	$(.26\times10^{-7})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-7

EXPERIMENT NO. 7*

Sample Size = 2100

$\bar{\rho}$ = Low

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
15	36	1.111	.019	.0246	.126	.126	.003	$.3\times10^{-3}$	1.22	.11	.085	.065	.195
					(.0233)	(.0233)	$(.36\times10^{-5})$	$(.85\times10^{-8})$					
15	50	1.2	.027	.097	.0297	.03	$.3\times10^{-3}$	$.79\times10^{-5}$	1.49	.021	.015	.020	.05
					(.0296)	(.0297)	$(.248\times10^{-5})$	$(.58\times10^{-8})$					
15	50	1.3	.052	.2467	.0262	.0262	$.792\times10^{-4}$	$.75\times10^{-6}$	1.12	.02	.015	.0125	.0325
					(.0585)	(.0585)	$(.48\times10^{-5})$	$(.11\times10^{-7})$					
15	50	1.4	.105	.379	.054	.054	$.243\times10^{-3}$	$.444\times10^{-5}$	1.17	.049	.039	.020	.07
					(.125)	(.125)	$(.35\times10^{-4})$	$(.24\times10^{-6})$					
15	26	1.5	.26	.523	.209	.209	.0021	$.6\times10^{-4}$	.637	.2	.175	.050	.26
					(.369)	(.369)	$(.57\times10^{-3})$	$(.86\times10^{-5})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-8

EXPERIMENT NO. 8

Sample Size: 2100

$\bar{\rho}$ = Intermediate

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
15	36	1.111	.04	.029	.167	.167	.0013	$.405\times10^{-5}$	.818	.16	.138	.047	.21
					(.045)	(.045)	$(.11\times10^{-5})$	$(.97\times10^{-9})$					
15	50	1.2	.075	.072	.061	.0614	$.27\times10^{-3}$	$.26\times10^{-5}$	.576	.055	.045	.020	.08
					(.0793)	(.0794)	$(.14\times10^{-5})$	$(.96\times10^{-9})$					
15	50	1.3	.144	.155	.0675	.0675	$.22\times10^{-3}$	$.19\times10^{-5}$	.57	.065	.055	.020	.085
					(.154)	(.154)	$(.54\times10^{-5})$	$(.7\times10^{-9})$					
15	50	1.4	.259	.254	.0799	.0799	$.3\times10^{-3}$	$.43\times10^{-5}$	.83	.075	.065	.020	.1
					(.279)	(.279)	$(.19\times10^{-4})$	$(.71\times10^{-7})$					
15	26	1.5	.461	.334	.193	.191	.0023	$.11\times10^{-3}$	.99	.185	.155	.055	.25
					(.525)	(.524)	$(.26\times10^{-3})$	$(.41\times10^{-5})$					

TABLE IV-9

EXPERIMENT NO. 9*

Sample Size: 2100

$\overline{\rho}$ = High

n	k	E_j	V_*	V^*-V_*	E(v)	$\overline{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
15	36	1.111	.066	.023	.293	.294	.0017	$.16x10^{-4}$	.22	.29	.26	.055	.345
					(.0725)	(.0726)	$(.85x10^{-6})$	$(.18x10^{-}\ )$					
15	20	1.2	.124	.0613	.32	.32	.0012	$.91x10^{-5}$	.22	.315	.295	.04	.36
					(.136)	(.136)	$(.45x10^{-5})$	$(.21x10^{-8})$					
15	50	1.3	.248	.095	.32	.32	.0013	$.35x10^{-5}$	.075	.315	.295	.045	.365
					(.278)	(.278)	$(.12x10^{-4})$	$(.3x10^{-8})$					
15	50	1.4	.441	.111	.335	.335	$.4x10^{-3}$	$-.37x10^{-7}$	-.001	.33	.31	.04	.37
					(.478)	(.478)	$(.11x10^{-4})$	$(-.5x10^{-10})$					
15	26	1.5	.709	.102	.171	.171	.0028	$.245x10^{-3}$	1.65	.155	.135	.053	.235
					(.726)	(.726)	$(.297x10^{-6})$	$(.27x10^{-6})$					

*Values in parenthesis correspond to non-standardized V(x)

TABLE IV-10

EXPERIMENT NO. 10*

Sample Size: 2100

$-1 \leq \rho_{ij} \leq 1$

n	k	E_j	V_*	V^*-V_*	E(v)	$\bar{v}$	$S^2(v)$	$M^3(v)$	Sk	M	q	Q	v^-
15	50	1.2	.001	.135	.018	.018	$.135 \times 10^{-3}$	$.22 \times 10^{-6}$	1.39	.009	.007	.008	.028
					(.0034)	(.0034)	$(.25 \times 10^{-5})$	$(.54 \times 10^{-8})$					
15	50	1.3	.00105	.298	.021	.021	$.14 \times 10^{-3}$	$.24 \times 10^{-5}$	1.38	.012	.007	.012	.032
					(.0073)	(.0073)	$(.13 \times 10^{-4})$	$(.63 \times 10^{-7})$					
15	50	1.4	.003	.484	.029	.029	$.31 \times 10^{-3}$	$.77 \times 10^{-5}$	1.41	.02	.015	.02	.050
					(.017)	(.017)	$(.77 \times 10^{-4})$	$(.95 \times 10^{-6})$					
15	26	1.5	.014	.776	.063	.062	.0016	$.13 \times 10^{-4}$	2.05	.042	.03	.04	.11
					(.063)	(.062)	$(.94 \times 10^{-3})$	$(.59 \times 10^{-4})$					

*Values in parenthesis correspond to non-standardized V(x)

FIGURE IV — I

RESPONSE OF Q TO CHANGES IN EXPERIMENTAL CONDITIONS

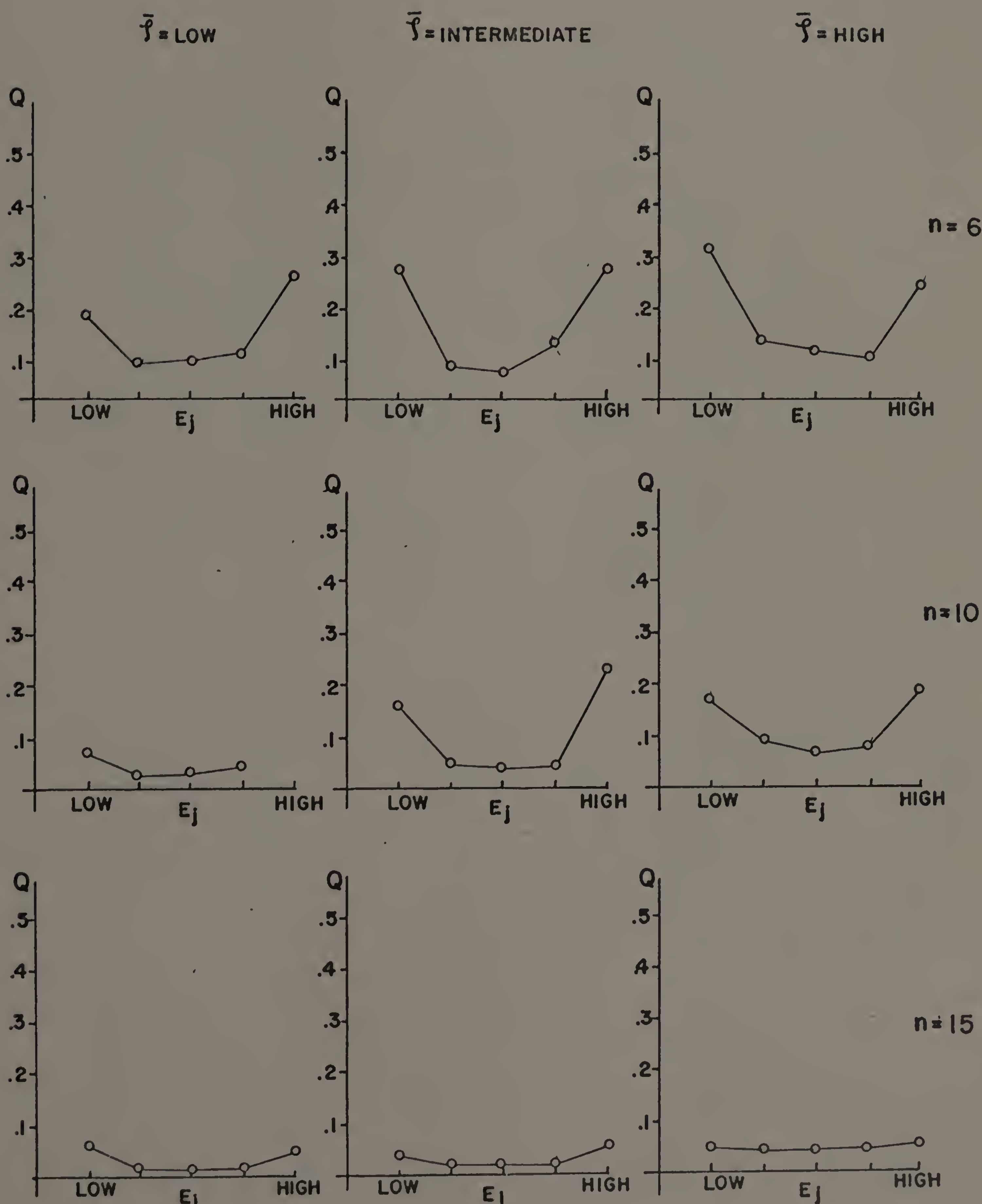


FIGURE IV — I

RESPONSE OF Q TO CHANGES IN EXPERIMENTAL CONDITIONS

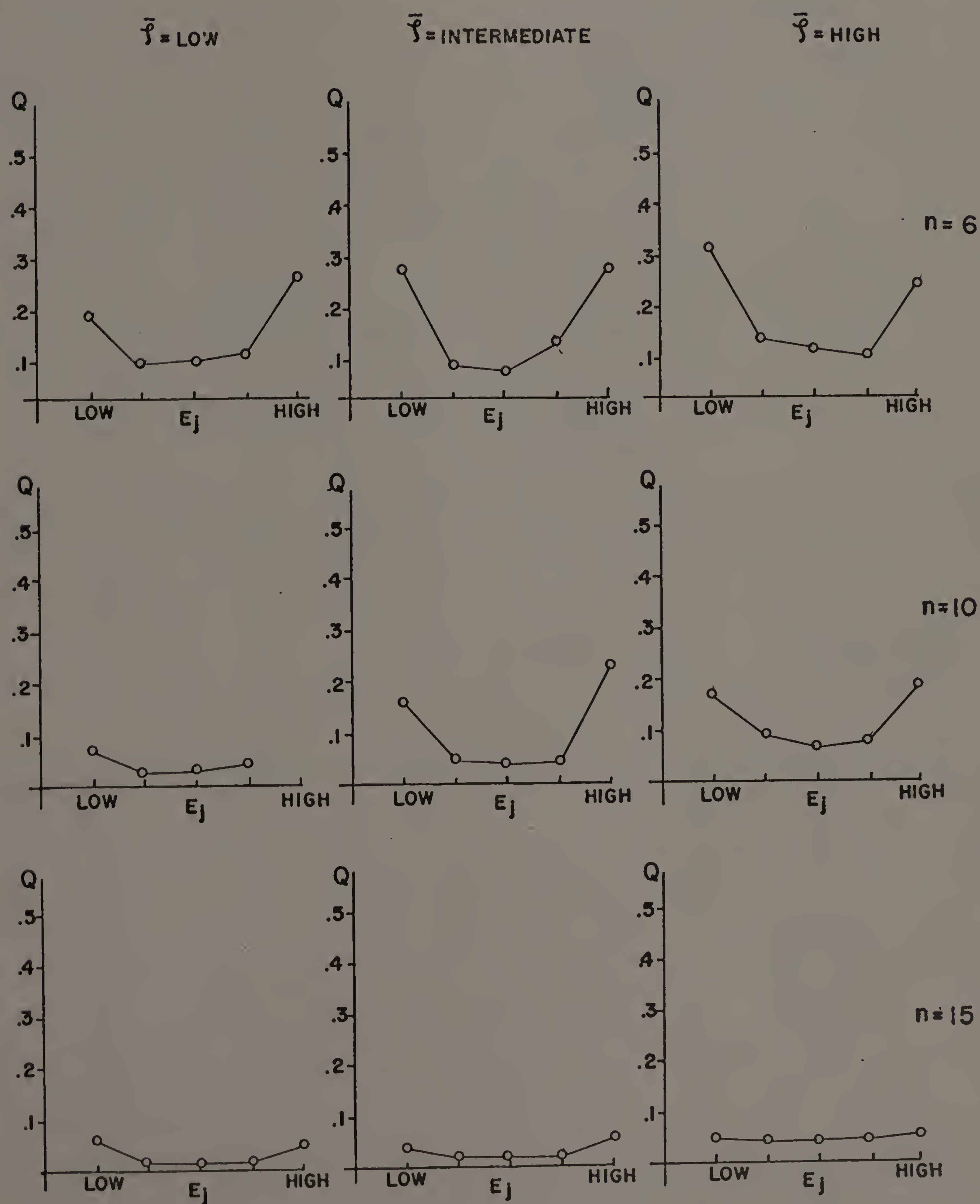


FIGURE IV – 2

RESPONSE OF $S^2(v)$ TO CHANGES IN EXPERIMENTAL CONDITIONS

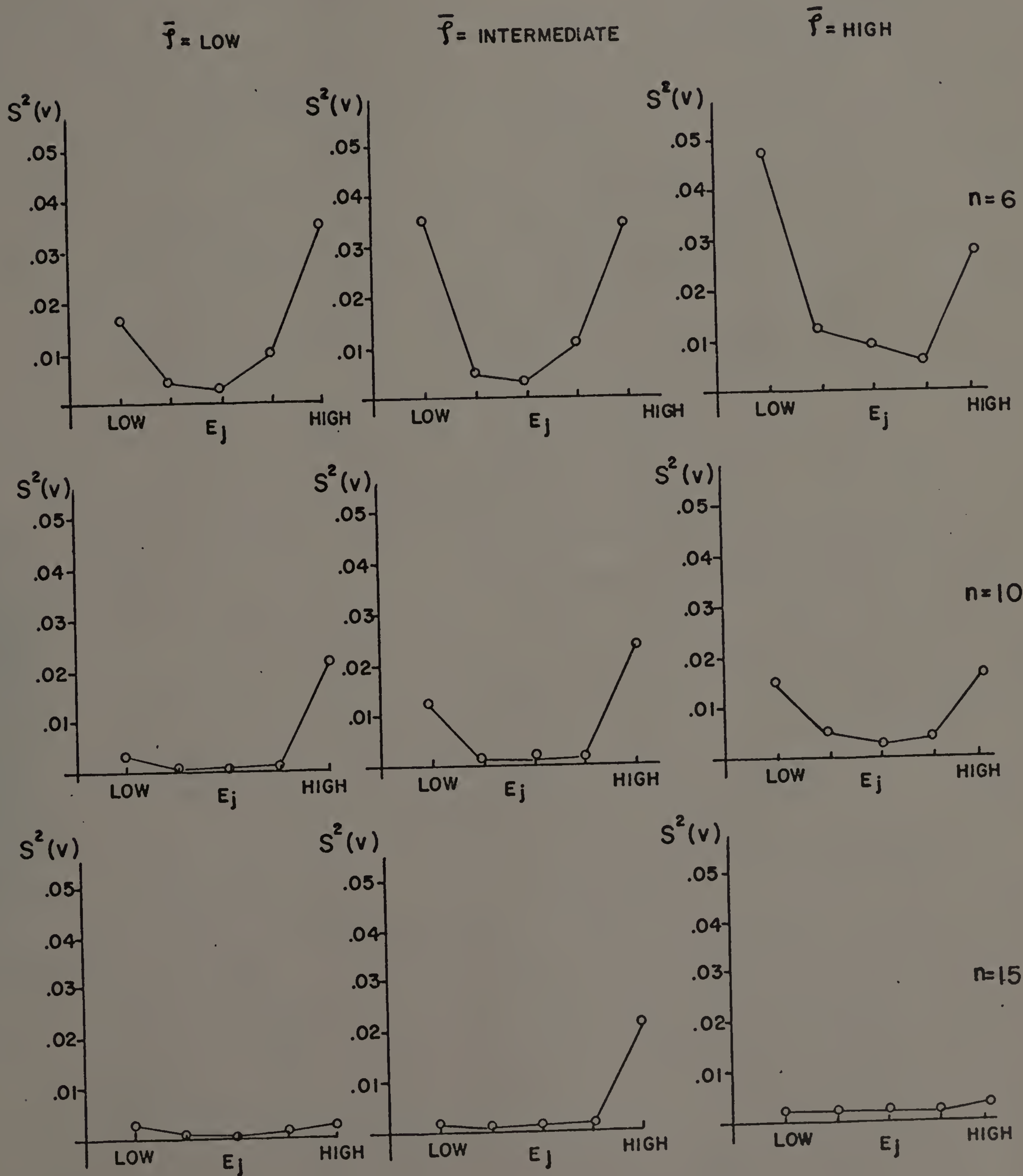


FIGURE IV – 3

RESPONSE OF E(V) TO CHANGES IN EXPERIMENTAL CONDITIONS

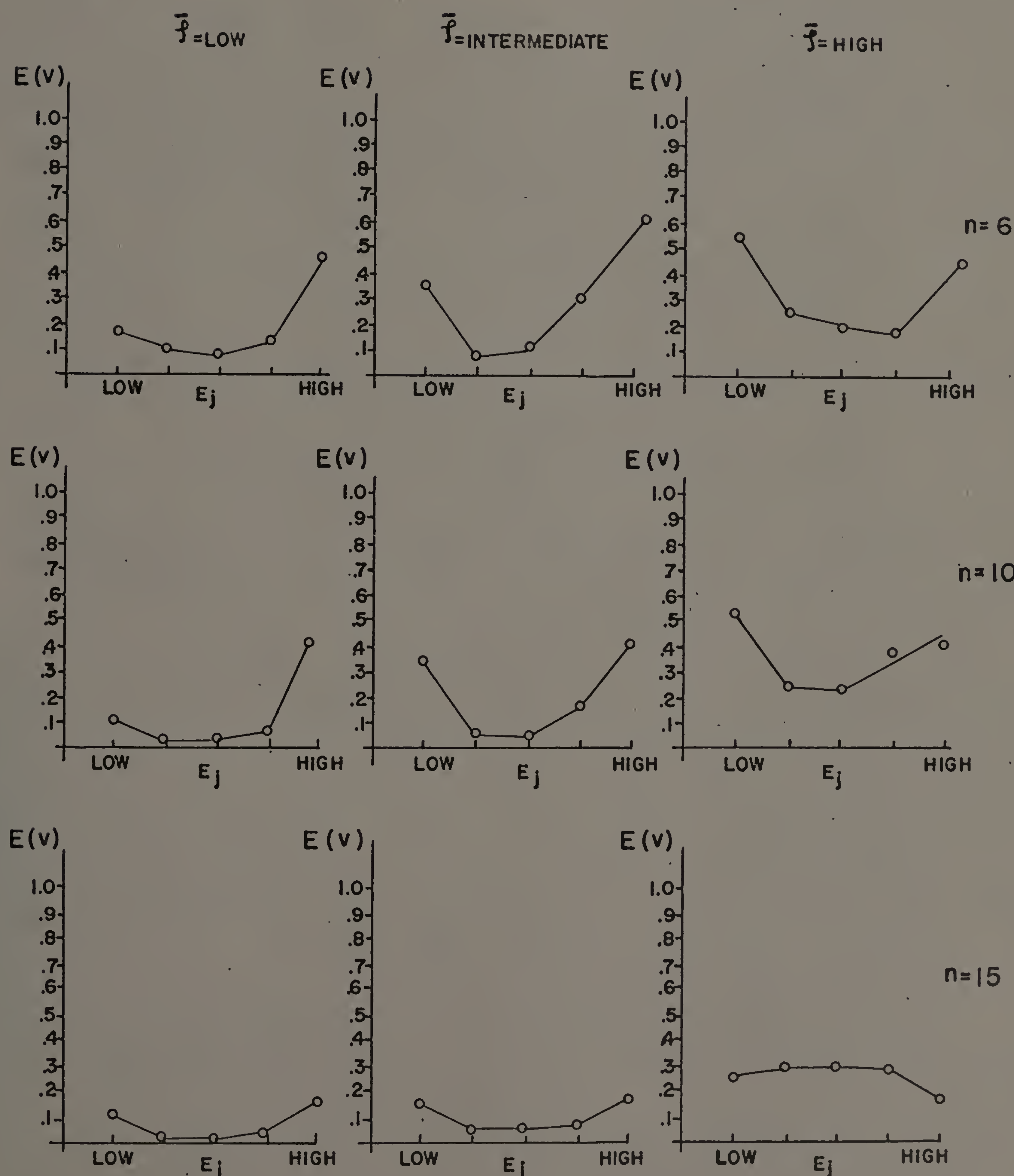


FIGURE IV – 4

RESPONSE OF q TO CHANGES IN EXPERIMENTAL CONDITIONS

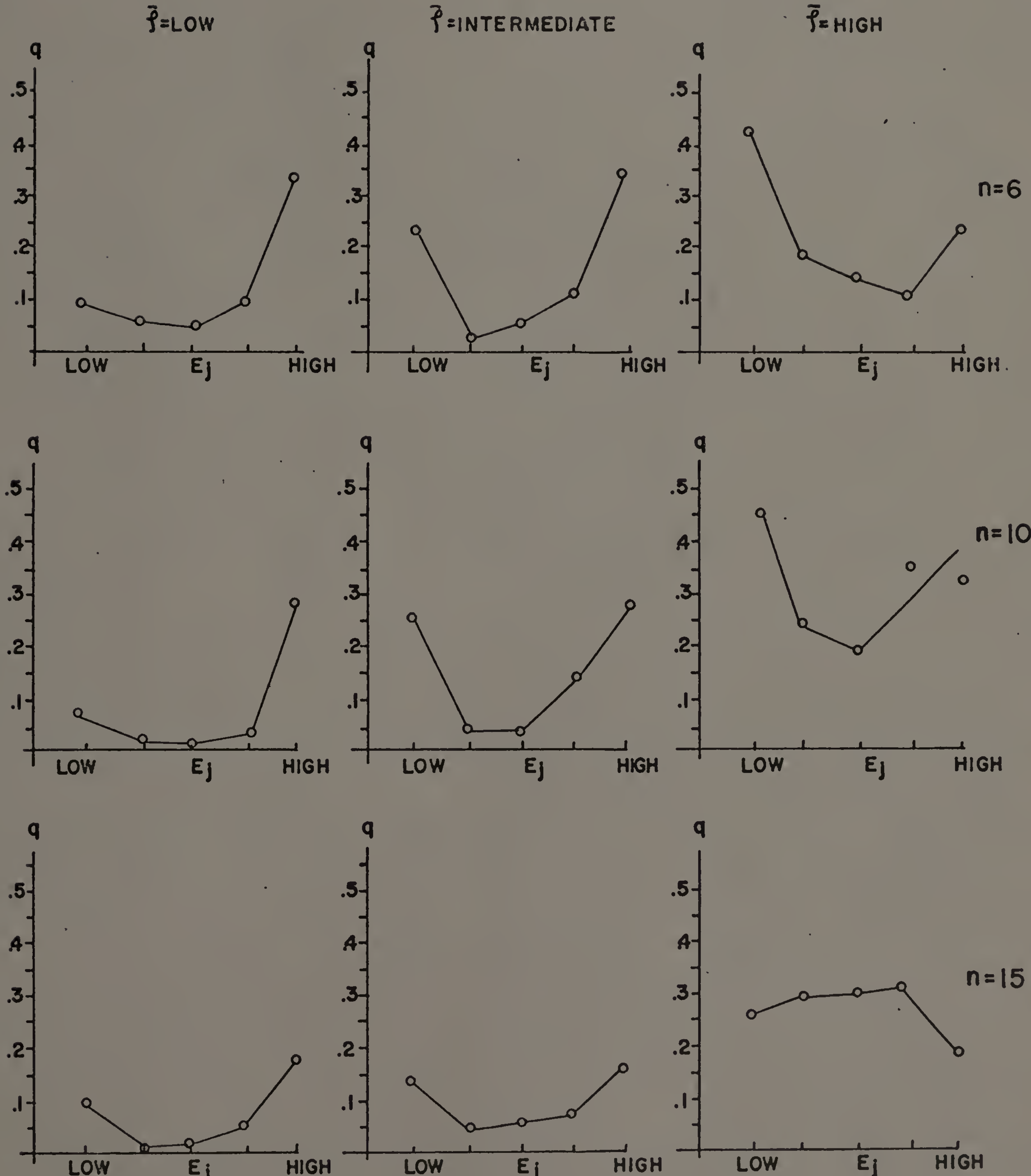


FIGURE IV — 5

RESPONSE OF $\bar{V}$ TO CHANGES IN EXPERIMENTAL CONDITIONS

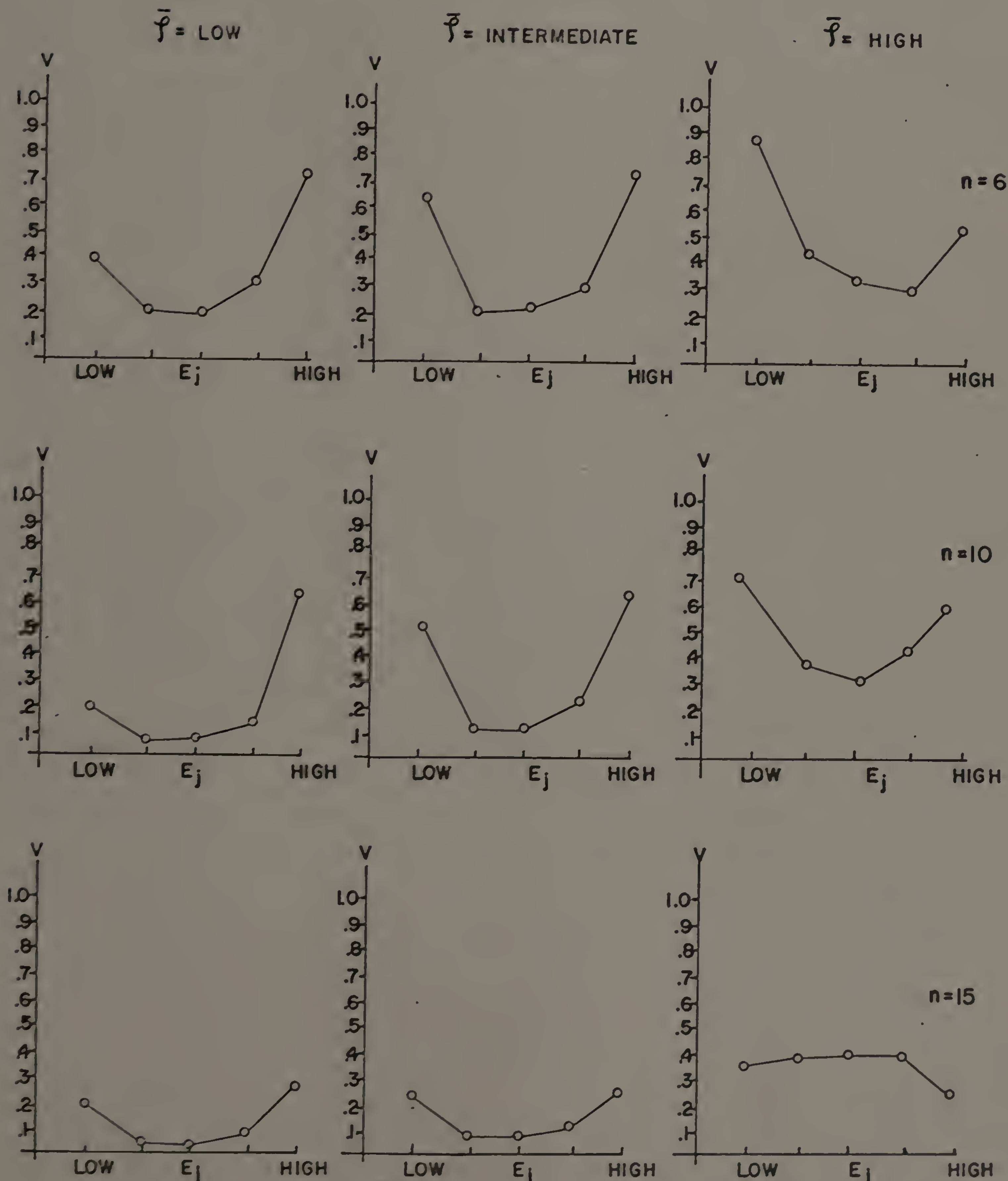


FIGURE IV-6

RESPONSE OF Sk TO CHANGES IN EXPERIMENTAL CONDITIONS

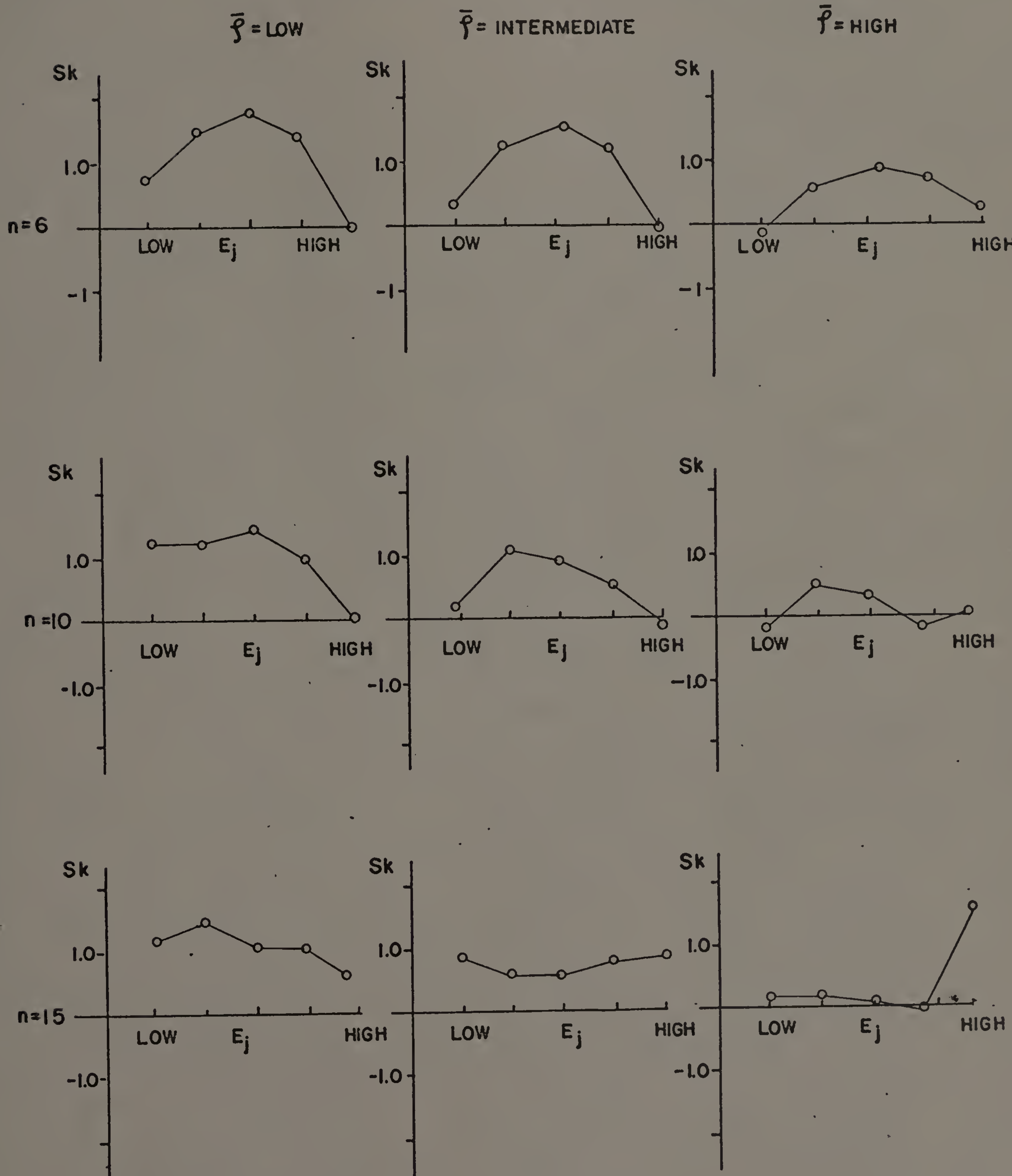


FIGURE IV - 7

COMPOSITE DESCRIPTION OF THE BEHAVIOR OF $f(v \mid E_j)$

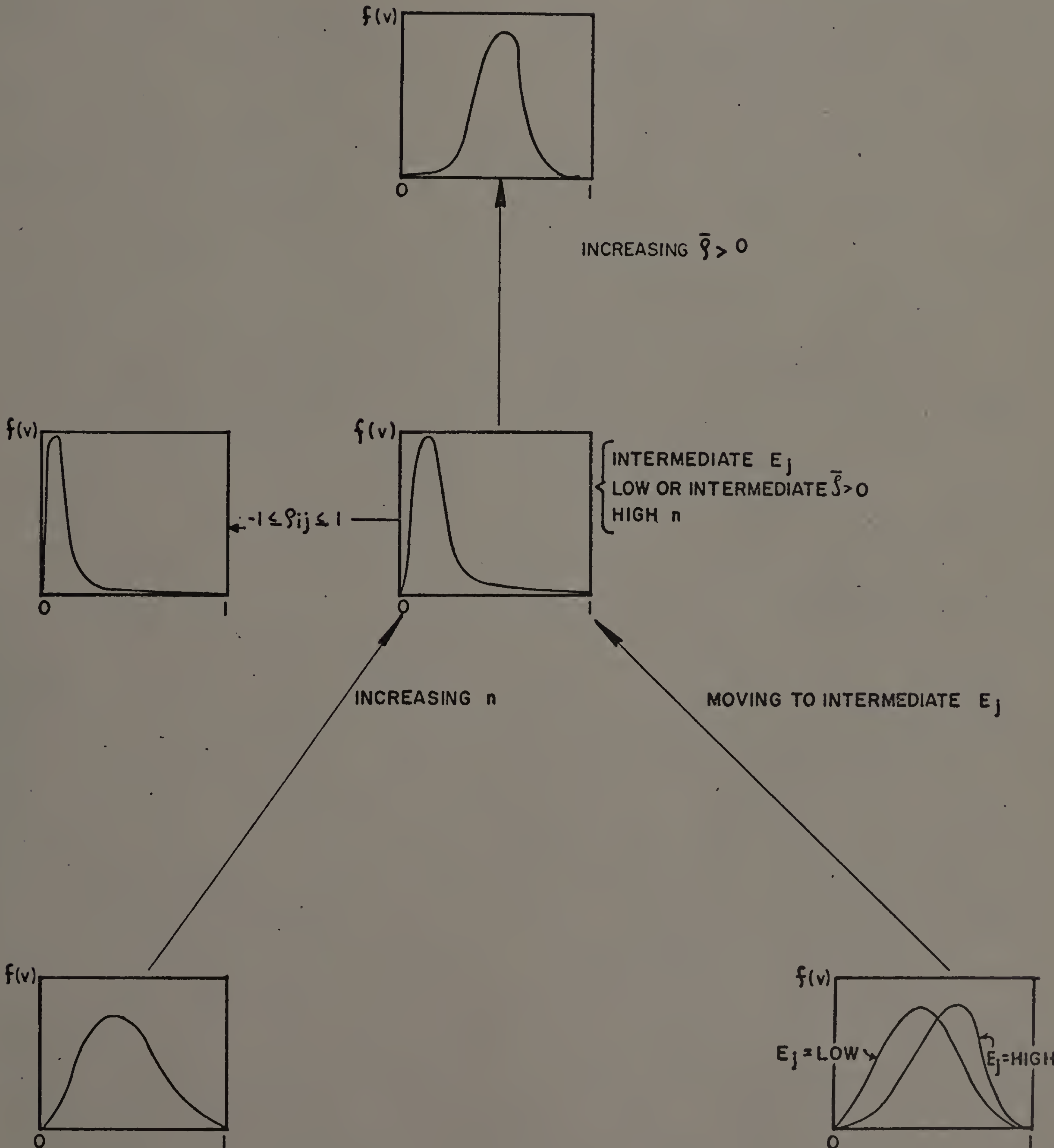
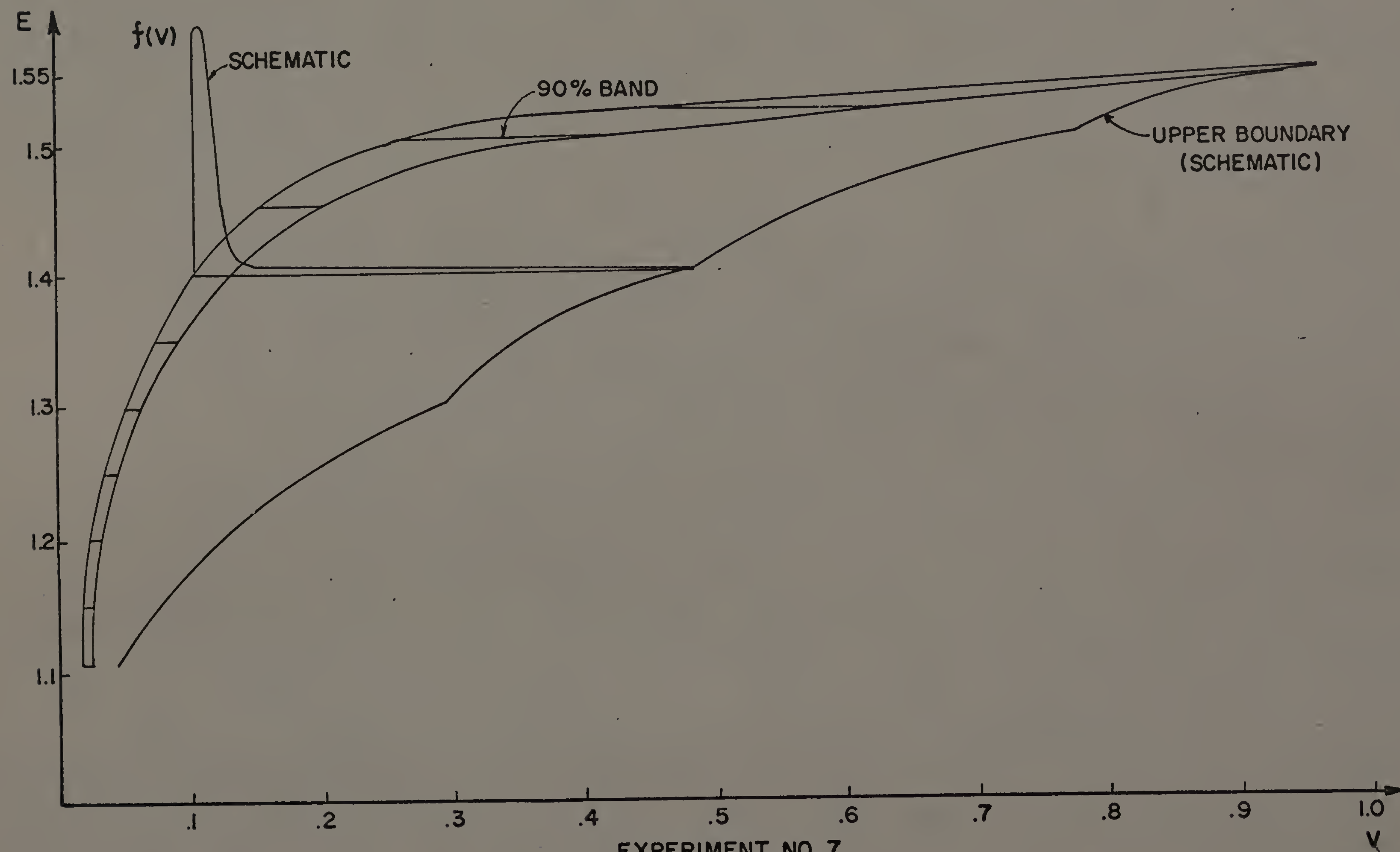


FIGURE IV - 8

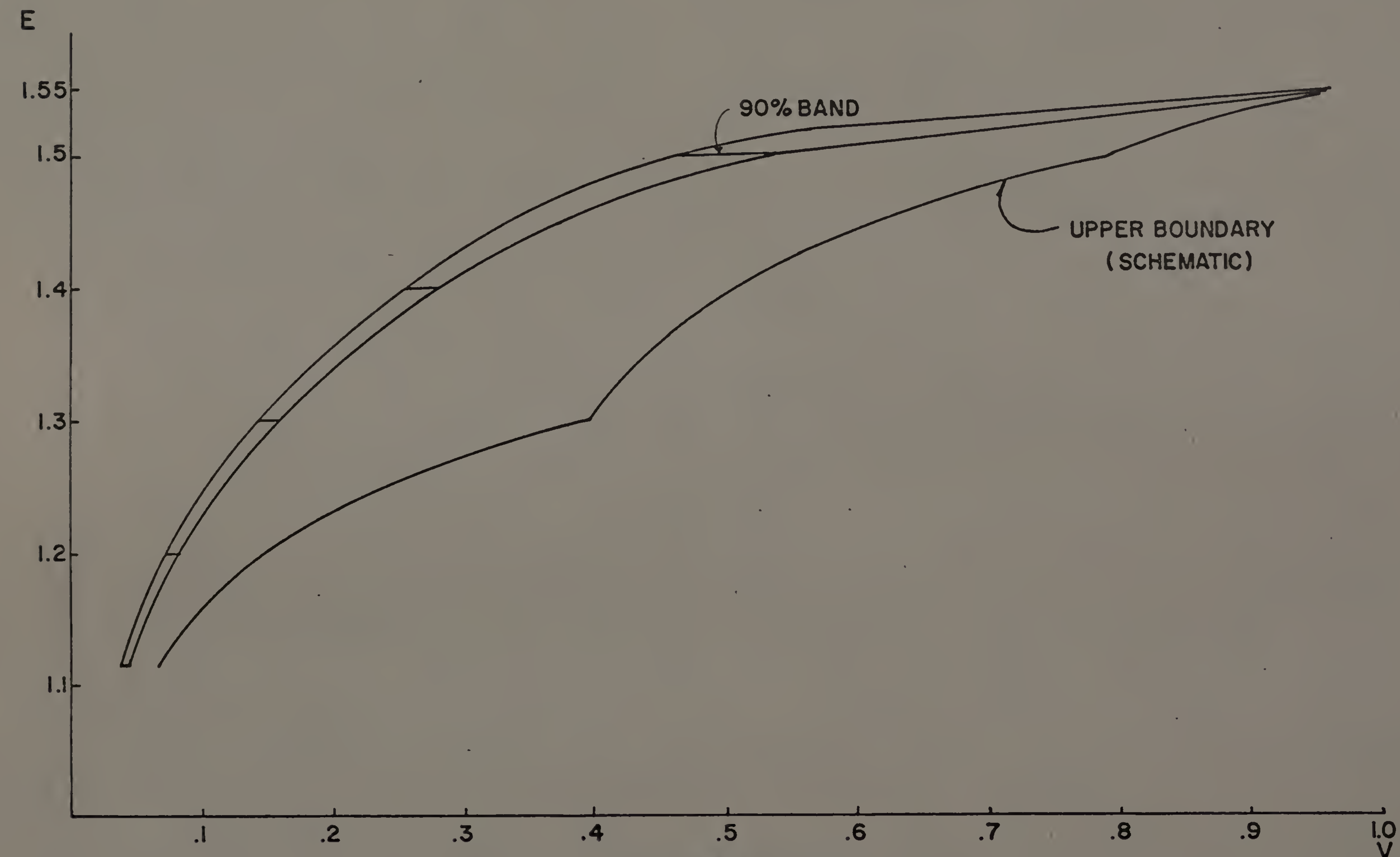
OPPORTUNITY LOCUS AND 90% PROBABILITY BAND



EXPERIMENT NO. 7

FIGURE IV - 9

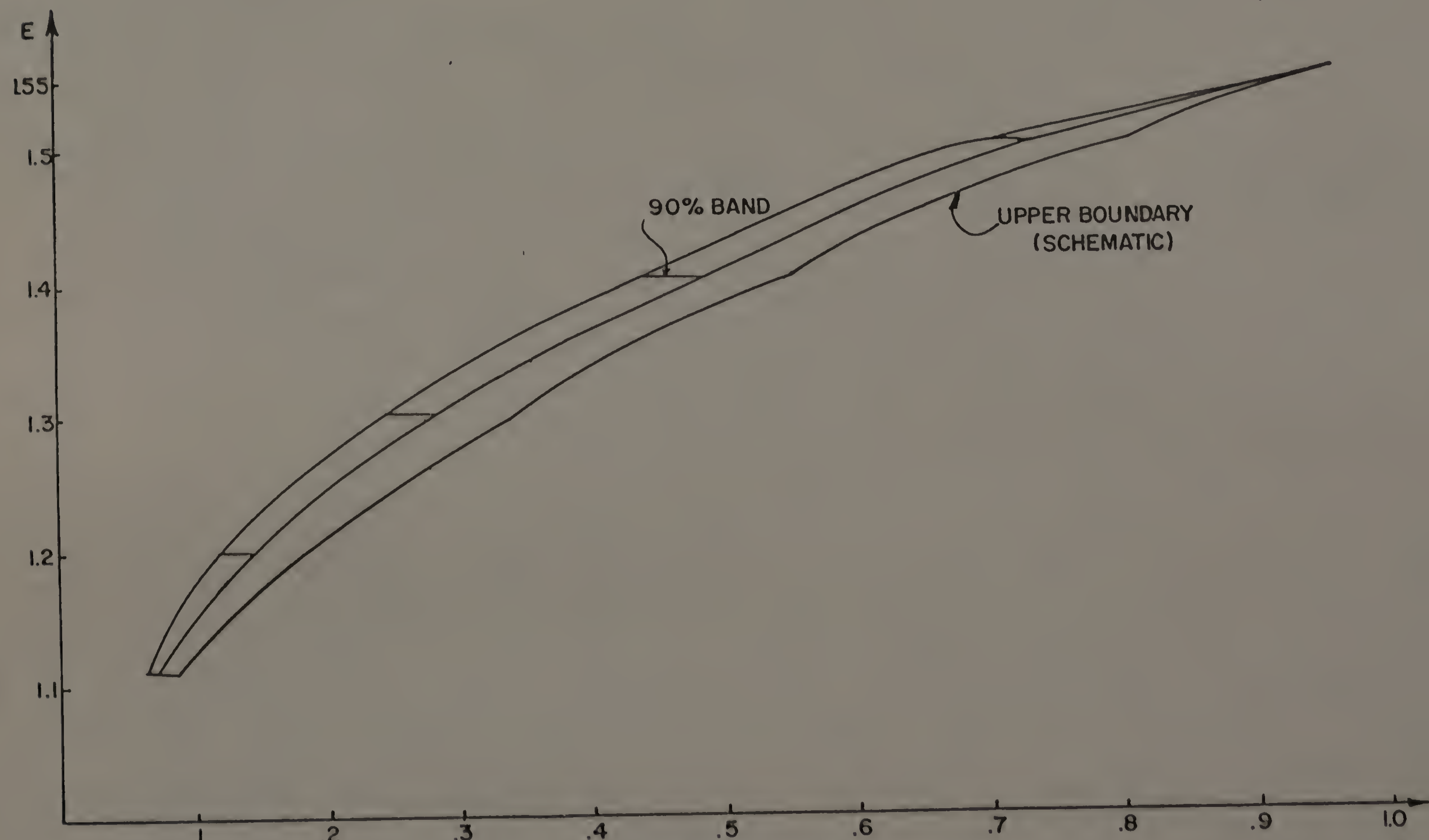
OPPORTUNITY LOCUS AND 90% PROBABILITY BAND



EXPERIMENT NO.8

FIGURE IV-10

OPPORTUNITY LOCUS AND 90% PROBABILITY BAND



EXPERIMENT NO. 9

Analysis of the Results of the Experiments

Dispersion of the distributions. The behavior of the two measures of dispersion Q and $s^2(v)$ as it is summarized in figures IV-1 and IV-2 confirms the statement made earlier that they respond similarly to changes in the experimental conditions. The observation of their patterns of behavior leads to the formulation of the following conclusions about the dispersion of the sample distributions:

(1) For fixed levels of n and $\bar{\rho}$, the density of the distributions in the neighborhood of their means can be expected to increase as one moves towards the extremes of the range of values feasible for E_j. This change becomes negligible for larger values of n, the size of the universe of investment prospects. This conclusion corroborates the analytically derived conclusion that as the number of extreme points in the region increases, the variance of the distribution becomes negligible. It is interesting to notice that for universes as small as the one composed by fifteen investment prospects, the dispersion of a large sample distribution of variances is already very small; an interval of less than ten percent of $[V_*,V^*]$, centered around the median of the distributions, contains fifty percent of the observations. Observation of the values of $s^2(v)$ for the fifteen-prospect universe confirms the assertion that for large values of k, a high concentration of the observations may be expected to cluster in the

neighborhood of the mean of the distribution. It is again very reassuring that this is already evident for n=15.

(2) For fixed levels of expectation E_j and a given universe of investment prospects, an increase in the overall level of correlation between pairs of investment prospects leads to relative increases in the dispersion of the distributions. As interesting as this effect may be for the distributions of standardized variances, the importance of this effect is diminished by the fact that increases in the level of correlation will also lead to narrower intervals between V_* and V^*. An illustration of this latter effect may be observed by reference to figures IV-8, IV-9 and IV-10 which depict the mean-variance opportunity loci corresponding to an investment opportunity universe of size ten, and respectively to low, intermediate and high levels of correlation between investment prospects.

In conclusion, it can be asserted that for investment opportunity universes of a moderate or large size ($n \geq 15$), it appears that the conditional distribution of variances corresponding to random portfolios, yielding ex ante a fixed expected return, will be concentrated in the immediate neighborhood of the distributional mean; and that in the limit, the distribution will tend to a spike located on the mean. This effect is already pronounced and observable for n=15.

<u>Location of the mean of the distribution</u>. The analysis of Figure IV-3 indicates that while for intermediate and low

levels of correlation between pairs of prospects, the size of the investment opportunity universe (a proxi for the number of extreme points k) has an overriding influence on the position of $E(v)$, which drastically approaches the minimum variance as n increases; it can be observed that for high levels of correlation, $E(v)$ remains comparatively large regardless of the size of the universe. This behavior corroborates the conclusions reached in Chapter II after the heuristic analysis of the similarity between the vector $E(x)$ and the upper and lower bounds x^* and x_*.

It can also be observed, in agreement with that analysis, that as E_j moves towards the extremes of its feasible range, $E(v)$ moves away from the minimum variance. The significance of this effect diminishes in importance for large values of n.

In conclusion, given the present analysis and the conclusion of the previous section, it is reasonable to expect that for intermediate to low levels of correlation between pairs of prospects, there is a high probability that portfolios drawn from a population of investment prospects of moderate or large size ($n > 15$), and expected to yield ex ante a return E_j, will map on the mean-variance plane at a relatively small distance from the Markowitz efficient frontier. How high this probability may be expected to be depends on the behavior of the tails of the distribution which will be analyzed subsequently.

<u>Behavior of the tails of the distribution</u>. The first

conclusion that must be drawn from the observation of figures IV-4 and IV-5 is that the lengths of the two tails of the distribution move in almost perfect negative correlation in response to changes in the experimental conditions. This follows from the fact that the length of the first quartile q, and that of the 90% probability band v^- are almost perfectly correlated descriptive variables. This effect is consistent with the previous conclusions about the dispersion of the distributions and the location of the high density interval, and constitutes in effect a cross validation of the previous conclusions. This indicates that the distribution responds to changes in the experimental variables by "sliding" the high density interval from left to right or from right to left, preserving the composition of this interval. The analysis of figures IV-4 and IV-5 leads to the following conclusions about the "sliding" effect:

(1) For low and intermediate levels of correlation between pairs of investment prospects, the left tail of the distribution can be expected to be short and fat. Although for intermediate levels in the range of E_j, the length of the tail is almost negligible, as E_j moves towards the extremes of this range the length of the left tail can no longer be neglected, e.g. the probability of $v < q$ is very small for extreme values of E_j. This is in perfect harmony with the heuristic analysis of the similarity between E(x) and respectively x* and x_* which was made in Chapter II.

(2) For a high positive level of correlation between prospects, both tails are long and narrow, so that the probability of a random portfolio mapping on the mean variance plane in the neighborhood of either V^* or V_* is negligible.

An interesting conclusion that can be derived from this and the previous sections, and which will be useful for the discussion of the implications of this research is the following. Given the characteristics about the dispersion. location of the mean and tail behavior which have been reached here, assume that portfolios are generated at random for expectations spanning the feasible range of E_j's and that these portfolios are plotted on the mean-variance, or mean-standard deviation plane. A descriptive least-square polynomial is fitted to these points using the conditional variance or standard deviation as the independent variable, and the expectation E_j as the dependent variable. The graphical representation of this polynomial function for the relevant range may be expected to be closely parallel to the Markowitz-efficient frontier for intermediate values in the range of E_j and to swing downwards and away from the frontier for extreme values of E_j. It is very plausible then that a quadratic regression constitutes an appropriate approximation. This conclusion was reached after the examination of figures IV-8, IV-9 and IV-10, where for a fixed investment universe and at three levels of correlations, a 90% probability band has been constructed such that the sample probability $P(V(\tilde{x}) \varepsilon 90\% \text{ Band} | E_j)$

= .9. Observe that this band is very narrow in the neighborhood of the Markowitz-efficient frontier, and also that its upper bound is roughly parallel to the frontier in all three cases.

In view of the strong conclusions derived above, the analysis of the behavior of the skewness coefficient in response to the experimental conditions seems almost redundant. It is nonetheless useful since it provides a cross validation for the conclusions reached.

Analysis of the skewness of the sample distributions. The following observations can be made from the analysis of the effect of experimental conditions on the coefficient of skewness Sk:

(1) The distributions are positively skewed for large investment universes ($n > 15$) and for intermediate levels in the feasible range of E_j's. This behavioral pattern is even more accentuated for low levels of correlation between pairs of investment prospects.

(2) The distributions are nearly symmetrical and even negatively skewed at the extremes of the feasible range of E_j's for small values of n. These patterns are more pronounced when the overall level of correlations approaches unity.

These observations are in perfect agreement with the conclusions reached before and do not require further comment. It remains to analyze the behavior of the distributions when the covariance matrix contains negative elements.

<u>The effect of negative covariances</u>. All observations and conclusions made in the immediately preceding sections refer to experimental conditions in which the covariances between pairs of investment prospects are positive. Experiment No. 10 has been designed to observe the effect of allowing covariances to take positive and negative values indiscriminately. From the results shown in Table IV-10, it can be immediately observed that the descriptive variables respond to experimental conditions in the same form of those corresponding to Table IV-7, which may be used as a control.

The conclusion is that in comparison with a situation in which the covariances are positively small; everything else being equal, the incidence of negative covariances will cause the sample distributions of the standardized variable v to be more positively skewed, with a lower expectation and a smaller dispersion about their means and (or) their medians.

The incidence of negative correlations is unlikely in the stock market where all securities are roughly correlated to the market. But it is conceivable that other investment problems which may be regarded as portfolio problems will exhibit this characteristic.

The conclusions reached in this chapter have been summarized into a composite perspective in figure IV-7 which attempts to present a pictorial description of the behavior of the sample distributions in response to changes in the experimental conditions. The following section will provide a summary of the conclusions.

Summary of the Conclusions

The tentatively defined causality relationships of Chapter II and the empirical support they have received from the results of the experiments described in this chapter lead to the following conclusion about the conditional distributions of variances corresponding to randomly selected portfolios which are expected to yield ex ante a return E_j:

(1) If the randomly selected portfolios described above are drawn from an investment universe of moderate or large size ($n > 15$), for values of E_j not at the extremes of their feasible range; and if the investment prospects are such that pairwise correlations between their returns are not inordinately high for all pairs, then there is a very high probability that these portfolios will map on the mean-variance plane within a very narrow band drawn from the mean-variance efficient frontier.

(2) This probability will be reduced for portfolios whose expectations are at the extremes of the feasible range; if correlations between pairs of investment prospects are uniformly high; or if the size of the investment universe becomes very small.

(3) For investment universes of moderate and large size ($n > 15$) the dispersion of the conditional distributions about $V(E(x|E_j))$ is very small, so that the highest incidence of ob-

servations in a sample may be expected in the neighborhood of the mean.

This conclusion, although weakly stated, becomes a very strong statement when it is realized that the investment universe made of all the securities in the New York Stock and American exchanges exceeds one thousand prospects, that correlations between returns of pairs of securities are typically positively moderate, and that even the loosest diversification constraints prevent the incidence of E_j at the extremes of its range.

It is the coincidence of these conditions with the observations made here what makes the results of this chapter worthy of consideration, since they suggest an absence of variety in the investment process, as conceptualized by the mean-variance model, which may have wide ranging implications for the formulation of normative and positive conclusions based on the assumptions of the mean-variance paradigm. Some of these implications will be discussed in Chapter VI.

Chapter V analyzes the assumptions underlying the previous experiments and demonstrates the robustness of the present conclusions under an alternative random selection process.

Footnotes

1. All covariance matrices were tested to be positive definite and non-singular by evaluation of their principal minors.

2. The results of the experiments are independent of distributional characteristics and errors in the estimation of the parameters, since both u and C are assumed to be the true expectation and covariance matrix of the n-variate distribution of wealth relatives.

3. On occasions the minimum level $E_j=1.1$ could not be used since the efficient frontier turned at a higher value. This should not be disturbing since other low values of E_j fulfill the requirement that it be low.

4. Experiment No. 10 allows p_{ij} to take freely positive and negative values. This is of less interest since securities in the stock market are typically positively correlated.

CHAPTER V

A CRITIQUE OF THE RANDOM PROCESS[1]

Introduction

This chapter is devoted to the analysis of the random process which was formulated in Chapter II, and which was used for the Monte Carlo generation of random portfolios corresponding to the experiments described in the previous chapter.

The main concern of this analysis is related to the property of the linear transformation

(1) $$x = Yw$$

by which a point x in $\{x|E_j\}$ may not always be uniquely determined by a vector w. This introduces the suspicion that the high density observed in the neighborhood of the conditional expectation of the sample distributions may be due to overdetermination of "double counting".

A secondary concern, although of some utilitarian value, is to establish a correspondence between the random process adopted in this research and an alternative intuitively acceptable method for selection of random points in $\{x|E_j\}$.

The Problem of Multiple Determination

In addition to the linear transformation defined in (1) above, let the augmented matrix Y_x be defined as

(2) $$Y_x = (Y,x)$$

where x is a point in $\{x|E_j\}$.

The vector x in E^n is said to be uniquely represented by the vector w in E^k if and only if[2]

(3) $$\text{rank } (Y) = \text{rank } (Y_x) = k$$

Under any other circumstances in which the rank of Y equals the rank of Y_x, there is an infinite number of solutions in w for the system of simultaneous linear equations (1), and thus x will not be uniquely determined by any one w.

The problem arises due to the nature of the linear transformation which expresses x as a convex linear combination of the extreme points in $\{x|E_j\}$ which is

(4) $$x = Yw$$

$$\sum_{i=1}^{k} x_i = 1$$

$$w \geq 0$$

The constraints impose pecularities on the system that are not entirely explained by the characteristics of (1). For example, if $x=y_i$, one of the extreme points of the convex set, then there must be a unique feasible solution to (1) given the constraints in (4) since, by definition, an extreme point cannot be represented by a convex linear combination of other points in the set. The same can be shown to be true for points in the edge of the convex set $\{x|E_j\}$ which are uniquely determined by a convex linear combination of two adjacent extreme points given the constraints in (4).

In terms of the random process defined in Chapter II, these considerations mean that compared with interior points, the points in the edges of the convex set have a negligible probability of being selected. This is in contradiction with the stated purpose that all points in $\{x|E_j\}$ be given identical probabilities of selection.

In effect, given the Monte Carlo generation process described in Chapter III, points in the edges of the set will never be selected since the uniform random variates generated by the pseudorandom number generator are in the open interval (0,1) and not in [0,1]; and therefore, values for $w_i = 0$ or $w_i = 1$ can never occur.

Due to this property of the pseudo random process, the random vectors $\tilde{w}$ are not selected from the closed set defined in (4), but from the open set

$$(5) \qquad \sum_{i=1}^{k} w_i = 1$$

$$w > 0$$

so that all random vectors $\tilde{w}$ are interior points of the convex set defined in (4).

It will be illustrative to consider that the system (1) can be rewritten as

$$(6) \qquad x = Y_n w_n + Y_{(k-n)} w_{(k-n)}$$

where Y_n is an n-th order non-singular matrix[3] whose columns are any n extreme points y_i; w_n is the corresponding n-column vector. $Y_{(k-n)}$ and $w_{(k-n)}$ correspond to the remaining (k-n) extreme points.

Since Y_n is non-singular, then solutions for (6) can be obtained for

$$w_n = Y_n^{-1}x - Y_n^{-1}Y_{(k-n)}w_{(k-n)} \quad (7)$$

An infinite number of solutions can be obtained assigning arbitrary values to $w_{(k-n)}$. If at least two of these solutions are feasible in terms of the constraints in (5), then there will be an infinite number of $\tilde{w}$'s which will determine a single point x; namely, all $\tilde{w}$'s in the line between the two feasible solutions. There is still the possibility that for a given x in the interior of $\{x|E_j\}$ there is a unique feasible solution; while for others there is an infinite number of solutions.

The question could be resolved if it could be shown that there exists an infinite number of solutions feasible under constraints (5) for every interior x, since in this case the probability of selection would be equal for all interior points. Or, conversely, if it could be shown that given the constraints in (5) there is a unique feasible solution for every point in $\{x|E_j\}$.

In the absence of any of these proofs, the possibility remains that some interior points have a greater probability of being selected than others. Rather than delving into the mathematical solution of this issue, a test has been devised to provide an indication of the nature of this problem as it related to the conclusions of Chapter IV. An examination of Figures B-1 through B-50 in Appendix B will immediately identify a prime suspect for relatively excessive overdetermination. As it had been noticed before, the distributions of sample conditional variances "pack" in the neighborhood of the variance corresponding to $E(\tilde{x}) = (1/k) \sum_{i=1}^{k} y_i$, and this effect has been shown to be more accentuated for large values of k. That this behavioral observation is due to the fact that "more" w's will yield values in the neighborhood of E(x) than in the neighborhood of any other points in the set could easily be detected by observing the relative sample frequency in which E(x) appears for vectors w such that $w_i \neq (1/k)$ and i-1,...,k; and compare it with the relative frequency in which a different point x_1 that has been generated by the vector w_1 will appear as a consequence of vectors $w \neq w_1$.

Since the process is strictly continuous, the probability of an observation x = E(x), or $x = x_1$ is exactly equal to zero. For this reason, the frequencies should be computed for vectors in "the neighborhood of" E(x) or x_1, which are not the direct result of values of w "in the neighborhood of" either $w = \{1/k\}$ or w_1. The concept "in the neighborhood of" can easily be accomplished by discretizing the comparisons as it was done in the test that follows.

Let the vectors i(x) and i(w) be such that their elements are defined by

$$\{i(x)_j\} = \{.01\mathrm{INT}(\ 100x_j + .5)\} \tag{8}$$
$$\{i(w)_1\} = \{.01\mathrm{INT}(\ 100w_1 + .5)\}$$

where the operator INT(*) truncates the argument to its integer component.

The computation of the frequency of double counting, this is the frequency in which the same x results from different values of w in a sample of a given size, can now be accomplished by means of the following logical relations.

Let x_1 be the vector under scrutiny, and w_1 be the vector used for its generation; then for any couple (x_2,w_2)

(9) If $i(w_1) \neq i(w_2)$ and $i(x_1) = i(x_2)$ $\rightarrow$"1"

If $I(w_1) = i(w_2)$ or (if $i(w_1) \neq i(w_2)$ and $i(x_1) \neq i(x_2)$) $\rightarrow$"0"

where "1" is a command to increment the frequency of double counting by one and "0" is a command to examine the next pair (x_2,w_2) in the sample.

The investment universe with the greatest concentration of sample observations in the neighborhood of the variance corresponding to E(x) is that of fifteen investment prospects. For this reason, this universe is the prime suspect for the observation of the double counting effect. The test described in (9) was performed for samples of 500 observations each corresponding to levels of expectation E_j = 1.1, 1.2, 1.3, 1.4, 1.5.

In these tests, ten x variates corresponding to each level of E_j were selected in such a way that nine of them were different from $E(\tilde{x})$; the tenth vector was $E(\tilde{x})$. These vectors served as x_1's for the test described above. The vectors used for the generation of the first nine were used as their corresponding w_1's. The w_1 corresponding to E(x) was {1/k}. Each of the ten vectors for each level of E_j was then compared by means of (9) with the vectors in the sample. The w_2's corresponding to the vectors in the sample were those randomly generated vectors which yielded the sample variates. This test was repeated for the ten security universe.

The results obviate the need for the construction of formal statistical tests. The frequency of double counting resulting from these comparisons was exactly zero for all vectors and all levels of E_j.[4] The conclusion is straightforward: The high density of the sample distributions in the neighborhood of the variance corresponding to $E(\tilde{x})$ is not due to overrepresentation of E(x) by randomly generated vectors from the set defined by (5).

It remains to establish the correspondence between the results of Chapter IV and the results of a behaviorally plausible method for generating random portfolios.

A Behaviorally Plausible Random Generation Process

The implications for the conclusions of this research of establishing a correspondence between the results of Chapter IV and those of a behaviorally plausible model are merely utilitarian. It will help to get the point across.

It is granted that to claim that individuals choose portfolios in any way that resembles the process formulated in Chapter II would be to stretch the limits of academic naivete. But it is also fair to state that the model proposed in that chapter is not a behavioral model, but simply a method for generating random portfolios. Nonetheless, it can be anticipated that unless a relationship is established between this method and some other method that can be easily understood; the results of the previous chapter may be met with criticism on the grounds that the random generation process does not resemble the behavior of an investor selecting portfolios on information regarding expectations alone.

For this reason it is suggestive to examine the properties of a random selection process that because of its simplicity <u>could</u> be used without computational requirements other than a calculator and a pencil. If a correspondence is established between the results of such a process and those of the previous chapter, the alternative method provides an attractive way of cross-validating the conclusions of Chapter IV.

The following process has been designed for the replication of experiments Four, Five and Six, of Chapter IV which correspond to a ten-prospect universe at three levels of correlation between the returns of investment prospects.

<u>A discrete model for the generation of random portfolios</u>. This model imposes the requirement that the random vector x is such that

$$(10) \qquad x_i = .1I \qquad i=1,\ldots,n$$

$$I = 1,2,3,\ldots,9$$

$$\sum_{i=1}^{k} x_i = 1$$

$$E_{j1} \geq x'u \geq E_{j2}$$

$$x \geq 0$$

where x and u are as defined before, and $[E_{j2},E_{j1}]$ is an expectation interval.

In the experiments described above the expectation constraint was dropped so that random portfolios can span the whole range of feasible expectations. The generation of random vectors was accomplished by selecting at random (with equal probability) any one of the unordered portfolio compositions feasible under the constraints, and randomly assigning the weights in the composition to the n assets in such a way that the assignment probabilities are equal. Table I in Appendix F depicts all feasible unordered compositions; and program DISCR in Appendix D shows the method for random generation.

The results corresponding to the three experiments are summarized in tables V-1 and V-2 of this chapter. It was decided to describe the sample distributions corresponding to each set of experimental conditions merely by their medians and their interquartile range. These two descriptive parameters adequately describe the response of the distributions to changes in the experimental conditions. Table 1 contains the medians of the sample distributions corresponding to high, intermediate and low levels of correlation and to eight different intervals $[E_{j2},E_{j1}]$; the corresponding values for the interquartile range are presented in Table V-2.

Discussion of the results. Before discussing the results of tables V-1 and V-2, it will be of great interest to examine the bias introduced by the use of the discretized method. It can be observed in Table 1 of Appendix F that the feasible unordered portfolio configurations are such that only two of them have more than eight non-zero elements while thirty-one have configurations containing no more than six non-zero elements and twenty-one have configurations with no more than four non-zero elements.

The result of the higher incidence of configurations with fewer non-zero elements is that a high proportion of the random portfolios are expected to have compositions different to the composition of the minimum variance portfolio for a given level of expectation; and also, that the dispersion of the sample distributions is comparatively large.

The results shown in tables V-1 and V-2 conform to this prediction. When compared with experiments 4, 5 and 6 of the previous chapter, the distributions resulting from the discretized process have higher expectations and a proportionally larger dispersion. What is important is that the patterns of behavior that were predicted in Chapter IV are preserved for this method of generation of random portfolios.

Table V-1

Medians of the Sample Distributions
Discrete Random Generation Method

	$\bar{\rho}$ = Low			$\bar{\rho}$ = Intermediate			$\bar{\rho}$ = High		
$E_{j2}-E_{j1}$	V_*	V^*	M	V_*	V^*	M	V_*	V^*	M
1.1-1.15	.020	.096	.048	.035	.096	.060	.059	.096	.075
1.15-1.2	.024	.135	.057	.045	.14	.079	.086	.186	.117
1.2-1.25	.028	.221	.075	.064	.221	.105	.13	.221	.173
1.25-1.3	.041	.29	.094	.090	.29	.140	.17	.344	.245
1.3-1.35	.054	.36	.10	.121	.36	.180	.258	.360	.332
1.35-1.4	.09	.56	.21	.165	.56	.290	.34	.529	.45
1.4-1.45	.12	.69	.32	.215	.69	.42	.443	.690	.59
1.5-1.55	.3	.962	.80	.460	.962	.845	.81	.962	.90

TABLE V-2

Interquartile Range of the Sample Distributions
Discrete Random Generation Process

	$\bar{\rho}$ = Low		$\bar{\rho}$ = Intermediate		$\bar{\rho}$ = High	
$E_{j2}-E_{j1}$	V^*-V_*	I.Q.R.	V^*-V_*	I.O.R.	V^*-V_*	I.O.R.
1.1-1.15	.076	.016	.061	.011	.037	.016
1.15-1.2	.111	.023	.095	.017	.100	.029
1.2-1.25	.193	.028	.157	.026	.091	.020
1.25-1.3	.25	.030	.20	.033	.174	.040
1.3-1.35	.3	.053	.24	.045	.102	.054
1.35-1.4	.47	.12	.395	.090	.189	.074
1.4-1.45	.57	.14	.465	.123	.247	.070
1.5-1.55	.66	.015	.5	.013	.152	.031

From Table V-1 it can be observed that the medians of the sample distributions remain in the neighborhood of the minimum variance for intermediate and low levels of correlation, with the exception of the extremes of the range of feasible expectations. Furthermore, it can be observed in Table V-2 that the dispersion of the distributions, as measured by their interquartile range, remains a fairly constant proportion of the total range (V^*-V_*).for the three levels of correlation and at fixed levels of expectation. It also can be observed that as one moves away from the extremes of the range of feasible expectations, the dispersion of the distribution decreases as a proportion of (V^*-V_*); except for the case of high correlations in which it remains uniformly small.

Finally, the medians corresponding to the distributions in the high correlation case remain at some distance of either extreme as was predicted in the previous chapter; and also as predicted, the medians corresponding to either very high or very low levels of expectation may be found in the neighborhood of the maximum variance for those levels of expectation.

In conclusion, this experiment supports fully the behavioral conclusions reached at the end of Chapter IV; and although the medians and dispersions of the distributions resulting from the discretized method are uniformly higher than those corresponding to Experiments 4, 5 and 6, the main effects have been preserved.

Conclusion

With the help of a rigorous test, it was ascertained that there is no evidence of "double counting" of vectors in the neighborhood of $E(\tilde{x})$, or of a selected number of other vectors at any level of expectation. How rigorous the test was is left for other minds to evaluate since the same mind who developed the model can never be expected to devise the "most rigorous test available." Furthermore, three experiments were replicated using a discretized method for the generation of the random portfolios which does not share any of the characteristics of the method described in Chapters II and III. The results of this replication are consistent with the conclusions of Chapter IV, and provide evidence that the patterns of behavior predicted in that chapter are preserved under the discretized method. Since the latter method does not rely on a linear transformation, these results cannot be attributed to the overdetermination effect; and therefore, presumably it has no bearing at all on the conclusions reached in the previous chapter.

Finally, the discretized model is a believable procedure that can be used by "the men on the street," and since its results correspond behaviorally to those of the random process formulated in Chapters II and III, it can be expected that more precise methods for random selection of portfolios which impose an expectation constraint and totally disregard variance, will yield portfolios which are more likely to be near the Markowitz frontier than elsewhere on the mean-variance opportunity locus.

The following chapter will examine the implications of this and the previous chapter for some aspects of the so-called mean-variance portfolio theory.

Footnotes

1. I am indebted to Professors Pao Lun Cheng and Walt McKibben of the University of Massachusetts at Amherst for their valuable comments and suggestions in relation to this chapter.

2. G. Hadley, Linear Algebra, Reading (Addison-Wesley Publishing Company), 1964, p. 172.

3. Ibid., p. 171.

4. Computer printouts are available on request.

5. Frequency distributions of sample variances are available on request.

CHAPTER VI

IMPLICATIONS OF THE RESULTS

Introduction

The results presented in Chapters IV and V have wide-ranging and perhaps unsettling implications for different aspects of what is commonly known as the mean-variance portfolio theory.

This chapter will examine the implications of the results in relation to a model which calls for the mechanical management of an "active," high-turnover portfolio devoted to capitalize on the detection of incorrectly priced securities. It will be shown that ex ante, Markowitz prescriptive models cannot dominate models suggested by the structure of the opportunity locus implied by the results of the Monte Carlo experiments. This presentation borrows considerably from Treynor and Black,[1] but it is somewhat unrelated. The only assumption maintained from their formulation is that mutual funds may decide what proportion of total wealth W_a is to be invested in the "active" portfolio, independently of the decision of how much wealth to invest on a well diversified "market" portfolio.

The final sections of the chapter are devoted to the analysis of the empirical implications of the structure implied by the results of the Monte Carlo experiments. Structure, as used in this paper, must be distinguished from the usual econometric meaning of the word which refers to systems of equations. Structure here is reminiscent of what biologists call "strong equilibrium" and relates to the high density which can be observed in the mean variance opportunity locus in the neighborhood of the conditional expectation of variance for fixed levels of expectation.

The philosophical points of view sponsored in these latter sections have been strongly influenced by Basmann,[2,3] although the conceptual transfer applies to circumstances not examined by him. These sections will examine old and new empirical research intended for the validation of the implications of the two parameter mean-variance model. It will be demonstrated that the structural characteristics of the opportunity locus can provide equally plausible explanation for the empirical observations; and furthermore, that they explain observations which are not predicted by the two parameter model.

It is suggested that the real contribution of this research may be in these demonstrations. The real test of the mean variance model is in confrontation to empirical evidence; and so far, the mean variance paradigm has shown admirable robustness and withstood the battery of tests, as it was indicated in Chapter I. This chapter attempts to provide an explanation for the robustness of the paradigm which is independent of any of its assumptions. The arguments presented in this chapter give an indication that the robustness of the mean-variance model may be the result of the structure of the opportunity locus, and thus not at all an attribute of the model.

Implications Concerning the Utilitarian Value of Prescriptive Models

This analysis will be confined to the management of an active portfolio which is composed of incorrectly priced securities. It will be assumed, perhaps unrealistically, that long and short selling is not allowed.

In this conceptualization, the return of interest is the "differential return," which is not explained by fluctuations of the market. The return of a security can thus be defined as

(1) $$R_i = R_{im} + R_{ie}$$

where R_{im} is the return explained by the market fluctuations and R_{ie} is the differential return, which results from incorrect prices of the security.

It can be assumed that R_{im} and R_{ie} are independently distributed random variables since the factors affecting market fluctuations do not have a bearing on the fact that a security is underpriced or overpriced. It may also be assumed that in general, the factors affecting the price of any one security to be above or below its equilibrium value are independent of those affecting other securities.

It follows that

(2) $$\text{Cov}(R_{im}, R_{ie}) = 0 \qquad \text{for } i\text{-}1,\ldots\ldots\ldots,n$$

and that

(3) $$\text{Cov}(R_{ie}, R_{je}) = 0 \qquad \text{for } i \neq j.$$

Given the assumptions of the mean-variance paradigm, the universe of n securities can be characterized by the expectation u_e of the multivariate distribution of differential returns, and by their covariance matrix C_e. This is all the information required for the selection of mean-variance efficient active portfolios. These portfolios will again be defined by the vector x of proportions of W_a invested in each security for the holding period.

Assuming that the investor is a mean-variance utility maximizer, he will attempt to choose portfolios that minimize ex ante variance for a given level of expectation. Thus he will select his portfolio in such a way that

(4) $$V_e = x'C_e x \text{ is a minimum}$$

subject to $$x'u_e = E_j$$

and $$\sum_{i=1}^{n} x_i = 1$$

$$x \geq 0$$

Unfortunately the true values of u_e and C_e are unknown, and only the estimates $\hat{u}_e$ and $\hat{C}_e$ are available. Under these circumstances it is worthwhile to examine what the best course is for a mean-variance utility maximizer. Assuming that the investor is more confident of his estimate of u_e than he is of his estimate of C_e, which is not an unreasonable assumption since the estimation of the covariance matrix is a non-trivial subjective exercise, the portfolio problem may be reformulated by asserting that the investor would like to minimize portfolio variance (unknown or subject to great estimation errors), for a fixed level of estimated return which he is fairly confident of achieving. This investor will then select x hoping that

(5) $$V_e = x'C_e x \text{ is a minimum (but it is unknown)}$$

$$\text{subject to } \quad x'\hat{u}_e = E_j$$

$$\text{and} \quad \sum_{i=1}^{n} x_i = 1$$

$$x \geq 0$$

The portfolio opportunity set is then defined as $\{x | E_j = x'\hat{u}_e\}$.

Three selection modes are proposed for the solution of problem (5); the first, which will be labeled "the Markowitz prescriptive model," dictates that the investor must select x such that $x'\hat{C}_e x$ is a minimum in the portfolio opportunity set defined above. The second method, described as the "constrained random selection model," will select portfolios randomly in the portfolio opportunity set. Finally, a different prescriptive model called the "constrained average selection model" will dictate that the investor must select the portfolio

(6) $$x = (1/k) \sum_{i=1}^{k} y_i$$

where y_i's are the extreme points of the convex set $\{x | E_j = x'\hat{u}_e\}$.

Given the assumed characteristics of C_e, an investment universe of fifteen or more securities, and that the investor does not choose E_j at the extremes of the range implied by $\hat{u}_e$, the first two alternatives will yield portfolios with a very high probability of being close to the benchmark V_e. The third alternative will yield a portfolio which is close to the benchmark V_e with probability one. Clearly, if the investor is a risk averter there is no reason why he should not prefer the constrained average selection mode proposed in (6).

The previous presentation is very idealized and it carries very strong normative overtones which may be distasteful. It nonetheless shows that the mean variance normative model is very vulnerable in its own backyard. If mean-variance analysis is an acceptable normative criterion for the selection of portfolios, then it does not follow that a mean variance prescriptive model based on estimates is the best available prescriptive tool under the assumptions of the mean-variance paradigm.

From the practical point of view there are some real advantages derived from utilizing selection techniques based entirely on expectations. It was observed in Chapter I that mutual funds fail to convert detection of wrongly priced securities into differential return because the rate at which management absorbs information is often slower than the rate at which the market will absorb it. A successful active portfolio must be then constantly reevaluated as a result of information received from the field. It was suggested in Chapter I that this is probably a task that mechanical models for portfolio management could perform successfully if they are supported by an efficient information system whose one priority is to convert information into action at as rapid a rate as possible. In this context, a mechanical selection model such as the one presented in (6) above would obviate painful and time-consuming estimates of C_e and concentrate on the relevant task which is the detection of differential returns and their conversion into profit.

The latter remarks are very speculative and are contingent on the usefulness of mean-variance prescriptive models for the same purpose. If mean-variance prescriptive models are useful, then the selection mode described in (6) above is just as useful from the theoretical point of view of ex ante minimization of expected utility; and more useful from a practical point of view since it requires much less information.

In the next sections two models will be presented to analyze the empirical implications of the results of the previous chapters.

Implications of the Underlying Structure of the Opportunity Locus

It was seen in Chapter IV that for investment universes of moderate to large size, the mean of the conditional distribution of variances for a fixed level of expectation is in the immediate neighborhood of the Markowitz frontier, provided that E_j is not at the extremes of the feasible range and that the correlations are moderate. For the extremes of the feasible range of E_j it has been shown that the conditional expectation will be in the neighborhood of the maximum variance attainable for those levels. These characteristics are labeled "structural characteristics" of the mean-variance opportunity locus.

The structural characteristics will be used in the following sections to provide an alternative explanation to the empirical findings of Farrar[4] and Sharpe.[5] The purpose of this analysis is to provide a background for the discussion of the methodological and empirical implications of the structural characteristics. Since validation of the implications of the two parameter model has been greatly improved both in its formulation and its theoretical content, the last sections of the chapter are devoted to the analysis of the implications of the structural characteristics for the empirical results presented in one of the most complete and up-to-date tests of the model.[6]

Analysis of Farrar's Results. As it was stated in Chapter I, Farrar made the claim that mutual funds are mean-variance isomorphic since the mappings of their portfolios, on the historically derived standard deviation-average return plane, are consistently above but "close to" the efficient frontier.[7]

An alternative explanation for these observations may be proposed here. It will be assumed for the sake of argument that mutual funds select their portfolios as if they are return achievers, selecting random portfolios for a fixed level of expectation E_j.

The problem, which is expected to be resolved by the empirical evidence, is to choose the "correct alternative" given the choice of

H_1 : "Mutual Funds select portfolios as if they were mean-variance utility maximizers"

H_2 : "Mutual Funds select portfolios as if they were return achievers, selecting random portfolios for a fixed level E_j"

The design of the proper test involves the analysis of the observable characteristics that would result if either H_1 or H_2 are maintained. To this effect, define $\emptyset$ as the subset of feasible values of E_j such that E_j is not at either extreme, and $\emptyset_c$ as its complement.

Were all of Farrar's mutual funds in the range of expectations contained in $\emptyset$, a full-fledged identification problem would arise since for this range both H_1 and H_2 predict that portfolios are in the immediate neighborhood of the efficient frontier. The problem would then be outside the scope of statistical hypothesis testing.

A proper test must be conducted in the range of expectations for which the two hypotheses predict observations which are at odds with one another. For observations contained in $\emptyset_c$, the hypothesis H_1 predicts "Mutual fund portfolios are closer to the efficient frontier than to the locus of maximum variances;" H_2 predicts that "Mutual funds are closer to the maximum variance locus than to the efficient frontier." This is a proper test which may be achieved through traditional statistical testing procedures if the distribution of random variances is known.

Farrar's data do not warrant his conclusion that mutual funds are mean-variance isomorphic. Casual examination of his Chart III[8] shows that as E_j increases so does the relative distance between the frontier and the mapping of the fund portfolio; with the highest return portfolios literally touching the locus of maximum variances.

Sharpe's early study of Mutual funds. In his early attempt to validate his own theory of capital asset prices, Sharpe[9] mapped the portfolios of thirty-four open-end mutual funds on the historically estimated standard deviation-average return plane. Cross-sectionally, he regressed standard deviation on average return and average return on standard deviation and concluded that although the linear models provided adequate representation of the data (correlations of the order of .83), the relationship between average return and standard deviation does not appear to be linear. Moreover, the quadratic relations provided a slightly better fit. Observation of the data,[10] give an indication of why the quadratic relationships may be a better fit. For intermediate and low levels of E_j, the funds pack close together, while for large returns it is evident that there is an upward shift in the standard deviation.

Interestingly, the results of Chapter IV indicate that this might be precisely the effect implied by the structure of the mean variance opportunity locus. Thus, if H_1 and H_2 are defined as above, and it is added to H_1 that investors may borrow and lend at the given rate of interest, it is possible to predict the consequences $\emptyset_1$ and $\emptyset_2$ that would be expected if the hypotheses are maintained. If the relationship between standard deviation and return is defined by

$$\sigma_j = A_1 + A_2 E_j + A_3 E_j^2 + \tilde{e} \tag{7}$$

it follows that

$$\emptyset_1 = \begin{cases} A_1 = -A_2 r_f \\ A_2 > 0 \\ A_3 = 0 \end{cases} \tag{8}$$

and

$$
(9) \qquad \theta_2 = \begin{cases} A_1 \lessgtr -A_2 r_f \\ A_2 > 0 \\ A_3 > 0 \end{cases}
$$

where r_f is the given risk-free rate of interest.

The only conclusion used by Sharpe is that average return and standard deviation are positively related ($A_2 > 0$), and it is precisely on this observable result in which H_1 and H_2 are empirically indistinguishable. That A_3 may be greater than zero has not been documented by Sharpe, but it can be inferred from the upward swing of the data, and by the fact that the quadratic fit was slightly better than the linear fit.

The previous two sections have presented suggestive indications that the structure of the opportunity locus explains the empirical results better than the two parameter model. Unfortunately the empirical studies here discussed are dated in the sense that they do not incorporate the assumptions and qualifications that have been added to the model in the last decade. For this reason, a different model has been constructed in the next section for the comparison of its predictions with the recent predictions of the two parameter model provided by Fama and McBeth.[11]

<u>Formulation of an alternative structural model</u>. To establish a correspondence between the implications of the findings of Chapters IV and V with the modern formulation of the two parameter model, it is of interest to formulate tentatively a structural relationship between expected return and var-

iability of the form

$$(10) \qquad \tilde{E}_p = a_1 + a_2\sigma_p + a_3\sigma_p^2 + \ldots\ldots + a_{n+1}\sigma_p^n + \tilde{\xi}$$

where $\tilde{E}_p$ is a random variable denoting the expectation of portfolios with standard deviation σ_p.

Unfortunately the conditional distributions of $\tilde{E}_p$ given a fixed standard deviation have not been studied, but an analysis of figures IV-8, IV-9 and IV-10 provided a strong indication that they may also be highly concentrated in the neighborhood of the efficient frontier. Moreover, since a high concentration of conditional variances may be expected in the neighborhood of the conditional expected variance for a level of expectation E_j, then the descriptive regression of E_j on $E(\sigma|E_j)$ will provide an indication of the shape of the regression of E_p on σ_p. The regression of E_j on $E(\sigma|E_j)$ is expected to be parallel to the efficient frontier, as drawn in figure IV-8, for the wide range of intermediate expected returns, and turn downwards at the extremes. Since the frontier is positively sloped and rising at a decreasing rate, then the structural relationship proposed in (10) may be approximated by

$$(11) \qquad \tilde{E}_p = \hat{a}_1 + \hat{a}_2\sigma_p + \hat{a}_3\sigma_p^2 + \tilde{\xi}$$

where $\hat{a}_1$, $\hat{a}_2>0$ and $\hat{a}_3<0$ are least-square estimates.

Assume now that a time series relationship is formulated between the return of a well diversified portfolio p and a market factor I which may be represented by the "market portfolio." Then

$$(12) \qquad \tilde{R}_{pt} = A_p + B_p\tilde{I}_t + \tilde{e}_{pt}$$

where $\tilde{R}_{pt}$ is the return of portfolio p, A_p and B_p are constant through time and I_t and e_{pt} are stationary and independent. The ex ante variance of the portfolio may be defined as

$$(13) \qquad \sigma_p^2 = B_p^2\sigma_I^2 + \sigma_{ep}^2$$

where B_p is the responsiveness of the returns to fluctuations in the market, σ_I is the standard deviation of the market factor, and σ_{pe} is the standard deviation of the residual return not explained by the market.

By substitution of the terms in (13), (11) can be rewritten as

$$(14) \qquad E_p = \hat{a}_1 + \hat{a}_2B_p\sigma_I(1+(\sigma_{ep}^2/B_p^2\sigma_I^2))^{1/2} + \hat{a}_3(B_p^2\sigma_I^2+\sigma_{ep}^2) + \xi$$

The term in the square root can be expanded into a Taylor series as

$$(15) \qquad (1+q)^{1/2} = 1 + \tfrac{1}{2}q - (1/2^3)q^2 + (1/2^4)q^3 + q_n$$

where q_n is the remainder with terms of order greater than q^3.

Depending on the size of the $q=(\sigma_{ep}/B_p\sigma_I)^2$, the remainder can be neglected. Black, Jensen and Scholes have shown that σ_{ep} is a small proportion of σ_p,[12] so that $q<1$, and thus q_n can be ignored. (14) can now be rewritten as

$$(16) \qquad E_p = \hat{a}_1 + (\hat{a}_2B_p\sigma_I/2^4)(q^3-q^2) + \hat{a}_2B_p\sigma_I + \hat{a}_3\sigma_I^2B_p^2$$
$$+ (\hat{a}_3+(\hat{a}_2/B_p\sigma_I))\sigma_{ep}^2 + \xi$$

Since a random variable can always be represented as the sum of its expectation plus a random disturbance, then

$$(17) \qquad \tilde{R}_p = E_p + \tilde{\eta}_p$$

and thus, adding a t to the subscript to represent the next time period

$$(18) \qquad \tilde{R}_{pt} = \hat{a}_1 + (\hat{a}_2 B_p \sigma_I / 2^4)(q^3 - q^2) + \hat{a}_2 B_p \sigma_I + \hat{a}_3 \sigma_I^2 B_p^2 + (\hat{a}_3 + (\hat{a}_2 / 2B_p \sigma_I))\sigma_{ep}^2 + \xi + \tilde{\eta}_{p,t}$$

If $\hat{\beta}_{p,t-1}$ is an unbiased estimate of $B_{p,t-1}$, it must also be an unbiased estimate of $B_{p,t}$ since B_p is assumed constant. Levy[13] states that for large portfolios, $\hat{\beta}_{p,t}$ is stationary. It follows that a predictive model can be formulated, provided that σ_{ep}^2 is also stationary, and that s_{ep}^2 is its estimate. This model is

$$(19) \qquad \tilde{R}_{pt} = \hat{a}_1 + (\hat{a}_2 B_p \sigma_I / 2^4)(q^3 - q^2) + (\hat{a}_2 \sigma_I)\beta_{p,t-1} + (\hat{a}_3 \sigma_I^2) \quad \beta_{p,t-1}^2 + (\hat{a}_3 + (\hat{a}_2 / 2B_p \sigma_I)) s_{ep,t-1}^2 + \tilde{\delta}_{p,t} + \xi$$

The structural model (20) is now directly comparable with Fama and McBeth's proposed test of the positive implications of the mean-variance model.[14] They propose the following model for the test[15]

$$(20) \qquad R_{p,t} = \hat{\gamma}_0 + \hat{\gamma}_1 \beta_{p,t-1} + \hat{\gamma}_2 \overline{\beta^2}_{p,t-1} + \hat{\gamma}_3 \overline{s^2}_{p,t-1} + \phi_{p,t}$$

where

$$(21) \qquad \overline{\hat{\beta}^2_{p,t-1}} = \hat{\beta}^2_{p,t-1} - 2 \sum_{i=1}^{n} \sum_{j\neq 1}^{n} (x_i x_j)_p \hat{\beta}_i \hat{\beta}_j$$

and x_i is the proportion of portfolio p invested in the i-th security, $\hat{\beta}_i$ is the regression estimate of the responsiveness of the i-th security to fluctuations in the market. Similarly,

$$(22) \qquad \overline{s^2_{p,t-1}} = s^2_{ep,t-1} - 2 \sum_{i=1}^{n} \sum_{j\neq 1}^{n} (x_i x_j)_p \hat{\mathrm{Cov}}(e_i, e_j)$$

where e_i is the return of the i-th security not explained by market fluctuations. These discrepancies require a reformulation of the structural model in (19) which must be rewritten as

$$(23) \qquad R_{p,t} = \theta_1 + (\hat{a}_2 \sigma_I)\hat{\beta}_{p,t-1} + (\hat{a}_3 \sigma_I^2)\overline{\hat{\beta}^2_{p,t-1}} + (\hat{a}_3 + (\hat{a}_2/2B_p\sigma_I))\overline{s^2_{p,t-1}} + \delta_{p,t}$$

and

$$(24) \qquad \theta_1 = \hat{a}_1 + (\hat{a}_2 B_p \sigma_I (q^3 - q^2)) + (\hat{a}_2 \sigma_I)(2 \sum_{i=1}^{n} \sum_{j\neq i}^{n} (x_i x_j)_p B_i B_j) + (\hat{a}_3 + \hat{a}_2/2B_p\sigma_I)(2 \sum_{i=1}^{n} \sum_{j\neq i}^{n} (x_i x_j)_p \mathrm{Cov}(e_i, e_j)) + \xi$$

B_i is the responsiveness of individual securities to market fluctuations. Comparison of (20) and (23) allows for the prediction of consequences resulting from the structural model in (23). This set of consequences will be labeled $\emptyset_2$ as before, and the set of consequences proposed by Fama and McBeth will be labeled $\emptyset_1$. $\emptyset_1$ and $\emptyset_2$ are summarized below.

(25) $\hat{\gamma}_1 > 0$ Return is positively related to risk

$\hat{\gamma}_2 = 0$

$\hat{\gamma}_3 = 0$ No other measure of risk systematically affects returns

and from (20) and (23)

(26) $\hat{\gamma}_1 = (\hat{a}_2\sigma_I) > 0$

$\hat{\gamma}_2 = (\hat{a}_3 + (\hat{a}_2/2B_p\sigma_I)) \lesseqgtr 0$

$\hat{\gamma}_3 = (\hat{a}_3\sigma_I^2) < 0$

Judging from (25) and (26), the crucial test between the two models is the observation of $\hat{\gamma}_3$ which is the consequence for which the two models are at odds in their prediction of consequences. Weaker tests will result from the empirical observation of $\hat{\gamma}_2$, and no discrimination is possible by merely observing the sign of $\hat{\gamma}_1$. If it is observed nonetheless that $\hat{\gamma}_1$ is correlated to the market variance, this will constitute additional support for the structural model.

From the analysis of the results of Fama and McBeth as they relate to $\emptyset_2$, it can be concluded that the mean-variance model does not fare very well. They state that "there are some variables in addition to $\hat{\beta}_p$ that systematically affect period by period returns. Some of the omitted variables are apparently related to $[\overline{\hat{\beta}_p^2}]$ and to the [average standard deviation of residual returns of the securities in the portfolio]. But the latter are almost surely proxies since there is <u>no economic rationale</u> for their presence in ....the risk-return model."[16]

Perhaps not an economic rationale, but a structural explanation may be provided. It is very suggestive that for their Panel D which estimated the coefficients in (20), $\hat{\gamma}_3$ when significant (t<-2) was less than zero, and that out of nine non-overlapping periods, the averages of $\hat{\gamma}_3$ were negative for seven periods.[17] Furthermore, they state also that declines in the averages of $\hat{\gamma}_1$ "are matched by quite a noticeable downward shift"[18] in the market variance.

Moreover, it can be observed from their data that $\hat{\gamma}_2$ is not consistently equal to zero and that it is fairly well correlated with $\hat{\gamma}_3$.[19] The latter is not surprising since $\overline{s^2_{ep}}$ increases with B_p[20] and therefore the responsiveness of the model to changes in this variable must be reflected more by $\hat{a}_3$ than by $(\hat{a}_2/2B_p\sigma_I)$, causing $\hat{\gamma}_2$ and $\hat{\gamma}_3$ to be correlated.

In summary, the only definite conclusion reached by Fama and McBeth which is not conflicting with their own data is that $\hat{\gamma}_1$ is greater than zero. Unfortunately this is also predicted by the structural model. It is clear from the previous discussions that the absence of "an economic rationale" does not justify pushing conflicting evidence under the rug. Fama and McBeth's conclusion that they "cannot reject the hypothesis that no other measure of risk in addition to [$\hat{\beta}_p$] systematically affects average returns"[21] is completely unwarranted except for semantic interpretations of the word "systematic". An alternative explanation has been provided here which predicted the discrepancies between the theory and the observations. This explanation is entirely based on plausible structural characteristics of the data utilized to validate the implications of the mean variance model. Future tests of the two parameter model should demonstrate that the relationships observed are not merely due to the structure of the data which is empty of any behavioral or economical content. A fair guess resulting from the results discussed here is that if the structural relationships are eliminated, then the empirical results may show that there is no substance to the implications of the mean variance model. This is left as an open question for future research.

Summary and Conclusions

It has been argued in this chapter that there is a strong indication that mean-variance portfolio theory can contribute very little in its positive and normative implications that is not imbedded in the nature of the mean-variance opportunity locus.

Footnotes

1. J.L. Treynor and F. Black, "How to Use Security Analysis to Improve Portfolio Selection," _Journal of Business_, 46 (January 1973), 66-68.

2. R.L. Basmann, "On the Empirical Testability of "Explicit Causal Chains," Against the Class of "Interdependent" Models," Institute Paper No. 104, _Institute for Research In the Behavioral, Economic and Management Sciences_, Purdue University, 1965.

3. ____________, "On the Causal Interpretations of Non-Triangular Systems of Economic Relations," _Econometrica_, 31 (1963), 439-448.

4. D.E. Farrar, _The Investment Decision Under Uncertainty_, Chicago (Markham), 1967.

5. W.F. Sharpe, "Risk Aversion in the Stock Market: Some Empirical Evidence," _Journal of Finance_, 20 (May 1965), 416-422.

6. E.F. Fama and J.D. McBeth, "Risk Return and Equilibrium: Empirical Tests," _Journal of Political Economy_, 81 (May/June 1973), 607-636.

7. Farrar, _op_. _cit_., p. 74.

8. _Ibid_., p. 75.

9. Sharpe, _op_. _cit_.

10. W.F. Sharpe, _Portfolio Theory and Capital Markets_, New York (McGraw Hill), 1970, 165.

11. Fama and McBeth, _op_. _cit_.

12. F. Black, M. Jensen and M. Scholes, "The Capital Asset Pricing Model: Some Empirical Tests," in M. Jensen (ed.), _Studies in the Theory of Capital Markets_, New York (Praeger), 1972, 89.

13. R.A. Levy, "On the Short Term Stationarity of Beta Coefficients," _Financial Analyst Journal_, 27 (November-December 1971), 55-62.

14. Fama and McBeth, _op_. _cit_., p. 632.

15. There is a slight discrepancy in that I have used the weighted average of the residual variances ($\bar{s}^2_{p,t-1}$) while Fama and McBeth used the weighted average of the standard deviations. However, the responsiveness of $R_{p,t}$ to changes in the average residual standard deviation should be of the same nature as the responsiveness of $R_{p,t}$ to changes in average residual variance since they are almost perfectly correlated.

16. Fama and McBeth, op. cit., p. 633.

17. Ibid., p. 623.

18. Ibid., p. 628.

19. Ibid., p. 623.

20. Ibid., p. 629.

21. Ibid., p. 633.

C O N C L U S I O N

The results of ten Monte Carlo experiments corresponding to investment universes of six, ten and fifteen investment prospects indicate that the conditional distribution of variances corresponding to randomly selected portfolios which are expected to yield ex ante a fixed return E_j have the following properties:

(1) Randomly selected portfolios will map on the mean-variance plane in the neighborhood of the Markowitz efficient frontier with a very high probability, provided that the investment universe is large $(n>15)$; that E_j is not at the extremes of its feasible range and that the correlations between pairs of investment prospects is not near one.

(2) For portfolios expected to yield ex ante returns E_j at the extremes of the feasible range of expectations, there is a high probability that they will map towards the center of the interval between the minimum and maximum variances, and even in the neighborhood of the maximum variance for that level of expectation.

(3) When the correlations between returns of pairs of prospects are high, a great proportion of the randomly selected portfolios may be expected in the center of the interval between the maximum and minimum variances for the level of expectation E_j.

(4) For investment universes of moderate and large size, the dispersion of the conditional distribution about $V(E(x|E_j))$ is very small and it may be expected to decrease as the size of the universe is increased.

Based on these conclusions and on the fact that the correlations between pairs of securities in the stock market is of the order of .5, it has been suggested that a quadratic polynomial fit between variability and average returns may be an adequate description of the mapping of a group of portfolios in the average return-variability plane. This description has been treated as a structural property of the mean variance opportunity locus, and it has been used to demonstrate that the results of two published attempts to validate two-parameter models of the asset pricing mechanism could be explained by the structure implicit on the data used for empirical validation.

This demonstration poses methodological problems which have apparently never been considered before in relation to the empirical validation of the capital asset pricing model. The validation of this model is contingent upon evidence that the observed relations are not a consequence of the structural properties of the mean-variance opportunity locus since they are merely properties of the data and have no theoretical content.

It has also been suggested that the mean-variance normative model may be of little value when it is based on estimates of the parameters of the multivariate distribution of returns. The characteristics of the opportunity locus indicate that under these conditions, two alternative selection modes which are based exclusively on information about the expectation of the multivariate distribution of returns will not be dominated ex ante by the mean-variance selection mode.

The implications of the results of this dissertation are apparently novel and it is hoped that they will stimulate further analysis of the properties of the mean variance locus. It has been demonstrated that they have a bearing on the usefulness of the mean-variance prescriptive model; and more importantly, on the tests of the validity of its implications. These should be sufficient reasons to motivate the analysis of the structural properties of the opportunity locus.

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APPENDIX A

Expectations and Covariances Corresponding to Experiments 1 through 10

Experiment No. 2

$$u = [1.55,\ 1.45,\ 1.35,\ 1.25,\ 1.15,\ 1.05]$$

$$C = \begin{bmatrix} .9613 & .0676 & .0516 & .0302 & .0175 & .0133 \\ .0676 & .6867 & .0387 & .0283 & .0163 & .0132 \\ .0516 & .0387 & .3591 & .0157 & .0103 & .0076 \\ .0302 & .0283 & .0157 & .221 & .0136 & .0090 \\ .0175 & .0163 & .0103 & .0136 & .0904 & .0044 \\ .0133 & .0132 & .0076 & .009 & .0099 & .0453 \end{bmatrix}$$

Experiment No. 2

$u = [1.55,\ 1.45,\ 1.35,\ 1.25,\ 1.15,\ 1.05]$

$$C = \begin{bmatrix} .9613 & .3380 & .2581 & .151 & .0878 & .0665 \\ .3380 & .6867 & .1939 & .1417 & .0817 & .0663 \\ .2581 & .1939 & .3591 & .0785 & .0514 & .0378 \\ .1510 & .1417 & .0785 & .2210 & .068 & .0452 \\ .0878 & .0817 & .0514 & .0680 & .0904 & .0221 \\ .0665 & .0663 & .0378 & .0452 & .0221 & .0453 \end{bmatrix}$$

Experiment No. 3

$u = [1.55, 1.45, 1.35, 1.25, 1.15, 1.05]$

$$C = \begin{bmatrix} .9613 & .6422 & .4904 & .2869 & .1668 & .1263 \\ .6422 & .6867 & .3684 & .2692 & .1552 & .1260 \\ .4904 & .3684 & .3591 & .1491 & .0976 & .0718 \\ .2869 & .2692 & .1491 & .221 & .1292 & .0859 \\ .1668 & .1552 & .0976 & .1292 & .0904 & .0420 \\ .1263 & .1260 & .0718 & .0859 & .0420 & .0453 \end{bmatrix}$$

Experiment No. 4

u = [1.55, 1.5, 1.45, 1.4, 1.35, 1.25, 1.20, 1.15, 1.10, 1.05]

$$C = \begin{bmatrix} .9613 & .0663 & .0675 & .407 & .0515 & .0301 & .0358 & .0175 & .0184 & .0132 \\ .0663 & .8137 & .0583 & .0221 & .0227 & .0308 & .0144 & .0177 & .0196 & .0227 \\ .0675 & .0583 & .6867 & .0602 & .0305 & .0216 & .0269 & .0142 & .0131 & .0104 \\ .0407 & .0221 & .0602 & .5512 & .0107 & .0116 & .0199 & .0214 & .0069 & .0142 \\ .0515 & .0227 & .0305 & .0107 & .3591 & .0211 & .0132 & .0118 & .013 & .0088 \\ .0301 & .0308 & .0216 & .0116 & .0211 & .221 & .0152 & .0066 & .0037 & .0041 \\ .0358 & .0144 & .0269 & .0199 & .0132 & .0152 & .1395 & .0095 & .0073 & .0073 \\ .0175 & .0177 & .0142 & .0214 & .0118 & .0066 & .0095 & .0904 & .0049 & .0048 \\ .0184 & .0196 & .0131 & .0069 & .0130 & .0037 & .0073 & .0049 & .0613 & .0049 \\ .0132 & .0227 & .0104 & .0142 & .0088 & .0041 & .0073 & .0048 & .0049 & .0453 \end{bmatrix}$$

Experiment No. 5

u = [1.55, 1. 0, 1.45, 1.4, 1.35, 1.25, 1.2, 1.15, 1.1, 1.05]

$$C = \begin{bmatrix} .9613 & .3316 & .3379 & .2038 & .2579 & .1509 & .1794 & .0877 & .0922 & .0664 \\ .3316 & .8137 & .2915 & .1105 & .1135 & .1541 & .0724 & .0888 & .0982 & .0720 \\ .3379 & .2915 & .6867 & .3014 & .1526 & .1084 & .1364 & .0710 & .0656 & .0522 \\ .2038 & .1105 & .3014 & .5512 & .0536 & .0584 & .0998 & .1073 & .0349 & .0713 \\ .2579 & .1135 & .1526 & .0536 & .3591 & .1056 & .066 & .0594 & .0625 & .044 \\ .1509 & .1541 & .1084 & .0584 & .1056 & .221 & .0763 & .0332 & .0186 & .0205 \\ .1794 & .0724 & .1364 & .0998 & .066 & .0763 & .1395 & .0477 & .0365 & .0365 \\ .0877 & .0888 & .0710 & .1073 & .0594 & .0332 & .0477 & .0904 & .0245 & .0243 \\ .0922 & .0982 & .0656 & .0349 & .0625 & .0186 & .0365 & .0245 & .0613 & .0245 \\ .0664 & .0720 & .0522 & .0713 & .044 & .0205 & .0365 & .0243 & .0245 & .0453 \end{bmatrix}$$

Experiment No. 6

u = [1.55, 1.5, 1.45, 1.4, 1.35, 1.25, 1.20, 1.15, 1.10, 1.05]

$$C = \begin{bmatrix}
.9613 & .7886 & .7398 & .6197 & .5335 & .4058 & .3384 & .2593 & .2185 & .1852 \\
.7886 & .8137 & .6649 & .5405 & .4566 & .3698 & .2876 & .2362 & .2016 & .1701 \\
.7398 & .6649 & .6867 & .5576 & .4297 & .3349 & .2822 & .219 & .1802 & .1557 \\
.6197 & .5405 & .5576 & .5512 & .3479 & .2762 & .2453 & .2025 & .1504 & .1419 \\
.5335 & .4566 & .4297 & .3479 & .3591 & .2475 & .1962 & .1567 & .134 & .1118 \\
.4058 & .3698 & .3349 & .2762 & .2475 & .221 & .1572 & .1188 & .0951 & .0831 \\
.3384 & .2876 & .2822 & .2453 & .1962 & .1572 & .1395 & .102 & .0828 & .0727 \\
.2593 & .2362 & .219 & .2025 & .1567 & .1188 & .102 & .0904 & .0656 & .0579 \\
.2185 & .1802 & .108 & .1504 & .134 & ..0951 & .0828 & .0656 & .0613 & .0482 \\
.1852 & .1701 & .1557 & .1419 & .1118 & .0831 & .0727 & .0579 & .0482 & .0453
\end{bmatrix}$$

Experiment No. 7

$$u = [1.55, 1.52, 1.48, 1.44, 1.41, 1.37, 1.33, 1.30, 1.27, 1.23, 1.19, 1.15, 1.11, 1.08, 1.05]$$

$$C = \begin{bmatrix}
.9613 & .0829 & .08 & .074 & .072 & .066 & .0581 & .0487 & .043 & .0402 & .034 & .0286 & .0263 & .0232 & .0205 \\
.0829 & .8934 & .0767 & .0753 & .071 & .0647 & .0554 & .0552 & .042 & .0389 & .0349 & .028 & .0244 & .0218 & .0185 \\
.08 & .0767 & .8137 & .0682 & .0649 & .0583 & .0474 & .0462 & .0363 & .0342 & .0313 & .0261 & .0242 & .0218 & .0175 \\
.074 & .0753 & .0682 & .6867 & .0614 & .0586 & .048 & .0439 & .0364 & .0327 & .0286 & .0246 & .0211 & .0196 & .0164 \\
.072 & .071 & .0649 & .0614 & .5512 & .0543 & .0457 & .0416 & .0325 & .0315 & .0273 & .0227 & .0204 & .0187 & .0158 \\
.066 & .0647 & .0583 & .0586 & .0543 & .9832 & .0418 & .0373 & .0325 & .0294 & .0248 & .0208 & .0191 & .0168 & .0153 \\
.0581 & .0554 & .0474 & .048 & .0457 & .0418 & .3559 & .0318 & .0274 & .0244 & .0222 & .0179 & .0154 & .0149 & .0125 \\
.0487 & .0552 & .0462 & .0439 & .0416 & .0373 & .0318 & .2987 & .0242 & .0215 & .0204 & .0159 & .0144 & .0131 & .0110 \\
.043 & .042 & .0363 & .0364 & .0325 & .0325 & .0274 & .0242 & .221 & .0182 & .0168 & .0134 & .0118 & .0106 & .0092 \\
.0402 & .0389 & .0342 & .0327 & .0315 & .0294 & .0244 & .0215 & .0182 & .1702 & .0144 & .0124 & .011 & .0098 & .0082 \\
.034 & .0349 & .0313 & .0286 & .0273 & .0248 & .0222 & .0204 & .0168 & .0144 & .1395 & .0113 & .0095 & .0089 & .0076 \\
.0286 & .028 & .0261 & .0246 & .0227 & .0208 & .0179 & .0159 & .0134 & .0124 & .0113 & .0904 & .0081 & .0072 & .0062 \\
.0263 & .0244 & .0242 & .0211 & .0204 & .0191 & .0154 & .0144 & .0118 & .011 & .0095 & .0081 & .0723 & .0065 & .0056 \\
.0232 & .0218 & .0218 & .0196 & .0187 & .0168 & .0149 & .0131 & .0106 & .0098 & .0089 & .0072 & .0065 & .0613 & .0049 \\
.0205 & .0185 & .0175 & .0164 & .0158 & .0153 & .0125 & .0110 & .0092 & .0082 & .0076 & .0062 & .0056 & .0049 & .0453
\end{bmatrix}$$

Experiment No. 8

u = [1.55, 1.52, 1.48, 1.44, 1.41, 1.37, 1.33, 1.30, 1.27, 1.23, 1.19, 1.15, 1.11, 1,08, 1.05]

	.9613	.3672	.3543	.3298	.3199	.2924	.2577	.2159	.1907	.1779	.1506	.1267	.1166	.103	.0912
	.3672	.8934	.3398	.3335	.3147	.2865	.2456	.2314	.1861	.1726	.1546	.1241	.1081	.0968	.0823
	.3543	.3398	.8137	.3019	.2876	.2582	.2098	.2049	.1608	.1514	.1387	.1159	.1071	.0969	.0775
	.3298	.3335	.3019	.6867	.2717	.2597	.2127	.1946	.1614	.1447	.1268	.1093	.0937	.0868	.0725
	.3199	.3147	.2876	.2717	.5512	.2404	.2024	.1845	.1442	.1396	.1208	.1005	.0906	.0829	.0699
	.2924	.2865	.2582	.2597	.2404	.4832	.185	.1654	.1439	.1303	.1102	.0922	.0848	.0744	.0679
C=	.2577	.2456	.2098	.2127	.2024	.185	.3559	.1408	.1212	.1081	.0986	.0794	.0684	.0660	.0554
	.2159	.2314	.2049	.1946	.1845	.1654	.1408	.2987	.1071	.0952	.0903	.0707	.0637	.0583	.0487
	.1907	.1861	.1608	.1614	.1442	.1439	.1212	.1071	.221	.0808	.0747	.0594	.0525	.0473	.0407
	.1779	.1726	.1514	.1447	.1396	.1303	.1081	.0952	.0808	.1702	.0638	.0551	.0488	.0433	.0365
	.1506	.1546	.1387	.1268	.1208	.1102	.0986	.0903	.0747	.0638	.1395	.0503	.0421	.0394	.0339
	.1267	.1241	.1159	.1093	.1005	.0922	.0794	.0707	.0594	.0551	.0503	.0904	.036	.032	.0278
	.1166	.1081	.1071	.0937	.0906	.0848	.0684	.0637	.0525	.0488	.0421	.036	.0723	.0291	.025
	.103	.0968	.0969	.0868	.0829	.0744	.0660	.0583	.0473	.0433	.0394	.032	.0291	.0613	.022
	.0912	.0823	.0775	.0725	.0699	.0679	.0554	.0487	.0407	.0365	.0339	.0278	.025	.022	.0453

Experiment No. 9

u = [1.55, 1.52, 1.48, 1.44, 1.41, 1.37, 1.33, 1.30, 1.27, 1.23, 1.19, 1.15, 1.11, 1.08, 1.05]

C=

.9613	.7345	.7086	.6597	.6399	.5849	.5154	.4318	.3815	.3559	.3013	.2535	.2333	.2061	.1824
.7345	.8934	.6797	.6672	.6295	.5732	.4914	.4629	.3722	.3453	.3094	.2482	.2162	.1937	.1646
.7086	.6797	.8137	.6039	.5752	.5164	.4197	.4099	.3216	.3028	.2775	.2319	.2143	.1938	.1551
.6597	.6672	.6039	.6867	.5434	.5194	.4255	.3892	.3229	.2895	.2536	.2186	.1874	.1736	.1451
.6399	.6295	.5752	.5434	.5512	.4808	.4048	.369	.2885	.2794	.2417	.201	.1813	.1658	.1398
.5849	.5732	.5164	.5194	.4808	.4832	.3701	.3309	.2879	.2607	.2205	.1844	.1697	.1488	.1359
.5154	.4914	.4197	.4255	.4048	.3701.	1359	.2816	.2425	.2163	.1973	.1589	.1369	.132	.1109
.4318	.4629	.4099	.3892	.369	.3309	.2816	.2986	.2143	.1904	.1806	.1415	.1274	.1167	.0974
.3815	.3722	.3216	.3229	.2885	.2879	.2425	.2143	.221	.1616	.1496	.1188	.1051	.0947	.0815
.3559	.3453	.3028	.2895	.2794	.2607	.2163	.1904	.1616	.1702	.1277	.1102	.0977	.0867	.073
.3013	.3094	.2775	.2536	.2417	.2205	.1973	.1806	.1496	.1277	.1395	.1006	.0842	.0789	.0679
.2535	.2482	.2319	.2186	.201	.1844	.1589	.1415	.1188	.1102	.1006	.0904	.0712	.064	.0556
.2333	.2162	.2143	.1874	.1813	.1697	.1369	.1274	.1051	.0977	.0842	.0712	.0723	.0582	.0501
.2061	.1937	.1938	.1736	.1658	.1488	.132	.1167	.0947	.0867	.0789	.064	.0582	.0613	.0441
.1824	.1646	.1551	.1451	.1398	.1359	.1109	.0974	.0815	.073	.0679	.0556	.0501	.0441	.0453

Experiment No. 10

u = [1.55, 1.52, 1.48, 1.44, 1.41, 1.37, 1.33, 1.30, 1.27, 1.23, 1.19, 1.15, 1.11, 1.08, 1.05]

C=

.9613	-.7886	.2771	-.2528	.014	-.024	-.036	.039	-.016	.0775	-.079	.028	.04	.062	.062
-.7886	.8934	-.5486	.437	.0322	-.0952	.0591	-.0942	-.047	-.055	-.053	-.034	-.035	.047	-.052
.2771	-.5486	.8137	-.571	-.1237	.2436	-.1779	.1479	.0644	-.0132	-.054	.069	-.074	-.0106	.024
-.2528	.437	-.571	.6867	-.1335	-.0597	.1513	-.1495	-.025	.0316	-.0334	-.013	-.009	.024	-.04
.014	.0322	-.1237	-.1335	.5512	-.398	.0471	-.008	-.049	-.031	.056	-.059	.024	-.022	.002
-.024	-.0952	.2436	-.0597	-.398	.4832	-.2334	.191	-.013	.026	-.0035	.0168	-.0255	.0135	-.004
-.036	.0591	-.1779	.1513	.0471	-.2339	.3559	-.2768	.0436	-.0264	-.0223	.0212	.01	.0045	.0041
.039	-.0942	.1479	-.1495	-.008	.191	-.2768	.2987	-.1205	.0907	.0224	-.014	-.019	.0008	.0039
-.016	-.047	.0644	-.025	-.049	-.013	.0436	-.1205	.221	-.154	-.011	.0137	-.0035	.0098	-.024
.0775	-.055	-.0132	.0316	-.031	.026	-.0264	.0907	-.154	.1702	-.0452	.0218	.0051	-.0108	.0187
-.079	-.053	-.054	-.0334	.056	-.0035	-.0223	.0224	-.011	-.0452	.1395	-.089	-.0012	-.0069	-.0022
.028	-.034	.069	-.013	-.059	.0168	.0212	-.014	.0137	.0218	-.089	.0904	-.0348	..0298	.0083
.04	-.035	-.074	-.009	.024	-.0255	.01	-.019	-.0035	.0051	-.0012	-.0348	.0723	-.0544	0
.062	.047	-.0106	.024	-.022	.0135	.0045	.0008	.0098	-.0108	-.0069	.0298	-.0544	.0613	-.021
.062	-.052	.024	-.04	.002	-.004	.0041	.0039	-.024	.0187	-.0022	.0083	0	-.021	.0453

APPENDIX B

SAMPLE CONDITIONAL DISTRIBUTIONS

CORRESPONDING TO EXPERIMENTS 1 THROUGH 10

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.1$
EXPERIMENT NO. 1

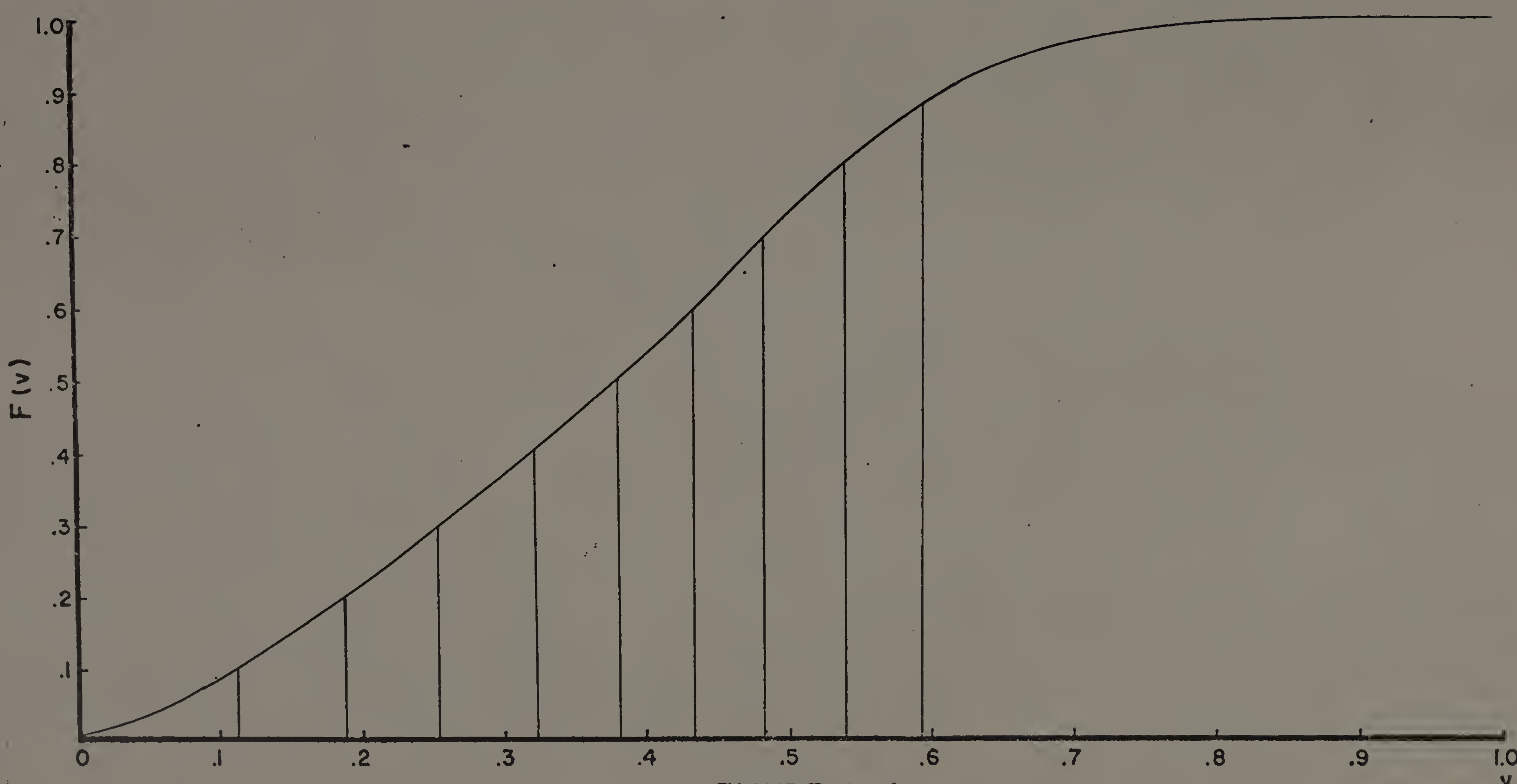

FIGURE B-1

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 1

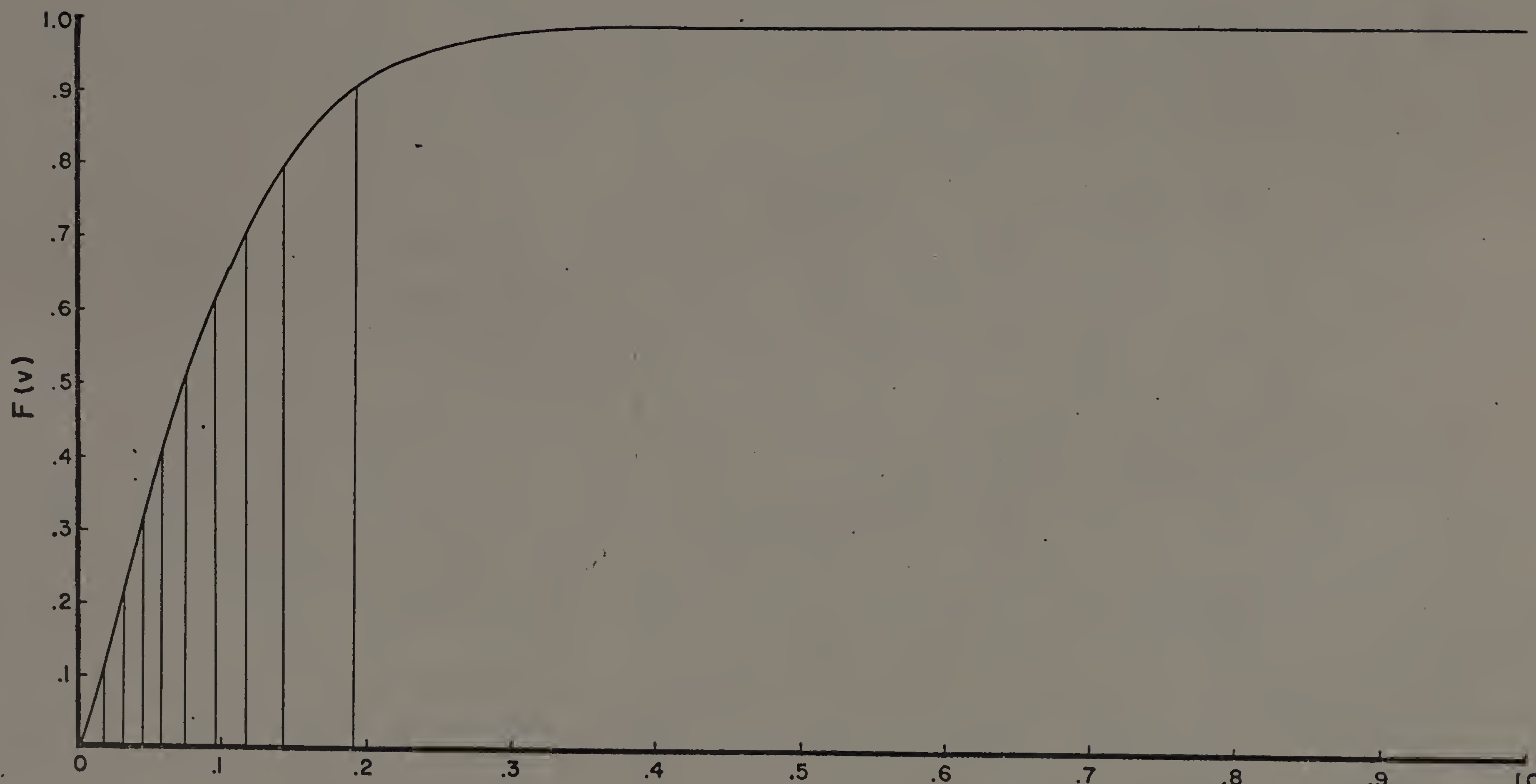

FIGURE B-2

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 1

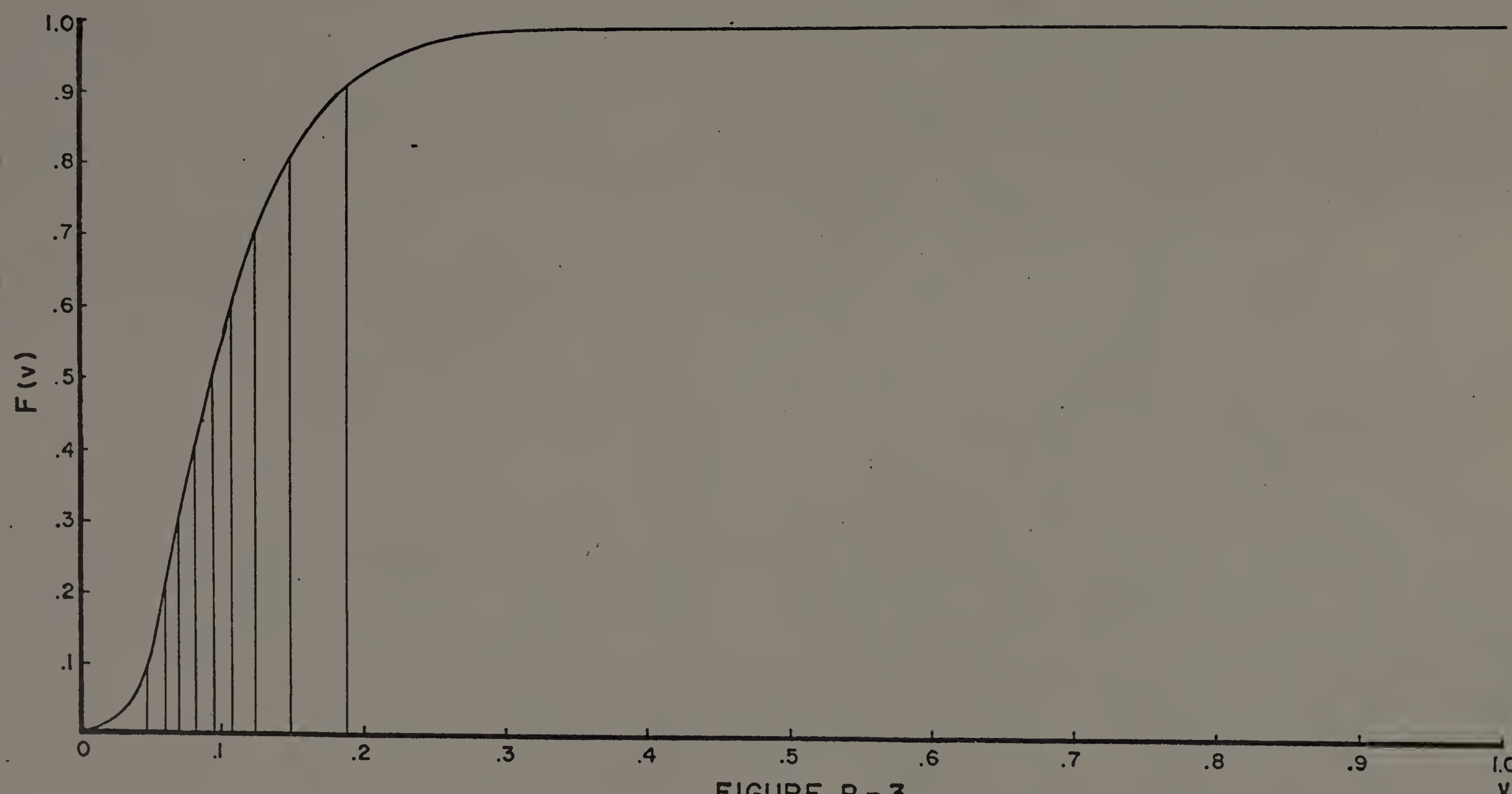

FIGURE B-3

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 1

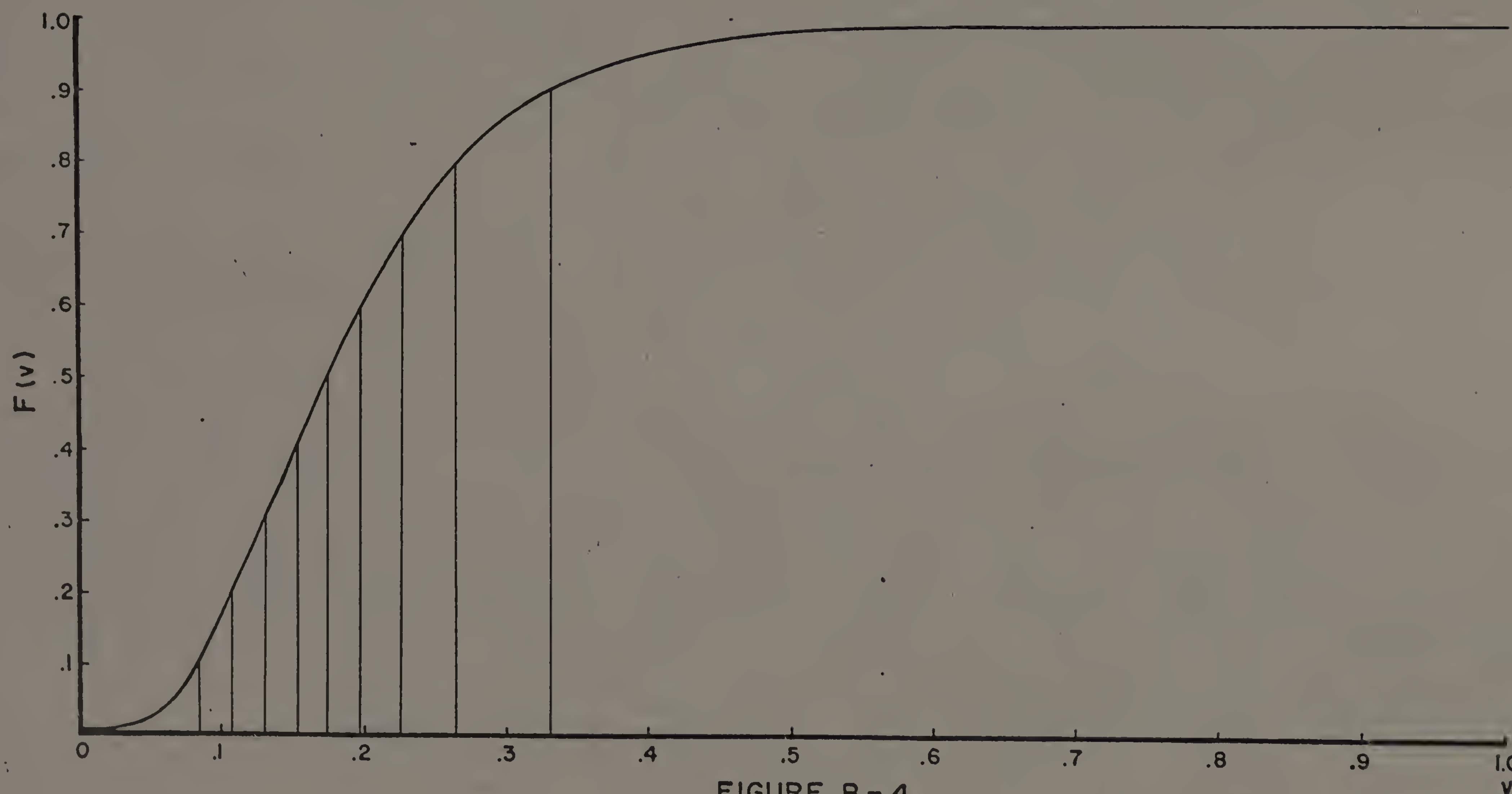

FIGURE B-4

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 1

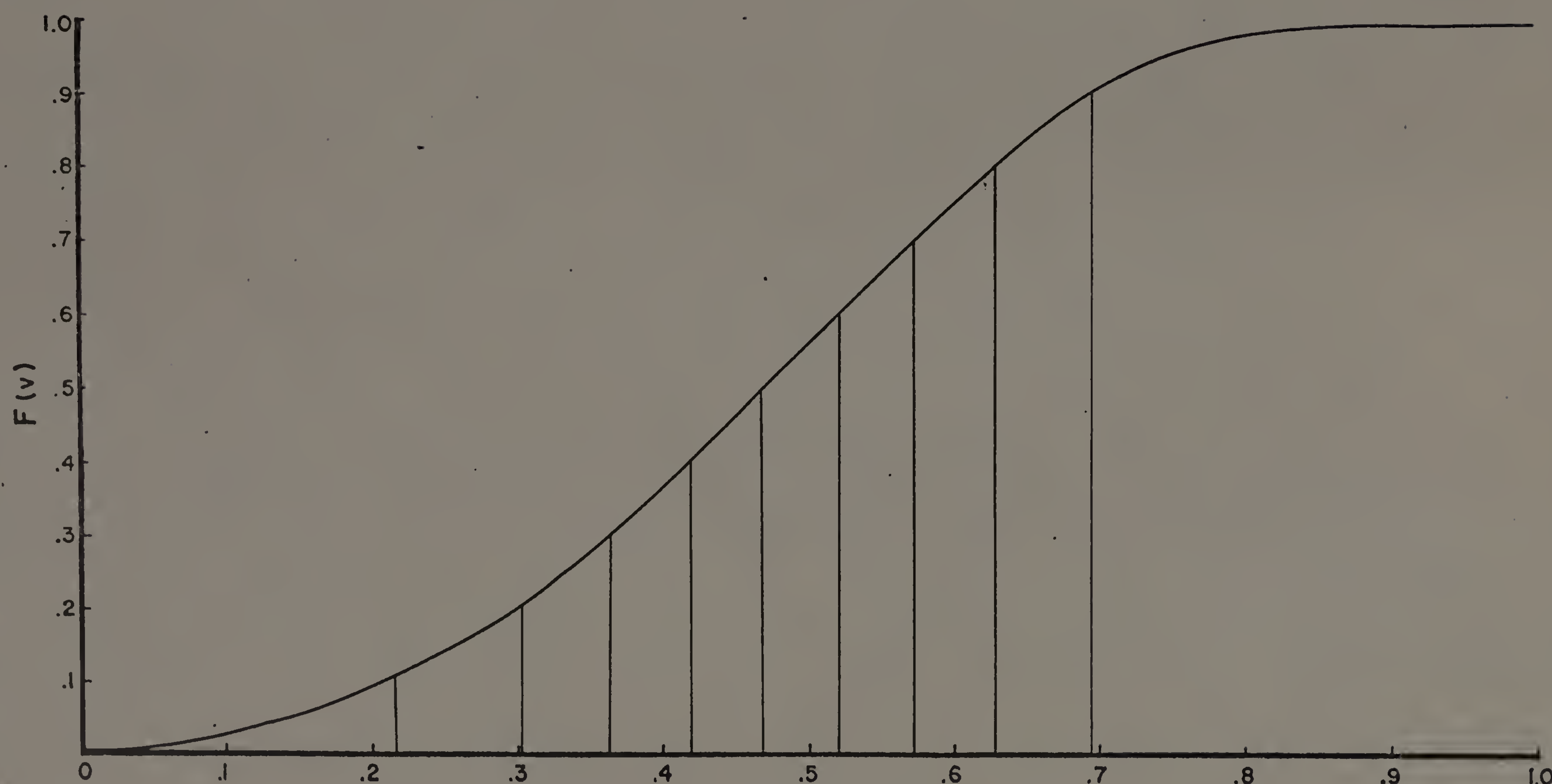

FIGURE B-5

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.13$
EXPERIMENT NO. 2

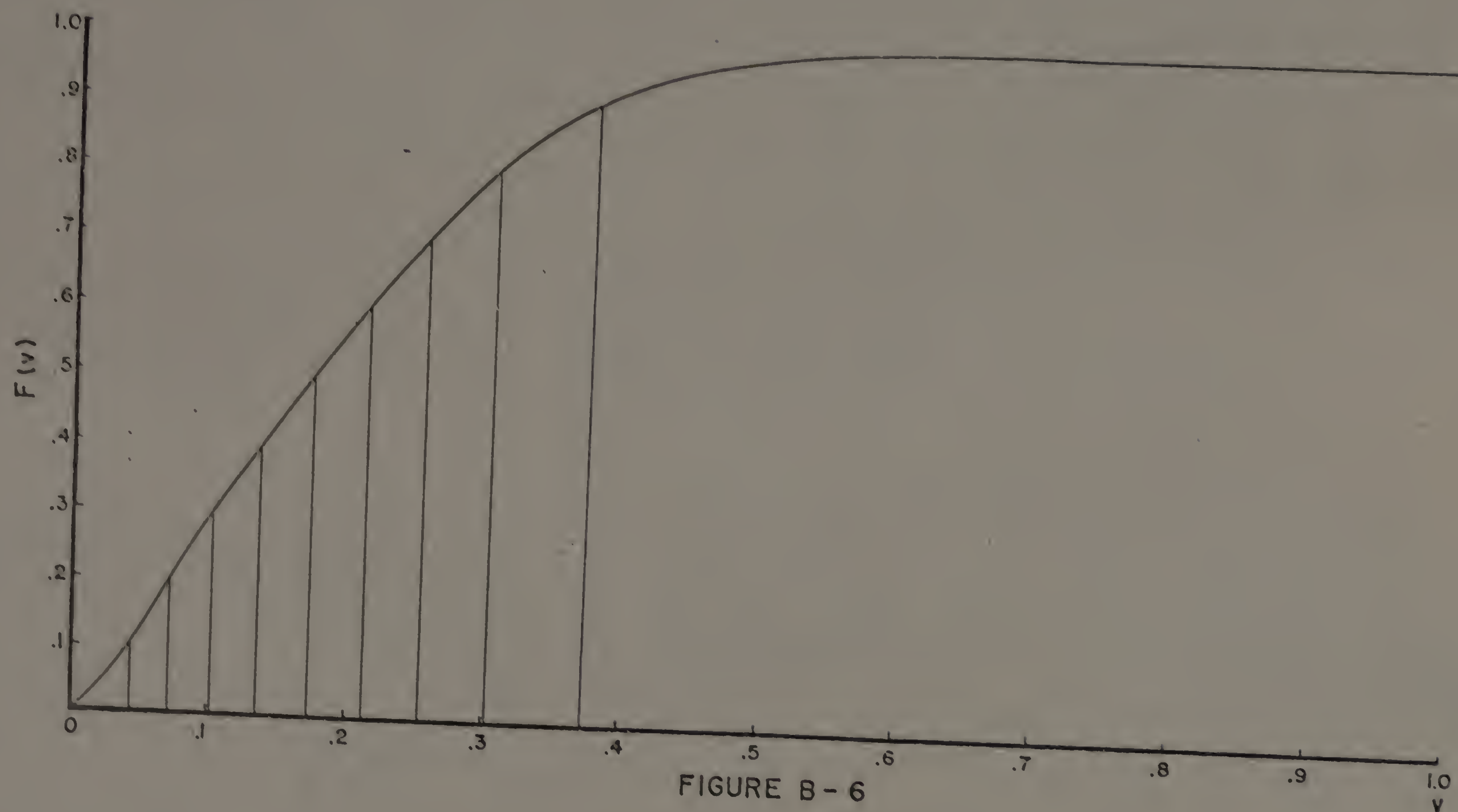

FIGURE B-6

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.20$
EXPERIMENT NO. 2

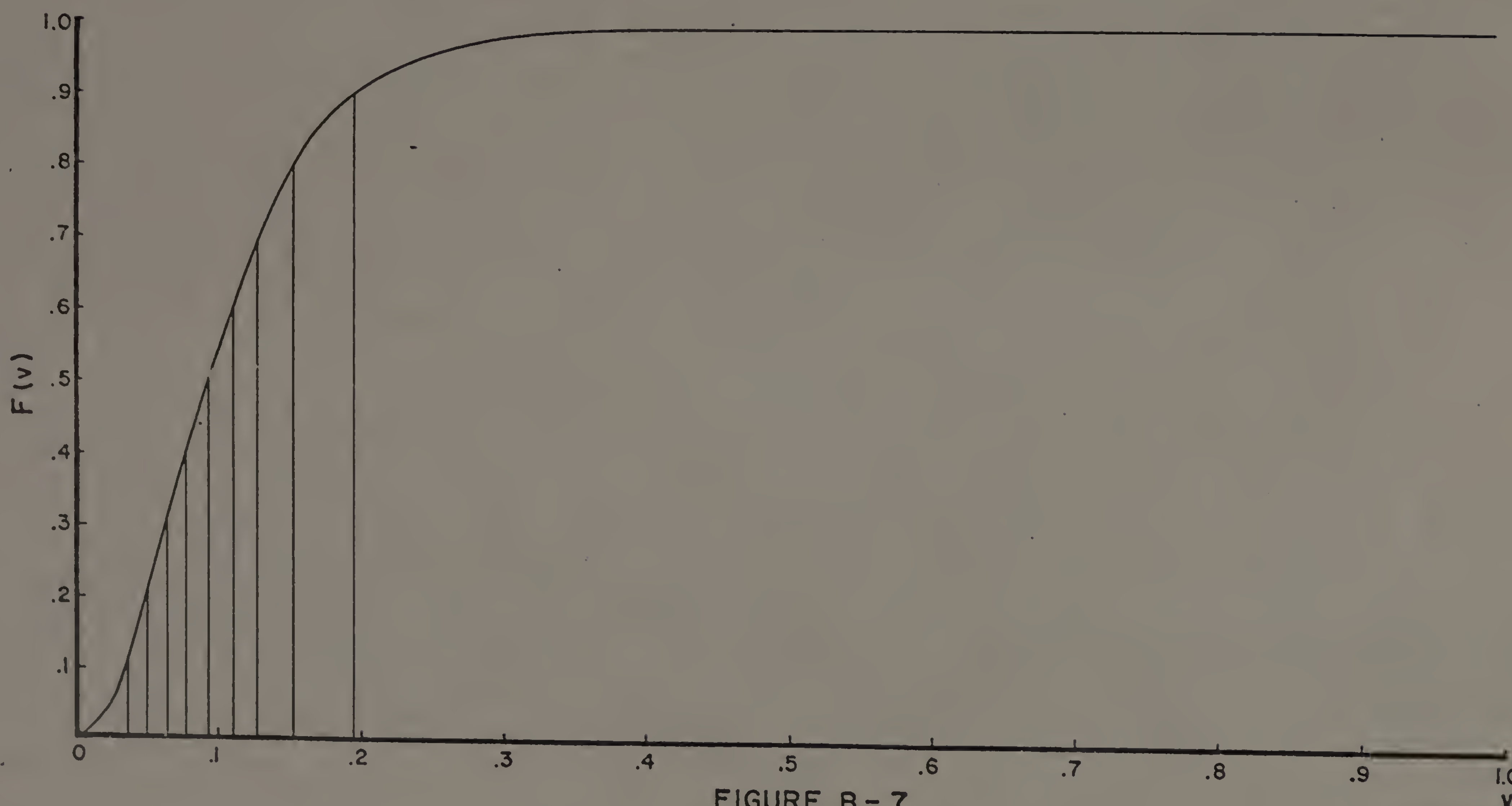

FIGURE B-7

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 2

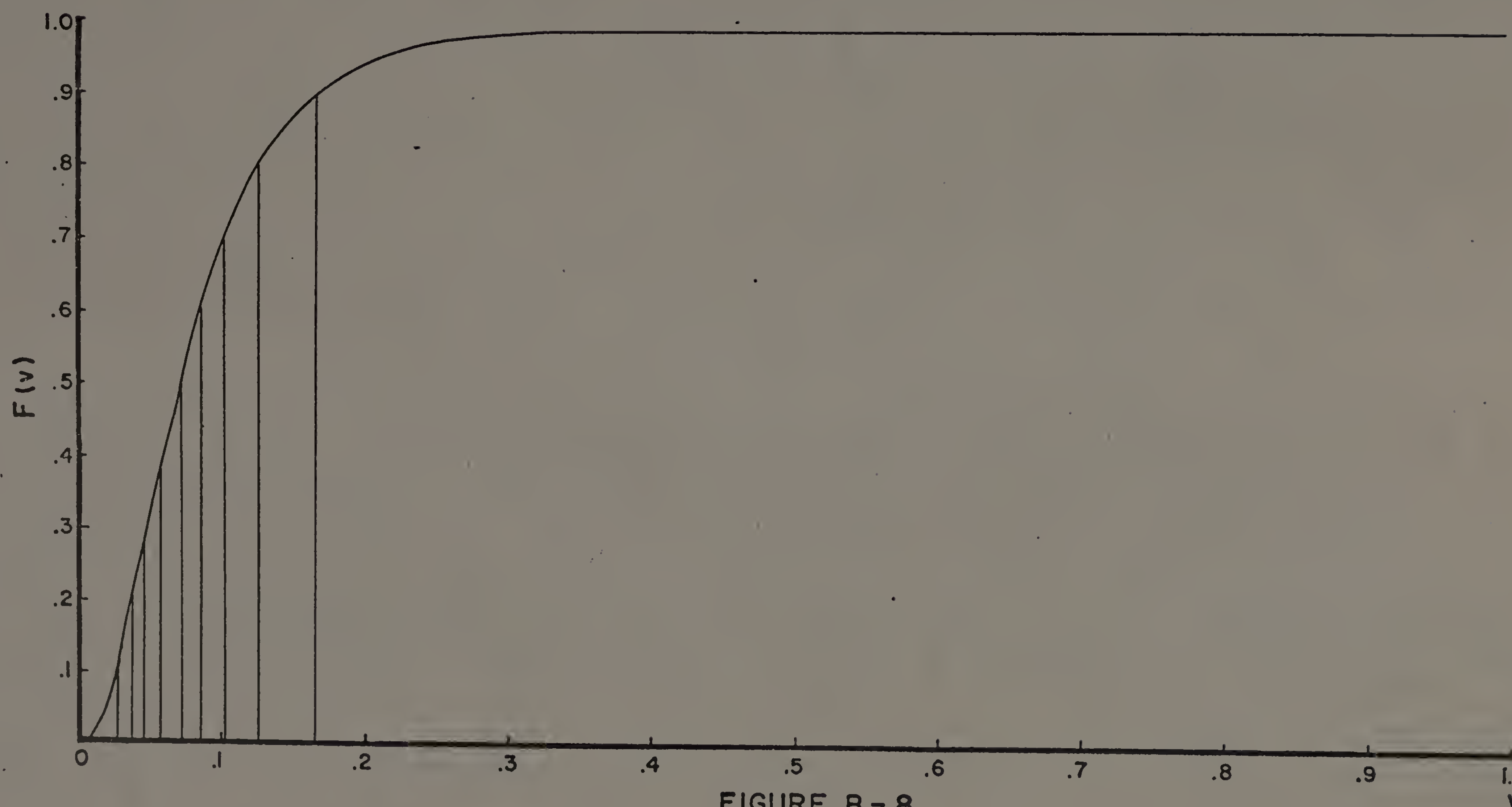

FIGURE B-8

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.35$
EXPERIMENT NO. 2

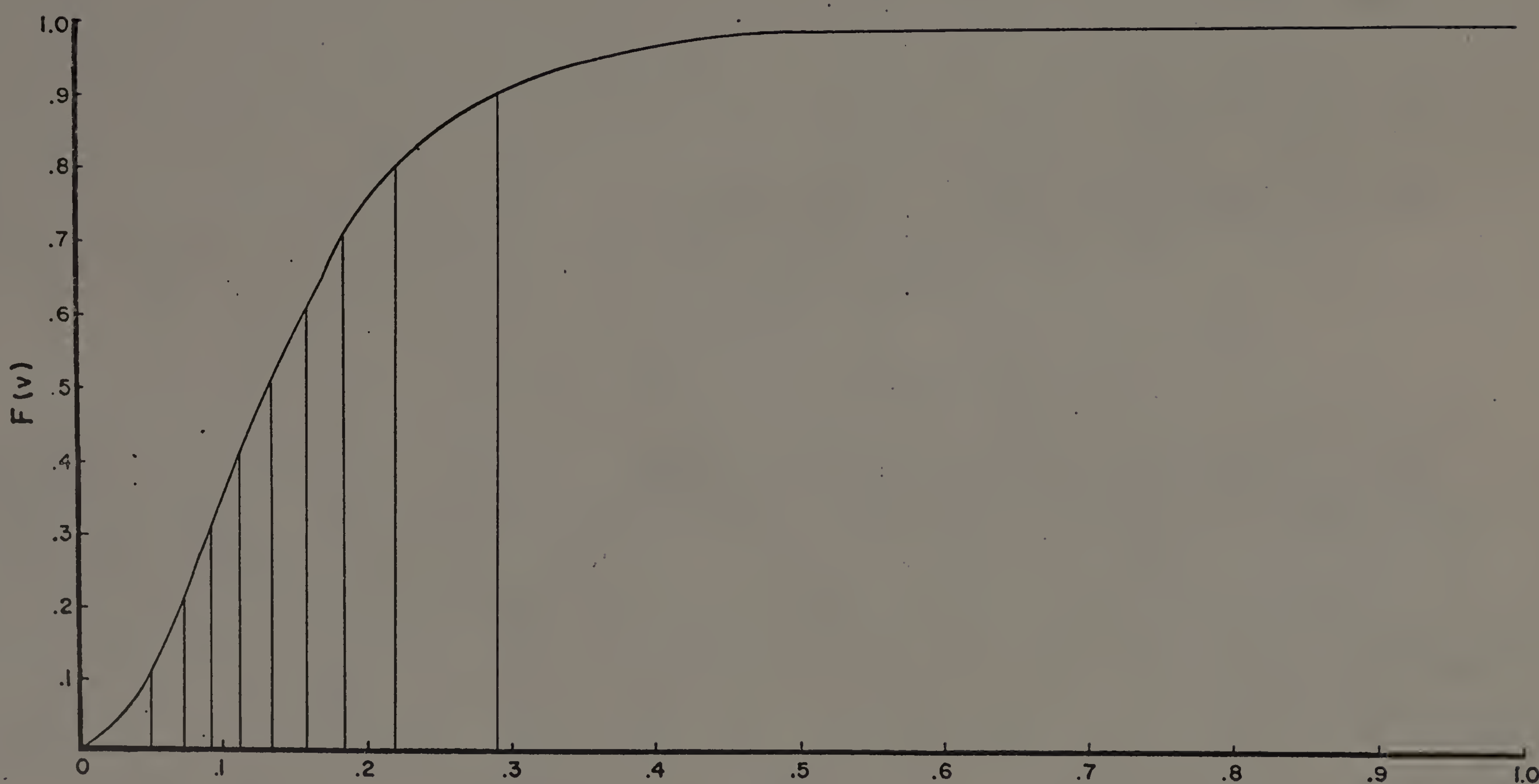

FIGURE B-9

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 2

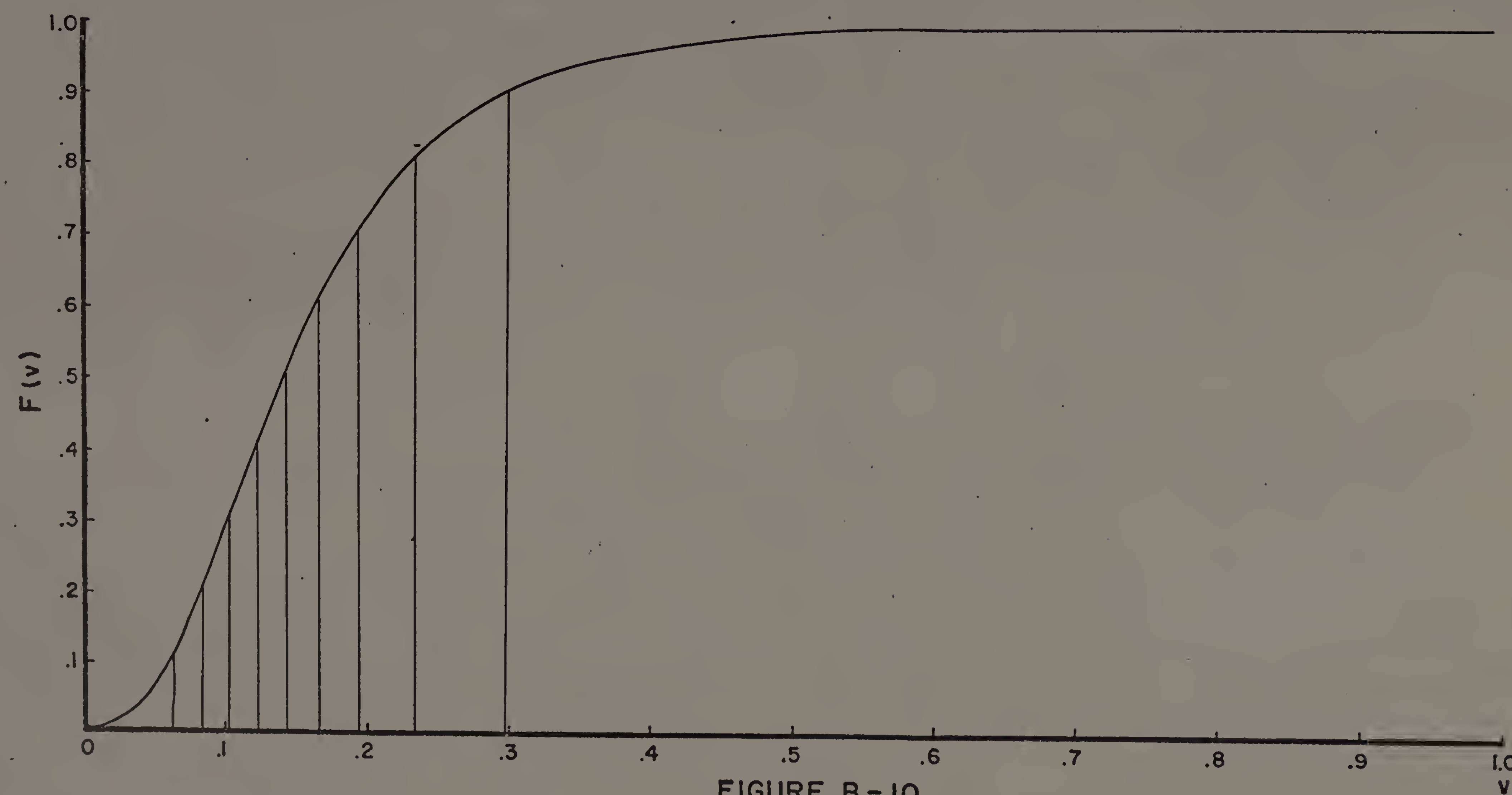

FIGURE B-10

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 2

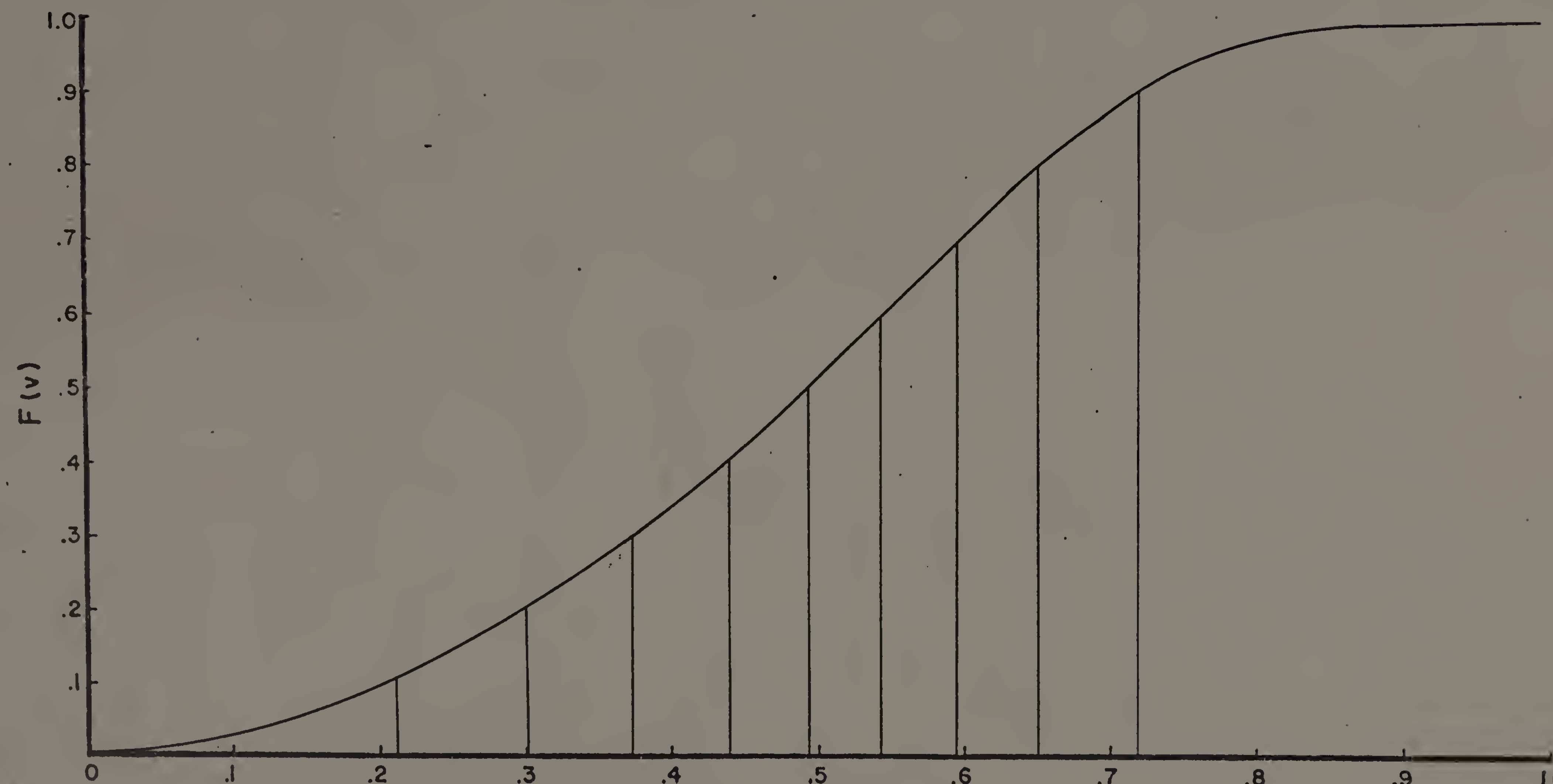

FIGURE B-11

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.1$
EXPERIMENT NO. 3

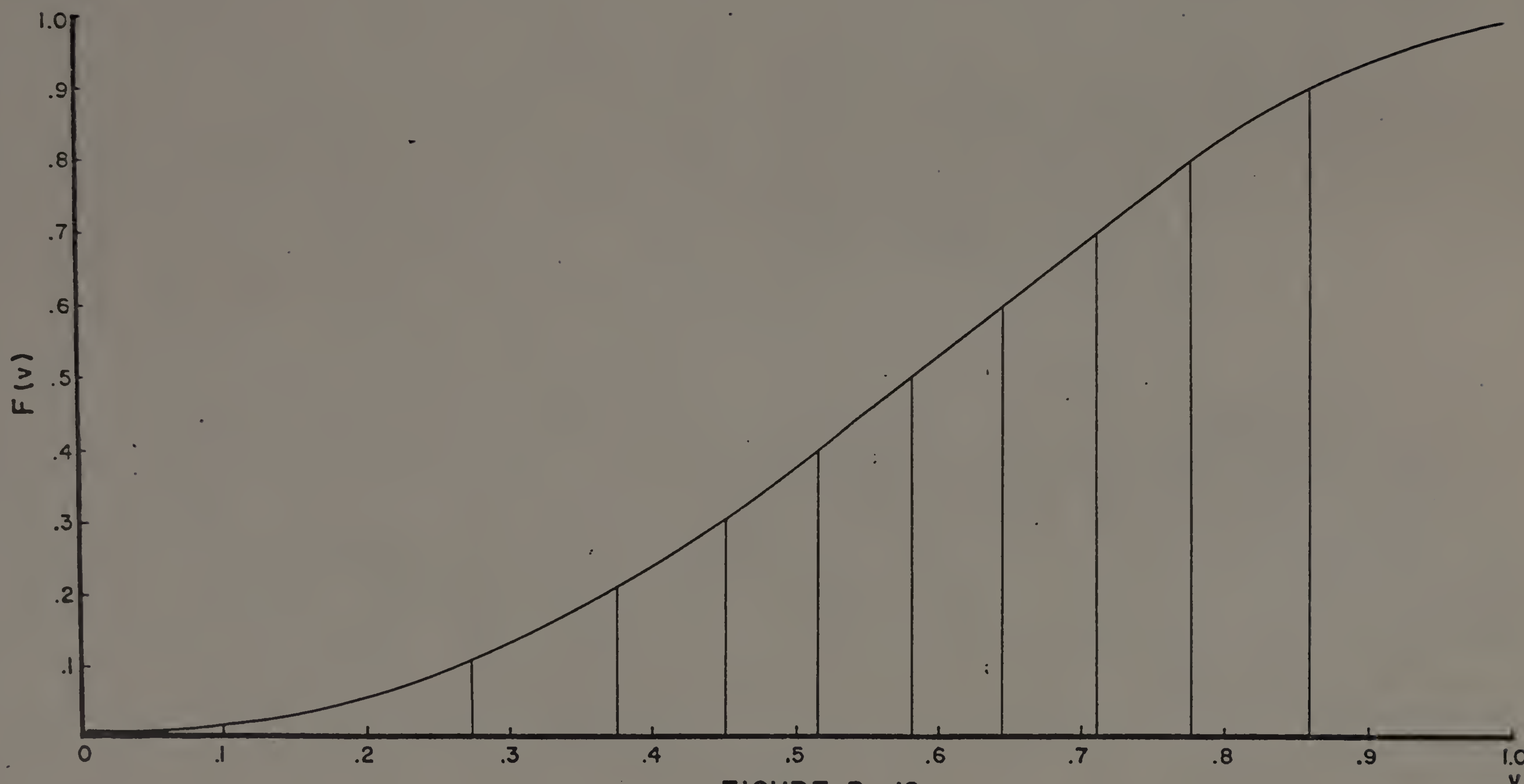

FIGURE B-12

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 3

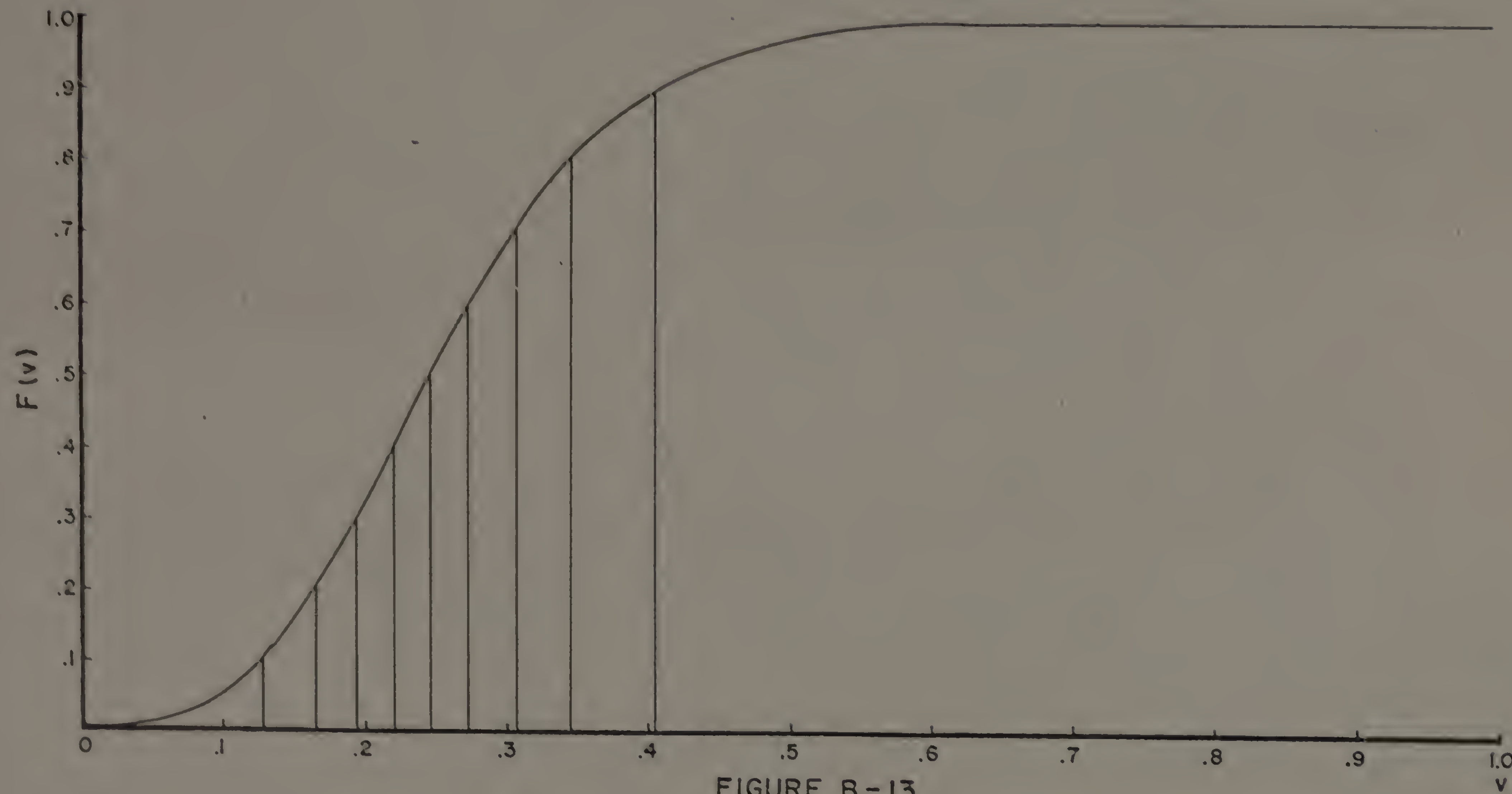

FIGURE B-13

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 3

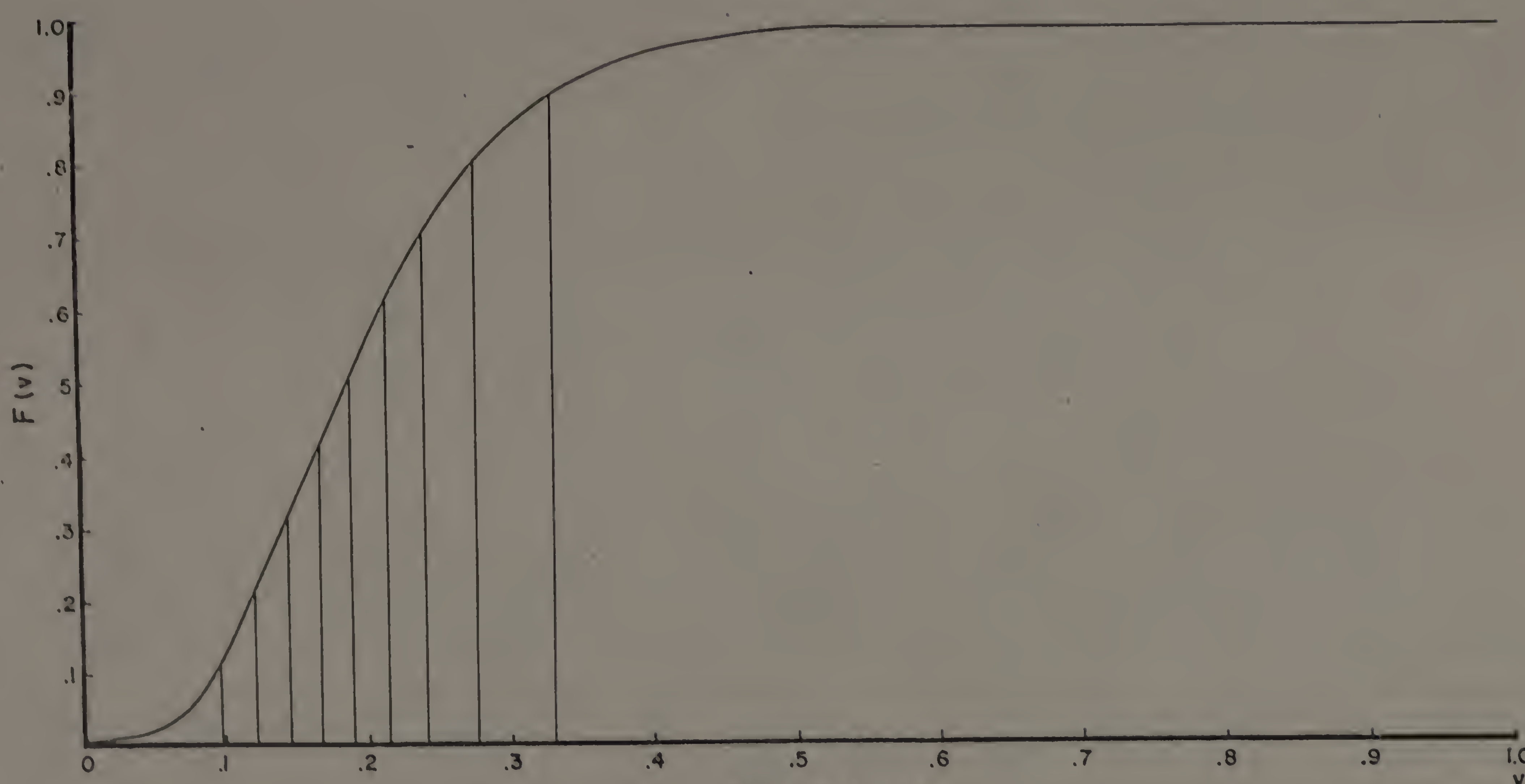

FIGURE B-14

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 3

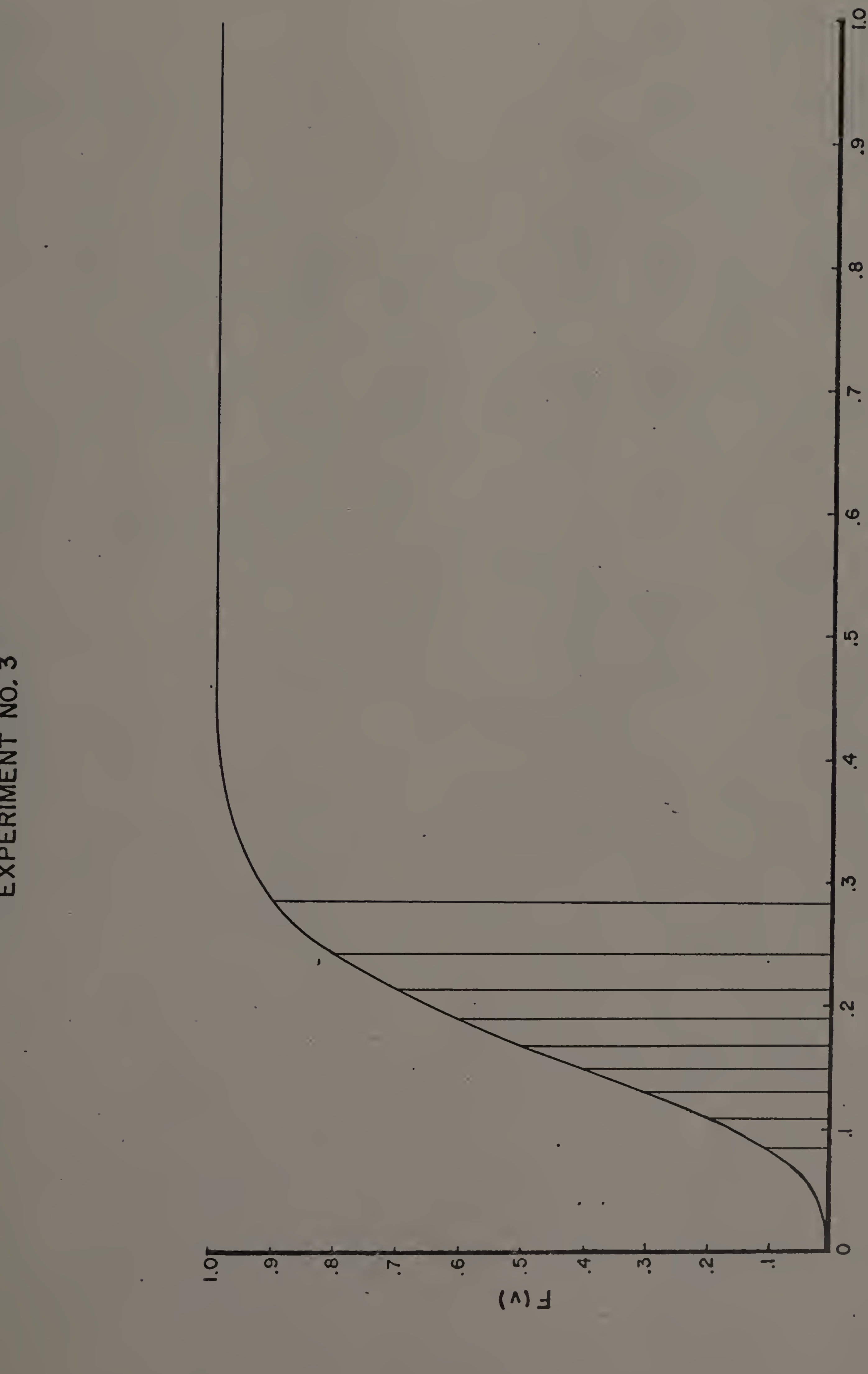

FIGURE B-15

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 3

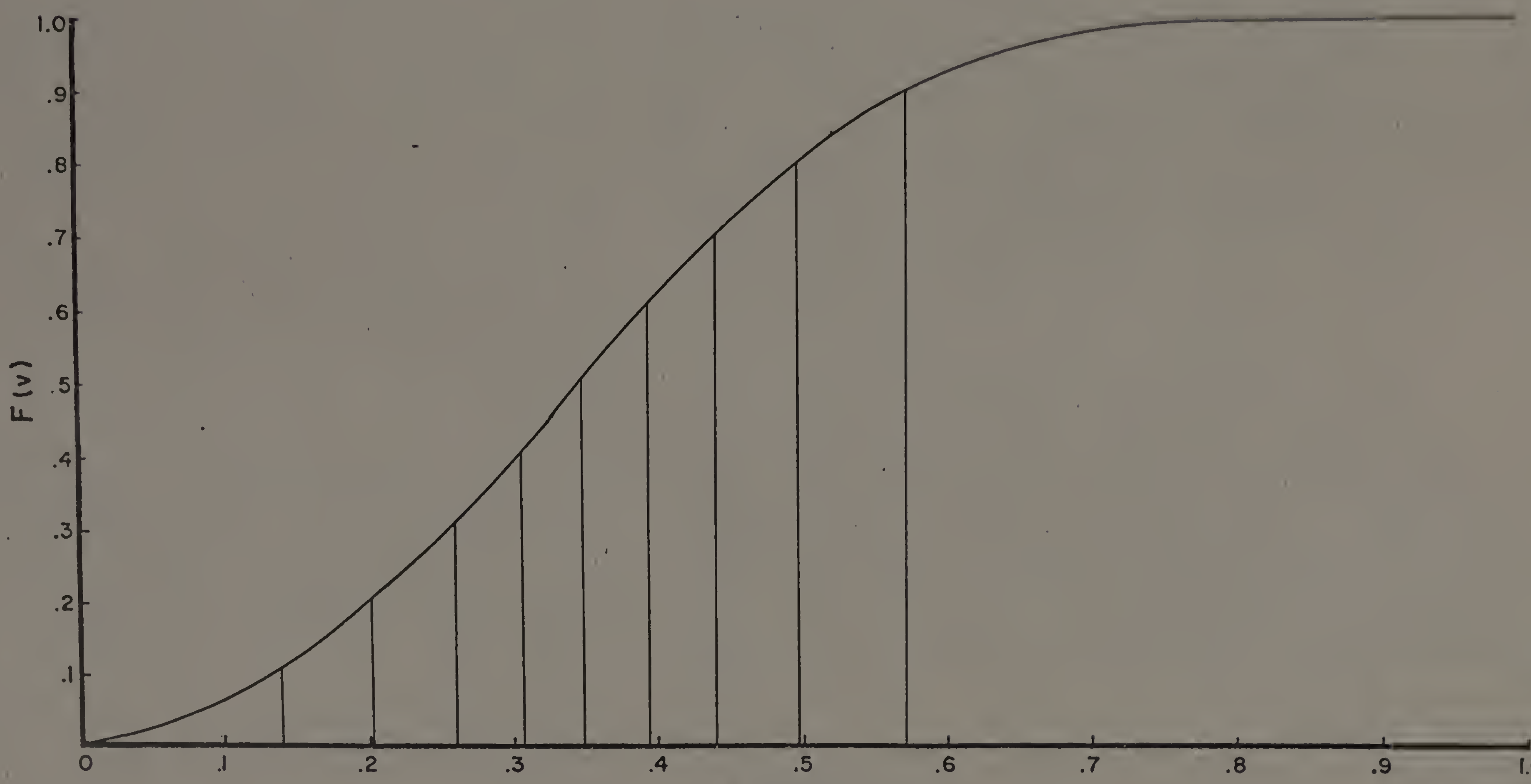

FIGURE B-16

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.13$
EXPERIMENT NO. 4

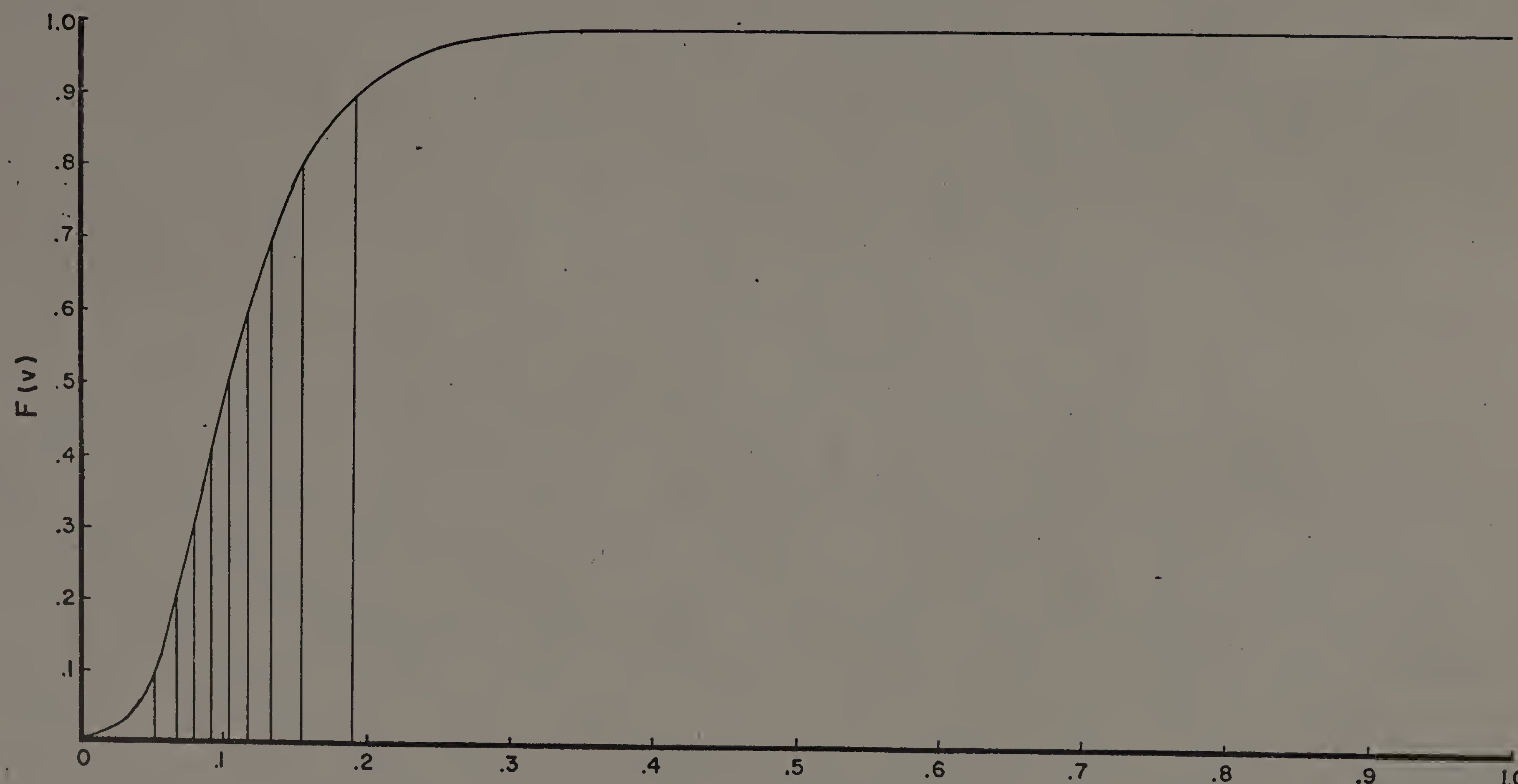

FIGURE B-17

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 4

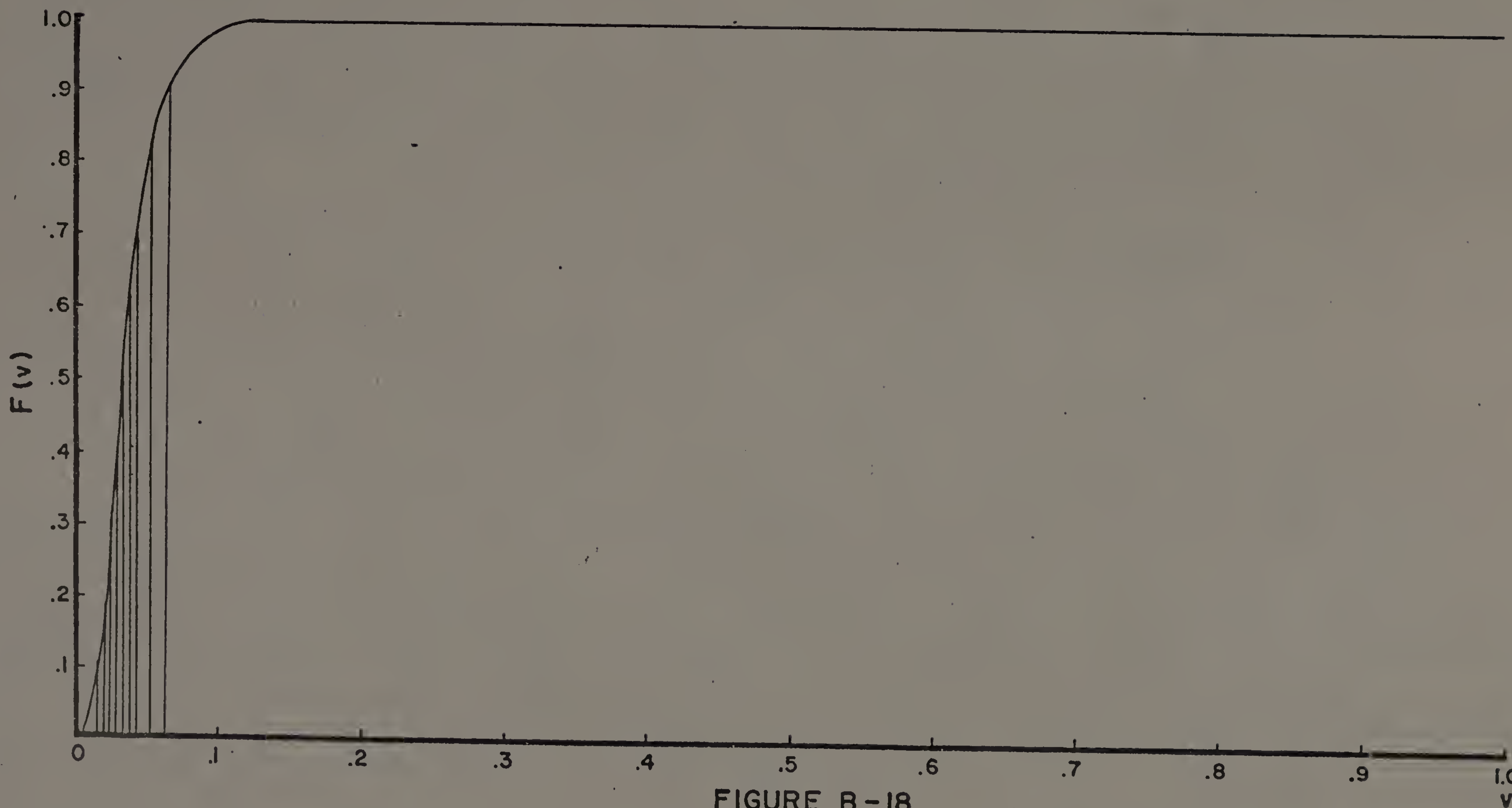

FIGURE B-18

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 4

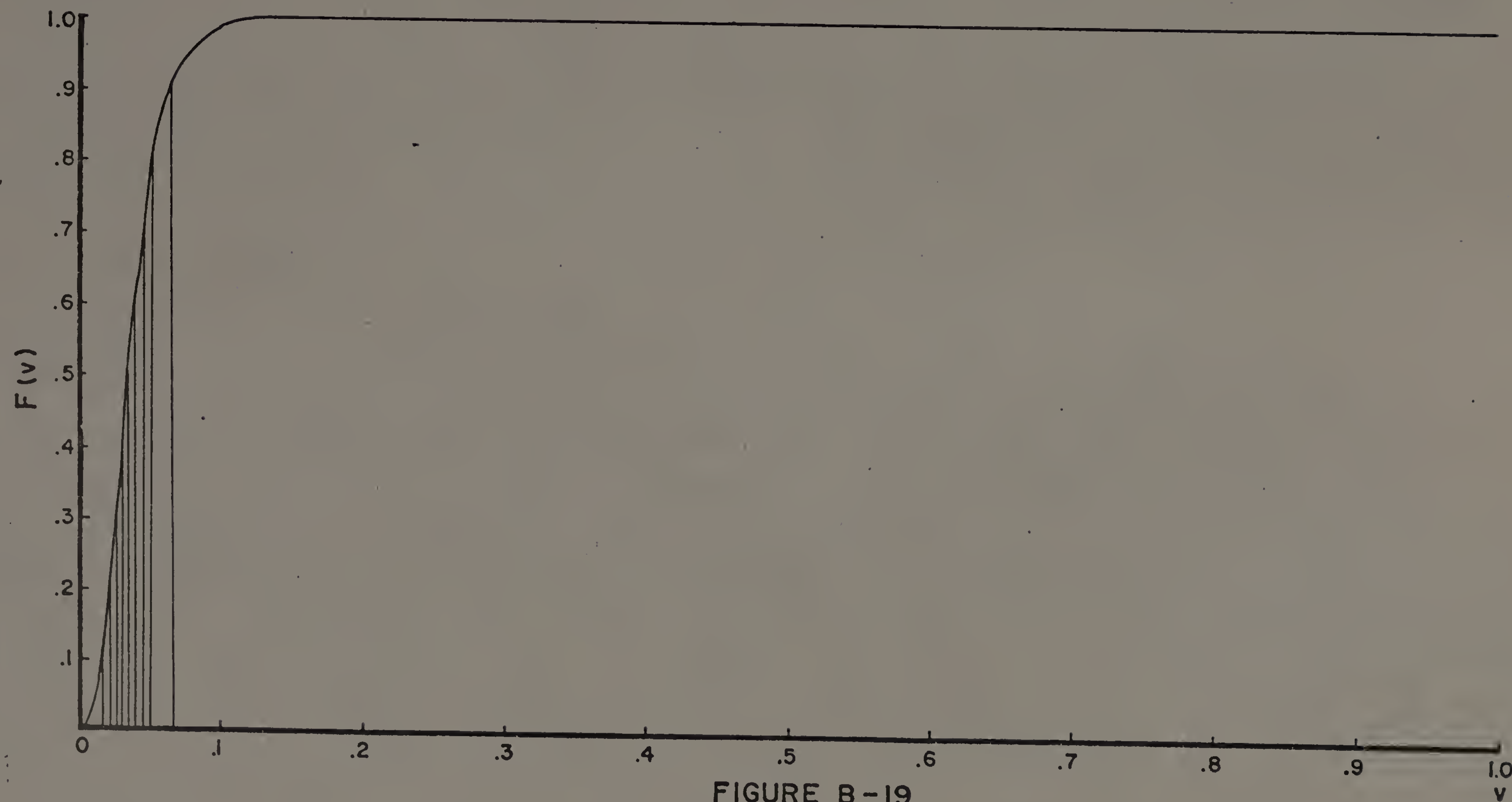

FIGURE B-19

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 4

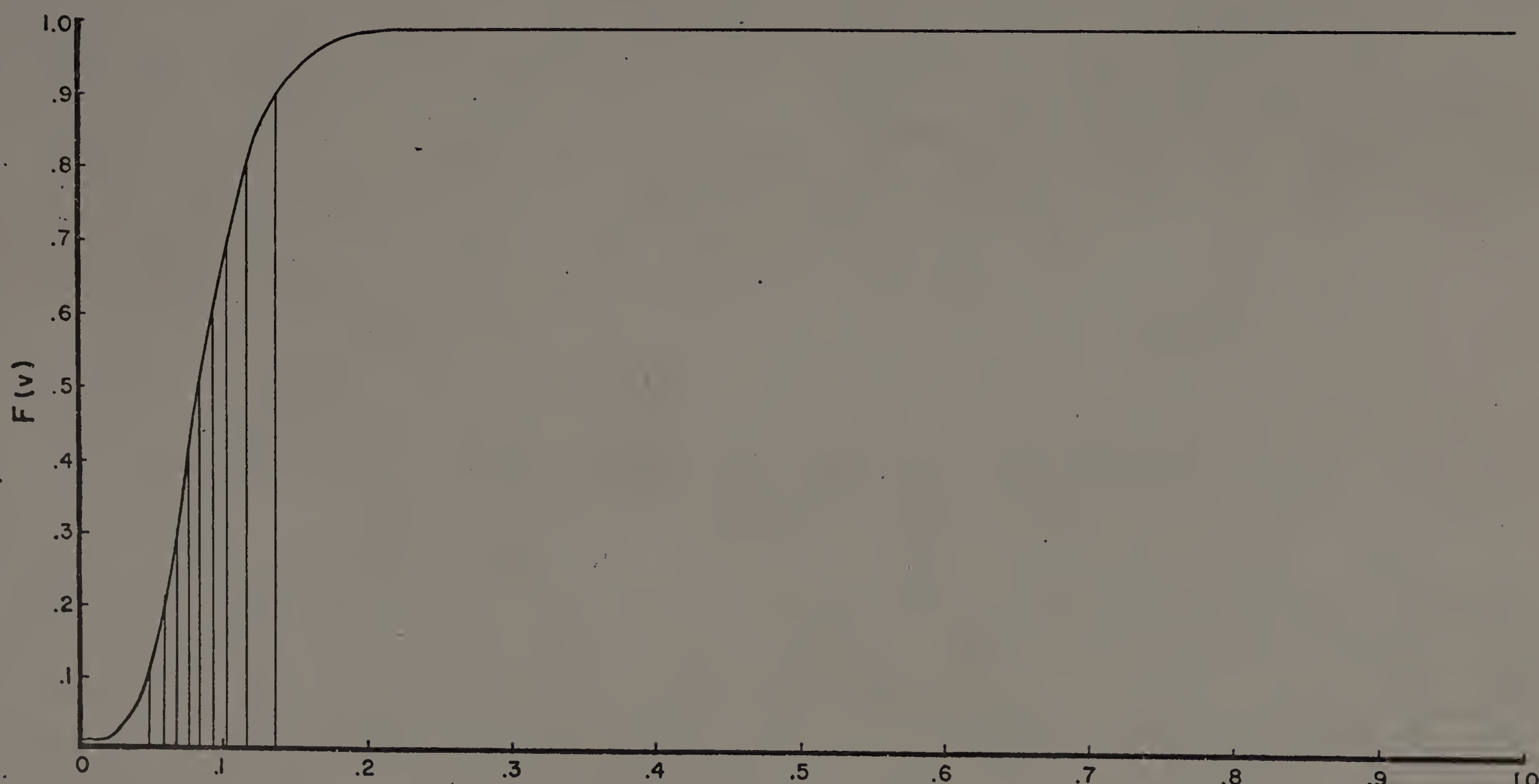

FIGURE B-20

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 4

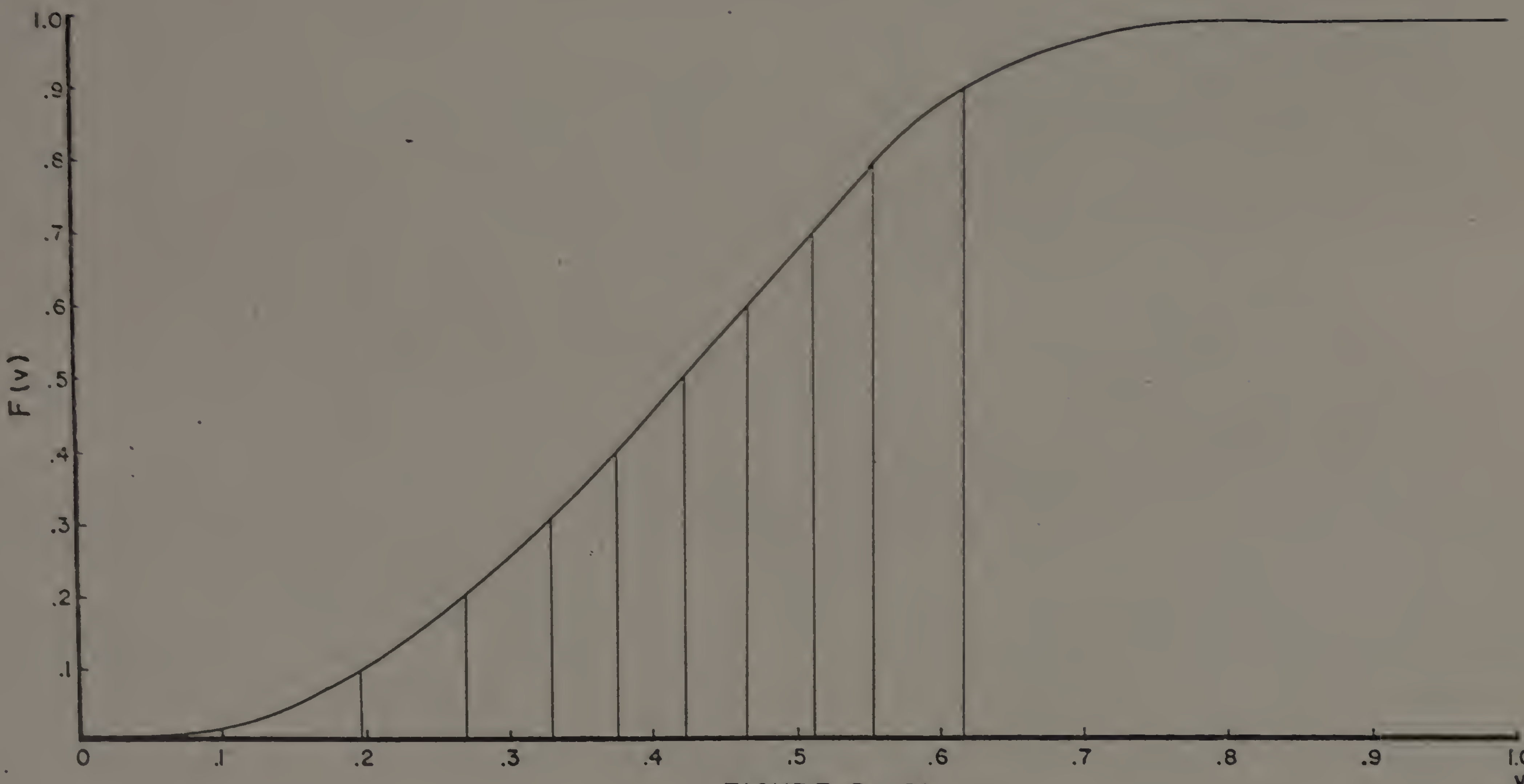

FIGURE B-21

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.1$
EXPERIMENT NO. 5

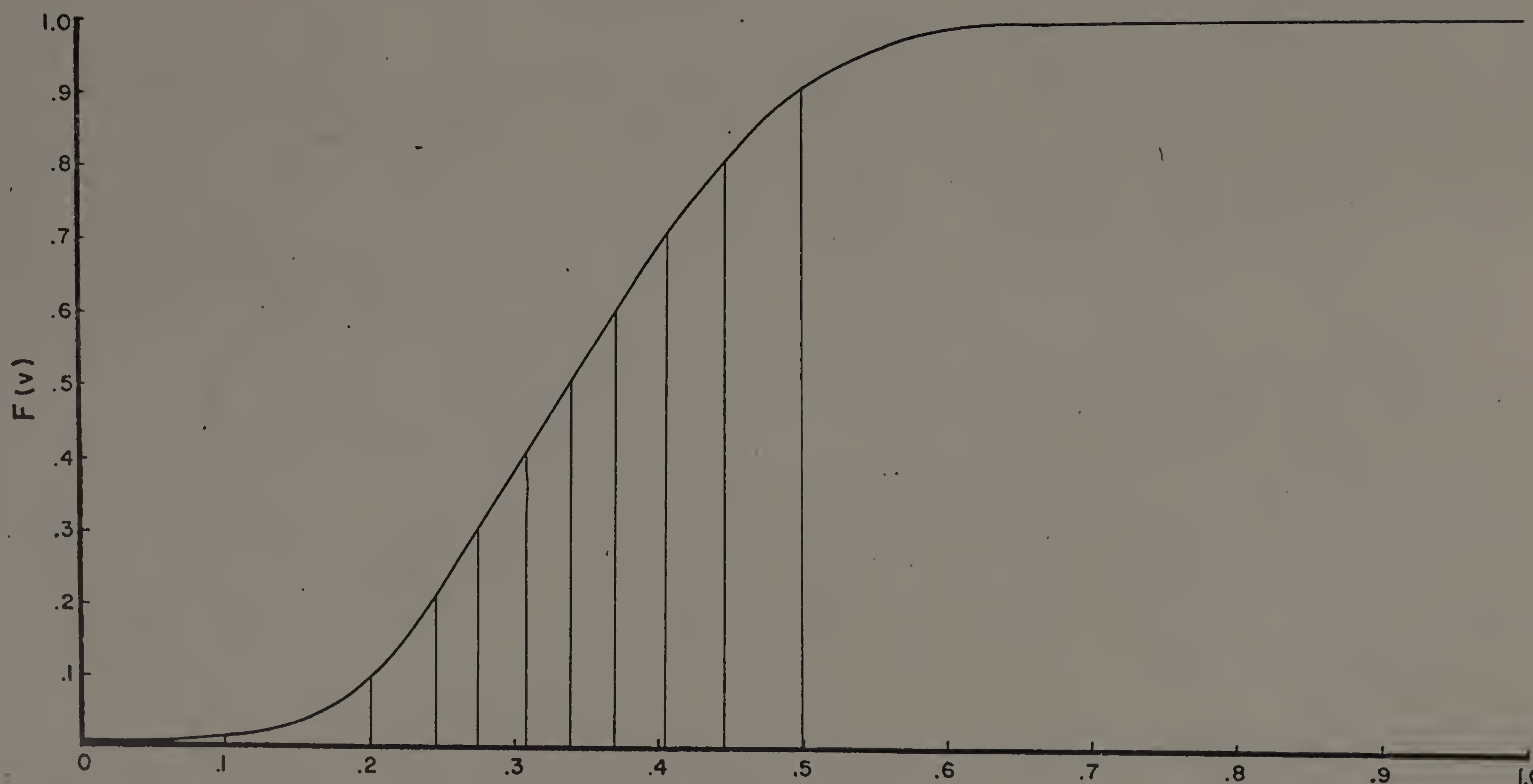

FIGURE B-22

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 5

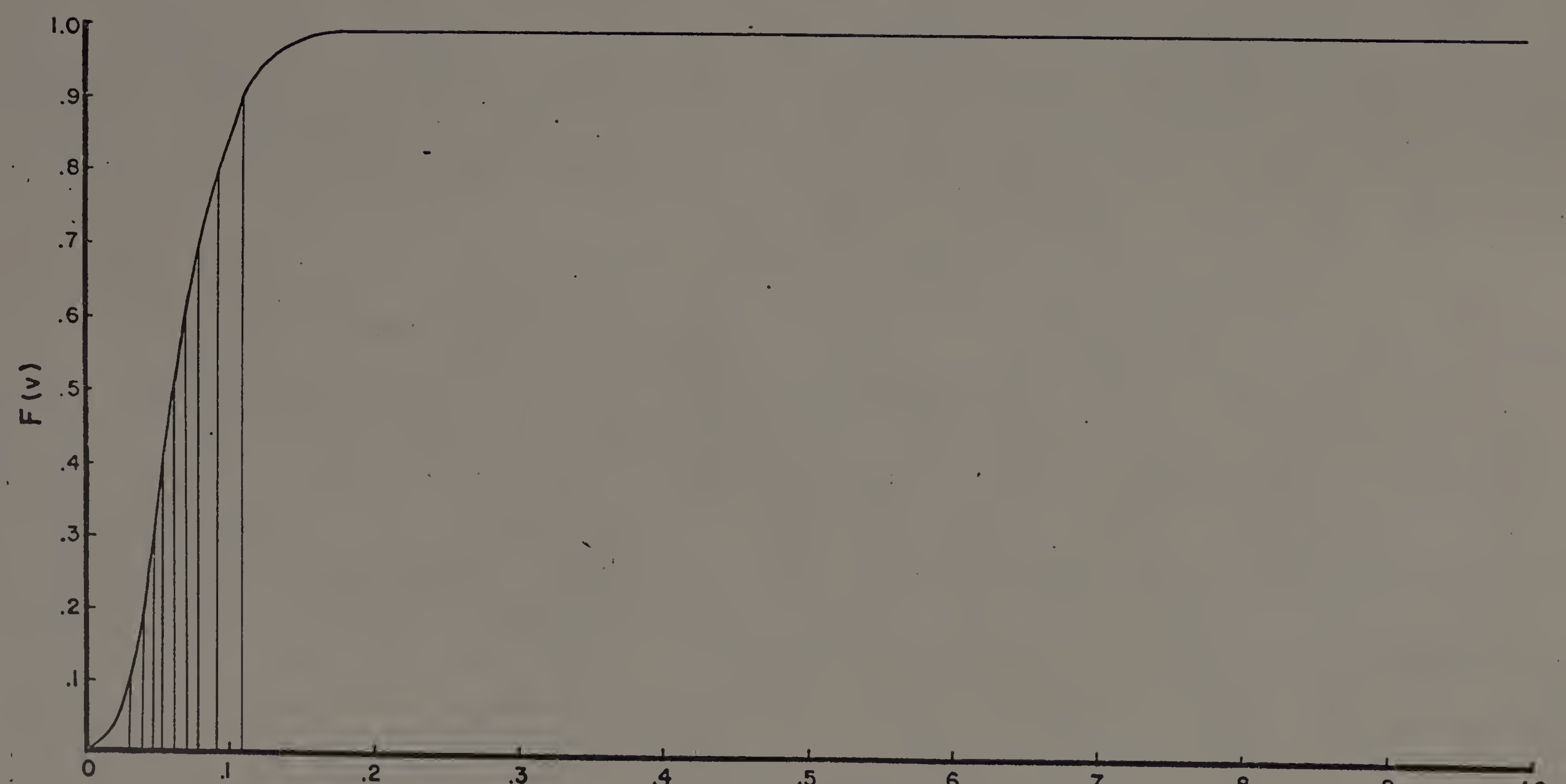

FIGURE B-23

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j=1.3$
EXPERIMENT NO. 5

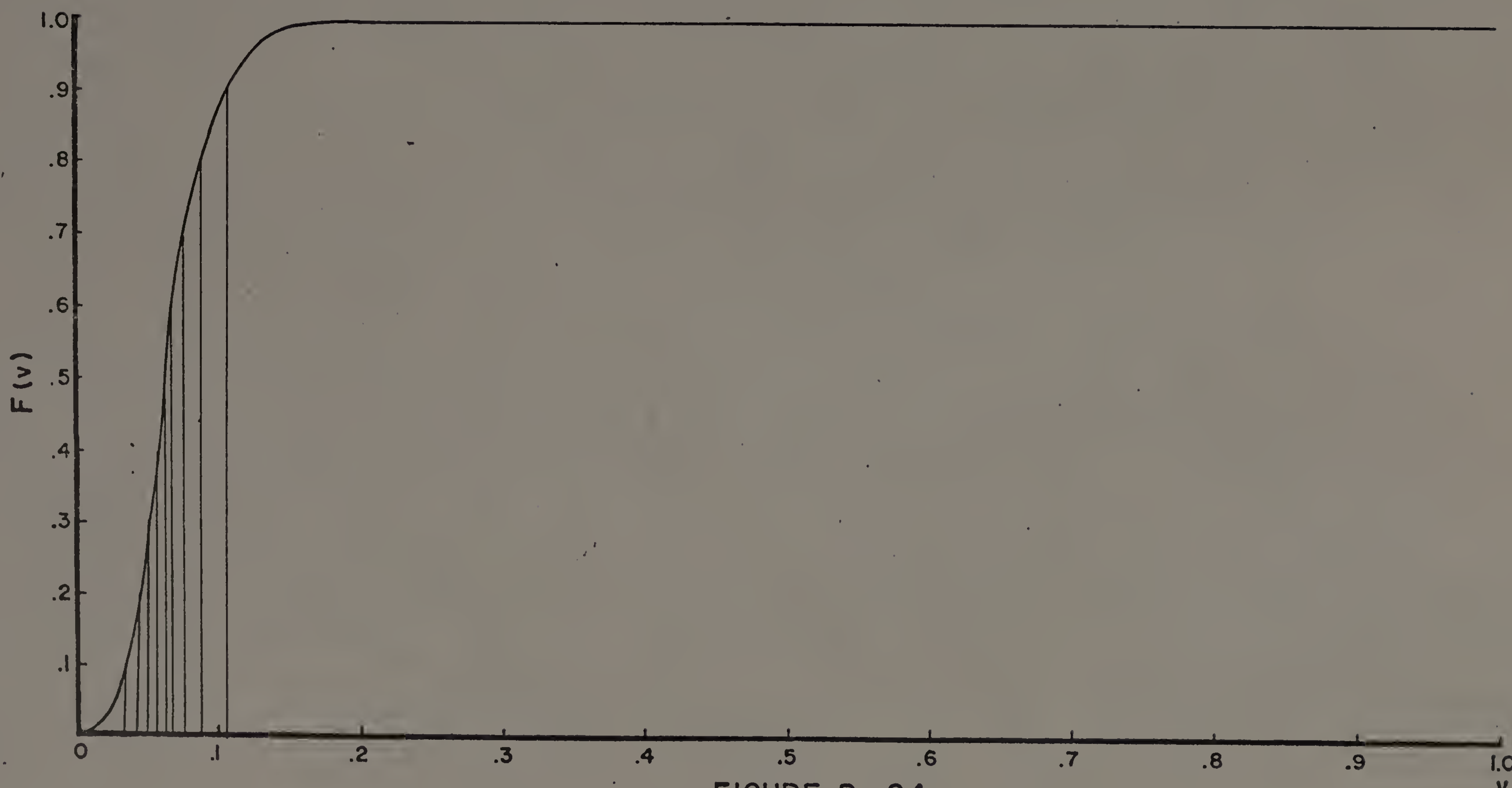

FIGURE B-24

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 5

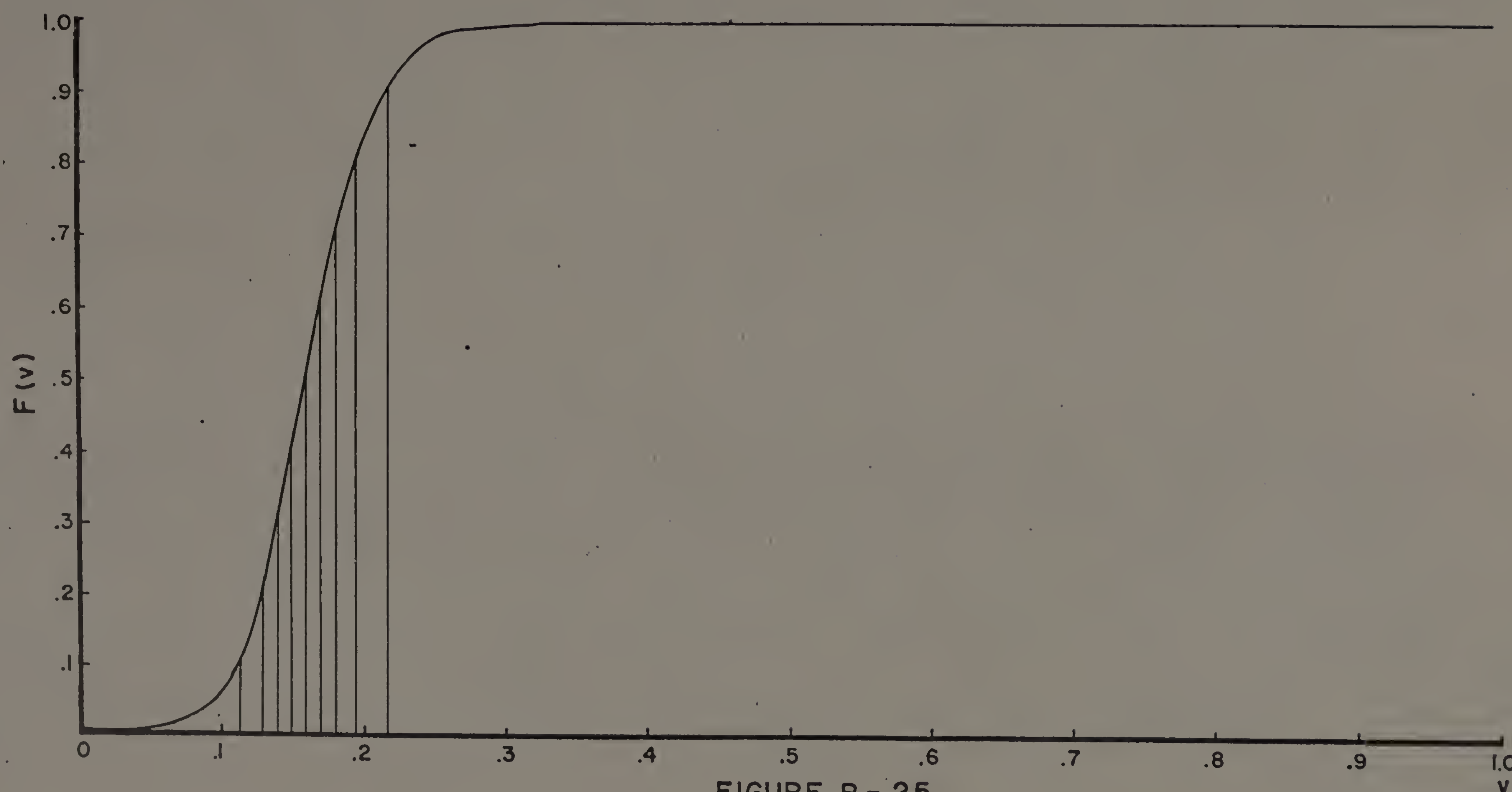

FIGURE B-25

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 5

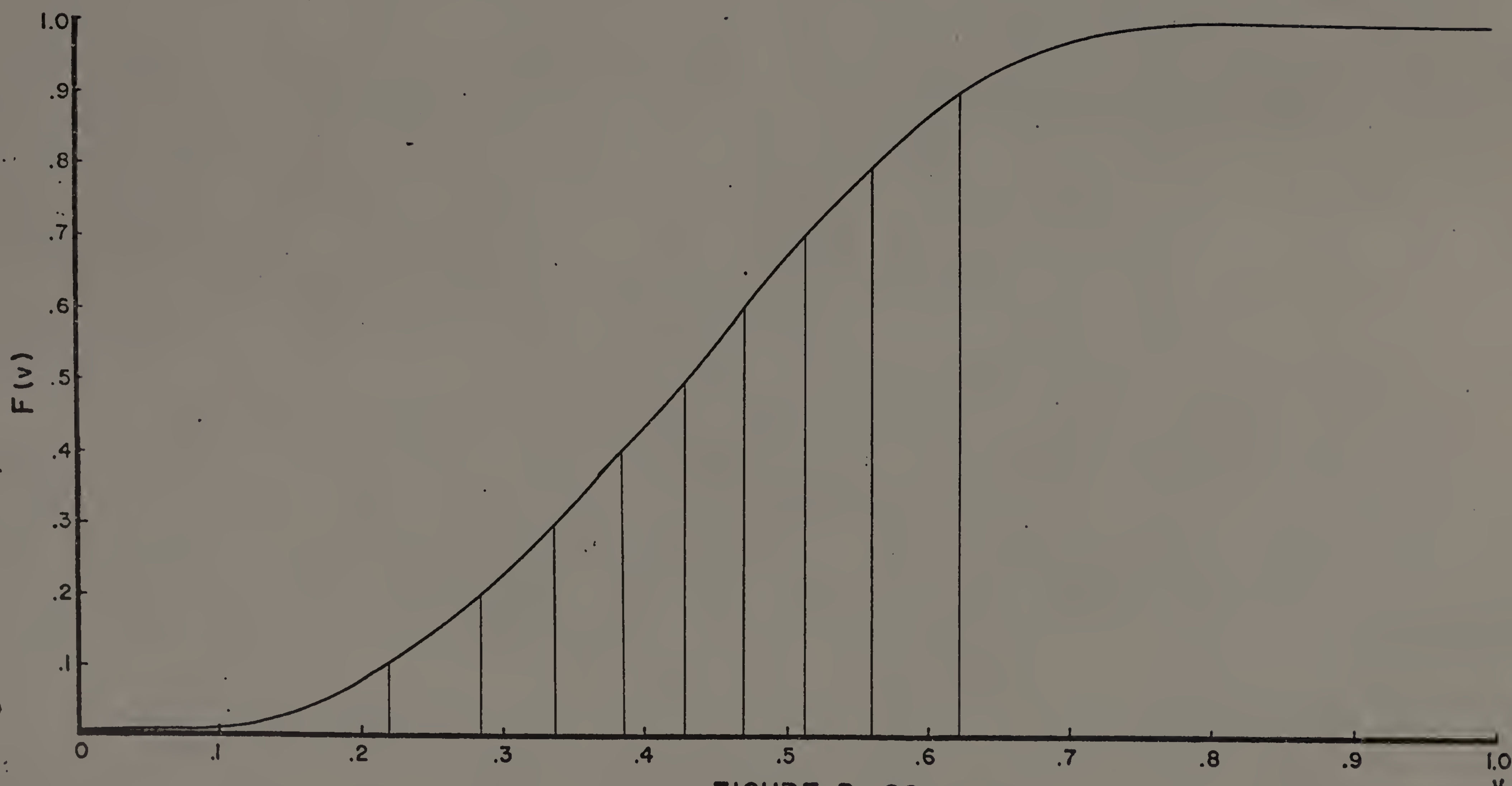

FIGURE B-26

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.1$
EXPERIMENT NO. 6

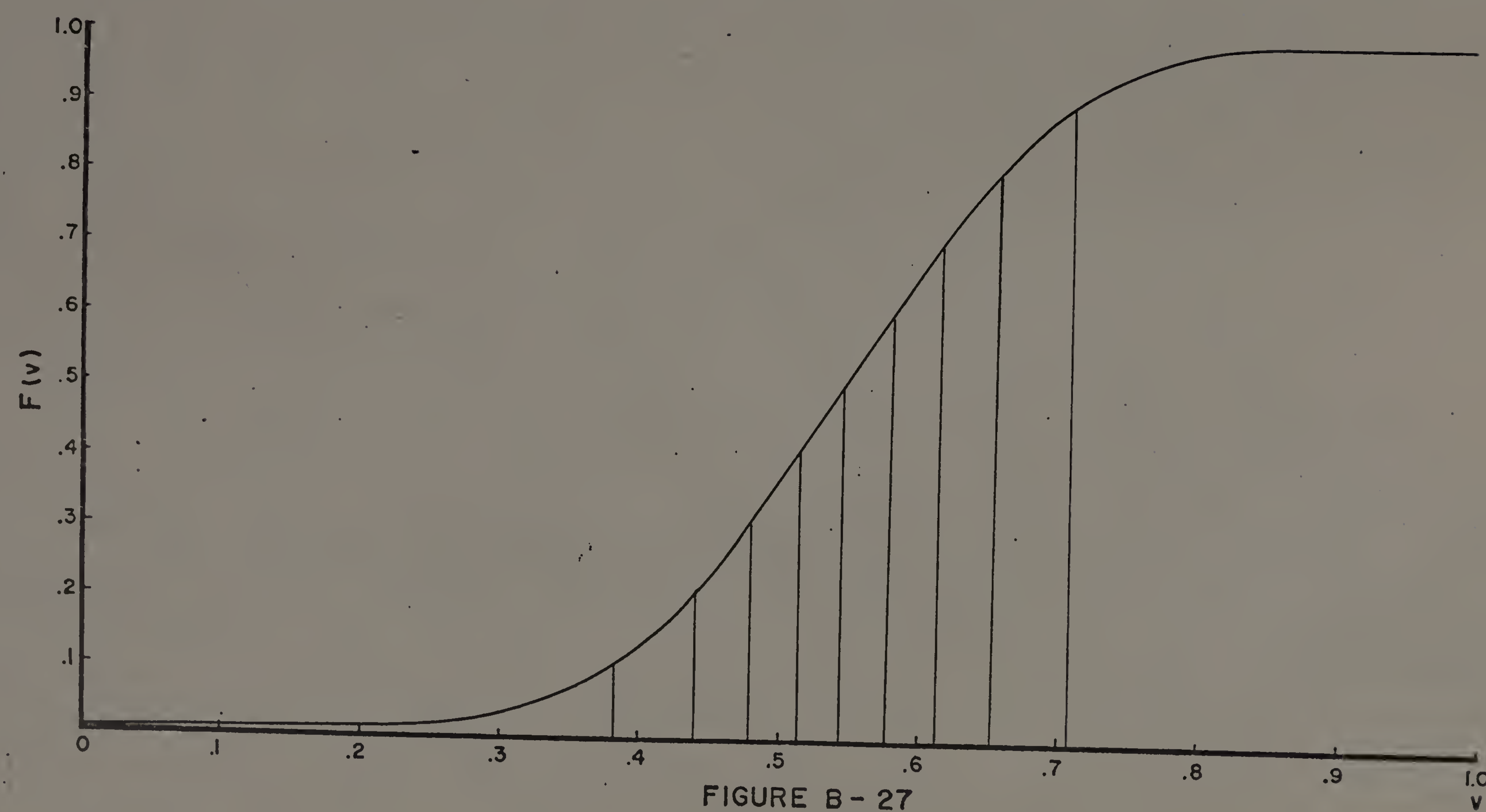

FIGURE B-27

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 6

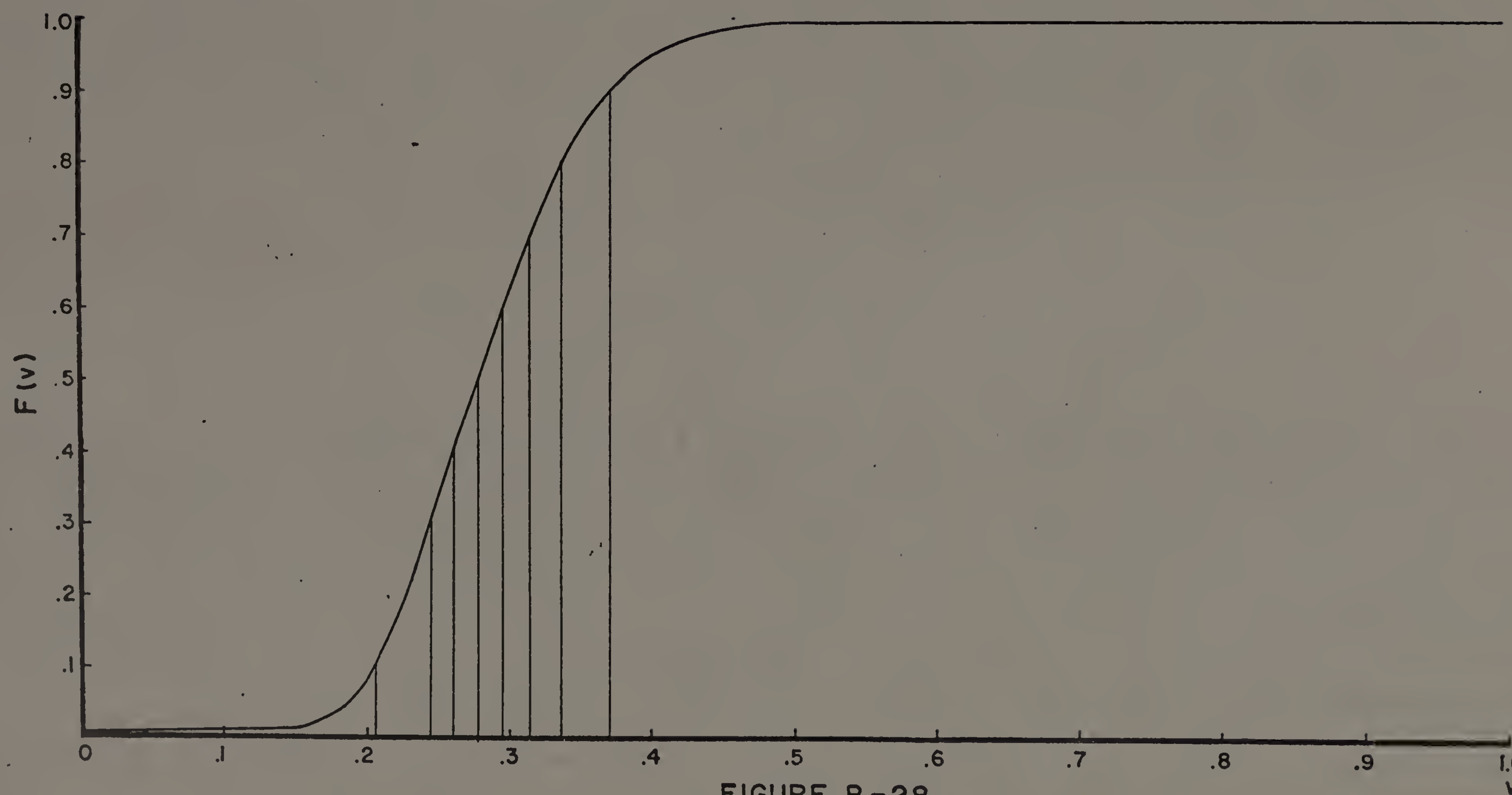

FIGURE B-28

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 6

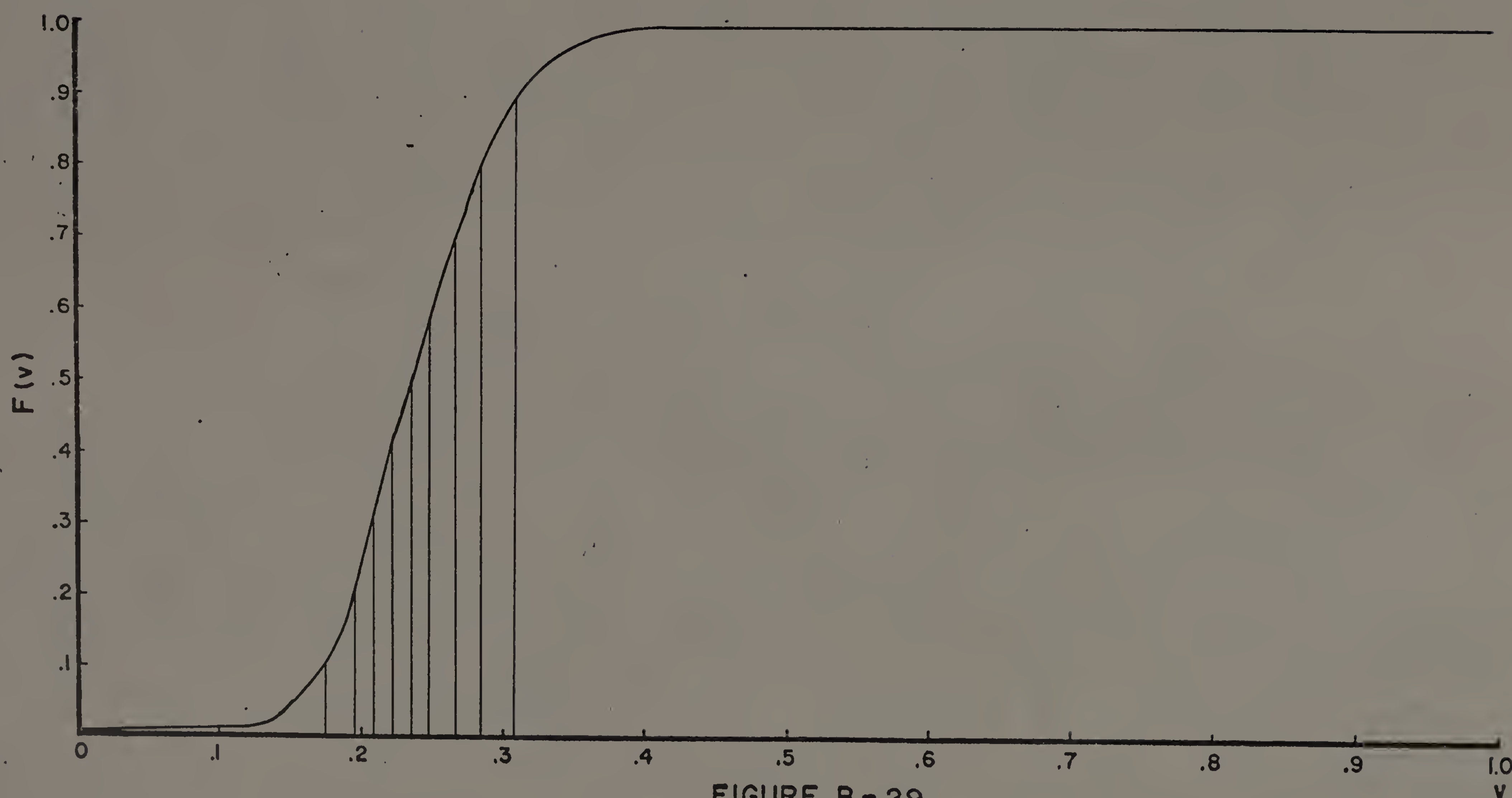

FIGURE B-29

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 6

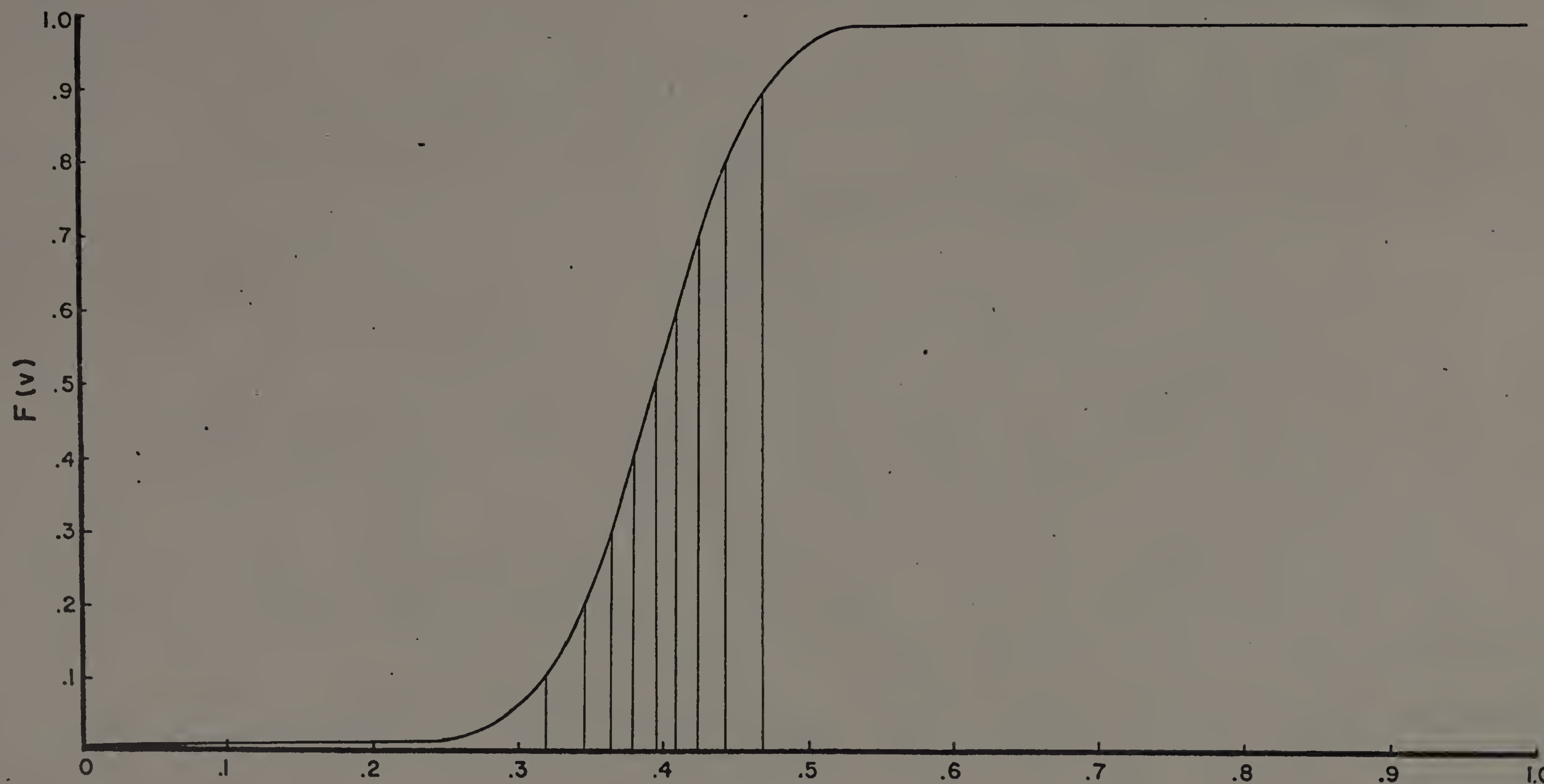

FIGURE B-30

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 6

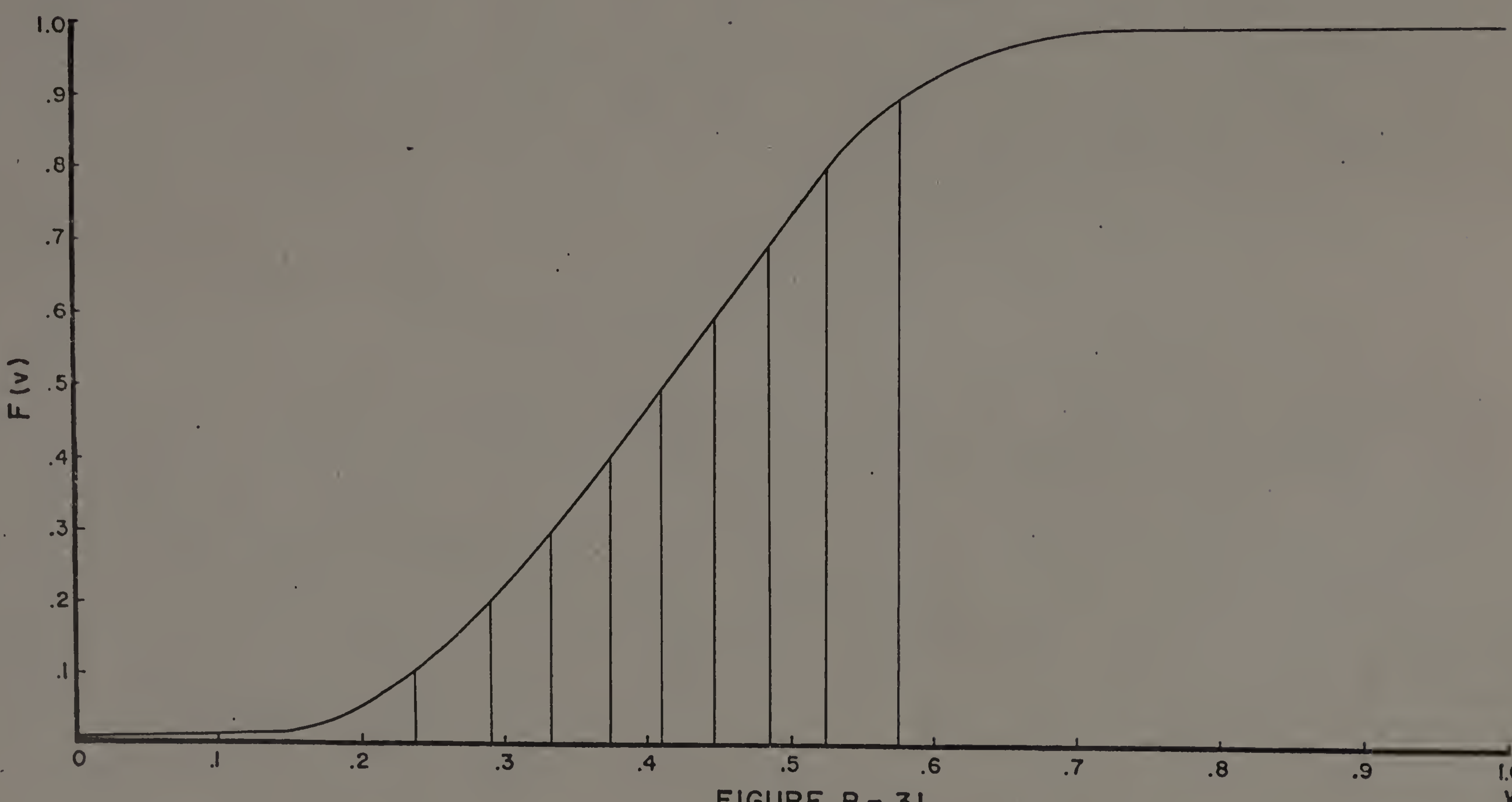

FIGURE B-31

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.111$
EXPERIMENT NO. 7

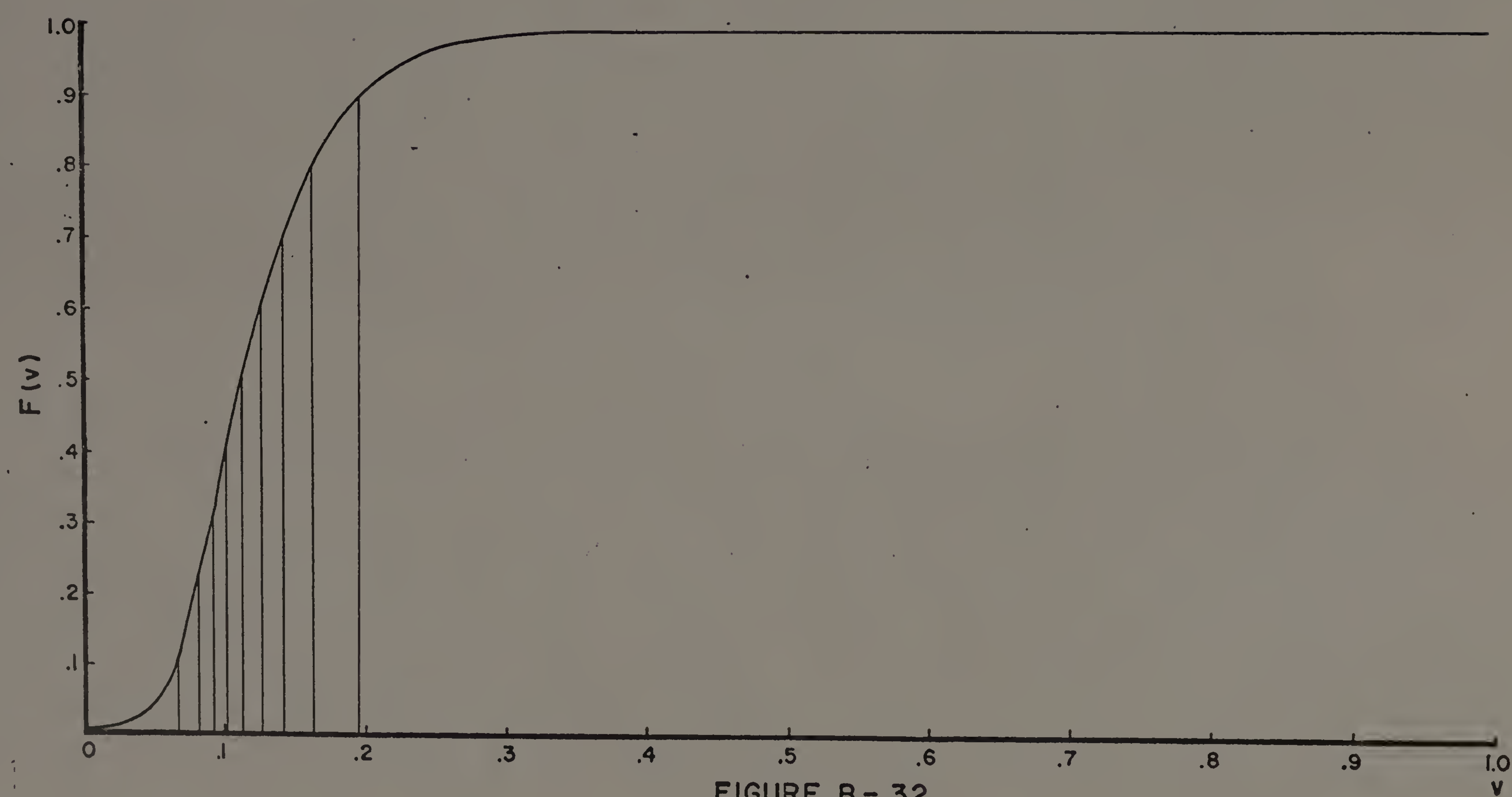

FIGURE B - 32

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 7

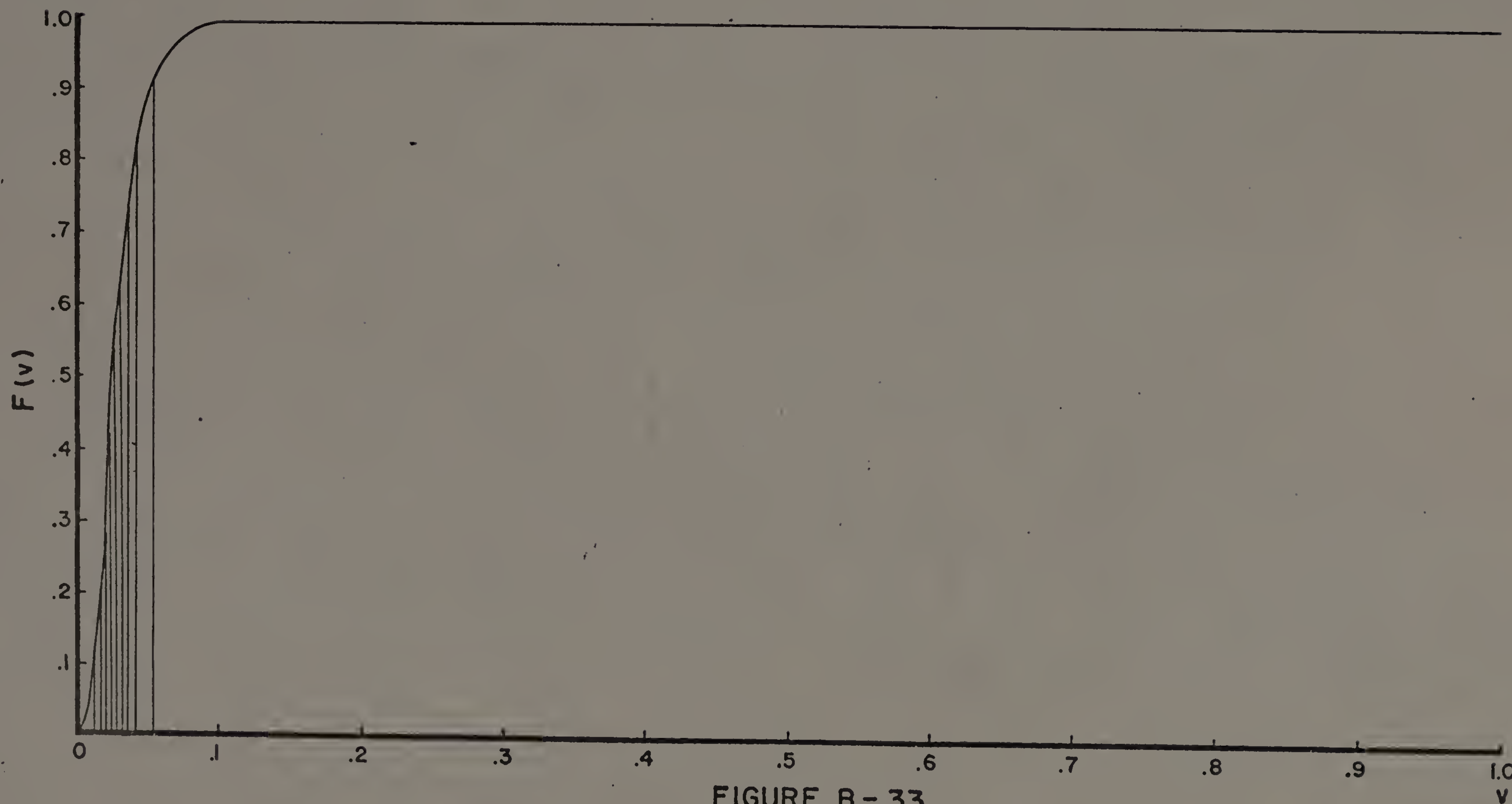

FIGURE B-33

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j=1.3$
EXPERIMENT NO. 7

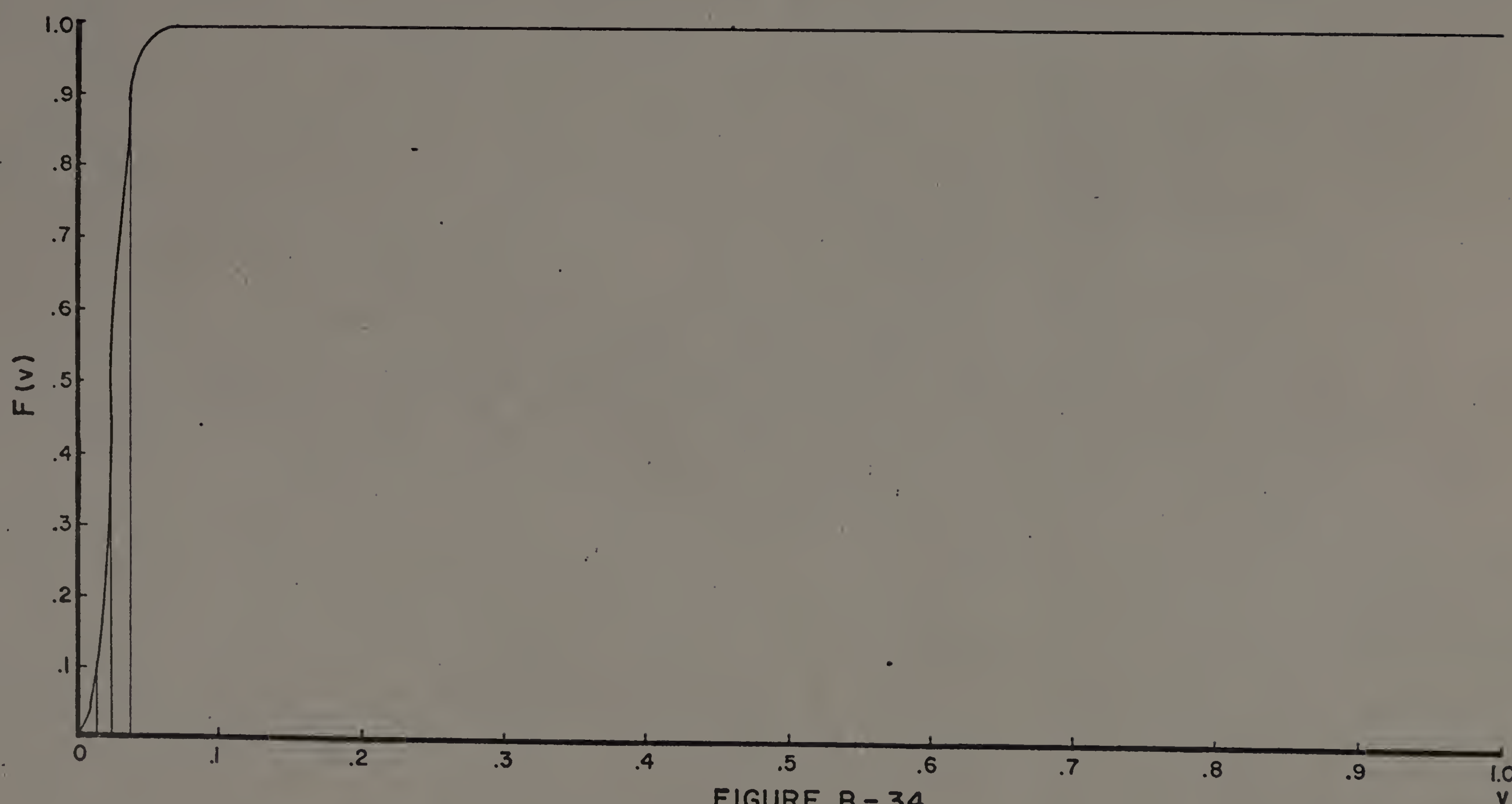

FIGURE B-34

SAMPLE DISTRIBUTION OF $v = (V(\tilde{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 7

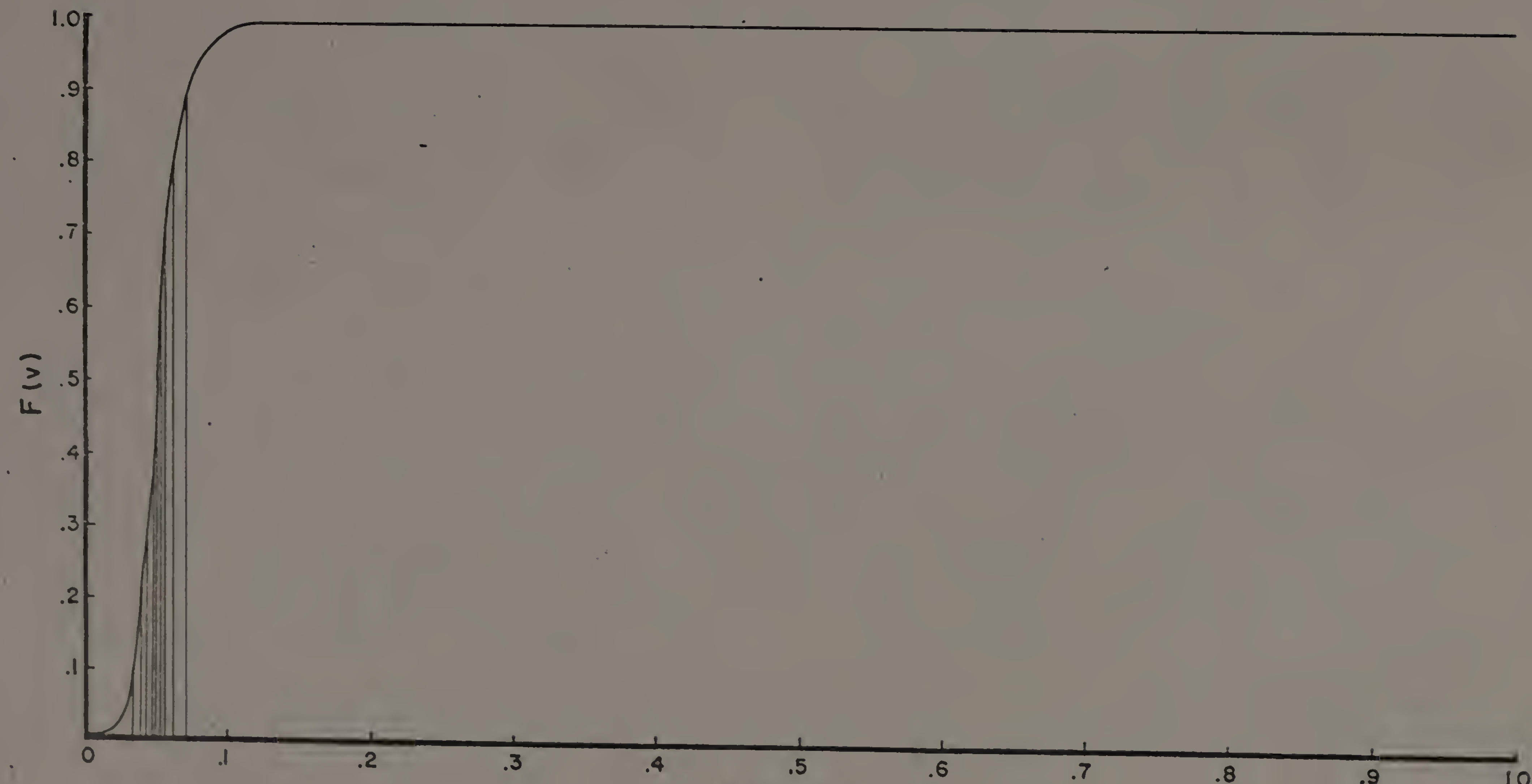

FIGURE B-35

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 8

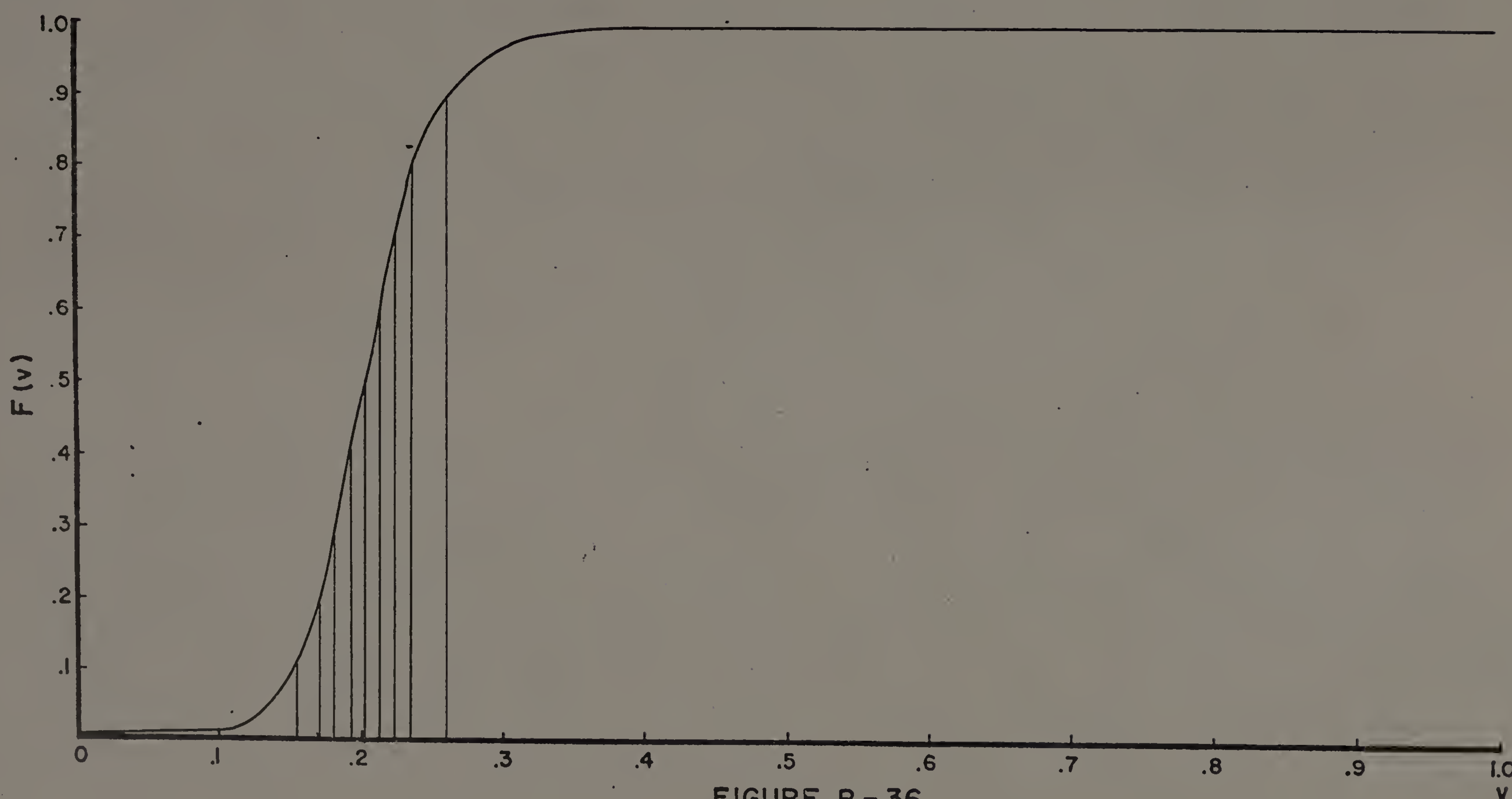

FIGURE B-36

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.111$
EXPERIMENT NO. 8

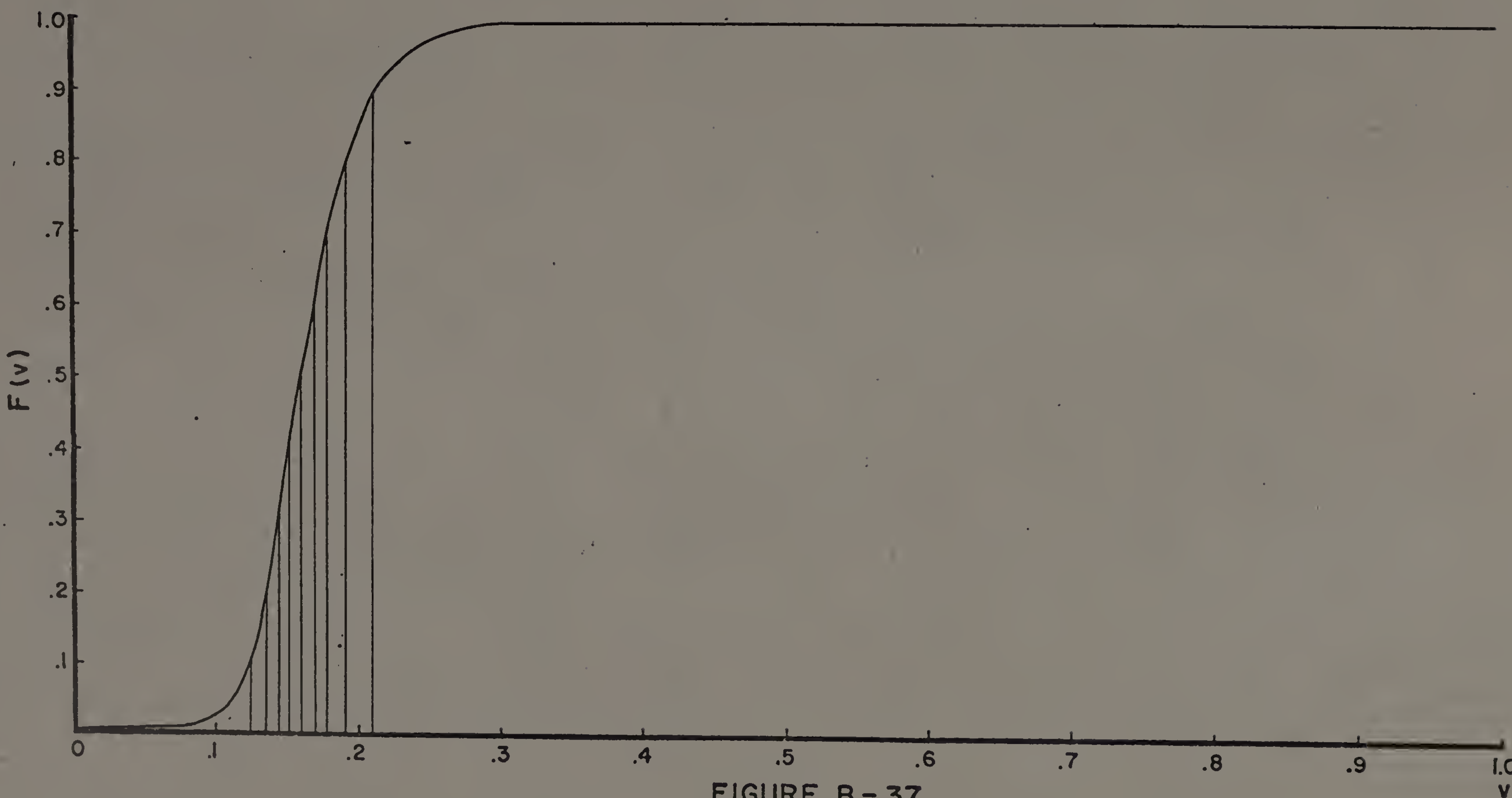

FIGURE B-37

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 8

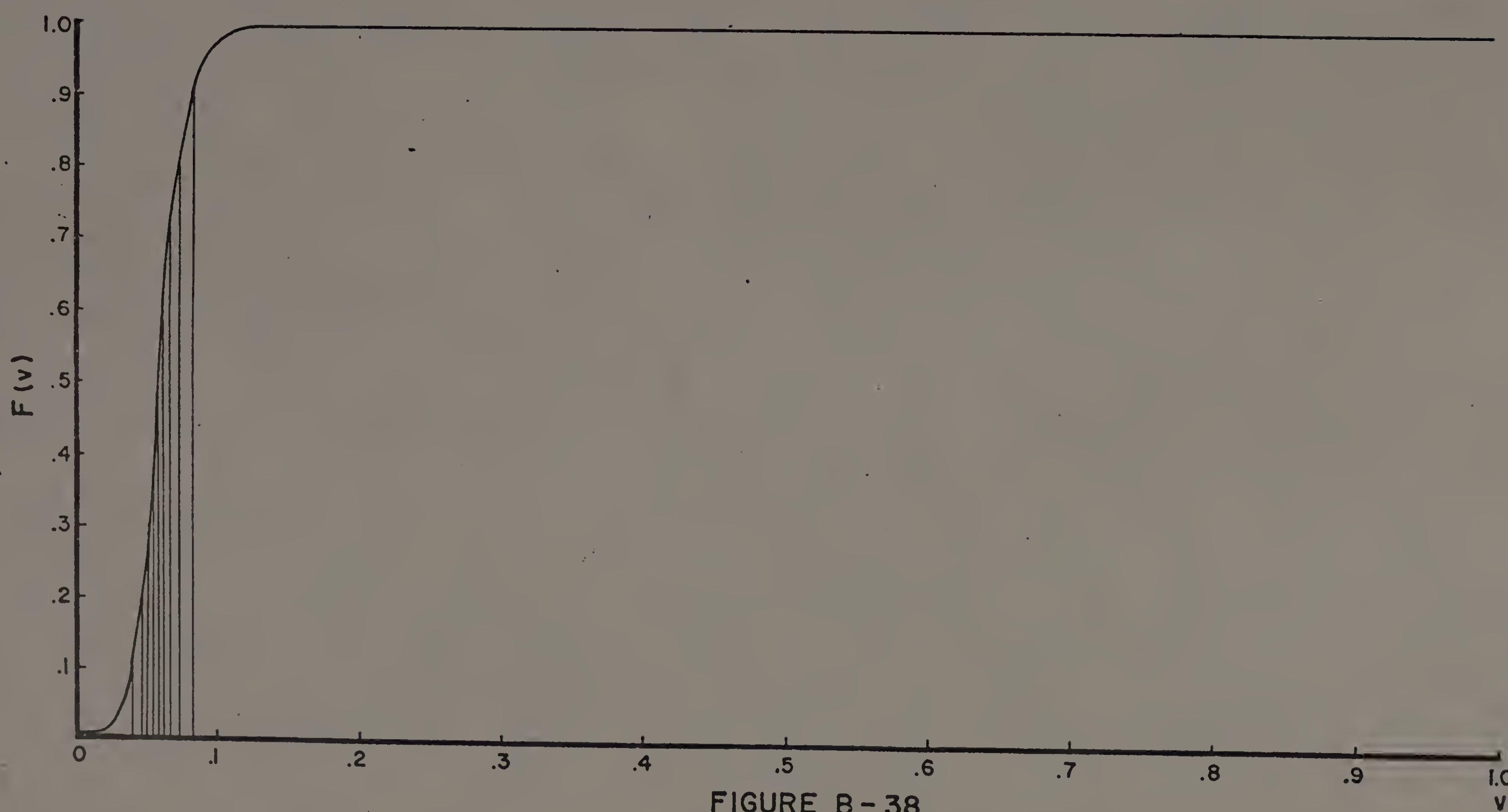

FIGURE B-38

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 8

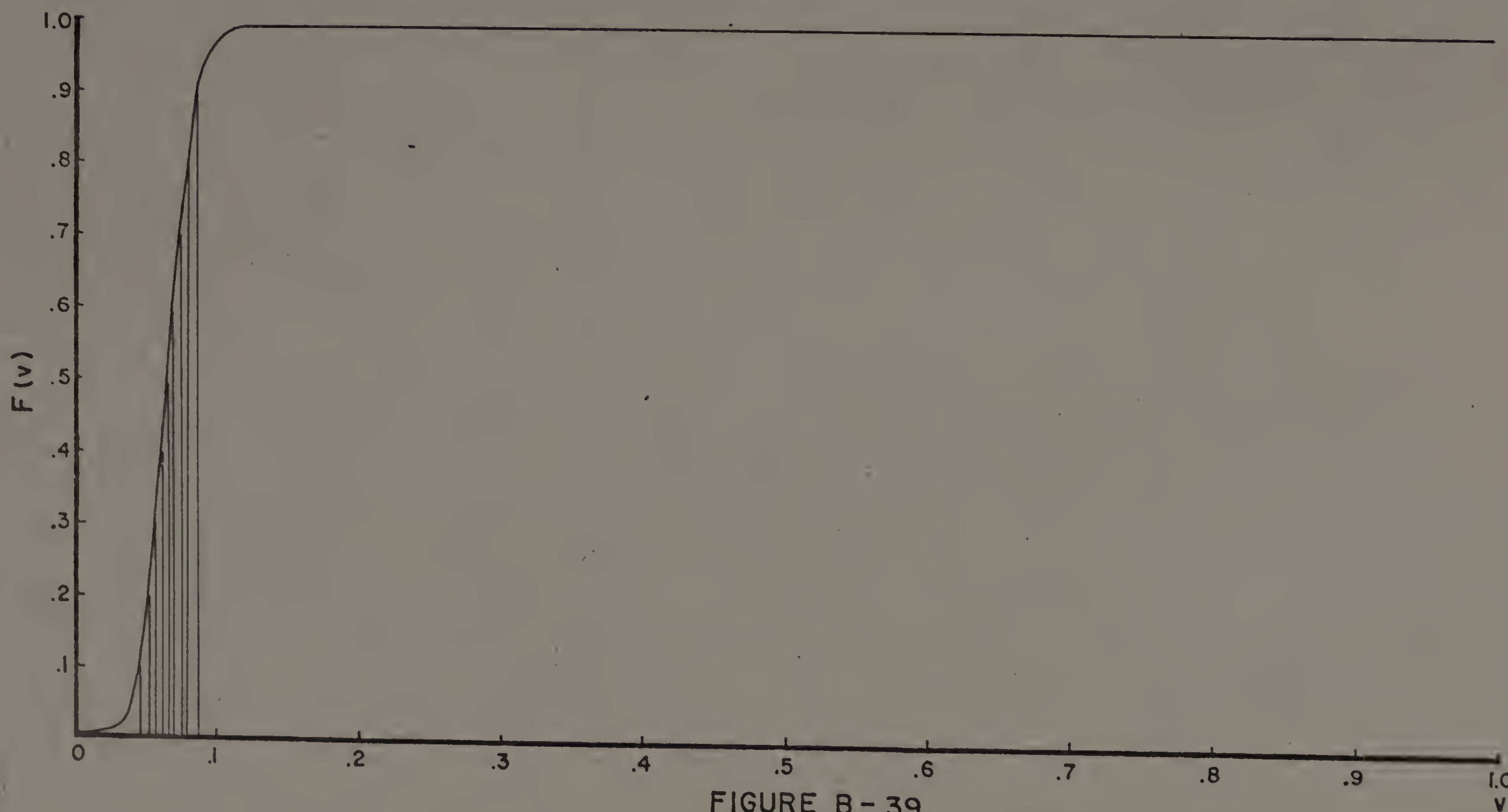

FIGURE B-39

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 8

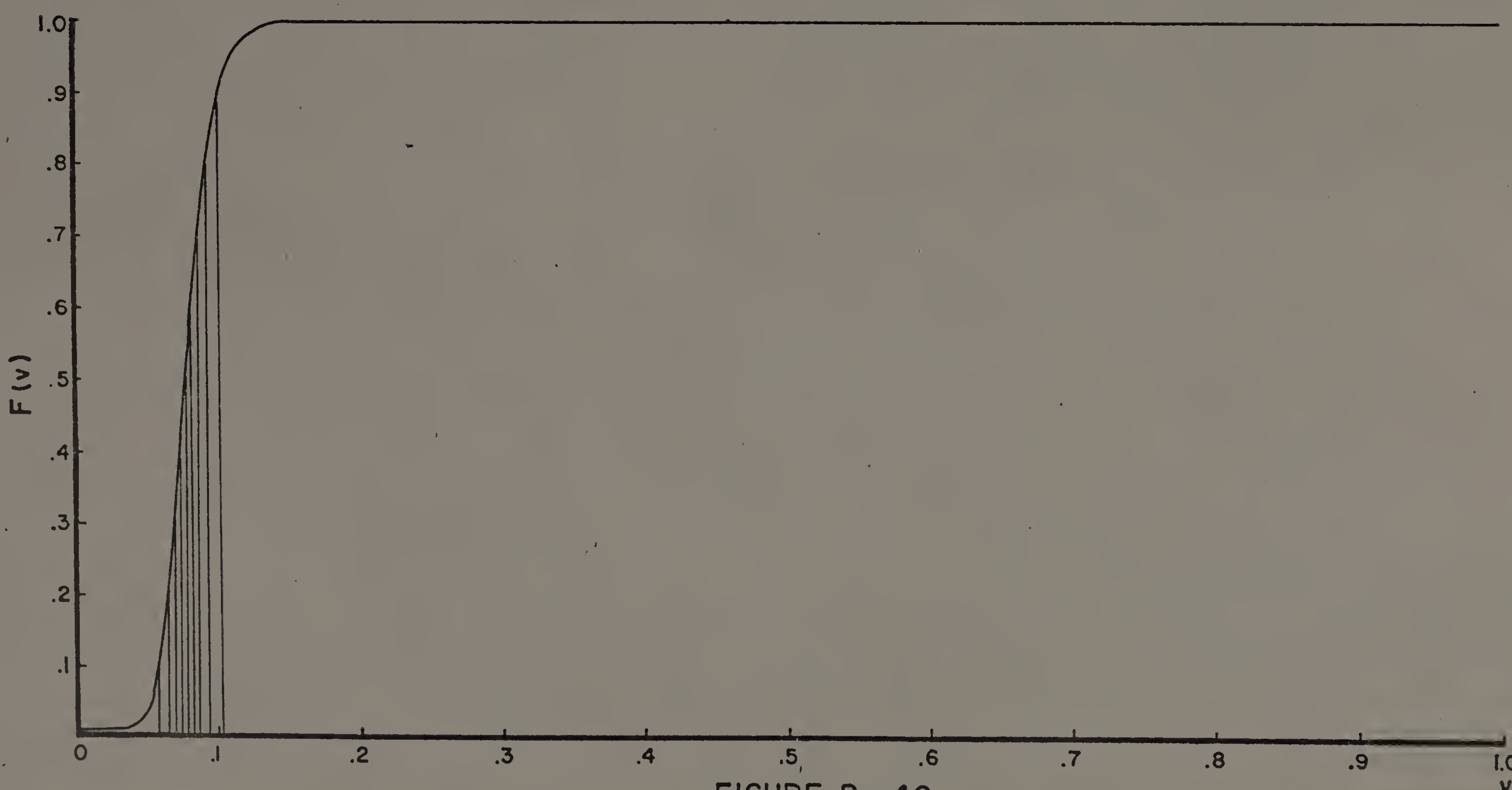

FIGURE B-40

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 8

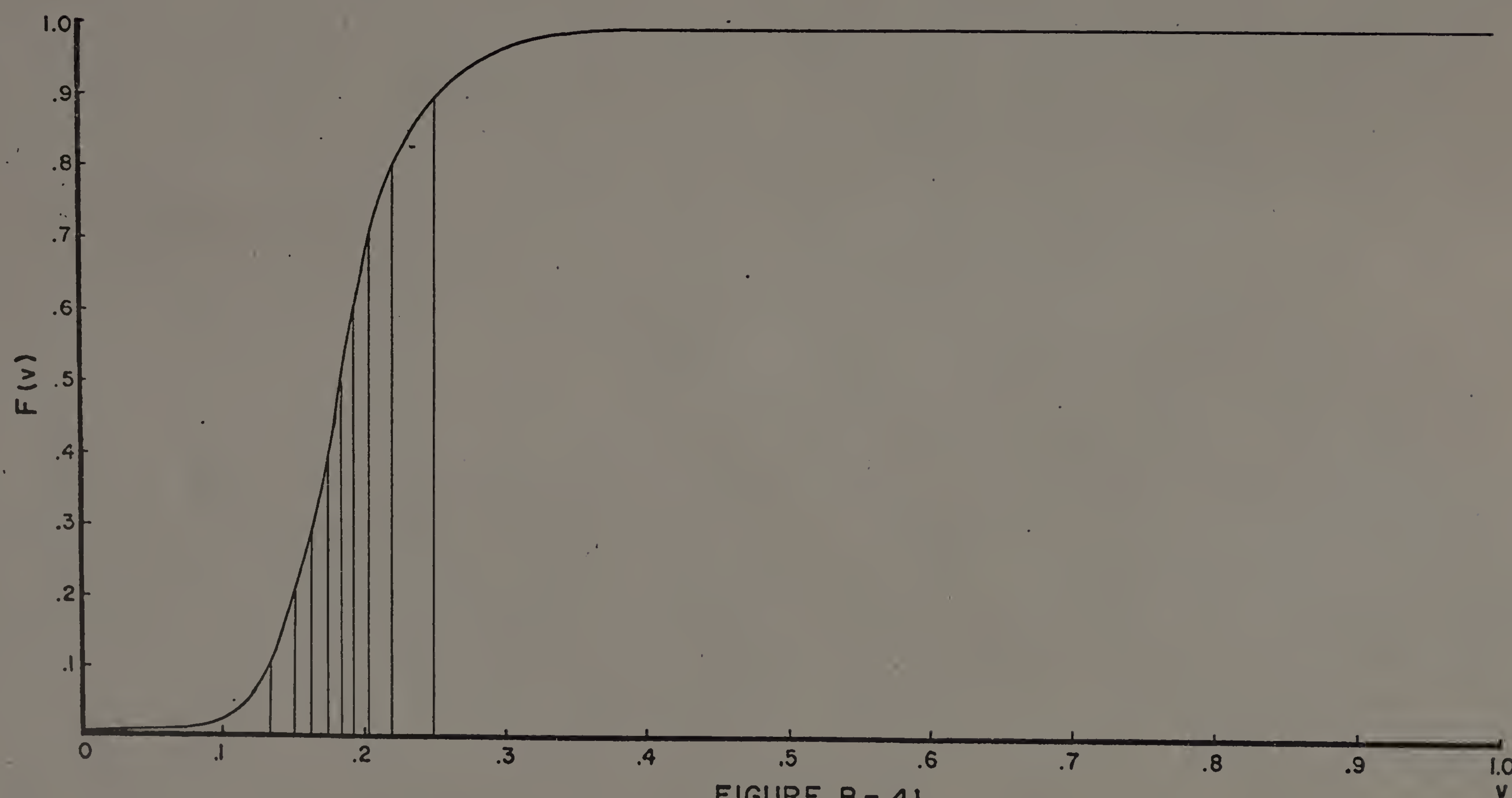

FIGURE B-41

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.111$
EXPERIMENT NO. 9

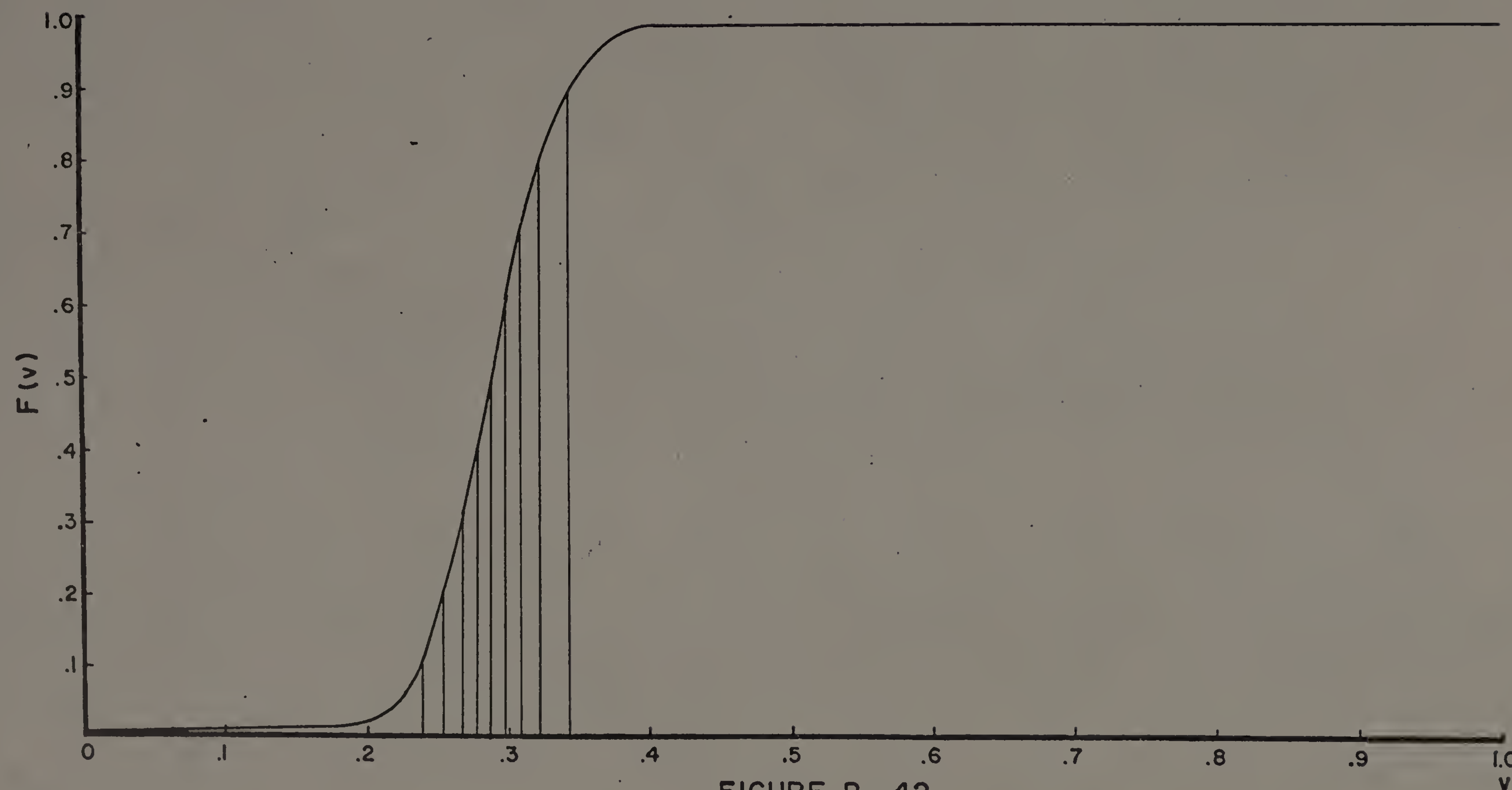

FIGURE B-42

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 9

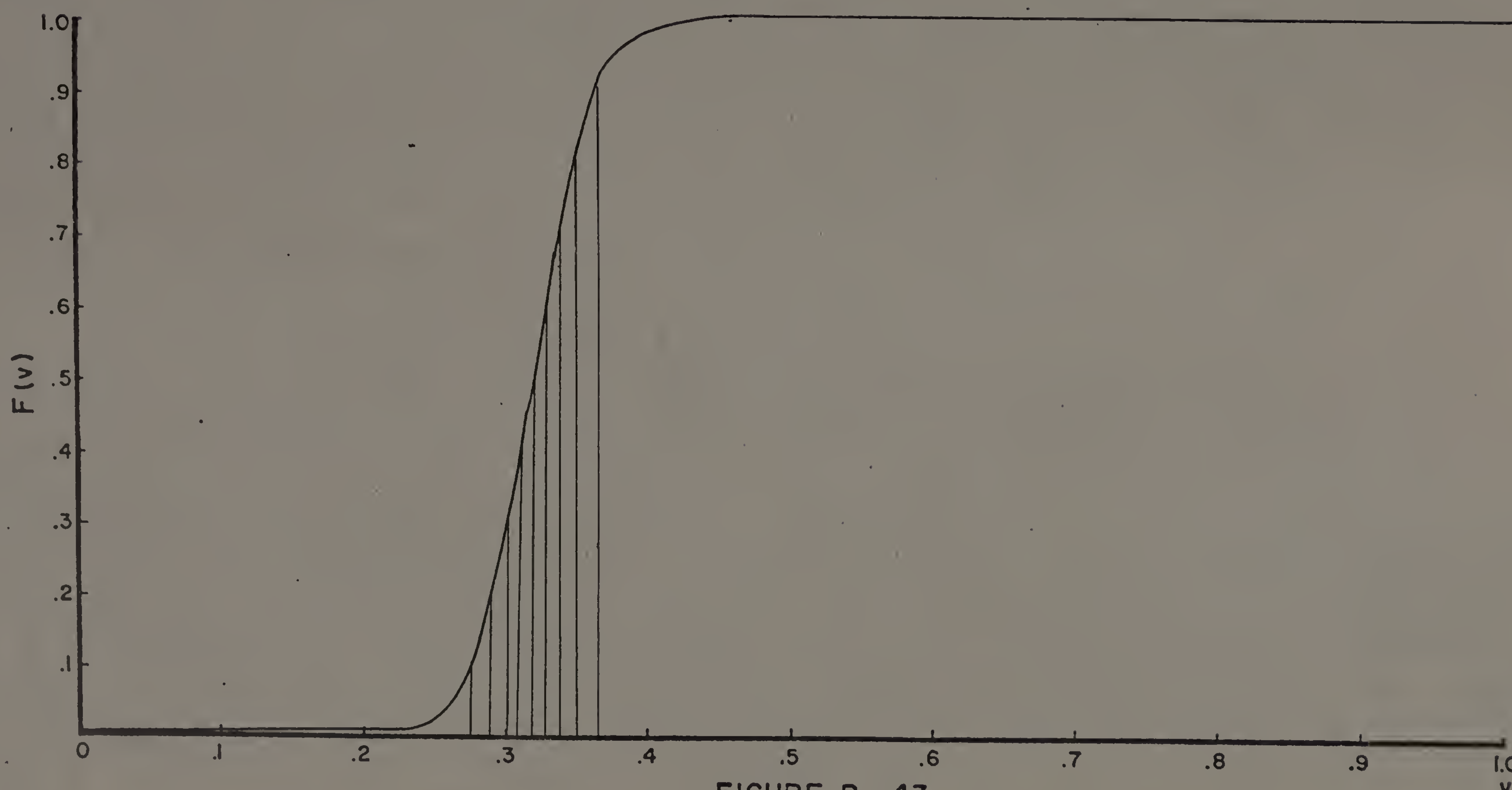

FIGURE B-43

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 9

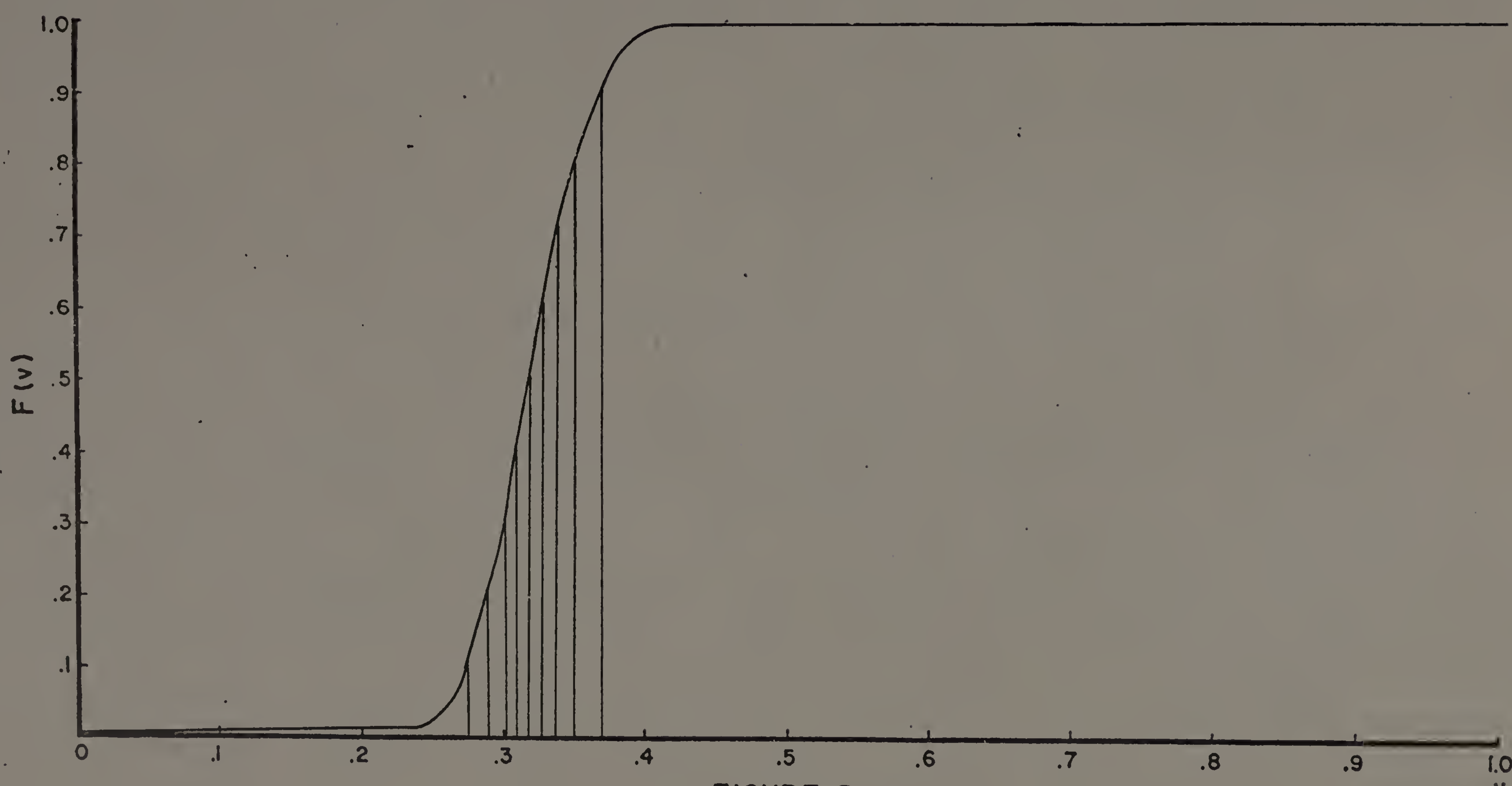

FIGURE B-44

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 9

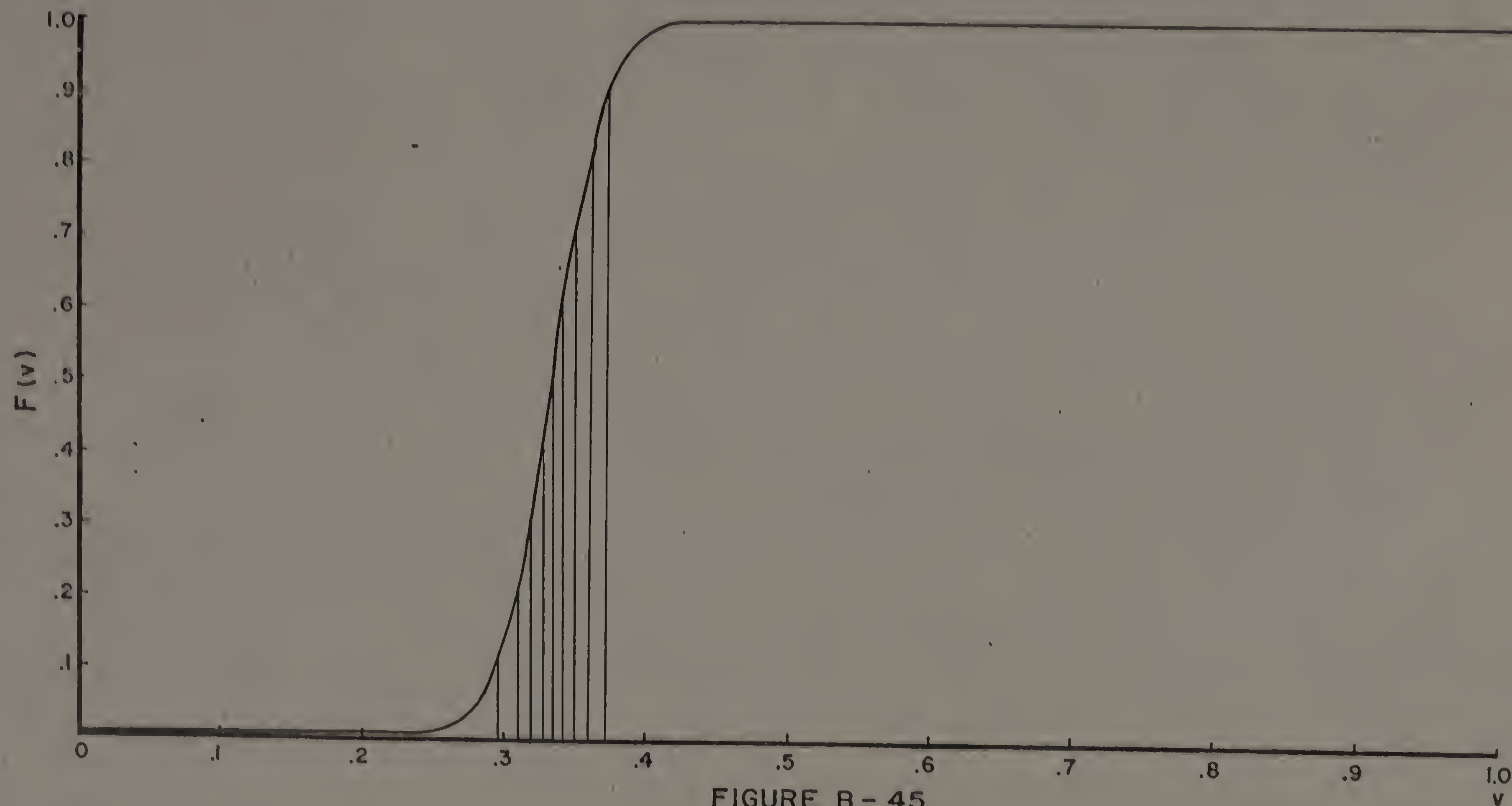

FIGURE B-45

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 9

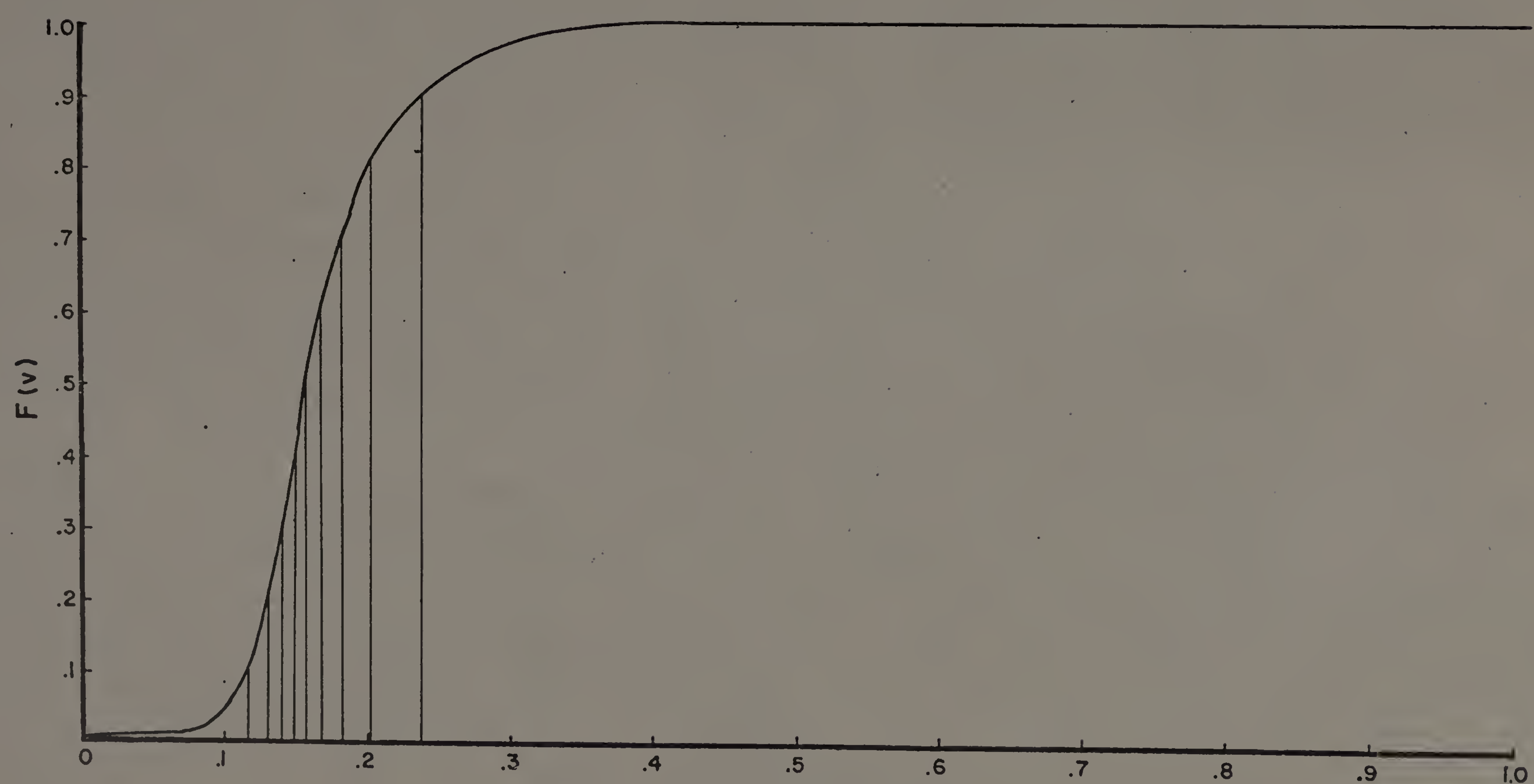

FIGURE B-46

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.2$
EXPERIMENT NO. 10

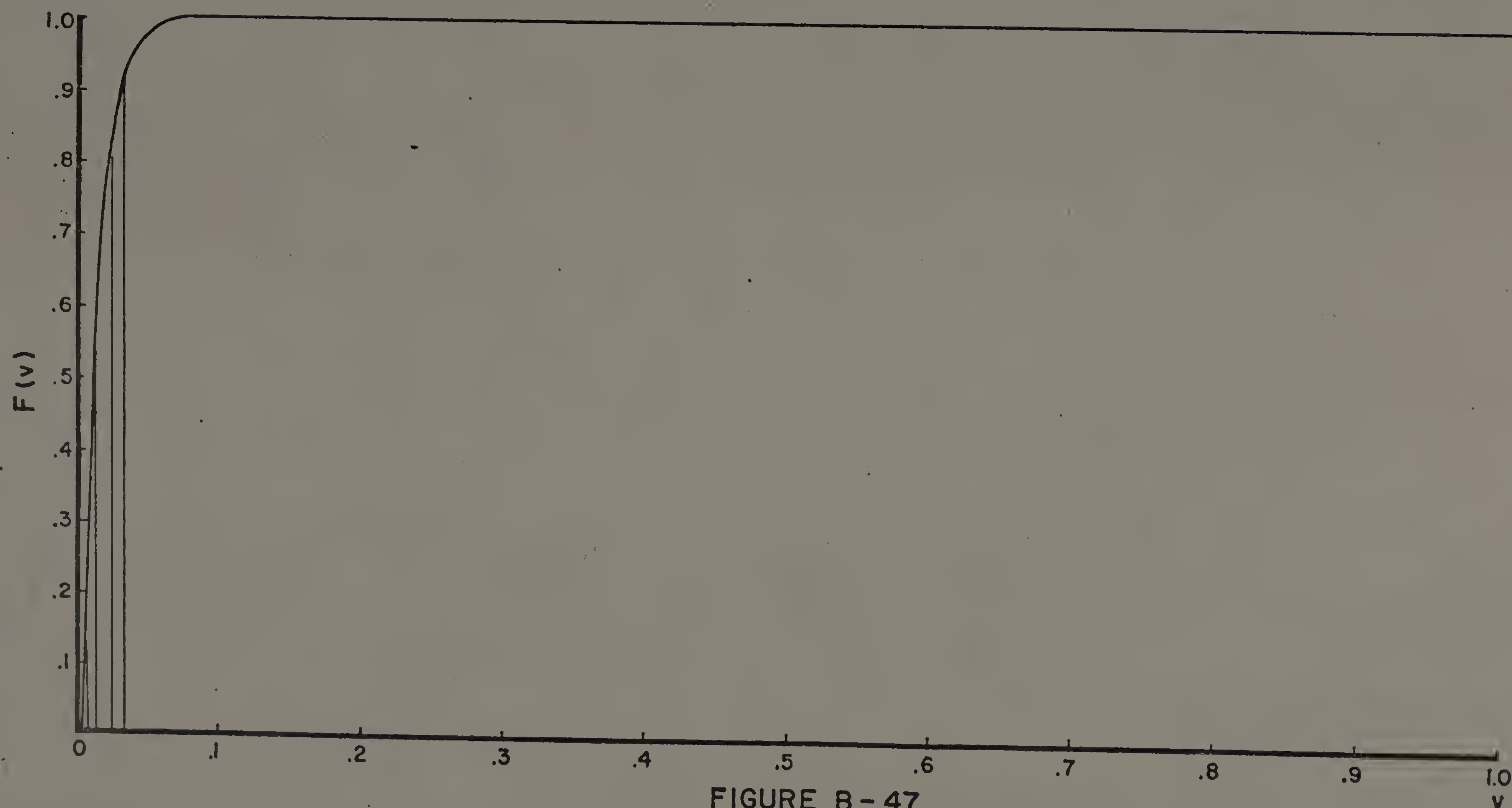

FIGURE B-47

SAMPLE DISTRIBUTION OF $v = (V(\bar{x}) - V_*)/(V^* - V_*)$
CONDITIONAL ON $E_j = 1.3$
EXPERIMENT NO. 10

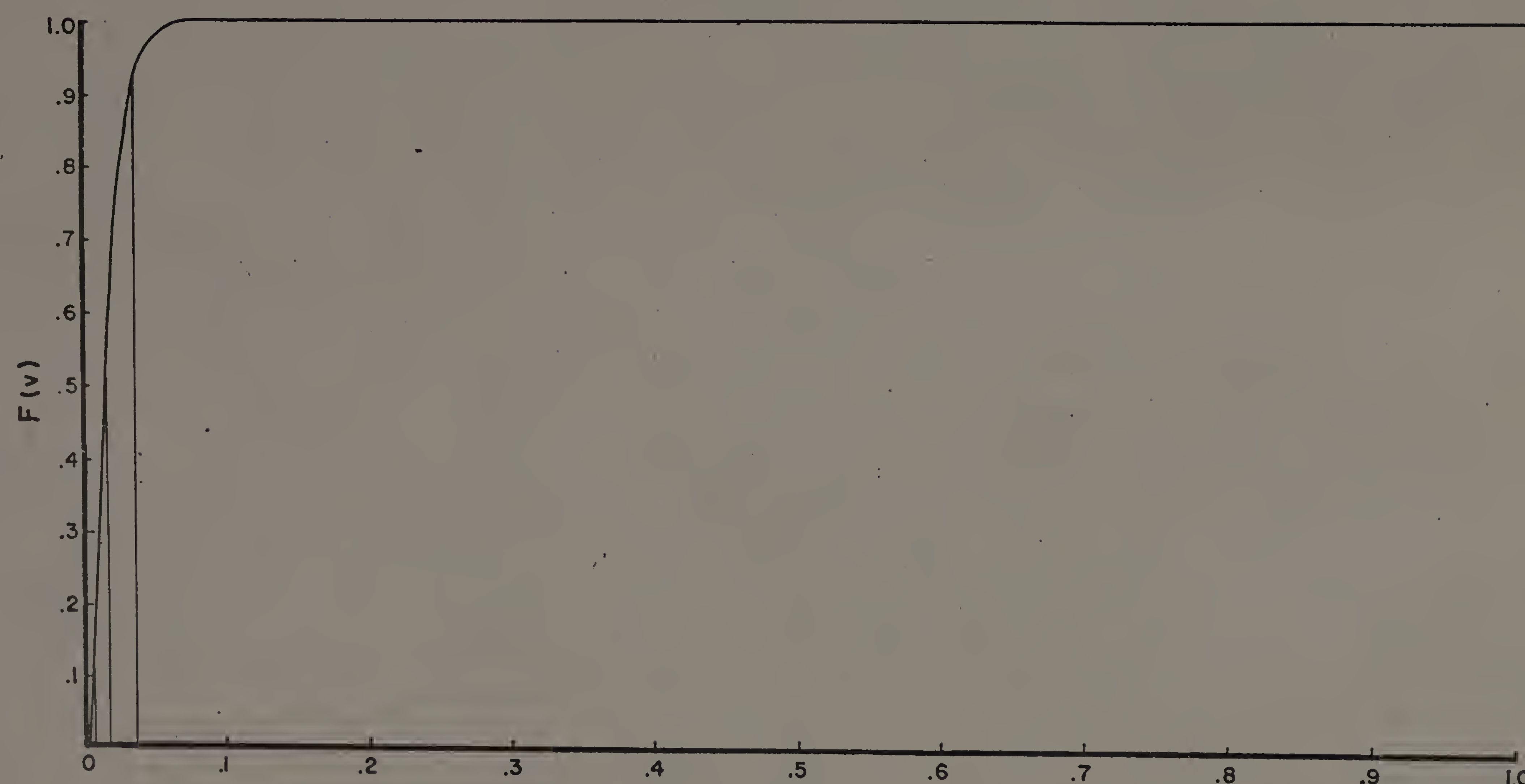

FIGURE B-48

SAMPLE DISTRIBUTION OF $v=(V(\tilde{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.4$
EXPERIMENT NO. 10

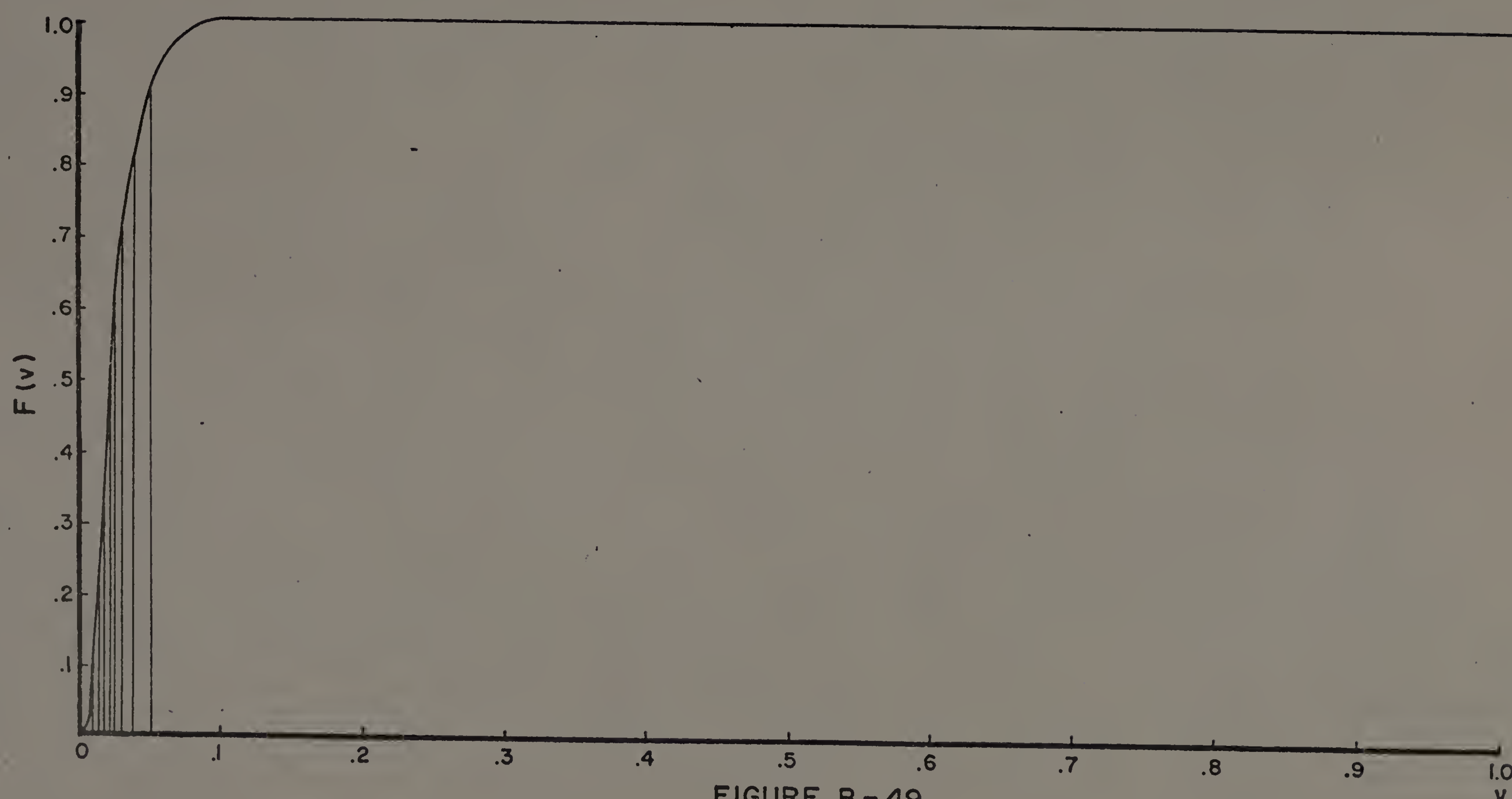

FIGURE B-49

SAMPLE DISTRIBUTION OF $v=(V(\bar{x})-V_*)/(V^*-V_*)$
CONDITIONAL ON $E_j = 1.5$
EXPERIMENT NO. 10

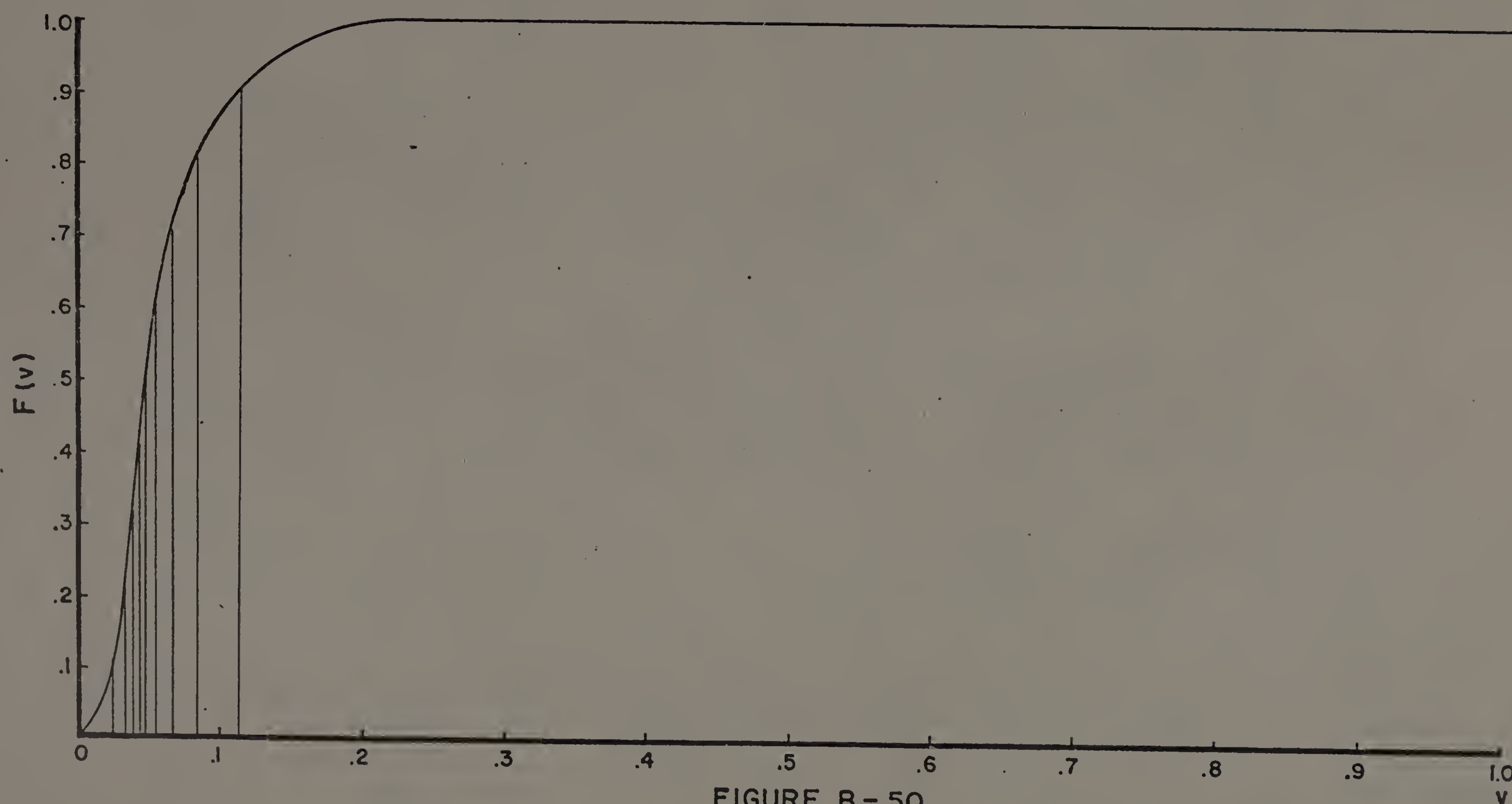

FIGURE B-50

APPENDIX C

Appendix C

Two programs were used for the generation of random portfolios and for the analysis of the distribution of their variances, conditional on a level of expectation E_j.

The first program is the program BASIX which identified all the basic feasible solutions for the convex region (3) using the heuristic procedure described in Chapter III. After identifying all k basic feasible solutions, program BASIX generates a sample of random portfolios according to the procedure described in (1) and (2) of Chapter III. This sample is stored in the disk file VECT for the analysis of the sample distribution of its corresponding variances.

The second program is the program VARAN which analyzes the distribution of variances corresponding to the sample of random vectors generated by the program BASIX. Program VARAN calculates the expectation of the conditional distribution of variances, the mean, variance, third moment and skewness coefficient of its sample distribution; and the corresponding values of these statistics for the sample distribution of the standardized variable v which is defined in Chapter IV. In addition, the program calculates the cumulative sample distribution for intervals of length equal to $.01(V^*-V_*)$. These approximations of the cumulative sample distribution were used for their graphic representation in Appendix B.

PROGRAM BASIX

```
10 REM PROGRAM  CALCULATES BASIC FEASIBLE SOLUTIONS
15 REM AND GENERATES RANDOM VECTORS X FOR REGION (3)
20 DIM E(20,1),R(100),W(1,100),X(100,20),Y(1,20)
21 RANDOMIZE
27  OPEN 5,"VECT"
29 OPEN 6,"BASE"
31 INPUT N1,U1
35 MAT READ E(N1,1)
40 FOR I=1 TO N1
45 IF E(I,1) < U1 THEN  60
50 IF E(I,1)=U1 THEN 75
55 GO TO 80
60 J1=I-1
65  N2=J1*(N1-J1)
70  I=N1
72 GO TO 80
75 J1=I-1
77 N2=0
79 I=N1
80 NEXT I
85 IF N2>0  THEN 150
90  N2=1+J1*(N1-J1-1)
95 K1=J1+2
100 IF K1>N1 THEN 140
102 IF J1=0 THEN 140
105 K2=J1+1
110 MAT X=ZER(N1,1)
115 X(K2,1)=1
120 MAT WRITE (6) X
121 MAT B=CON(2,1)
122 MAT A=CON(2,2)
123 B(1,1)=U1
125 GO TO 180
140 MAT  X=ZER(N1,1)
142  K2=J1+1
143 PRINT "UNIQUE SOLUTION"
144 X(K2,1)=1
145 MAT PRINT X
146 GO TO 999
150 K1=J1+1
151 MAT B=CON(2,1)
152 B(1,1)=U1
160 MAT A=CON(2,2)
180 FOR I=1 TO J1
```

```
185 FOR J=K1 TO N1
190 A(1,1)=E(I,1)
195 A(1,2)=E(J,1)
200 MAT C=INV(A)
205 MAT Y=C*B
210 MAT X=ZER(N1,1)
215 X(I,1)=Y(1,1)
220 X(J,1)=Y(2,1)
225 MAT WRITE (6) X
230 NEXT J
235 NEXT I
236 REWIND 6
237 MAT READ (6) X(N2,N1)
238 WRITE  (5) N2
240 MAT W=ZER(1,N2)
241 PRINT "INPUT SAMPLE SIZE"
242 INPUT K9
245 FOR I=1 TO K9
247 N5=0.
250 FOR J=1 TO N2
255 N6=RND(1)
260 R(J)=-LOG(N6)
265 N5=N5+R(J)
270 NEXT J
275 FOR J=1 TO N2
280 W(1,J)=R(J)/N5
285 NEXT J
290 MAT Y=W*X
295 MAT WRITE (5) Y
300 NEXT I
305 CLOSE 5
310 CLOSE 6
999 END
```

PROGRAM VARAN

```
10 REM PROGRAM ANALYSES DISTRIBUTION OF VARIANCES
15 DIM X(100,20),V(20,20),Y(1,20),D(100,1),Z(20,1),P(100,1)
18 DIM W(20,1)  ,T(1,100)
21 OPEN 4,"DIFL"
22 OPEN 5,"VECT"
25 OPEN  6,"BASE"
28 READ (4) N1,U1,M1
30 INPUT N1,U1,M1,M2,N5
33 M3=M1-M2
35 MAT READ (4) V(N1,1)
36 MAT READ (4) V(N1,N1)
40 READ (5) N2
60 MAT P=ZER(100,1)
61 M5=0
62 M4=0.
63 M6=0
65 FOR I=1 TO N5
70 MAT READ (5) Y(1,N1)
75 MAT Z=TRN(Y)
80 MAT W=Y*V
85 MAT A=W*Z
90 J=1+INT(100*(A(1,1)-M2)/M3)
91 IF J>=1 THEN 95
92 J=1
93 GO TO 105
95 IF J<=100 THEN 105
100 J=100
105 P(J,1)=1+P(J,1)
110 M5=M5+(A(1,1)↑2)
111  M4=M4+A(1,1)
112 M6=M6+(A(1,1)↑3)
120 NEXT I
122 M4=M4/N5
124 M6=M6-3*M5*M4+2*N5*(M4↑3)
125 M5=M5-N5*(M4↑2)
170 M5=M5/(N5-1)
175 M6=M6/(N5-1)
180 S1=M6/(M5↑(3/2))
185 MAT T=CON(1,N2)
190 MAT READ (6) A(N2,N1)
195 MAT Y=T*A
200 MAT Z=TRN(Y)
```

```
205 MAT W=Y*V
210 MAT A=W*Z
215 T1=A(1,1)
217 T2=0
220 REWIND 6
225 FOR I=1 TO N2
230 MAT READ (6) Y(1,N1)
235 MAT Z=TRN(Y)
240 MAT W=Y*V
245 MAT A=W*Z
250 T2=T2+A(1,1)
255 NEXT I
260 T1=(T1+T2)/(N2*(N2+1))
265 PRINT "FOR A SAMPLE OF SIZE=",N5
270 PRINT
275 PRINT "SUBJECT TO EXPECTATION=",U1
280 PRINT "MAXIMUM VARIANCE=",M1
290 PRINT "MINIMUM VARIANCE=",M2
295 PRINT
300 PRINT "SAMPLE MEAN=",M4,(M4-M2)/M3
305 PRINT "TRUE MEAN=",T1,(T1-M2)/M3
320 PRINT "SAMPLE VARIANCE=",M5,M5/(M3↑2)
325 PRINT "THIRD MOMENT=",M6,M6/(M3↑3)
330 PRINT "SKEWNESS=",S1
335 PRINT
345 PRINT "CUMULATIVE DISTRIBUTION AT .01 INTERVALS"
346 PRINT
350 MAT D=ZER(100,1)
360 D(1,1)=P(1,1)
362 PRINT (D(1,1)/N5)
365 FOR I=2 TO 100
380 J=I-1
385 D(I,1)=D(J,1)+P(I,1)
390 PRINT (D(I,1)/N5)
395 NEXT I
400 PRINT
408 CLOSE 4,"DIFL"
410 CLOSE 5
415 CLOSE 6
999 END
```

APPENDIX D

Program DISCR

This program is written in Extended Basic, compatible with the UMASS Time Sharing System at the University of Massachusetts.

The program selects at random a configuration from the table shown in Appendix F; and given the unordered configuration, it assigns randomly the weights on the configuration to the n investment prospects. This defines a random portfolio in the region spanning the range of feasible values for E_j. The random portfolios are stored in the disc file "EX7", for the analysis of the distributions.

Program DISCR

```
[illegible] =Y(U,V)
[illegible]
[illegible] FOR J=1 TO N
[illegible] IF X(J,I)>0 THEN [illegible]
[illegible] J0=J0+1
[illegible] IF [illegible] THEN [illegible]
[illegible] ( ,I)=J
[illegible] GO TO [illegible]
[illegible] NEXT [illegible]
[illegible]
[illegible] FOR J=1 TO [illegible]
[illegible] IF X(J,I)=0 THEN [illegible]
[illegible] (J,I)
[illegible] (J,I)=Y( ,I)
[illegible]
[illegible] [illegible] (0) [illegible]
[illegible]
[illegible]
[illegible]
```

Appendix E

Derivation of $V(V(\tilde{x}))$

The random variable $V(\tilde{x})$ was defined in Chapter II as

(1) $$V(\tilde{x}) = \sum_{i=1}^{k} \sum_{j=1}^{k} w_i w_j Q_{ij},$$

where $Q_{ij} = y_i' C y_j$.

The random variable $V(\tilde{x})^2$ is then

(2) $$V(\tilde{x})^2 = (\Sigma\Sigma w_i w_j Q_{ij})^2$$

$$= \sum_{i=1}^{k} w_i^4 Q_{ij}^2 + 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} w_i^3 w_j Q_{ij} Q_{ii}$$

$$+ \sum_{i=1}^{k} \sum_{j\neq}^{k} w_i^2 w_j^2 Q_{ii} Q_{jj} + 2 \sum_{i=1}^{k} \sum_{j\neq i}^{k} w_i^2 w_j^2 Q_{ij}^2$$

$$+ 2 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq i\neq j}^{k} w_i^2 w_j w_p Q_{jp} Q_{ii}$$

$$+ 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq i\neq j}^{k} w_i^2 w_j w_p Q_{ij} Q_{ip}$$

$$+ \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} \sum_{q\neq p,j,i}^{k} w_i w_j w_p w_q Q_{ij} Q_{pq}$$

From (14) in Chapter II it follows that

(3) $$E(w_i^4) = 4!(k-1!/(k+3)!$$

$$E(w_i^3 w_j) = 3!(k-1!/(k+3)!$$

$$E(w_i^2 w_j^2) = 2!2!(k-1)!/(k+3)!$$

$$E(w_i^2 w_j w_p) = 2!(k-1)!/(k+3)!$$

Then $E(V(\tilde{x})^2)$ may be expressed as

(4) $$E(V(\tilde{x})^2) = (k-1)!/(k+3)![24 \sum_{i=1}^{k} Q_{ii}^2 + 24 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}Q_{ii}$$

$$+ 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ii}Q_{jj} + 8 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}^2$$

$$+ 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j\neq i}^{k} Q_{jp}Q_{ii} + 8 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j\neq i}^{k} Q_{ij}Q_{ip}$$

$$+ \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} \sum_{q\neq p,j,i}^{k} Q_{ij}Q_{qp}].$$

The variance $V(E(\tilde{x}))$ corresponding to the vector $E(\tilde{x})$ is

(5) $$V(E(\tilde{x})) = \frac{1}{k^2} \sum_{i=1}^{k} \sum_{j=1}^{k} Q_{ij},$$

it follows from (5) by analogy with (2) that

(6) $$k^4 V(E(\tilde{x}))^2 = [\sum_{i=1}^{k} Q_{ii}^2 + 4 \sum_{i=1}^{k} \sum_{i\neq j}^{k} Q_{ij}Q_{ii}$$

$$+ \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ii}Q_{jj} + 2 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}^2$$

$$+ 2 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} Q_{jp}Q_{ii} + 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} Q_{ij}Q_{ip}$$

$$+ \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} \sum_{q\neq p,j,i}^{k} Q_{ij}Q_{pq}].$$

From the relations in (6) it is possible to rewrite (4) as

(7) $$E(V(\tilde{x})^2) = (k-1)!/(k+3)!\,[k^4 V(E(\tilde{x}))^2$$

$$+ 23 \sum_{i=1}^{k} Q_{ii}^2 + 20 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}Q_{ii}$$

$$+ 3 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ii}Q_{jj} + 6 \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}^2$$

$$+ 2 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j}^{k} Q_{jp}Q_{ii} + 4 \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} Q_{ij}Q_{ip}]$$

Define

(8) $$\overline{Q_{ii}^2} = \frac{1}{k} \sum_{i=1}^{k} Q_{ii}^2$$

(9) $$\overline{Q_{ij}Q_{ii}} = \frac{1}{k(k-1)} \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}Q_{ii}$$

(10) $$\overline{Q_{ij}^2} = \frac{1}{k(k-1)} \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ij}^2$$

(11) $$\overline{Q_{ii}Q_{jj}} = \frac{1}{k(k-1)} \sum_{i=1}^{k} \sum_{j\neq i}^{k} Q_{ii}Q_{jj}$$

(12) $$\overline{Q_{jp}Q_{ii}} = \frac{1}{k(k-1)(k-2)} \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} Q_{jp}Q_{ii}$$

(13) $$\overline{Q_{ij}Q_{ip}} = \frac{1}{k(k-1)(k-2)} \sum_{i=1}^{k} \sum_{j\neq i}^{k} \sum_{p\neq j,i}^{k} Q_{ij}Q_{ip}$$

It follows from (7) and (8) through (13) that

(14) $$E(V(\tilde{x})^2) = [k^3/(k+3)(k+2)(k+1]V(E(\tilde{x}))^2$$

$$+ [23\, \overline{Q_{ii}^2}/(k+3)(k+2)(k+1)]$$

$$+ [(k-1)(20\overline{Q_{ij}Q_{ii}}+6\overline{Q_{ij}^2}+3\overline{Q_{ii}Q_{jj}})/(k+3)(k+2)(k+1)]$$

$$+ [(k-1(k-2)(2\overline{Q_{jp}Q_{ii}}+4\overline{Q_{ij}Q_{ip}})/(k+3)(k+2)(k+1)]$$

From (14) it can be observed that'

(15) $$\lim_{k\to\infty} E(V(\tilde{x})^2) = V(E(\tilde{x}))^2$$

The variance of $V(\tilde{x})$ is

(16) $$V(V(\tilde{x})) = E(V(\tilde{x})^2) - E(V(\tilde{x}))^2$$

$$= E(V(\tilde{x})^2)-(k^2/(k+1)^2)\ V(E(x))^2-(2k/(k+1^2)\bar{V}V(E(\tilde{x}))$$

$$- (1/(k+1)^2)\bar{V}^2$$

From (15) and (16) it can be observed that

(17) $$\lim_{k\to\infty} V(V(\tilde{x})) = \operatorname*{Lim}_{k\to\infty} E(V(\tilde{x})^2) - \lim_{k\to\infty} E(V(\tilde{x}))^2$$

$$= V(E(\tilde{x}))^2 - V(E(\tilde{x}))^2$$

$$= 0^+$$

To illustrate the effect, assume that $k=10^3$, then

(18) $$V(V(\tilde{x})) = [k^3/(k+3)(k+2)(k+1)]V(E(x))^2$$

$$+[(k-1)(k-2)/(k+3)(k+2)(k+1)](2\overline{Q_{jp}Q_{ii}}+4\overline{Q_{ij}Q_{ip}})$$

$$-(k/k+1)^2V(E9x))^2-[2k/(k+1)^2]\bar{V}V(E(\tilde{x}))$$

$$+ \quad (10^{-6})$$

where $o(10^{-6})$ means terms of order of magnitude less than 10^{-6}.

It follows that for a large k

(19) $$V(V(\tilde{x}))<[(k-1)(k-])/(k+3)(k+2)(k+1)](2\overline{Q_{jp}Q_{ii}}+4\overline{Q_{ij}Q_{ip}})$$

$$- [2k/(k+1)^2]\bar{V}V(E(\tilde{x})) + o(10^{-6})$$

For $k = 10^3$

$$V(V(\tilde{x}))<10^{-3}[(2\overline{Q_{jp}Q_{ii}} + 4\overline{Q_{ij}Q_{ip}}) - 2\bar{V}V(E(\tilde{x}))] + o(10^{-6})$$

If the term in the brackets is negative, the variance is of order of magnitude less than 10^{-6}; and in any event very small since $\overline{Q_{jp}Q_{ii}}$, $\overline{Q_{ij}Q_{ip}}$ are of equal or smaller order than $\bar{\nabla}V(E(\tilde{x}))$

Appendix F

TABLE F-1

Feasible Unordered Configurations in the Discrete Model For Random Generation of Portfolios

.9,.1	.4,.4,.1,.1
.8,.2	.4,.3,.1,.1,.1
.8,.1,.1	.4,.2,.2,.1,.1
.7,.3	.4,.2,.1,.1,.1,.1
.7,.2,.1	.4,.1,.1,.1,.1,.1,.1
.7,.1,.1,.1	.3,.3,.3,.1
.6,.4	.3,.3,.2,.1,.1
.6,.3,.1	.3,.3,.1,.1,.1,.1
.6,.2,.2	.3,.2,.1,.1,.1,.1,.1
.6,.2,.1,.1	.3,.1,.1,.1,.1,.1,.1,.1
.6,.1,.1,.1,.1	.2,.2,.2,.2,.2
.5,.5	.2,.2,.2,.2,.1,.1
.5,.4,.1	.2,.2,.2,.1,.1,.1,.1
.5,.3,.2	.2,.2,.1,.1,.1,.1,.1,.1
.5,.3,.1,.1	.2,.1,.1,.1,.1,.1,.1,.1,.1
.5,.2,.2,.1	.1,.1,.1,.1,.1,.1,.1,.1,.1,.1
.5,.2,.1,.1,.1	
.2,.1,.1,.1,.1,.1	
.4,.4,.2	
.4,.3,.3	
.4,.2,.2,.2	
.4,.3,.2,.1	