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LINEAR STABILITY OF RADIALLY SYMMETRIC EQUILIBRIUM SOLUTIONS TO THE SINGULAR LIMIT PROBLEM OF THREE-COMPONENT ACTIVATOR-INHIBITOR MODEL

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Abstract. We show linear stability or instability for radially symmetric equilibrium solutions to the system of interface equation and two parabolic equations arising in the singular limit of three-component activator-inhibitor models.

1. Introduction and statement of main results

We are interested in the system of equations

$$V_{\Gamma(t)} = W(v_1, v_2) - (N-1)\alpha H \qquad \text{on } \Gamma(t),\ t > 0, \tag{1.1}$$

$$\theta_1 \frac{\partial v_1}{\partial t} = \Delta v_1 + G_1^+(v_1)\chi_{\Omega^+(t)} + G_1^-(v_1)\chi_{\Omega^-(t)} \qquad \text{in } \mathbb{R}^N,\ t > 0, \tag{1.2}$$

$$\theta_2 \frac{\partial v_2}{\partial t} = \Delta v_2 + G_2^+(v_2)\chi_{\Omega^+(t)} + G_2^-(v_2)\chi_{\Omega^-(t)} \qquad \text{in } \mathbb{R}^N,\ t > 0. \tag{1.3}$$

Here $\Omega^+(t) \subset \mathbb{R}^N$ is a bounded domain, $\Gamma(t) = \partial\Omega^+(t)$ is an embedded surface called an interface, $\Omega^-(t) = \mathbb{R}^N \setminus \overline{\Omega^+(t)}$, H is the mean curvature at each point of $\Gamma(t)$, and $V_{\Gamma(t)}$ is the normal velocity of $\Gamma(t)$ in the direction of $\Omega^-(t)$. Furthermore, θ_1 and θ_2 are nonnegative constants, α is a positive constant, and χ_A denotes the characteristic function of a subset $A \subset \mathbb{R}^N$. Throughout this paper, we assume that $N \geq 2$. We make the following assumptions on $G_j^\pm$ and W.

- **(G)** $G_j^\pm \in C^1(\mathbb{R})$, $\frac{dG_j^\pm}{dv_j}(v_j) < 0$, and there exist $\underline{v}_j, \overline{v}_j$ such that $G_j^+(\overline{v}_j) = 0$, $G_j^-(\underline{v}_j) = 0$, where $-\infty < \underline{v}_j < \overline{v}_j < \infty$, for each $j = 1, 2$.
- **(W)** $W \in C^1(\mathbb{R}^2)$, $W_{v_1}(v_1, v_2) < 0$, and $W_{v_2}(v_1, v_2) < 0$.

A typical example satisfying the assumptions (G) and (W) is $G_j^\pm(v_j) = \pm 1 - v_j$, and $W(v_1, v_2) = -(av_1 + bv_2 + c)$, where a, b, c are constants with $a, b > 0$.

This problem (1.1), (1.2) and (1.3) can be derived formally by taking the singular limit of the following three-component activator-inhibitor model (or

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propagator-controller model):

$$\begin{cases} \dfrac{1}{\alpha}\dfrac{\partial u}{\partial t} = \Delta u + \dfrac{1}{\varepsilon^2}\left(u - u^3 + \dfrac{\sqrt{2}\varepsilon}{3\alpha}W(v_1, v_2)\right), \\ \theta_1 \dfrac{\partial v_1}{\partial t} = \Delta v_1 + f_1(u, v_1), \\ \theta_2 \dfrac{\partial v_2}{\partial t} = \Delta v_2 + f_2(u, v_2). \end{cases}$$

Here $f_j(u, v_j)$ is a function that is monotonically decreasing in v_j, and monotonically increasing in u, θ_1 and θ_2 are nonnegative constants, ε is a small parameter, and α is a given constant. When ε is sufficiently small, the phase domains $\{u \sim 1\}$ and $\{u \sim -1\}$ are formed, and the thin layered region appear between them. The internal transition layer has a width of order ε. The discontinuity surface, which is often called the sharp interface, appears in the limit $\varepsilon \to 0$. The evolution of the interface is governed by not only the inhibitors v_1 and v_2 but also its mean curvature.

Heijster and Sandstede [9] studied travelling spots that bifurcate from radially symmetric stationary spots of three-component FitzHugh–Nagumo system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon^2 \Delta u + u - u^3 - \varepsilon(av + bw + c), \\ \theta_1 \dfrac{\partial v}{\partial t} = \Delta v + u - v, \\ \theta_2 \dfrac{\partial w}{\partial t} = d^2 \Delta w + u - w. \end{cases} \tag{1.4}$$

It is suggested that the supercritical drift bifurcation does not occur in two-component FitzHugh–Nagumo system. The existence and stability of planar radially symmetric spots of (1.4) was studied in [8]. Taniguchi [7] studied the linear stability of spherical interfaces in an equilibrium ball in a two-phase boundary problem. Internal layered patterns and sharp interfaces arising in reaction-diffusion systems including two-component or three-component FitzHugh–Nagumo type have been studied extensively in recent years (see [1, 3, 4, 5, 6, 10] and references therein).

Radially symmetric equilibrium solutions. Denote by $\Gamma(R) = \{x \in \mathbb{R}^N : |x| = R\}$ the radially symmetric interface. To consider the radially symmetric stationary solutions to (1.1), (1.2) and (1.3), we define the following functions. For each $j = 1, 2$ and $R > 0$, let $V_j(r, R)$ be the unique solution to

$$\begin{cases} -\Delta_r v = G_j^+(v(r))\chi_{\{r<R\}} + G_j^-(v(r))\chi_{\{r>R\}}, \quad 0<r<\infty \\ v_r(0,R) = 0, \quad v(+\infty, R) = \underline{v}_j \end{cases} \tag{1.5}$$

where

$$\Delta_r := \partial_r^2 + \frac{N-1}{r}\partial_r$$

and $r = |x|$. It is known that for each $j = 1, 2$ the solution $V_j(r, R)$ satisfies $\frac{\partial V_j}{\partial r}(r, R) < 0$ for all $r > 0$. See [2]. We then define the functions $Z_j(R) := V_j(R, R)$ for $j = 1, 2$. Then define $h(R) := W(Z_1(R), Z_2(R))$ and

$$U(R) := h(R) - \frac{(N-1)\alpha}{R}.$$

We see that $U(R_0) = 0$ if and only if $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$ is a radially symmetric equilibrium solution to (1.1)–(1.3).

The linearized eigenvalue problem. Let $\Phi_n(\xi)$, $\xi \in S^{N-1}$ be any spherically harmonic function of degree n. Then

$$-\Delta_{S^{N-1}}\Phi_n = \kappa_n \Phi_n \quad \text{on } S^{N-1}, \tag{1.6}$$

where $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1} and $\kappa_n = n(n+N-2)$, $n = 0, 1, 2, \ldots$. Our linearized eigenvalue problem around the radially symmetric equilibriums is the following:

(1.7)

$$\begin{cases} \lambda_n = -\sum_{j=1}^{2} P_j(R_0)[\partial_r V_j(R_0, R_0) + z_{j,n}(R_0)] + \dfrac{\alpha(N-1-\kappa_n)}{R_0^2}, \\ \left(-\Delta_r + \dfrac{\kappa_n}{r^2} + g_j(r, R_0) + \theta_j \lambda_n\right) z_{j,n}(r) = Q_j(R_0)\delta_{R_0}(r) \quad (j = 1, 2) \end{cases}$$

for each $n = 0, 1, \ldots$, where $\delta_{R_0}(r)$ denotes the Dirac delta function concentrated at $r = R_0$, and

$$\begin{aligned} P_j(R) &:= -\frac{\partial W}{\partial v_j}(V_1(R,R), V_2(R,R)) > 0, \\ Q_j(R) &:= G_j^+(V_j(R,R)) - G_j^-(V_j(R,R)) > 0, \\ g_j(r,R) &:= -\frac{dG_j^+}{dv_j}(V_j(r,R))\chi_{\{r<R\}} - \frac{dG_j^-}{dv_j}(V_j(r,R))\chi_{\{r>R\}} > 0 \end{aligned}$$

for each $j = 1, 2$ and $R > 0$. When $(\lambda_n, z_{1,n}, z_{2,n})$ solves (1.7), we call λ_n an eigenvalue of mode n. We can approximate the solutions near $(\Gamma(R_0),$

$V_1(r, R_0)$, $V_2(r, R_0)$) as in

$$\Gamma(t) = \{[R_0 + \eta\rho(\xi)e^{\lambda t}]\xi + O(\eta^2) : \ \xi \in S^{N-1}\},$$
$$v_1(x,t) = V_1(r, R_0) + \eta w_1(x)e^{\lambda t} + O(\eta^2), \tag{1.8}$$
$$v_2(x,t) = V_2(r, R_0) + \eta w_2(x)e^{\lambda t} + O(\eta^2)$$

with a small parameter η, $\lambda = \lambda_n$ and

$$(\rho(\xi), w_1(x), w_2(x)) = (\Phi_n(\xi), z_{1,n}(r)\Phi_n(\xi), z_{2,n}(r)\Phi_n(\xi)).$$

See Appendix A for the derivation of this eigenvalue problem.

To state our main results, we define a function

$$f(R) = -\frac{(N+1)h(R)}{(N-1)R} + \sum_{j=1}^{2} P_j(R)Q_j(R)[\phi_{j,1}(R,R) - \phi_{j,2}(R,R)], \tag{1.9}$$

where $\phi_{j,1}(r, R)$ $(j = 1, 2)$ is the unique solution to the equation

$$\begin{cases} (-\Delta_r + \frac{N-1}{r^2} + g_j(r,R))\phi = \delta_R, \\ \phi(\infty, R) = 0, \quad \phi_r(0, R) = 0, \end{cases} \tag{1.10}$$

for $R > 0$, and $\phi_{j,2}(r, R)$ $(j = 1, 2)$ is the unique solution to the equation

$$\begin{cases} (-\Delta_r + \frac{2N}{r^2} + g_j(r,R))\phi = \delta_R, \\ \phi(\infty, R) = 0, \quad \phi_r(0, R) = 0, \end{cases} \tag{1.11}$$

for $R > 0$. Let $\phi_{j,0}(r)$ $(j = 1, 2)$ be the unique solution to the equation

$$\begin{cases} (-\Delta_r + g_j(r,R_0))\phi = \delta_{R_0}, \\ \phi(\infty, R_0) = 0, \quad \phi_r(0, R_0) = 0, \end{cases} \tag{1.12}$$

where $R_0 > 0$ is a solution to $U(R_0) = 0$. Our main result gives criteria for the stability of equilibrium solutions.

Theorem 1.1. *Suppose that $R_0 > 0$ satisfies $U(R_0) = 0$. Then* (1.1)–(1.3) *has an equilibrium solution* $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$. *Suppose also that $\theta_j \geq 0$ $(j = 1, 2)$ satisfies*

$$\frac{1}{R_0^{N-1}} \sum_{j=1}^{2} P_j(R_0)Q_j(R_0)\left(\int_0^\infty r^{N-1}|\phi_{j,0}|^2\, dr\right)\theta_j < 1. \tag{1.13}$$

Then we have the following:

(1) *If $U'(R_0) < 0$ and $f(R_0) < 0$, then the equilibrium solution* $(\Gamma(R_0)$, $V_1(r, R_0)$, $V_2(r, R_0))$ *to* (1.1)–(1.3) *is linearly stable.*
(2) *If either $U'(R_0) > 0$ or $f(R_0) > 0$, then the equilibrium solution* $(\Gamma(R_0)$, $V_1(r, R_0)$, $V_2(r, R_0))$ *to* (1.1)–(1.3) *is linearly unstable.*

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by using some of the results in [2] and [7]. In Section 3, we give an example of both stable and unstable radially symmetric equilibrium solutions.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We regard a radially symmetric function as a function of $r = |x|$. We define

$$C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N) := \{u \in C^\infty_0(\mathbb{R}^N) \mid u \text{ is a radially symmetric function}\}.$$

Let L^2_{rad} be the completion of $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2 := \int_0^\infty r^{N-1}|u|^2\,dr.$$

For each $\kappa \geq 0$, let $H^1_{\mathrm{rad},\kappa}$ be the completion of $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{H^1_{\mathrm{rad},\kappa}} := \int_0^\infty r^{N-1}(|u_r|^2 + |u|^2) + \kappa r^{N-3}|u|^2\,dr.$$

We regard $v \in L^2_{\mathrm{rad}}$ as an element of $(H^1_{\mathrm{rad},\kappa})'$ such that

$$\langle u, v\rangle = \int_0^\infty r^{N-1}u(r)\overline{v(r)}\,dr$$

for $u \in H^1_{\mathrm{rad},\kappa}$. For $j = 1, 2$, let $L_j(\kappa, \lambda)$ be a linear operator from $H^1_{\mathrm{rad},\kappa}$ to $(H^1_{\mathrm{rad},\kappa})'$ such that

$$\langle u, L_j v\rangle = \int_0^\infty \left[r^{N-1}\frac{du}{dr}\cdot\frac{d\overline{v}}{dr} + r^{N-3}\kappa u\overline{v} + r^{N-1}(g_j + \theta_j\overline{\lambda})u\overline{v}\right] dr \tag{2.1}$$

for all $u, v \in H^1_{\mathrm{rad},\kappa}$. For smooth v, we have

$$L_j(\kappa, \lambda)v = -\Delta_r v + \left(\frac{\kappa}{r^2} + g_j(r, R) + \theta_j\lambda\right)v. \tag{2.2}$$

For $j = 1, 2$, let $u_j(\cdot, \kappa, \lambda)$ be the unique solution to the equation

$$L_j(\kappa, \lambda)u_j = \delta_{R_0}, \quad u_j(\cdot, \kappa, \lambda) \in H^1_{\mathrm{rad},\kappa} \tag{2.3}$$

for $\kappa \geq 0$ and $\operatorname{Re}\lambda \geq 0$. Then for $j = 1, 2$, we have $\phi_{j,0}(r) = u_j(r, 0, 0)$, $\phi_{j,1}(r, R_0) = u_j(r, N-1, 0)$, and $\phi_{j,2}(r, R_0) = u_j(r, 2N, 0)$.

Let $R_0 > 0$ be a number such that $U(R_0) = 0$, and $(\Gamma(R_0)$, $V_1(r, R_0)$, $V_2(r, R_0))$ be the associated equilibrium solution. Assume that $(\lambda_n$, $z_{1,n}(r)$,

$z_{2,n}(r))$ solves (1.7). Since $u_j(r,\kappa,\lambda)$ $(j=1,2)$ satisfy the equation (2.3), we have $z_{j,n}(r)=Q_j u_j(r,\kappa_n,\lambda_n)$ for $j=1,2$. Hence, we obtain

$$-\sum_{j=1}^{2}(P_j\partial_r V_j(R_0,R_0)+P_jQ_ju_j(R_0,\kappa_n,\lambda_n))+\frac{\alpha(N-1-\kappa_n)}{R_0^2}-\lambda_n=0.$$

Now we define

(2.4)

$$F(\kappa,\lambda):=-\sum_{j=1}^{2}(P_j\partial_r V_j(R_0,R_0)+P_jQ_ju_j(R_0,\kappa,\lambda))+\frac{\alpha(N-1-\kappa)}{R_0^2}-\lambda,$$

and

$$E(\kappa):=-\sum_{j=1}^{2}(P_j\partial_r V_j(R_0,R_0)+P_jQ_ju_j(R_0,\kappa,0))+\frac{\alpha(N-1-\kappa)}{R_0^2}. \tag{2.5}$$

We have the following:

Lemma 1. *For every $\kappa>0$, there holds $E''(\kappa)<0$.*

Proof. From (2.5), we get

$$E'(\kappa)=-\frac{\alpha}{R_0^2}-\sum_{j=1}^{2}P_j(R_0)Q_j(R_0)\frac{\partial u_j}{\partial\kappa}(R_0,\kappa,0).$$

Therefore we have

$$E''(\kappa)=-\sum_{j=1}^{2}P_j(R_0)Q_j(R_0)\frac{\partial^2 u_j}{\partial\kappa^2}(R_0,\kappa,0). \tag{2.6}$$

Notice that

$$\begin{aligned}R_0^{N-1}\frac{\partial^2 u_j}{\partial\kappa^2}(R_0,\kappa,0)&=\left\langle\frac{\partial^2 u_j}{\partial\kappa^2}(\cdot,\kappa,0),\delta_{R_0}\right\rangle\\&=\left\langle\frac{\partial^2 u_j}{\partial\kappa^2}(\cdot,\kappa,0),L_j(\kappa,0)u_j(\cdot,\kappa,0)\right\rangle\\&=\left\langle L_j(\kappa,0)\frac{\partial^2 u_j}{\partial\kappa^2}(\cdot,\kappa,0),u_j(\cdot,\kappa,0)\right\rangle\end{aligned}$$

for $j=1,2$. Using

$$L_j(\kappa,0)\left(\frac{\partial^2 u_j}{\partial\kappa^2}(\cdot,\kappa,0)\right)=-\frac{2}{r^2}\frac{\partial u_j}{\partial\kappa}(\cdot,\kappa,0)$$

and

$$L_j(\kappa,0)\left(\frac{\partial u_j}{\partial\kappa}(\cdot,\kappa,0)\right)=-\frac{1}{r^2}u_j(\cdot,\kappa,0),$$

we compute

$$\begin{aligned}&\left\langle L_j(\kappa,0)\frac{\partial^2 u_j}{\partial\kappa^2}(\cdot,\kappa,0),u_j(\cdot,\kappa,0)\right\rangle\\&=2\left\langle\frac{\partial u_j}{\partial\kappa}(\cdot,\kappa,0),-\frac{1}{r^2}u_j(\cdot,\kappa,0)\right\rangle\\&=2\left\langle\frac{\partial u_j}{\partial\kappa}(\cdot,\kappa,0),L_j(\kappa,0)\frac{\partial u_j}{\partial\kappa}(\cdot,\kappa,0)\right\rangle>0\end{aligned}$$

for $j=1,2$. It follows from (2.6) that $E''(\kappa)<0$ as desired. □

Lemma 2. *For all* $\kappa\geq 0$ *and* $\operatorname{Re}\lambda\geq 0$, *there holds*

$$\|u_j(\cdot,\kappa,\lambda)\|^2\leq\|u_j(\cdot,0,0)\|^2. \tag{2.7}$$

Proof. Assume that $\lambda=\lambda^R+i\lambda^I$ is an eigenvalue of (1.7) with $\lambda_R\geq 0$. Differentiating (2.3) with respect to λ^R, we have

$$-\left(\frac{\partial u_j}{\partial\lambda^R}\right)''-\frac{N-1}{r}\left(\frac{\partial u_j}{\partial\lambda^R}\right)'+\frac{\kappa}{r^2}\frac{\partial u_j}{\partial\lambda^R}+g_j(r)\frac{\partial u_j}{\partial\lambda^R}+\theta_j u_j+\theta_j\lambda^R\frac{\partial u_j}{\partial\lambda^R}=0.$$

This implies that

$$L_j\left(\frac{\partial u_j}{\partial\lambda^R}\right)=-\theta_j u_j\quad(j=1,2). \tag{2.8}$$

Similarly we have

$$L_j\left(\frac{\partial u_j}{\partial\lambda^I}\right)=-i\theta_j u_j\quad(j=1,2). \tag{2.9}$$

Furthermore, we differentiate (2.3) with respect to κ, we obtain

$$L_j\left(\frac{\partial u_j}{\partial\kappa}\right)=-\frac{u_j}{r^2}\quad(j=1,2). \tag{2.10}$$

We show that for all $\lambda^I\neq 0$, there holds $\|u_j(\cdot,\kappa,\lambda^R+i\lambda^I)\|^2<\|u_j(\cdot,\kappa,\lambda^R)\|^2$ $(j=1,2)$. It follows from (2.9) that

$$\begin{aligned}\frac{\partial}{\partial\lambda^I}\|u_j(\cdot,\kappa,\lambda^R+i\lambda^I)\|^2&=2\operatorname{Re}\left\langle u_j,\frac{\partial u_j}{\partial\lambda^I}\right\rangle\\&=2\operatorname{Re}\left\langle\frac{i}{\theta_j}L_j\left(\frac{\partial u_j}{\partial\lambda^I}\right),\frac{\partial u_j}{\partial\lambda^I}\right\rangle\\&=-2\lambda^I\left\|\frac{\partial u_j}{\partial\lambda^I}\right\|^2\end{aligned}$$

for $j=1,2$. Therefore, $\|u_j(\cdot,\kappa,\lambda^R+i\lambda^I)\|^2<\|u_j(\cdot,\kappa,\lambda^R)\|^2$ $(j=1,2)$ for all $\lambda^I\neq 0$.

We consider the case of $\lambda^I = 0$. Since $R_0^{N-1} u_j(R_0, \kappa, \lambda^R) = \langle u_j, L_j u_j \rangle$, we find that $u_j(R_0, \kappa, \lambda^R) > 0$ holds for $j = 1, 2$. By $u_j(\infty, \kappa, \lambda^R) = 0$ and the maximum principle, we see that $u_j(r, \kappa, \lambda^R) > 0$ $(j = 1, 2)$ for all $r > 0$. It therefore follows from (2.8) and (2.10) that

$$L_j \left(\frac{\partial u_j}{\partial \lambda^R} \right) < 0, \quad L_j \left(\frac{\partial u_j}{\partial \kappa} \right) < 0$$

for $j = 1, 2$. By

$$\frac{\partial u_j}{\partial \lambda^R}(\infty, \kappa, \lambda^R) = \frac{\partial u_j}{\partial \kappa}(\infty, \kappa, \lambda^R) = 0$$

and the maximum principle, we obtain that

$$\frac{\partial u_j}{\partial \lambda^R}(r, \kappa, \lambda^R) < 0, \quad \frac{\partial u_j}{\partial \kappa}(r, \kappa, \lambda^R) < 0$$

for all $r > 0$ and $j = 1, 2$. Thus it follows from

$$\frac{\partial}{\partial \lambda^R} \|u_j(\cdot, \kappa, \lambda^R)\|^2 = 2 \left\langle u_j, \frac{\partial u_j}{\partial \lambda^R} \right\rangle,$$

$$\frac{\partial}{\partial \kappa} \|u_j(\cdot, \kappa, \lambda^R)\|^2 = 2 \left\langle u_j, \frac{\partial u_j}{\partial \kappa} \right\rangle,$$

that $\frac{\partial}{\partial \lambda^R} \|u_j(\cdot, \kappa, \lambda^R)\|^2 < 0$ and $\frac{\partial}{\partial \kappa} \|u_j(\cdot, \kappa, \lambda^R)\|^2 < 0$ for $j = 1, 2$. We conclude that $\|u_j(\cdot, \kappa, \lambda)\|^2 \le \|u_j(\cdot, 0, 0)\|^2$ $(j = 1, 2)$ for all $\kappa \ge 0$ and $\operatorname{Re} \lambda \ge 0$. This completes the proof. □

We consider the equation

$$F(\kappa, \lambda) = 0, \quad \operatorname{Re} \lambda \ge 0 \tag{2.11}$$

for each $\kappa \ge 0$.

Lemma 3. *Assume that* (1.13). *Then any solution* λ *of* (2.11) *with a nonnegative real part must be real.*

Proof. For $j = 1, 2$, we write $u_j(r, \kappa, \lambda) = u_j^R + i u_j^I$ where both u_j^R and u_j^I are real. We can calculate $u_j(R_0, \kappa, \lambda)$ $(j = 1, 2)$ as

$$R_0^{N-1} u_j(R_0, \kappa, \lambda) = \langle u_j, \delta_{R_0} \rangle = \langle u_j, L_j u_j \rangle. \tag{2.12}$$

Taking the imaginary part, we have $R_0^{N-1} u_j^I(R_0, \kappa, \lambda) = -\lambda^I \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2$ for $j = 1, 2$. We obtain from $\operatorname{Im} F(\kappa, \lambda) = 0$ that

$$\lambda^I \left[\frac{1}{R_0^{N-1}} \sum_{j=1}^{2} P_j Q_j \theta_j \|u_j(\cdot, \kappa, \lambda)\|^2 - 1 \right] = 0.$$

It follows from (2.7) and (1.13) that

$$\frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,\kappa,\lambda)\|^2 - 1 \le \frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,0,0)\|^2 - 1 < 0.$$

This implies $\lambda^I = 0$. Therefore the eigenvalue λ must be real. This completes the proof. □

Lemma 4. *For all $\kappa \ge 0$ and $\lambda \in \mathbb{R}$ with $\lambda \ge 0$, there holds*

$$F_\lambda(\kappa,\lambda) \le \frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,0,0)\|^2 - 1.$$

Proof. Let $\kappa \ge 0$ and $\lambda \ge 0$. From (2.1) and (2.8), we get

$$R_0^{N-1}\frac{\partial u_j}{\partial\lambda}(R_0,\kappa,\lambda) = -\theta_j\|u_j\|^2$$

for $j = 1, 2$. It then follows that

$$\begin{aligned} F_\lambda(\kappa,\lambda) &= -1 - \sum_{j=1}^{2} P_j(R_0)Q_j(R_0)\frac{\partial u_j}{\partial\lambda}(R_0,\kappa,\lambda) \\ &= -1 + \frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,\kappa,\lambda)\|^2 \\ &\le \frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,0,0)\|^2 - 1 \end{aligned}$$

as desired. □

Proposition 5. *Assume that* (1.13) *holds. Then*

(1) (2.11) *has a unique solution $\lambda > 0$ if $E(\kappa) > 0$.*
(2) (2.11) *has a unique solution $\lambda = 0$ if $E(\kappa) = 0$.*
(3) (2.11) *has no solution if $E(\kappa) < 0$.*

Proof. Note that by Lemma 3, any solution λ of (2.11) with a nonnegative real part is real. By Lemma 4, we have

$$F(\kappa,\lambda) \le F(\kappa,0) - A\lambda = E(\kappa) - A\lambda \tag{2.13}$$

for $\lambda \ge 0$ with

$$A := 1 - \frac{1}{R_0^{N-1}}\sum_{j=1}^{2} P_jQ_j\theta_j\|u_j(\cdot,0,0)\|^2 > 0.$$

The claims (2) and (3) follow from (2.13).

(1) Let $E(\kappa) > 0$. Then it follows from (2.13) that $F(\kappa, 0) > 0 > F(\kappa, \lambda)$ if $\lambda > E(\kappa)/A$. Therefore by the monotonicity of $F(\kappa, \cdot)$ on $[0, \infty)$, there exists a unique $\lambda_* > 0$ such that $F(\kappa, \lambda_*) = 0$. This completes the proof of (1). □

In order to study the stability of $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$, we need to determine the sign of $E(0)$ and $E(2N)$.

Lemma 6. *Assume $U(R_0) = 0$. Then there holds*

$$E(0) = U'(R_0).$$

Proof. Differentiating $U(R)$ with respect to R, we get

$$U'(R) = \frac{\partial W}{\partial v_1} \cdot \frac{dV_1}{dR}(R, R) + \frac{\partial W}{\partial v_2} \cdot \frac{dV_2}{dR}(R, R) + \frac{(N-1)\alpha}{R^2}. \tag{2.14}$$

Differentiating (1.5) with respect to R, we have

$$(-\Delta_r + g_j(r, R)) \frac{\partial V_j}{\partial R} = [G_j^+(V_j(r, R)) - G_j^-(V_j(r, R))]\delta_R(r) \tag{2.15}$$

for $j = 1, 2$. Substituting $R = R_0$ into (2.15), we obtain

$$(-\Delta_r + g_j(R_0, R_0))\frac{\partial V_j}{\partial R}(R_0, R_0) = Q_j(R_0)\delta_{R_0},$$

and thus

$$\frac{\partial V_j}{\partial R}(R_0, R_0) = Q_j(R_0)u_j(R_0, 0, 0) \tag{2.16}$$

for $j = 1, 2$. For each $j = 1, 2$ and all $R > 0$, we have

$$\frac{dV_j}{dR}(R, R) = \frac{\partial V_j}{\partial r}(R, R) + \frac{\partial V_j}{\partial R}(R, R).$$

Therefore, substituting $R = R_0$ into (2.14) and using the equations (2.15) and (2.16), we obtain

$$U'(R_0) = -\sum_{j=1}^{2}(P_j\partial_r V_j(R_0, R_0) + P_jQ_ju_j(R_0, 0, 0)) + \frac{(N-1)\alpha}{R_0^2}.$$

By the definition of $E(\kappa)$, we obtain the desired relation. This completes the proof. □

Lemma 7. *Assume $U(R_0) = 0$. Then*

$$E(2N) = f(R_0).$$

Proof. Differentiating (1.5) with respect to r, we have

$$L_j(N-1,0)\left(\frac{\partial V_j}{\partial r}\right) = -Q_j(R)\delta_R$$

for $j = 1, 2$. Thus we find that

$$-\frac{\partial V_j}{\partial r}(r,R) = Q_j(R)\phi_{j,1}(r,R) \tag{2.17}$$

for $j = 1, 2$. From $U(R_0) = 0$, we get

$$\alpha = \frac{R_0 h(R_0)}{N-1}.$$

Therefore, by using the definition of $E(\kappa)$, we obtain $E(2N) = f(R_0)$, where f is defined as in (1.9). This completes the proof. □

Completion of Proof of Theorem 1.1.

Case 1: Assume that $U'(R_0) > 0$. Then this means that $E(0) > 0$ by Lemma 6. Hence there exits a positive eigenvalue $\lambda_0 > 0$ of mode 0 by Proposition 5 (1).

Case 2: Assume that $f(R_0) > 0$. Then $E(2N) > 0$ by Lemma 7. By Proposition 5 (1), we see that there exists a positive eigenvalue $\lambda_2 > 0$ of mode 2.

Case 3: Assume that $U'(R_0) < 0$ and $f(R_0) < 0$, then we have $E(0) < 0$ and $E(2N) < 0$. Note that we have

$$u_j(r, N-1, 0) = \phi_{j,1}(r, R_0) \tag{2.18}$$

for $j = 1, 2$. Substituting $\kappa = N - 1$ into (2.5) and using the equations (2.17) and (2.18), we get $E(N-1) = 0$. Combining this fact and $E''(\kappa) < 0$, we see that $E(\kappa) \leq E(2N) < 0$ for all $\kappa \in [2N, \infty)$. Therefore $E(\kappa_n) < 0$ for all $n \neq 1$. By Proposition 5 (3), we see that $\operatorname{Re}\lambda_n < 0$ for all $n \neq 1$. Moreover by Proposition 5 (2), there exists no eigenvalue λ_1 of mode 1 such that $\operatorname{Re}\lambda_1 \geq 0$ and $\lambda_1 \neq 0$.

This completes the proof of Theorem 1.1.

3. An Example

In this section, we present an example to illustrate the existence and the stability of equilibrium solutions. If θ_1 and θ_2 are sufficiently small, the stability of equilibriums is determined by the eigenvalues λ_0 and λ_2.

Example 1. *Let $N = 3$. Consider the following problem:*

(3.1)
$$\begin{aligned} V_{\Gamma(t)} &= -kv_1 - (1-k)v_2 - 2\alpha H \quad \textit{on } \Gamma(t),\ t>0,\\ \theta_1 \frac{\partial v_1}{\partial t} &= \Delta v_1 + (1 - b^2 v_1 + c)\chi_{\Omega^+} + (-1 - b^2 v_1 + c)\chi_{\Omega^-} \quad \textit{in } \mathbb{R}^3,\ t>0,\\ \theta_2 \frac{\partial v_2}{\partial t} &= \Delta v_2 + (1 - b^2 v_2 + c)\chi_{\Omega^+} + (-1 - b^2 v_2 + c)\chi_{\Omega^-} \quad \textit{in } \mathbb{R}^3,\ t>0. \end{aligned}$$

Here, $b \in (0,\infty)$, $k \in (0,1)$, $c = 1 - 2e^{-2} \approx 0.72933$ are constants, and $\alpha > 0$ is a parameter. Assume that θ_1 and θ_2 satisfy

$$\theta_1 \geq 0, \quad \theta_2 \geq 0, \quad k\theta_1 + (1-k)\theta_2 \leq 2b^3. \tag{3.2}$$

Let $G_j^{\pm}(v_j) = \pm 1 - b^2 v_j + c$ $(j = 1,2)$ and $W(v_1,v_2) = -kv_1 - (1-k)v_2$. Then $G_j^{\pm}$ and W satisfy all the assumptions (G) and (W) in Section 1 with $\underline{v}_j = -b^{-2}(1-c) < 0$, $\overline{v}_j = b^{-2}(1+c) > 0$. We use the same notations $h(R)$, $U(R)$, $P_j(R)$, $Q_j(R)$, $V_j(r,R)$, $g_j(r,R)$, $\phi_{j,1}(r,R)$, and $\phi_{j,2}(r,R)$ as in Section 1.

The radially symmetric stationary problem of (3.1) such that $v_j(x)$ has a finite limit as $|x| \to \infty$ is given by

$$h(R) = \frac{2\alpha}{R}, \tag{3.3}$$

$$\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr}\right) v_j = (1 - b^2 v_j + c)\chi_{\{r<R\}} + (-1 - b^2 v_j + c)\chi_{\{r>R\}}, \tag{3.4}$$

$$v_j'(0) = 0, \tag{3.5}$$

$$\lim_{r\to\infty} v_j(r) = -b^{-2}(1-c), \tag{3.6}$$

where $h(R) = -kv_1(R) - (1-k)v_2(R)$ and $j = 1,2$. The explicit solution $v_j(r) = V_j(r,R)$ of (3.4)–(3.6) is

$$v_j(r) = \begin{cases} b^{-2}[1 + c - 2(1+bR)e^{-bR}(br)^{-1}\sinh(br)] & \text{if } r < R,\\ 2b^{-2}[bR\cosh(bR) - \sinh(bR)](br)^{-1}e^{-br} - b^{-2}(1-c) & \text{if } r > R \end{cases}$$

for $j = 1,2$. Therefore we get

$$h(R) = b^{-2}\left[\frac{1}{bR} - \frac{e^{-2bR}}{bR} - e^{-2bR} - c\right]. \tag{3.7}$$

We find that $h'(R) < 0$ for $R > 0$, and $h(b^{-1}) = 0$. Hence $h(R) > 0$ for $R \in (0, b^{-1})$ and $h(R) < 0$ for $R \in (b^{-1}, \infty)$.

We consider the equation (3.3), that is, $U(R) = 0$. Now $U(R) = 0$ if and only if

$$\alpha = \frac{R}{2} b^{-2} \left[\frac{1}{bR} - \frac{e^{-2bR}}{bR} - e^{-2bR} - c \right] =: F_0(R).$$

Then we see that $F_0''(R) < 0$ for $R > 0$, $\lim_{R \to 0+} F_0(R) = 0$, and $F_0(b^{-1}) = 0$. Hence there exits a unique $R_* \in (0, b^{-1})$ such that $F_0'(R_*) = 0$. We have $F_0'(R) > 0$ for $R \in (0, R_*)$, and $F_0'(R) < 0$ for $R \in (R_*, \infty)$. Let $\alpha_1 = F_0(R_*) > 0$. Then we have the following:

- $U(R) = 0$ has two solutions $R = R_1(\alpha), R_2(\alpha)$ for each $\alpha \in (0, \alpha_1)$, where $0 < R_1(\alpha) < R_2(\alpha)$, $R_1(\alpha)$ is monotonically increasing, $R_2(\alpha)$ is monotonically decreasing in $(0, \alpha_1)$, $\lim_{\alpha \to 0+} R_2(\alpha) = b^{-1}$, and $\lim_{\alpha \to \alpha_1 -} R_2(\alpha) = \lim_{\alpha \to \alpha_1 -} R_1(\alpha) = R_*$. Moreover $U'(R_1(\alpha)) > 0$ and $U'(R_2(\alpha)) < 0$ for $\alpha \in (0, \alpha_1)$.
- $U(R) = 0$ has exactly one solution $R = R_*$, and $U'(R_*) = 0$ for $\alpha = \alpha_1$.
- $U(R) = 0$ has no solution for each $\alpha \in (\alpha_1, \infty)$.

Next we consider the linear stability of these equilibriums. Note that we have $P_1(R) = k, P_2(R) = 1 - k, Q_j(R) = 2$, and $g_j(r, R) \equiv b^2$. For $j = 1, 2$, let $u_j(r, R, \kappa_n)$ be the unique solution to

$$\mathcal{L}_j(\kappa_n) u(r) = \delta_R, \quad u \in H^1_{\mathrm{rad},\kappa}$$

where $R > 0$, $\kappa_n = n(n+1)$, and the operator $\mathcal{L}_j(\kappa)$ is defined as in

$$\mathcal{L}_j(\kappa) = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\kappa}{r^2} + b^2.$$

Then for $j = 1, 2$, $u_j(r, R, \kappa_n)$ can be expressed as

$$u_j(r, R, \kappa_n) = \begin{cases} R\sqrt{R/r}\, I_{n+\frac{1}{2}}(br) K_{n+\frac{1}{2}}(bR) & \text{if } r < R \\ R\sqrt{R/r}\, I_{n+\frac{1}{2}}(bR) K_{n+\frac{1}{2}}(br) & \text{if } r > R, \end{cases}$$

where $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ are the modified Bessel functions of the first kind and the second kind, respectively. Since $\phi_{j,1}(r, R) = u_j(r, R, \kappa_1)$ and $\phi_{j,2}(r, R) = u_j(r, R, \kappa_2)$, we have

$$\phi_{j,1}(R, R) = R I_{\frac{3}{2}}(bR) K_{\frac{3}{2}}(bR),$$
$$\phi_{j,2}(R, R) = R I_{\frac{5}{2}}(bR) K_{\frac{5}{2}}(bR)$$

for $j = 1, 2$. We set $s = bR$. By using the expression (1.9) of $f(R)$, we obtain

$$f(R) = 2R(I_{\frac{3}{2}}(s)K_{\frac{3}{2}}(s) - I_{\frac{5}{2}}(s)K_{\frac{5}{2}}(s)) - \frac{2h(R)}{R}. \tag{3.8}$$

By the elementary computation, we get

$$\begin{aligned} &I_{\frac{3}{2}}(s)K_{\frac{3}{2}}(s) - I_{\frac{5}{2}}(s)K_{\frac{5}{2}}(s) \\ &= \frac{1}{2s^5e^{2s}}\left[(2s^2-9)e^{2s} + 2s^4 + 8s^3 + 16s^2 + 18s + 9\right]. \end{aligned} \tag{3.9}$$

Substituting this relation and (3.7) into (3.8), we have

$$f(R) = \frac{R}{s^5e^{2s}}[(2cs^3-9)e^{2s} + 2s^4 + 10s^3 + 18s^2 + 18s + 9].$$

Now we define

$$\begin{aligned} F_1(s) &= (2cs^3-9)e^{2s} + 2s^4 + 10s^3 + 18s^2 + 18s + 9, \\ F_2(s) &= s^5e^{2s} > 0 \end{aligned}$$

for $s > 0$. By the assumption $0 < c < 1$, we find that $F_1(s)$ has a unique zero point $s_0 > 0$ with $F_1(s_0) = 0$, $F_1(s) < 0$ for $s \in (0, s_0)$, and $F_1(s) > 0$ for $s \in (s_0, \infty)$. Since $F_1(1) = 53 - 7e^2 \approx 1.2766 > 0$, we see that $s_0 < 1$. On the other hand, if $\alpha = \alpha_1$, then $R = R_*$ is an equilibrium such that $E(0) = U'(R_*) = 0$. Since $E''(\kappa)$ is negative and $E(N-1) = 0$, we see that $f(R_*) = E(2N)$ should be negative. Hence $bR_* < s_0$. Therefore there exists a unique $\alpha_2 \in (0, \alpha_1)$ such that $R_2(\alpha_2) = b^{-1}s_0$.

Let $R_0 > 0$ be an equilibrium solution. Note that $\phi_{j,0}(r)$ $(j = 1, 2)$ is given by

$$\phi_{j,0}(r) = \begin{cases} R_0\sqrt{R_0/r}\,I_{\frac{1}{2}}(br)K_{\frac{1}{2}}(bR_0) & \text{if } r < R_0 \\ R_0\sqrt{R_0/r}\,I_{\frac{1}{2}}(bR_0)K_{\frac{1}{2}}(br) & \text{if } r > R_0. \end{cases}$$

Then the condition (1.13) becomes

$$\begin{aligned} &\left(K_{\frac{1}{2}}(bR_0)^2\int_0^{R_0} rI_{\frac{1}{2}}(br)^2\,dr + I_{\frac{1}{2}}(bR_0)^2\int_{R_0}^{\infty} rK_{\frac{1}{2}}(br)^2\,dr\right) \\ &\times(k\theta_1 + (1-k)\theta_2) < \frac{1}{2R_0}, \end{aligned}$$

that is,

$$k\theta_1 + (1-k)\theta_2 < \frac{2b^3}{e^{-2bR_0}(e^{2bR_0} - 1 - 2bR_0)}.$$

Therefore under the condition (3.2), we have the following:

- $R_0 = R_2(\alpha)$ is linearly unstable for $\alpha \in (0, \alpha_2)$.

- $R_0 = R_2(\alpha)$ is linearly stable for $\alpha \in (\alpha_2, \alpha_1)$.
- $R_0 = R_1(\alpha)$ is linearly unstable for $\alpha \in (0, \alpha_1)$.

We remark that numerical computations show that

$$R_* \approx 0.508739 \cdot b^{-1}, \qquad \alpha_1 \approx 0.0417721 \cdot b^{-3},$$
$$s_0 \approx 0.808191, \qquad \alpha_2 \approx 0.0257134 \cdot b^{-3}.$$

Appendix A. Derivation of the linearized eigenvalue problem

To approximate solutions near the stationary solution $(\Gamma(R_0), V_1(r, R_0), V_2(r, R_0))$, set

$$\begin{aligned} \Gamma(t) &= \{[R_0 + \eta\rho(\xi)e^{\lambda t}]\xi + O(\eta^2) : \ \xi \in S^{N-1}\}, \\ v_1(x,t) &= V_1(r, R_0) + \eta w_1(x)e^{\lambda t} + O(\eta^2), \\ v_2(x,t) &= V_2(r, R_0) + \eta w_2(x)e^{\lambda t} + O(\eta^2) \end{aligned} \tag{A.1}$$

with small parameter η. Here $\lambda \in \mathbb{C}$, while $\rho(\xi)$ and $w_j(x)$ $(j = 1, 2)$ are real valued functions on S^{N-1} and $\mathbb{R}^N$, respectively.

By substituting (A.1) into (1.1), (1.2), and (1.3), dividing both sides by $\eta e^{\lambda t}$, and sending η to 0, we obtain

$$\begin{aligned} \lambda\rho(\xi) = &-\sum_{j=1}^{2} P_j(R_0)[V_j'(R_0, R_0)\rho(\xi) + w_j(R_0\xi)] \\ &+\frac{\alpha}{R_0^2}[(N-1)\rho(\xi) + \Delta_{S^{N-1}}\rho(\xi)]. \end{aligned} \tag{A.2}$$

$$(-\Delta + g_j(R_0, R_0) + \theta_j\lambda)w_j = \rho(\xi)Q_j(R_0)\delta_{R_0} \quad (j = 1, 2). \tag{A.3}$$

Here $\Delta_{S^{N-1}}$ denotes the Laplace–Beltrami operator on S^{N-1}.

Since the set $\{\Phi_n\}_{n=0}^{\infty}$ of the spherically harmonic functions is complete for the continuous functions on S^{N-1}, we can expand $\rho(\xi)$, $w_1(x)$, $w_2(x)$ in a Fourier series:

$$\rho(\xi) = \sum_{n=0}^{\infty} \rho_n \Phi_n(\xi), \qquad w_j(x) = \sum_{n=0}^{\infty} w_{j,n}(r)\Phi_n(\xi) \quad (j = 1, 2). \tag{A.4}$$

Then we have

$$\begin{aligned} \lambda\sum_{n=0}^{\infty} \rho_n\Phi_n(\xi) = &-\sum_{j=1}^{2}\sum_{n=0}^{\infty} P_j(R_0)[V_j'(R_0, R_0)\rho_n\Phi_n(\xi) + w_{j,n}(R_0)\Phi_n(\xi)] \\ &+\sum_{n=0}^{\infty} \frac{\alpha(N-1-\kappa_n)}{R_0^2}\rho_n\Phi_n(\xi), \end{aligned}$$

$$\sum_{n=0}^{\infty}\left(-\Delta_r+\frac{\kappa_n}{r^2}+g_j+\theta_j\lambda\right)w_{j,n}(r)\Phi_n(\xi)=\sum_{n=0}^{\infty}\rho_n Q_j(R_0)\Phi_n(\xi)\delta_{R_0}$$

for $j=1,2$. Therefore for each n,

$$\lambda\rho_n=-\sum_{j=1}^{2}P_j(R_0)[V_j'(R_0,R_0)\rho_n+w_{j,n}(R_0)] + \frac{\alpha(N-1-\kappa_n)}{R_0^2}\rho_n,$$

$$\left(-\Delta_r+\frac{\kappa_n}{r^2}+g_j+\theta_j\lambda\right)w_{j,n}(r)=\rho_n Q_j(R_0)\delta_{R_0}\quad (j=1,2).$$

If $\rho_n\neq 0$ for some n, then setting $z_{j,n}=\frac{w_{j,n}}{\rho_n}$ $(j=1,2)$, $(\lambda,z_{1,n},z_{2,n})$ solves

$$\lambda=-\sum_{j=1}^{2}P_j(R_0)[V_j'(R_0,R_0)+z_{j,n}(R_0)] + \frac{\alpha(N-1-\kappa_n)}{R_0^2},$$

$$\left(-\Delta_r+\frac{\kappa_n}{r^2}+g_j+\theta_j\lambda\right)z_{j,n}(r)=Q_j(R_0)\delta_{R_0}\quad (j=1,2).$$

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